PERCOLATION ON THE NON-P.C.F. SIERPIŃSKI GASKET AND HEXACARPET

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Abstract. We investigate bond percolation on the non-p.c.f. Sierpiński gasket and the hexacarpet. With the use of the diamond fractal, we are able to bound the critical probability of percolation on the non-p.c.f. gasket from above by $\frac{\sqrt{5}-1}{2}$, or approximately 0.618. We then show how the two fractals are related via the barycentric subdivisions of a triangle: the two spaces exhibit duality properties although they are not themselves dual spaces. Finally, we conjecture that the hexacarpet has a critical probability less than 1, which would imply that both the hexacarpet and non-p.c.f. gasket have non-trivial critical probabilities of percolation.

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Contents

1. Introduction

Suppose we immerse a porous stone in a bucket of water. What is the probability that the center of the stone gets wet? Broadbent and Hammersley originally constructed a percolation model to answer such a problem in [6]. In two dimensions, their model can be described as the following: Let $\mathbb{Z}^2$ be the planar square lattice and fix some parameter $p \in [0,1]$. For each edge in $\mathbb{Z}^2$, we set the edge to be open with probability p and closed with probability $1-p$, independent of all other edges. The edges in the lattice represent inner pathways within the stone, and the parameter p is the proportion of such pathways that allow water to pass. The stone would then be modeled by a finite, connected sub-graph of $\mathbb{Z}^2$; we say that a vertex x gets wet only if there is an open path connecting x to the boundary of the connected sub-graph. Such a process is known as *bond percolation*, and it is the main focus of this paper.

The problem of bond percolation on $\mathbb{Z}^2$ has been especially well studied. In [14], Kesten first proved that the critical probability of bond percolation on $\mathbb{Z}^2$ is equal to $\frac{1}{2}$. Although the exact values of the critical probabilities, denoted p_c, are not known for $\mathbb{Z}^d$, $d \geq 3$, it is known that $0 < p_c < \frac{1}{2}$. (See [10].)

A fractal is said to be *post-critically fininte*, or p.c.f., (in the sense of Kigami [15]) if the number of n^{th}-level cells that can intersect at any point is bounded. Several finitely ramified non-p.c.f. analogs of the Sierpiński gasket were first introduced in [24]; Bajorin et. al. [2] provided some analysis on the non-p.c.f. gasket. Begue et. al. [3] showed the relationship between the hexacarpet and the barycentric subdivisions of a triangle and provided a study of simple random walks on these spaces. The main goal of our paper is to show the relationship between the non-p.c.f. carpet and the hexacarpet, and to bound the critical probabilities of bond percolation on these spaces.

We now introduce some notation and definitions that will be needed throughout the paper. Consider some graph $G = (V, E)$ where V is its set of vertices and E is the set of edges in G. Then define an ambient probability space, Ω, and define a probability measure, P, on Ω by

$$\Omega = [0,1]^E,$$

$$dP = \prod_{i=1}^{\infty} d\lambda_{[0,1]},$$

where $d\lambda_{[0,1]}$ is Lebesgue measure on $[0,1]$. For each $e \in E$, we take $\omega(e) \sim U[0,1]$. We then define the set of all open edges in G given parameter p as

$$E_p = \{e \in E : \omega(e) \leq p\}.$$

We say that $e \in E$ is *open* if $e \in E_p$. The advantage of using this definition is that we can vary the parameter p without having to resample the environment.

Let $G_p = (V, E_p)$. Then an open cluster is a connected subgraph of G_p, and we denote the open, connected cluster containing zero as O_{G_p}.

Definition 1.1. *The percolation probability is defined as*

$$\theta_G(p) = P(|O_{G_p}| = \infty)$$

Definition 1.2. *The critical probability of percolation on a given graph G is given by*

$$p_G = \sup\{p : \theta_G(p) = 0\}$$

It is clear from the proceeding definitions that

$$\theta_G(p) \begin{cases} = 0 & \text{if } p < p_G, \\ > 0 & \text{if } p > p_G. \end{cases} \tag{1.1}$$

Note that we make no claim we regarding the value of θ_G at p_G. In general, the value of θ_G is unknown at the critical probability. We will make no further comment as this question is outside of the interests of the following. See [10] and references therein for further detail.

