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Class of 2013 Maximal Unbalanced Families
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ABSTRACT

The purpose of this article is to develop the combinatorics of maximal unbalanced families of subsets of $[n] = \{1, ..., n\}$. Specifically, we will start by finding the number E_n of such families for $n \leq 9$. We will then prove lower and upper bounds on this number in terms of n – both bounds are of the form 2^{Cn^2} for some $C > 0$. After this, we will exploit symmetries among the families to show that $n \mid E_n$. Finally, we will relate our work to known results about threshold functions, which arise in electrical engineering, switching theory, and artificial intelligence. Maximal unbalanced families themselves correspond to the regions of a certain hyperplane arrangement, known as the restricted all-subset arrangement, that has arisen in various forms in physics, economics and psychometrics. In particular, our bounds answer a question posed in thermal field theory concerning the order of the number of regions of this arrangement.

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CHAPTER 1

INTRODUCTION

1.1 Definitions

Let n be a natural number, and let

$$[n] := \{1, 2, \dots, n\}.$$

For any $F \subseteq [n]$, define its **characteristic function** $\chi_F : [n] \to \{0, 1\}$ by

$$\chi_F(i) = \begin{cases} 1 & : i \in F \\ 0 & : i \notin F \end{cases}$$

Given finitely many subsets $F_1, F_2, ..., F_m$ of $[n]$, a **linear combination** of their characteristic functions $\chi_{F_1}, \chi_{F_2}, \dots, \chi_{F_m}$ is a function $\chi : [n] \to [0, 1]$ of the form

$$\chi = \alpha_1 \chi_{F_1} + \alpha_2 \chi_{F_2} + \cdots + \alpha_m \chi_{F_m},$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ are non-negative.

Now suppose $F_1, ..., F_m \subseteq [n]$ are nonempty. If there exists a linear combination χ of $\chi_{F_1}, \dots, \chi_{F_m}$ such that $\chi(i) = 1$ for each $i = 1, 2, \dots, n$, then the nonempty family $\mathcal{F} := \{F_1, ..., F_m\}$ is **balanced**; otherwise, it is **unbalanced**. We call a family $\mathcal{F}$ **maximally unbalanced** if it is unbalanced and every family $\mathcal{G}$ satisfying $\mathcal{F} \subsetneq \mathcal{G} \subset 2^{[n]}$ is balanced.

Instead of thinking of χ_F as a boolean function, we may view it as a boolean n-vector, where the i-th entry of the tuple is 1 if $i \in F$ and 0 if $i \notin F$. Call this n-vector $(\chi_F(1), \dots, \chi_F(n))$ the **characteristic vector** of F. We will abuse notation by using χ_F to refer to both the characteristic function and characteristic vector of

F. Using this correspondence, we can alternatively define a **linear combination** of characteristic functions $\chi_{F_1}, \ldots, \chi_{F_m}$ to be a function corresponding to any linear combination of the vectors $\chi_{F_1}, \ldots, \chi_{F_m}$. Furthermore, a family is **balanced** if and only if $(1, \ldots, 1)$ is a linear combination of its characteristic vectors.

Example 1.1.1. Suppose $n = 3$. The family

$$\{F_1 = \{1\}, F_2 = \{1, 2\}, F_3 = \{1, 3\}\}$$

is maximally unbalanced, for we have

$$\chi_{F_1} = (1, 0, 0), \chi_{F_2} = (1, 1, 0), \chi_{F_3} = (1, 0, 1).$$

Any linear combination χ of these vectors will be of the form $\alpha_1 \chi_{F_1} + \alpha_2 \chi_{F_2} + \alpha_3 \chi_{F_3} = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_3)$. Note that we must have $\alpha_2 = \alpha_3 = 1$. But then α_1 must be -1 in order for us to have $\chi(1) = 1$, which cannot happen because the α_i must be non-negative.