Remark 1.1. If H is a subgraph of G, then $O_{H_p} \subseteq O_{G_p}$ and so

$$\theta_G(p) = P(|O_{G_p}| = \infty) \geq P(|O_{H_p}| = \infty) = \theta_H(p)$$

for all p. Hence $p_G \leq p_H$. We will make use of this fact frequently.

The main result of our paper is:

Theorem 1.1. *Let S denote non-p.c.f. Sierpiński gasket, and let H denote the hexacarpet. Then $0 < p_S \leq \frac{\sqrt{5}-1}{2}$ and $\frac{3-\sqrt{5}}{2} \leq p_H < 1$.*

In Section 2, we introduce the diamond fractal and discuss the results of Hambly and Kumagai in [13]. In Section 3, we introduce the non-p.c.f. Sierpiński gasket and the barycentric subdivisions of a triangle. We provide an upper bound on the critical probability of percolation on the barycentric subdivisions of a triangle using an embedding argument and the diamond fractal; we then prove the relationship between the non-p.c.f. Sierpiński gasket and the barycentric subdivisions. We introduce the hexacarpet in Section 4, and we demonstrate the relationship between the non-p.c.f. gasket and the hexacarpet vis-à-vis the barycentric subdivisions of a triangle. Finally, the proof of Theorem 1.1 is provided in Section 5.

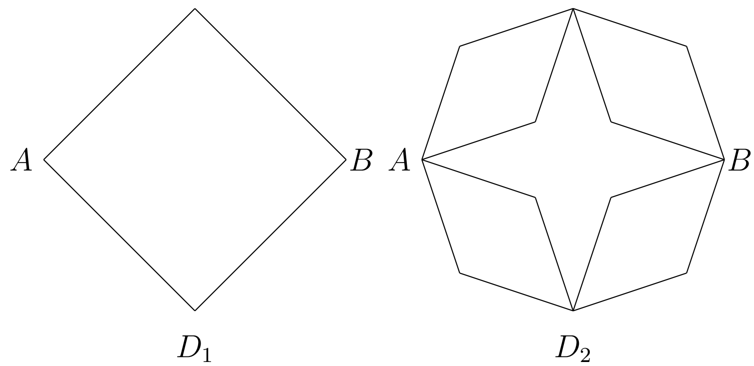

FIGURE 1. First and Second Graph Approximations of $D(2,2)$

2. Diamond Fractal

Percolation on the diamond fractal has already been studied by Hambly and Kumagai [13]. In this section, we will reproduce their results, as well as provide a more general derivation of the critical probability of percolation on the diamond fractal.

To construct the diamond fractal, begin with a line segment (A,B) denoted $D_0(m,n)$ where $m,n \geq 2$ denote the number of branches and edges, respectively. Then, the first approximation of the fractal, $D_1(m,n)$, is constructed by replacing (A,B) by m non-intersecting line segments that have been split into n sub-segments. More generally, $D_k(m,n)$ is generated by replacing each edge of $D_{k-1}(m,n)$ by a copy of $D_1(m,n)$. Figure 1 shows the first and second graph approximations of $D(2,2)$.

In this paper, we are interested mostly in the case $m = n = 2$, so when we write D_k, it should be understood to be the k^{th} graph approximation of $D(2,2)$.

Hambly and Kumagai first showed the critical probability of percolation on D to be

$$p_D = \frac{\sqrt{5}-1}{2} \tag{2.1}$$

(see [13]). Hambly and Kumagai only considered the case for $D(2,2)$, but the following method was implicit in their work and gives a general solution to finding the critical probability of percolation on $D(m,n)$ with $m,n \geq 2$.

Proposition 2.1. *The probability of not being able to cross $D_1(m,n)$ is given by*

$$P(\textit{no crossing}) = (1-p^n)^m,$$

where $p \in [0,1]$ is the probability of an edge being open and a crossing occurs if there is an open cluster containing both A and B as in Figure 1.