Example 1.1.2. Suppose $n = 3$. The family

$$\{F_1 = \{1\}, F_2 = \{1, 2\}\}$$

is unbalanced but is not maximally unbalanced, because the family in Example 1.1.1 is a proper superset of it and is unbalanced. In general, any proper subset of a maximal unbalanced family is unbalanced but not maximally unbalanced.

Proposition 1.1.3. *If $\mathcal{F}$ is unbalanced and $F \in \mathcal{F}$, then $[n] \setminus F \notin \mathcal{F}$.*

Proof. Suppose, for sake of contradiction, that $[n] \setminus F \in \mathcal{F}$. Then, since

$$\chi_F + \chi_{[n] \setminus F} = (1, 1, \ldots, 1),$$

$\mathcal{F}$ is balanced, which is a contradiction. □

Recall that $[n]$ has a total of 2^n subsets. By definition, any unbalanced family $\mathcal{F}$ of subsets of $[n]$ contains neither $\emptyset$ nor $[n]$, so there are $2^n - 2$ possible subsets of $[n]$ that can be elements of $\mathcal{F}$. Now each of these $2^n - 2$ subsets F can be paired with its complement $[n] \setminus F$. Then, by Proposition 1.1.3, each such pair can contribute at most one element to $\mathcal{F}$. Since the pairs are disjoint, there are $\frac{2^n-2}{2} = 2^{n-1} - 1$ pairs, so $\mathcal{F}$ contains at most this many elements.

This is an upper bound on the size of a maximal unbalanced family. In the next section we will show that each maximal unbalanced family has exactly $2^{n-1} - 1$ elements.

1.2 Maximal Unbalanced Families and the All-subset Arrangement

We will start with a few definitions. First, define a **linear hyperplane** to be an $(n-1)$-dimensional subspace H of $\mathbb{R}^n$. All such subspaces are of the form

$$H = \{v \in \mathbb{R}^n : \alpha \cdot v = 0\},$$

where α is a fixed nonzero vector in $\mathbb{R}^n$ and $\cdot$ is the **dot product**, defined by

$$(\alpha_1, \ldots, \alpha_n) \cdot (v_1, \ldots, v_n) = \sum_{i=1}^{n} \alpha_i v_i.$$

On the other hand, an **affine hyperplane** is a translate J of a linear hyperplane. All such translates are of the form

$$J = \{v \in \mathbb{R}^n : \alpha \cdot v = b\},$$

where again α is a fixed nonzero vector in $\mathbb{R}^n$ and $a \in \mathbb{R}$. Now define a closed **half-space** to be a set $\{x \in \mathbb{R}^n : x \cdot \alpha \geq c\}$ for some $\alpha \in \mathbb{R}^n, c \in \mathbb{R}$.

A set $\mathcal{A}$ of affine hyperplanes in a real vector space V is known as a **hyperplane arrangement in** V, and a **region of** $\mathcal{A}$ **in** V is a connected component of the complement X of the hyperplanes:

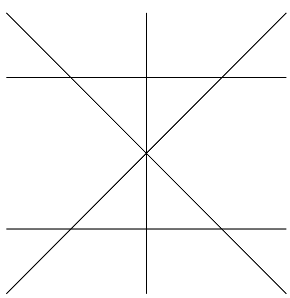
$$X := V - \bigcup_{H \in \mathcal{A}} H.$$

We will usually take $V = \mathbb{R}^n$, although this will not always be the case. Let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$ in $\mathbb{R}^n$, and let

$$r(\mathcal{A}) = \#\mathcal{R}(\mathcal{A})$$

denote the number of regions of $\mathcal{A}$ in $\mathbb{R}^n$.

Example 1.2.1. Let $n = 2$, so that the hyperplanes are lines in a plane. Suppose that the arrangement $\mathcal{A}$, which appears below, contains the hyperplanes $x = 0, y = 1, y = -1, x = y$, and $x = -y$. Then $r(\mathcal{A}) = 14$.



For the rest of this paper, all arrangements will be finite (i.e., finite sets of hyperplanes).

Let conv(X) denote the **convex hull** of X. That is, if $X = \{x_1, ..., x_m\}$ consists of m points in a vector space, then

$$\text{conv}\{X\} = \left\{ \sum_{i=1}^{m} \alpha_i x_i | (\forall i : \alpha_i \geq 0) \wedge \sum_{i=1}^{m} \alpha_i = 1 \right\}$$

The convex hull of any finite set of points in $\mathbb{R}^n$ is called a **polytope**, and the solution set of finitely many linear inequalities is called a **polyhedron**. The relationship between the two is summarized in the following proposition:

Proposition 1.2.2. *Polytopes in $\mathbb{R}^n$ are bounded polyhedra, and vice versa.*

Proof. This is Theorem 1.1 in [15]. We will omit the proof. □

The following will be referred to as the Alternative Theorem.

Theorem 1.2.3. *For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, either*

(1) $\mathcal{F}$ is balanced (i.e., $C_{\mathcal{F}} := conv\{\chi_S | S \in \mathcal{F}\}$ contains a constant vector $c\chi_{[n]}, 0 \leq c \leq 1$), or

(2) there is a $y \in \mathbb{R}^n$ such that $y \cdot \chi_{[n]} = 0$ and $y \cdot \chi_S > 0$ for all $S \in \mathcal{F}$,

but not both.

Proof. First we show the "not both" part. Suppose that (2) holds. Then y is perpendicular to $\chi_{[n]}$ but makes an acute angle with all of the χ_S. This means that $\chi_{[n]}$ cannot be a convex combination of the χ_S. Thus (1) does not hold.

Now we need to show that either (1) or (2) holds. So suppose that (1) fails and $\mathcal{F}$ is unbalanced. If $C := C_{\mathcal{F}}$ and

$$I := \{c\chi_{[n]} | 0 \leq c \leq 1\},$$

then $C \cap I = \emptyset$. Since C and I are polyhedra, we see that

$$C - I := \{x - z | x \in C, z \in I\}$$

is the convex hull of all differences of the vertices of C and I and is hence a polyhedron. Since $C \cap I = \emptyset$, it follows that $\vec{0} \notin C - I$. Hence there is a linear hyperplane H such that $C - I$ lies entirely on one side of H. Let $u \in \mathbb{R}^n$ be the normal vector of H pointing toward the side containing $C - I$. Then, for all $x \in C$ and $z \in I$, we have

$$(x - z) \cdot u > 0 \implies x \cdot u > z \cdot u.$$

In particular,

$$x \cdot u > \chi_{[n]} \cdot u \text{ and } x \cdot u > \vec{0} \cdot u = 0.$$

Thus if $r := \max\{0, \chi_{[n]} \cdot u\}$, then for all $x \in C$, we have $x \cdot u > r$, while $\chi_{[n]} \cdot u \leq r$. In particular, $\chi_S \cdot u > r$ for all $S \in \mathcal{F}$. Now set

$$y := u - \frac{\chi_{[n]} \cdot u}{n} \chi_{[n]}.$$

Then

$$y \cdot \chi_{[n]} = (u \cdot \chi_{[n]}) - \frac{\chi_{[n]} \cdot u}{n} (\chi_{[n]} \cdot \chi_{[n]}) = (u \cdot \chi_{[n]}) - (\chi_{[n]} \cdot u) = 0$$

and, for all $S \in \mathcal{F}$,

$$y \cdot \chi_S = (u \cdot \chi_S) - \frac{\chi_{[n]} \cdot u}{n} (\chi_S \cdot \chi_{[n]}) > r - \frac{r}{n}|S| = r \left(1 - \frac{|S|}{n}\right) \geq 0.$$