Proof. Consider each of the m branches to be n events in series. Then the probability of one branch being open is given by

$$P(\text{branch open}) = p^n,$$

and so the probability of the branch being closed is given by

$$P(\text{branch closed}) = 1 - P(\text{branch open}) = 1 - p^n.$$

For the event {no crossing} to occur, all m branches must be closed. Therefore

$$P(\text{no crossing}) = P(\text{branch closed})^m = (1 - p^n)^m.$$

□

Lemma 2.1. *The functions $f_{m,n}(p) = 1 - (1 - p^n)^m$ for $m, n \geq 2$ each have exactly one stationary point in $(0, 1)$.*

Proof. Consider the equation $g_{m,n}(p) = (1 - p) - (1 - p^n)^m$, then for any p for which $f_{m,n}(p) = p$, $g_{m,n}(p) = 0$. Note that

$$\frac{dg_{m,n}}{dp} = -1 + mnp^{n-1}(1 - p^n)^{m-1}$$

is equal to -1 for $p = 0, 1$, and $g_{m,n}(0) = g_{m,n}(1) = 0$. This implies that there exists some points $0 < a < b < 1$ for which $g_{m,n}(a) < 0$ and $g_{m,n}(b) > 0$. Since $g_{m,n}$ is a polynomial, it is a continuous function and so the Intermediate Value Theorem implies that there exists a point $p_c \in (a, b) \subset (0, 1)$ such that $g_{m,n}(p_c) = 0$, which is equivalent to $f_{m,n}(p_c) = p_c$.

To see that there is exactly one such point, we consider the problem in terms of solving

$$q(p) = 1 - p = (1 - p^n)^m = r(p).$$

It is clear that q is equal to r at $p = 0, 1$ and that both q and r are strictly decreasing for $p \in (0, 1)$ for all $m, n \geq 2$. Therefore there can only be one solution in $(0, 1)$. □

Theorem 2.1. *The critical probability of percolation on $D(m, n)$ is the stationary point of $f_{m,n}(p)$ in $(0, 1)$*

Proof. From Proposition 2.1, we know that the probability of crossing $D_1(m, n)$ with parameter p is given by $f_{m,n}(p)$. When we move from $D_1(m, n)$ to $D_2(m, n)$, each edge in $D_1(m, n)$ is replaced by $D_1(m, n)$ and so the probability of crossing $D_2(m, n)$ is equivalent to the probability of crossing $D_1(m, n)$ with parameter $f_{m,n}(p)$. That is, the probability of crossing $D_2(m, n)$ is $f^{\circ 2}_{m,n}(p) = f_{m,n}(f_{m,n}(p))$. One can easily check that the stationary point of $f_{m,n}$ in $(0, 1)$ is an attracting stationary point. And so, continuing on in this manner, we get that the critical probability of percolation is the stationary point of $f_{m,n}$ in the interval $(0, 1)$. (See [13] for further details.) □

Remark 2.1. Theorem 2.1 for $m = n = 2$ gives Hambly and Kumagai's result shown in Equation 2.1.

We now turn our attention to studying how the critical probability of percolation on $D(m,n)$ depends on m and n. Our reason for doing so will become clear in Section 3 following the proof of Theorem 3.1.

Proposition 2.2. *Let $p_c(m,n)$ denote the critical probability of percolation on $D(m,n)$. Then, for $m,n \geq 2$, $\frac{\partial p_c}{\partial m} < 0$ and $\frac{\partial p_c}{\partial n} > 0$.*

Proof. Consider the equation $g_{m,n}(p) = (1-p) - (1-p^n)^m$. Since we consider only $p \in (0,1)$ and $m \geq 2$,

$$\frac{\partial}{\partial m} g_{m,n}(p) = -(1-p^n)^m \ln(1-p^n) > 0.$$

Therefore $g_{m,n}(p) < g_{m',n}(p)$ for $m < m'$. For some m,n, we get a value p_c satisfying $g_{m,n}(p_c) = 0$. So for some $m' > m$,

$$0 = g_{m,n}(p_c) < g_{m',n}(p_c).$$

From Proposition 2.1, we know that $g_{m,n}(p) > 0$ for $p > p_c$. Therefore, $p_c(m',n) < p_c(m,n)$, and since $m' > m$,

$$\frac{\partial p_c}{\partial m} = \frac{\frac{1}{m^2}(1-p_c)^{\frac{1}{m}} \ln(1-p_c)}{np_c^{n-1} - \frac{1}{m}(1-p_c)^{\frac{1}{m}-1}} < 0.$$

Checking the signs on the numerator and denominator, we see that for $p_c \in (0,1)$,

$$np_c^{n-1} - \frac{1}{m}(1-p_c)^{\frac{1}{m}-1} > 0.$$

Computing $\frac{\partial p_c}{\partial n}$, we see that

$$\frac{\partial p_c}{\partial n} = -\frac{p_c \ln p_c}{np_c^{n-1} - \frac{1}{m}(1-p_c)^{\frac{1}{m}-1}} > 0,$$

giving the desired results. □

3. Barycentric Subdivisions and non-p.c.f Sierpiński Gasket

We now consider the iterated barycentric subdivisions of a triangle. The purpose of this discussion is to show how this structure is related to the non-p.c.f. Sierpiński gasket, which is introduced below. Theorem 3.2 shows how the critical probability of percolation on the iterated barycentric subdivision of a triangle can be used to bound the critical probability for the gasket from both above and below. We begin by introducing the iterated barycentric subdivision of a triangle and defining the limit space.

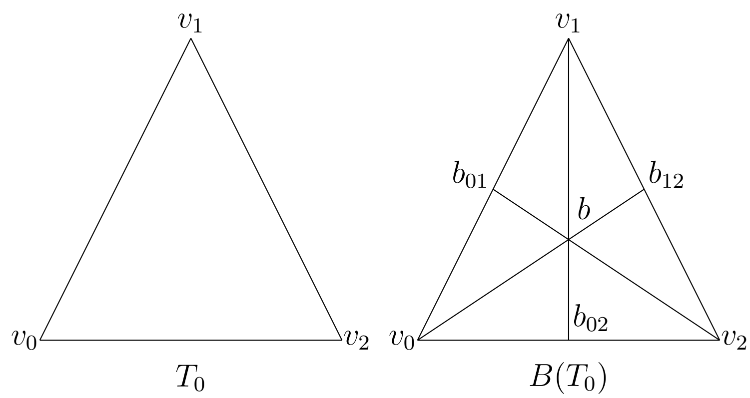

FIGURE 2. Barycentric Subdivision of T_0

Throughout the following discussion, we will adopt some of the notation from Begue [3]. Let T_0 be a triangle defined by its vertices (v_0, v_1, v_2).

Definition 3.1. *Let b_{ij}, $i, j = 0, 1, 2$, denote the midpoints of the edges of T_0 defined by (v_i, v_j), and let $b = \frac{1}{3}(v_0 + v_1 + v_2)$ be the barycenter of T_0. Denote the set of vertices $V = \{v_0, v_1, v_2, b, b_{01}, b_{02}, b_{12}\}$. We define the barycentric subdivision of T_0, denoted by $B(T_0)$, to be the set of all sub-triangles of T_0 of the form (v_i, b, b_k) where v_i is a vertex of T_0, b is the barycenter of T_0, and b_k is the midpoint of an edge of T_0 such that each (v_i, b, b_k) contains no points in V other than its own vertices.*

Example 3.1. Let T_0 be the triangle with vertices v_0, v_1, v_2 as shown in Figure 2; $B(T_0)$ is also shown. Note that the triangle (v_0, b, v_1) is not in $B(T_0)$ since b_{01} lies within the triangle but is not one of its vertices.

Definition 3.2. *We define the iterated barycentric subdivision of T_0 recursively as*

$$B^n(T_0) = B(B^{n-1}(T_0)).$$

Let T denote the structure resulting from performing barycentric subdivision of T_0 infinitely many times. That is, T is an infinite graph such that $B(T) = T$.

Example 3.2. For T_0 as in Figure 2, $B^2(T_0)$ is shown in Figure 3.

The following notation will be useful in the proof of the Theorem 3.1. Let $T_k = B^k(T_0)$. We label each of the 6 sub-triangles in T_1 in a clockwise direction: $0 = \{v_0, b_{01}, b\}$, $1 = \{b_{01}, v_1, b\}$, etc. For each sub-triangle, there are 3 vertices, 3 midpoints, and the barycenter; we label these points recursively where $i(v_0)$ is the lower left vertex of i, $i(b)$ is the barycenter of i, and so forth. We label the sub-triangles in T_2 in a similar manner. That is, for i a sub-triangle of T_1, we have $i0 = \{i(v_1), i(b_{01}), i(b)\}$ and label the other 5 sub-triangles of i in a clockwise manner.