Thus (2) holds. □

If $\mathcal{F}$ is an unbalanced family, then (2) holds, and in this case we consider a point $y \in \mathbb{R}^n$ as a possible **witness vector** to alternative (2) above. Now, given any such $y \in \mathbb{R}^n$, we now try to find the maximal family $\mathcal{F}$ for which alternative (2) holds with the given y. Let H_S be the hyperplane with normal vector χ_S. That is,

$$H_S = \left\{ x = (x_1, ..., x_m)^T \in \mathbb{R}^n : \sum_{i \in S} x_i = 0 \right\}.$$

Consider the arrangement

$$\mathcal{H}_n := \{H_S | \emptyset \neq S \subsetneq [n]\}$$

Then y satisfies (2) for a collection $\mathcal{F}$ if and only if $y \cdot \chi_{[n]} = 0, y \cdot \chi_S > 0$ for all $S \in \mathcal{F}$. That is, y must be on the same side of H_S as χ_S is for each $S \in \mathcal{F}$. Since y necessarily lies on the hyperplane H_n orthogonal to $\chi_{[n]}$, it is enough to consider the hyperplanes in H_n that are the intersections of the H_S with H_n. That is, it suffices to consider the hyperplanes $\bar{H}_S$ with normals $\bar{\chi}_S$ that are projections of the χ_S onto H_n. To find $\bar{\chi_S}$, note that it must be $\chi_S - c\chi_{[n]}$ for some $c \in \mathbb{R}$. Since H_n is the hyperplane $x_1 + ... + x_n = 0$, the sum of the coordinates of any point in H_n must be zero. Then, since the sum of the coordinates of χ_S is $|S|$ and the sum of the coordinates of $\chi_{[n]}$ is n, we must have $c = |S|/n$ in order to have $\bar{\chi_S} \in H_n$, and so

$$\bar{\chi}_S := \chi_S - \frac{|S|}{n}\chi_{[n]} = \left(1 - \frac{|S|}{n}\right)\chi_S - \frac{|S|}{n}\chi_{[n]\setminus S}.$$

Note that

$$\bar{\chi}_{[n]\setminus|S|} = \left(1 - \frac{|[n]\setminus S|}{n}\right)\chi_{[n]\setminus S} - \frac{|n\setminus S|}{n}\chi_S = \frac{|S|}{n}\chi_{[n]\setminus S} - \left(1 - \frac{|S|}{n}\right)\chi_S = -\bar{\chi}_S,$$

so $\bar{H}_S = \bar{H}_{[n]\setminus S}$, and we get an arrangement $\mathcal{A}_n^0$ of $2^{n-1} - 1$ hyperplanes in H_n:

$$\mathcal{A}_n^0 = \{H_S \cap H_{[n]} | \emptyset \neq S \subsetneq [n]\}.$$

In particular, for each $y \in H_n$, we have the unbalanced collection

$$\mathcal{F}_y := \{S \subset [n] | \bar{\chi}_S \cdot y > 0\}.$$

And since $\bar{\chi}_{[n]\setminus|S|} = -\bar{\chi}_S$, we have $\mathcal{F}_{-y} = \{[n] \setminus S | S \in \mathcal{F}_y\}$. We get from this the following:

Theorem 1.2.4. *Any maximal unbalanced collection $\mathcal{F}$ has $2^{n-1} - 1$ elements.*

Proof. From Theorem 1.2.3, there exists $y \in H_n$ such that $\mathcal{F} = \mathcal{F}_y$. If y is not on any hyperplane H_S, where $S \in \mathcal{F}$, then for all S exactly one of $\bar{H}_S \cdot y > 0$ or $\bar{H}_{[n]\setminus S} > 0$ holds. Then, for all S, $\mathcal{F}$ contains exactly one of S and $[n] \setminus S$. Since there are $2^n - 2$ possible sets S, $\mathcal{F}$ must have $\frac{2^n-2}{2} = 2^{n-1} - 1$ elements.

Now suppose y lies on some hyperplane H_R, where $R \subsetneq [n]$ is nonempty. Then we could perturb y within H_n such that it is no longer on H_R, but $\bar{\chi}_S$ is still positive for all $S \in \mathcal{F}$, so that $\mathcal{F}_y$ does not lose any elements. However, after y moves off of H_R, either $\bar{\chi}_R \cdot y > 0$ or $\bar{\chi}_{[n]\setminus R} \cdot y > 0$, so $\mathcal{F}_y$ gains either$R$ or $[n] \setminus R$ and stays unbalanced. But then $\mathcal{F}$ was not maximal, which is a contradiction. □

Example 1.2.5. Suppose $n = 3$. $\mathcal{H}_3$ is the arrangement consisting of the hyperplanes

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 + x_3 = 0$$

$$x_2 + x_3 = 0$$

and $\mathcal{A}_3^0$ is the arrangement consisting of the hyperplanes

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

within H_3, which is defined by $x_1 + x_2 + x_3 = 0$.

Define the n-dimensional **all-subset arrangement** $\mathcal{A}_n \subset \mathbb{R}^n$ by

$$\mathcal{A}_n := \{H_S : \emptyset \neq I \subseteq [n]\}.$$

Note that $\mathcal{A}_n = \mathcal{H}_n \cup \{H_n\}$. Note that $\mathcal{A}_n^0$ consists of the restrictions of the hyperplanes of $\mathcal{H}_n$ to H_n. We thus call $\mathcal{A}_n^0$ the **restricted all-subset arrangement**.

Theorem 1.2.6. *There is a bijection between the regions of $H_n \setminus \mathcal{A}_n^0$ in H_n and maximal unbalanced families of subsets of $[n]$.*

Proof. The bijection can be constructed as follows. In one direction, given any connected component C of $H_n \setminus \mathcal{A}_n^0$, choose a $y \in C$, and construct the maximal unbalanced family $f(C) := \mathcal{F}_y$. In the other direction, given a maximal unbalanced family $\mathcal{F}$, find a y for it that satisfies alternative (2) from Theorem 1.2.3, and let $g(\mathcal{F})$ be the connected component of $H_n \setminus \mathcal{A}_n^0$ that contains y.

Note that f is well defined because $\mathcal{F}_y$ is constant within each region. Also, g is well defined because any maximal unbalanced family $\mathcal{F}_y$ must have y on none of the H_S for $S \subsetneq [n]$, implying that $H_n \setminus \mathcal{A}_n^0$. Given that both f and g are well-defined, it is easy to see that f and g must be inverses of one another.

Now we need to show that both f and g are one-to-one. To show that f is one-to-one, suppose that $\mathcal{F}_{y_1} = f(C_1) = f(C_2) = \mathcal{F}_{y_2}$. Then y_1 and y_2 are in the same region, so C_1 and C_2 are the same connected component of $H_n \setminus \mathcal{A}_n^0$.

To show that g is one-to-one, suppose that $g(\mathcal{F}_1) = g(\mathcal{F}_2)$. Suppose y_1 and y_2 satisfy altervative (2) from Theorem 1.2.3 for $\mathcal{F}_1$ and $\mathcal{F}_2$, so that y_1 and y_2 are in the same region. Since $\mathcal{F}_y$ is constant on regions, y_1 must also satisfy alternative (2) for $\mathcal{F}_2$ and y_2 for $\mathcal{F}_1$. Since g is well defined, we may take $y := y_1 = y_2$; since $\mathcal{F}_1$ and $\mathcal{F}_2$ are maximal, this implies $\mathcal{F}_1 = \mathcal{F}_y = \mathcal{F}_2$. $\square$

As a result of the previous theorem, if we want to count maximal unbalanced families, we can instead count regions of the hyperplane arrangement $\mathcal{A}_n^0$ in H_n. The following theorem relates this number to the number of regions of $\mathcal{A}_n$.