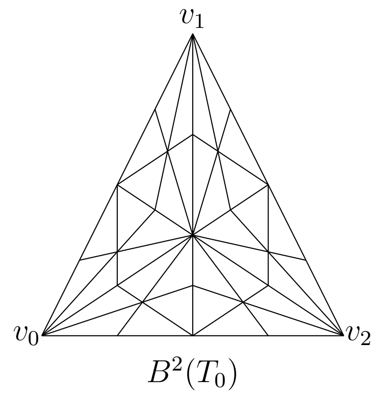

FIGURE 3. Repeated Barycentric Subdivision of T_0

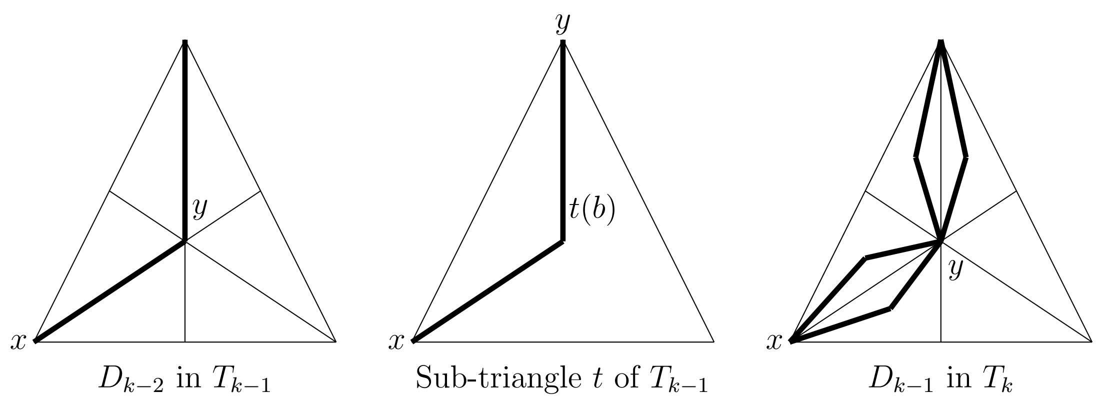

FIGURE 4. Embedding D_{k-1} into T_k

Theorem 3.1. *Let T be as defined above, and let p_T denote the critical probability of percolation on T. Then $p_T \leq \frac{\sqrt{5}-1}{2} < 1$.*

Proof. We will prove this result by embedding D_{k-1} into T_k. We begin by embedding D_1 into T_2. Note that there are in fact three copies of D_1 in T_2: $\{v_0, 1(b), b, 6(b)\}$, $\{v_1, 2(b), b, 3(b)\}$, and $\{v_2, 4(b), b, 5(b)\}$. Now suppose we have an embedding of D_{k-2} into T_{k-1}. The rule for identifying the new embedding of D_{k-1} in T_k is as follows: for each lowest-level sub-triangle t of T_{k-1} containing an edge $\{x, y\}$ of the embedded D_{k-2}, $\{x, y\}$ is replaced by two edges $\{x, t(b)\}$ and $\{t(b), y\}$. (See Figure 4.) That is, each edge of D_{k-2} in T_{k-1} is replaced by a cycle of edges in T_k, which is in fact a copy of D_1. We see that $D_{k-1} \subset T_k$ for any $k \geq 1$, and so $D \subset T$.

Since there are three copies of D embedded within T, we end up with a structure $\mathcal{D}$, depicted in Figure 5, where D represents the diamond fractal,

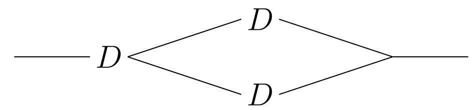

FIGURE 5. $\mathcal{D}$

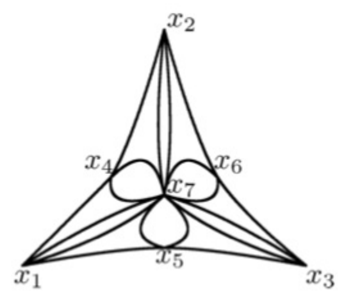

FIGURE 6. First-level approximation of S. Graphic from [2].

embedded within T. Since $D \subset \mathcal{D}$, we have

$$p_D \geq p_{\mathcal{D}} \geq p_T$$

(see Remark 1.1). We now just apply Equation 2.1 and we get the desired result. □

Remark 3.1. Our choice of using the $(2,2)$ diamond fractal was based upon to result of Proposition 2.2. That is, $D(2,2)$ has the highest critical probability among fractals whose first graph approximations can be embedded within T_2.

Remark 3.2. The critical probability of percolation on a fractal defined via their finite graph approximations can be defined through a limiting sequence of graphs such as $D_k \to D$ as in the previous section. Here we have a different limiting sequence, $T_k \to T$, where T is the result of repeated barycentric subdivision of a triangle and T_k is its k^{th} graph approximation.

We now introduce the non-p.c.f. Sierpiński gasket. Specifically we are interested in the simplest non-p.c.f. analog of the Sierpiński gasket first introduced by Teplyaev [24]. The non-p.c.f gasket can be constructed as a self-affine fractal in $\mathbb{R}^2$ using six affine contractions (see [24]); Figure 6 shows the first-level graph approximation of the non-p.c.f gasket.

Remark 3.3. The non-p.c.f. Sierpiński gasket is an analogue of the Sierpiński gasket in the same way that the diamond fractal is a non-p.c.f. analogue of the unit interval, which is p.c.f.

We would now like to examine the relationship between T and S, but first we need the following definition.

Definition 3.3. *Let $T_0 = (x, y, z)$ be a triangle. Then we define the boundary of T_0 to be the union of its sides, that is,*

$$\partial T_0 = (x, y) \cup (x, z) \cup (y, z),$$

and the interior of T_0 to be $\mathring{T}_0 = T_0 \setminus \partial T_0$.

We now introduce some notation that we will use in the following Lemma and Theorem. Let S denote the non-p.c.f. Sierpiński gasket and let T denote the limit space of repeated barycentric subdivisions of a triangle. Then we define four events as follows:

$$W = \{|O_T(2p)| = \infty\}$$
$$X = \{|O_S(p)| = \infty\}$$
$$Y = \{|O_T(p \text{ on } \partial T, p(2-p) \text{ in } \mathring{T})| = \infty\}$$
$$Z = \{|O_T(p)| = \infty\}$$

where $O_G(p)$ is the open cluster containing the origin in G given parameter p.

That is, W is the event that there exists an infinite open cluster in T given parameter $2p$, X is the event that there is an infinite open cluster in S given parameter p, Y is the event that there is an infinite open cluster in T given parameter p on the exterior and $p(2-p)$ on the interior, and Z is the event that there is an infinite open cluster in T given parameter p.

Remark 3.4. One can see by inspection that $T \subset S$. Consider Figure 6. We can insert T_1 into S_1 by counting only one of the 2 non-intersecting segments that connect each of x_1 through x_6 to x_7. If we label the sub-triangles of S clockwise from $S1$ to $S6$, then for S_2 we keep the same edges as before and now take only one of the two non-intersecting edges connecting $Si(x_1)$ through $Si(x_6)$ to $Si(x_7)$ for $i = 1, \ldots, 6$. Continuing through the iterations, we see that $T \subset S$.

Lemma 3.1. *Let T, S and X, Y be as defined above. Then $X = Y$.*

Proof. Note that as explained in the remark above, $T \subset S$, so the events X and Y lie within the same probability space.

Consider the relation between S and T. If we start with T, we can get to S by replacing each of the interior edges by two non-intersecting edges with the same endpoints. Rather than doing this, however, we instead assigned a probability $p(2-p)$ of being open to each of the interior edges in T. We choose $p(2-p)$ so that the probability of an interior edge being open in T is the same as the probability of at least one of the two corresponding edges being open in

S (given probability p of being open). Since these correspond exactly to the sets X and Y, the proof is complete. □

Theorem 3.2. *Let S and T be as above. Denote the critical probabilities of percolation on T and S as p_T and p_S, respectively. Then $p_T/2 \leq p_S \leq p_T$.*

Proof. From the definitions of the sets W, Y and Z, it is clear that

$$Z \subseteq Y \subseteq W.$$

This is due to the fact that an infinite cluster of open edges is more likely to exist given a higher probability of each edge being open.

If we apply Lemma 3.1, we see that

$$Z \subseteq X \subseteq W.$$

Taking the probability of each of these events and applying the definition of θ,

$$\theta_T(p) \leq \theta_S(p) \leq \theta_T(2p).$$

Now let $2p < p_T$, then $\theta_T(2p) = 0$ and so

$$\theta_S(p) \leq \theta_T(2p) = 0.$$

Therefore $p_S \geq p_T/2$. Now let $p > p_T$, then $\theta_T(p) > 0$. This gives

$$0 < \theta_T(p) \leq \theta_S(p),$$

and so $p_S \leq p_T$. Combining these yields $p_T/2 \leq p_S \leq p_T$. □

4. Hexacarpet

The hexacarpet has a natural definition as the pre-planar dual graph of the barycentric subdivisions of a triangle (see below: Definition 4.1 and Figure 8). We will first, however, define the hexacarpet in its own right in the manner of Begue [3]. We again adopt their notation: Denote