Theorem 1.2.7. *$\mathcal{A}_{n-1}$ and $\mathcal{A}_n^0$ are isomorphic. That is, there exists a bijection $f : L(\mathcal{A}_{n-1}) \to L(\mathcal{A}_n^0)$ such that for all $x, y \in L(\mathcal{A}_{n-1})$, $x \lessdot y$ if and only if $f(x) \lessdot f(y)$.*

Proof. Recall that $\mathcal{A}_{n-1}$ is an arrangement in $\mathbb{R}^{n-1}$. We can extend the hyperplanes in $\mathcal{A}_{n-1}$ to $\mathbb{R}^n$ by interpreting their equations as defining sets of points in $\mathbb{R}^n$ instead of in $\mathbb{R}^{n-1}$. In this way, we get an arrangement in $\mathbb{R}^n$ isomorphic to $\mathcal{A}_{n-1}$. Call this arrangement $\mathcal{B}$. Note that all of the hyperplanes in $\mathcal{B}$ have normals in the plane $x_n = 0$. Since the normals of $x_n = 0$ and H_n are not perpendicular, it suffices to show that $\mathcal{B}$ intersects H_0 in precisely the locations of the hyperplanes in $\mathcal{A}_n^0$.

We must show that $\mathcal{B}$ intersects H_0 in the same places as $\mathcal{A}_n$ does. Clearly all of the hyperplanes in $\mathcal{B}$ appear in $\mathcal{A}_n$. Of the hyperplanes in $\mathcal{A}$, only the hyperplanes whose equations involve x_n do not appear in $\mathcal{B}$. Let H_S be such a hyperplane. Since H_S and $H_{[n]\setminus S}$ intersect H_0 at the same place, we may consider $H_{[n]\setminus S}$ instead. But $H_{[n]\setminus S}$'s equation does not involve x_n, so it is in $\mathcal{B}$. □

Corollary 1.2.8. *For any arrangement $\mathcal{H}$ in $\mathbb{R}^n$, let $r'(\mathcal{H})$ denote the number of regions of $\mathcal{H}$ in H_n. Then*

$$\chi(\mathcal{A}_{n-1}, t) = \chi(\mathcal{A}_n^0, t) \text{ and } r(\mathcal{A}_{n-1}) = r'(\mathcal{A}_n^0)$$

Example 1.2.9. The equations of $\mathcal{A}_2$ are

$$x_1 = 0 \tag{1.1}$$

$$x_2 = 0 \tag{1.2}$$

$$x_1 + x_2 = 0 \tag{1.3}$$

$\mathcal{A}_3^0$ consists of the intersections of the following planes with H_3.

$$x_1 = 0 \tag{1.4}$$

$$x_2 = 0 \tag{1.5}$$

$$x_3 = 0 \tag{1.6}$$

$$x_1 + x_2 = 0 \tag{1.7}$$

$$x_1 + x_3 = 0 \tag{1.8}$$

$$x_2 + x_3 = 0 \tag{1.9}$$

When we restrict ourselves to H_3, equation (1.1) defines the same set of points as (1.5) and (1.10), (1.2) the same set as (1.6) and (1.9), and (1.3) the same set as (1.7) and (1.8).

1.3 Applications

Let E_n denote the number of maximal unbalanced families of subsets of $[n]$. It turns out that the sequence E_n and the hyperplane arrangements $\mathcal{A}_n$ and $\mathcal{A}_n^0$ arise in economics and physics.

1.3.1 Psychometrics and Economics

E_n is the number of possible ranking patterns generated by unfolding models of codimension 1 [5].

Suppose that we have a set of n objects labeled $1, ..., n$ and an individual who ranks them according to his preferences. In the **unfolding model**, we assume that the n objects are represented by points $\mu_1, ..., \mu_n$ in $\mathbb{R}^m$. Also, the individual is represented by a point $y \in \mathbb{R}^m$. This $\mathbb{R}^m$ is called the **joint space**. Now we say that the individual prefers i to j if and only if y is closer to μ_i than μ_j under the usual Euclidean distance. So y gives a ranking $(i_1 ... i_n)$, meaning i_1 is the individual's best object, i_2 is his second best object, and so on, if and only if y is closest to μ_{i_1}, second closest to μ_{i_2}, and so on.