$$\mathbf{X} = \{0, 1, 2, 3, 4, 5\},$$

$$\mathbf{X}^n = \{x_1 x_2 \cdots x_n : x_i \in \mathbf{X}\}.$$

Generally speaking, we call $\mathbf{X}$ an alphabet, and $\mathbf{X}^n$ is the set of all words of length n. We also define

$$\mathbf{X}^* = \bigcup_{n=0}^{\infty} \mathbf{X}^n \quad \text{and} \quad \Sigma = \prod_{i=1}^{\infty} \mathbf{X}.$$

Now let $x \in \mathbf{X}^*$ and $v \in \{0, 5\}^\omega$. Then we define the equivalence relationship $\sim$ as follows: If i is odd and $j \equiv i + 1 \mod 6$, then

$$xi3v \sim xj3v \qquad \text{and} \qquad xi4v \sim xj4v.$$

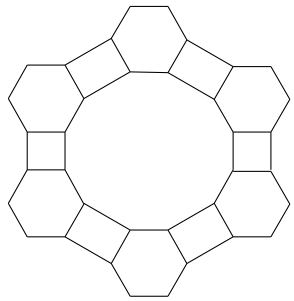

FIGURE 7. First Graph Approximation of the Hexacarpet

If i is even and $j \equiv i+1 \mod 6$, then

$$xi1v \sim xj1v \quad \text{and} \quad xi2v \sim xj2v.$$

We define the hexacarpet to be $H = \Sigma/\sim$. The first graph approximation is shown in Figure 7.

Our goal is to show that $p_H < 1$ by showing that it satisfies the two conditions given by Kozma in [16]. We successfully show the first in Lemma 4.1, but leave the second as Conjecture 4.1. We begin by reproducing Kozma's theorem without proof.

Theorem 4.1 (Kozma, [16])**.** *Let G be a planar graph with no vertex accumulation points such that*

1. *There exist numbers K and D such that for all $v \in G$ and for all $r \geq 1$, one has for the open ball $B(v,r)$ that the number of vertices satisfies $|B(v,r)| \leq Kr^D$.*
2. *There exist numbers $k, \varepsilon > 0$ such that for any finite non-empty set of vertices S, $|\partial S| \geq k|S|^{\varepsilon}$.*

Let p_G be the critical probability for independent bond percolation on G. Then $p_G < 1$.

Lemma 4.1. *The hexacarpet satisfies condition (1) in Theorem 4.1.*

Proof. Consider the hexacarpet in $\mathbb{R}^2$ where the length of each edge is exactly 1. Then each vertex is at least distance 1 from any other vertex, so if we place an open ball of radius of $\frac{1}{2}$ around each vertex, then these balls are all disjoint. Now let $r \geq 1$ and fix some vertex v. The ball $B(v,r)$ has area πr^2, and can contain at most $4r^2$ open balls of radius $\frac{1}{2}$. Hence $|B(v,r)| \leq 4r^2$ for any $r \geq 1$ and any vertex v. Then in condition (1) we can take $K = 4$ and $D = 2$. □

Conjecture 4.1. *The hexacarpet satisfies condition (2) in Theorem 4.1.*

Remark 4.1. Consider taking half-cuts of the k^{th} graph approximation of H; denote such a subgraph by X_n, and let X_n' be its complement in H. As n

increases, $|X_n|$ increases at a rate of $O(6^n)$ while $|\partial X_n|$ increases at a rate of $O(2^n)$. Our goal is to find $k, \varepsilon > 0$ such that $|\partial X_n| = k|X_n|^\varepsilon$. Taking the logarithm of both side and dividing by n,

$$\frac{\log |X_n|}{n} \leq \frac{\log k}{n} + \left(1 - \frac{1}{\varepsilon}\right) \frac{\log |\partial X_n|}{n}.$$

Taking $n \to \infty$, we have

$$\log 6 \leq \left(1 - \frac{1}{\delta}\right) \log 2.$$

Solving,

$$\delta \leq \frac{1}{1 - \frac{\log 2}{\log 6}}.$$

We believe this to be an equality for the hexacarpet. Our conjecture is based upon the fact that this same calculation on the pre-Sierpiński carpet gives the same value for δ calculated by Osada in [20].

We now turn our attention to the relationship between the hexacarpet and the barycentric subdivisions of a triangle. We remarked earlier that the hexacarpet is the pre-dual of the barycentric subdivisions of a triangle. That is, the dual graph of T is H, where *dual graph* is defined as follows in the manner of Grimmett.

Definition 4.1. *Let G be a planar graph, drawn in the plane such that the edges only intersect at vertices. We construct G_d, called the (planar) dual of G, in the following manner. In each face of G (including the infinite face if it exists) we place a vertex of G_d; for each edge e of G, we place a corresponding edge joining those two vertices of G_d which lie in the two faces of G abutting the edge e.*

It remains to show how bond percolation on T relates to bond percolation on H. We do so through the use of the following theorem, which comes from Bollobás and Riordan [5] and is presented without proof. We will first need the following definition.

Definition 4.2. *We say that a lattice G has k-fold symmetry if the rotation about the origin through an angle of $2\pi/k$ maps the plane graph G into itself.*

Theorem 4.2 (Bollobás and Riordan [5])**.** *Let G be a plane lattice with k-fold symmetry, $k \geq 2$, and let G_D be its dual. Then $p_G + p_{G_D} = 1$ for bond percolation.*

Theorem 4.3. *Let T and H be as before. Then $p_T + p_H = 1$.*

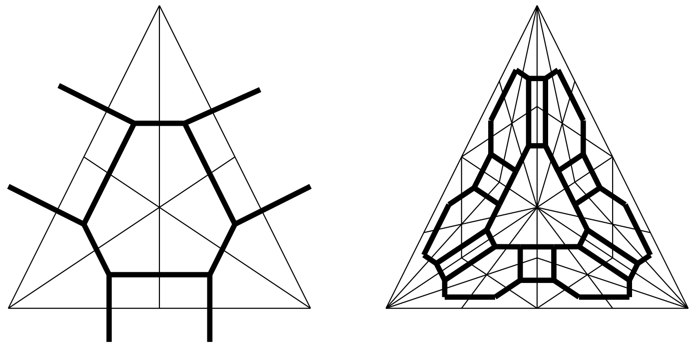

FIGURE 8. Dual Graphs

Proof. Consider the two dimensional triangular lattice. We define T'_n by replacing each triangle in the lattice with T_n. Since the triangular lattice is a planar lattice and we are not changing the underlying structure of the lattice, T'_n is also a planar lattice. Similarly, we define H'_n by replacing each hexagon in the hexagonal lattice by H_n and we again get a lattice. We can see in Figure 8 that if we tile T_n, then H'_n is clearly its dual since the hexagonal and triangular lattices are dual lattices. That is H'_n and T'_n are dual lattices for every n.

We now want to consider the relationship between percolation on T and $T' = \lim_{n\to\infty} T'_n$. We fix the origin x to be the lower left vertex of T. At x, T'_n consists of six copies of T_n sharing a common vertex x. Now let $B(x, 2^n)$ be a ball centered at x with a radius 2^n (in the graph metric). Then $T'_n \cap B(x, 2^n)$ is graph isomorphic to T_n and so percolation on T'_n is equivalent to percolation on T_n starting at the barycenter of T_1. We know that $p_T = \lim_{n\to\infty} p_{T_n}$, so it follows that $p_{T'_n} \to p_{T'}$. Hence $p_T = p_{T'}$. A similar argument for H'_n and H tells us that $p_H = p_{H'}$. This completes the proof. $\square$

5. PROOF OF THEOREM 1.1

Theorem 1.1. *Let S denote non-p.c.f. Sierpiński gasket, and let H denote the hexacarpet. Then $0 < p_S \leq \frac{\sqrt{5}-1}{2}$ and $\frac{3-\sqrt{5}}{2} \leq p_H < 1$.*

Proof. Assuming Conjecture 4.1 is true, Lemma 4.1 and Theorem 4.1 tell us that $p_H < 1$, so by Theorem 4.3, $p_T = 1 - p_H > 0$. Theorem 3.1 says that $p_t \leq \frac{\sqrt{5}-1}{2}$, and so applying Theorem 3.2,

$$0 < p_T/2 \leq p_S \leq p_T \leq \frac{\sqrt{5}-1}{2}.$$

Since $p_T < 1$, by Theorem 4.3,

$$p_H = 1 - p_T \geq 1 - \frac{1-\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}.$$

Given Conjecture 4.1, $p_H < 1$, which gives the desired result for H. □

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