In general, there are $n!$ rankings for n objects. However, not all of these rankings can be generated by the unfolding model.

Example 1.3.1. Suppose $m = 1$ and $n = 3$. Let $\mu_1 = 0$, $\mu_2 = 6$, $\mu_3 = 8$. If $y < 3$, then the ranking is (123). If $3 < y < 4$, then the ranking is (213). If $4 < y < 7$, then the ranking is (231). If $7 < y$, then the ranking is (321). Thus the rankings (132) and (312) are not realized by this unfolding model.

Call a ranking **admissible** if there exists a y that generates that ranking. A ranking is **inadmissible** otherwise. The set of all admissible rankings of the unfolding model with $\mu_1, ..., \mu_n$ is known as that model's **ranking pattern**. So in the above example, the ranking pattern is $\{(123), (213), (231), (321)\}$. In general, different choices of $\mu_1, ..., \mu_n$ will yield different ranking patterns, and the problem is to determine how many distinct ranking patterns there are for a particular n. If we restrict ourselves to unfolding models of codimension 1 (i.e., $n = m + 2$), then the result is E_n.

1.3.2 Thermal Field Theory

E_n is the number of independent real-time Green functions of quantum field theory produced when analytically continuing from Euclidean time/energy, where n is the number of energy/time variables [4, 3]. These functions are also known as

generalized retarded functions. T.S. Evans (after whom the numbers E_n are named) conjectured in 1994 that $E_n > O(N!)$ [4]. We will show in Chapter 3 that this conjecture is correct.

CHAPTER 2

CALCULATING E_N

2.1 Characteristic Polynomials and Zaslavsky's Theorem

A **partially ordered set**, or **poset**, is an ordered pair $(P, \leq)$, where P is a set and $\leq$ is a relation satisfying, for all $x, y, z \in P$,

(P1) (Reflexivity) $x \leq x$.

(P2) (Antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.

(P3) (Transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.

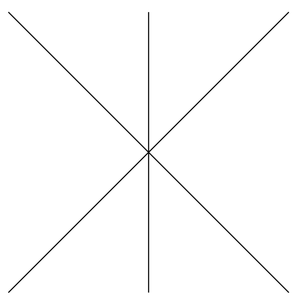
If there is an element $p \in P$ such that $p \leq x$ for all $x \in P$, then we denote p by $\hat{0}$. Likewise, if there is an element $q \in P$ such that $x \leq q$ for all $x \in P$, then we denote q by $\hat{1}$. If $x \leq y$ and $x \neq y$, then we can write $x < y$. We say that an element $y \in P$ **covers** x (denoted $x \lessdot y$) if $x < y$ and there is no $z \in P$ such that $x < z < y$. If $x \leq y$ in P, then the (closed) **interval** $[x, y]$ is defined by

$$[x, y] = \{z \in P : x \leq z \leq y\}.$$

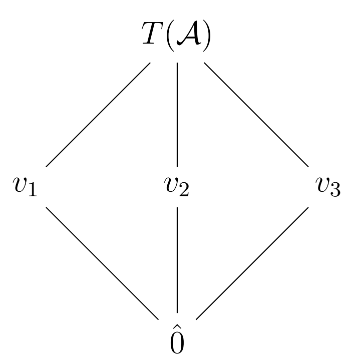
If $\mathcal{A}$ is an arrangement in a vector space V, let $L(\mathcal{A})$ be the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including V itself (as the intersection over the empty set). We can turn $L(\mathcal{A})$ into a poset by defining $x \leq y$ in $L(\mathcal{A})$ if $x \supseteq y$ as subsets of $\mathbb{R}^n$. We then call $L(\mathcal{A})$ the **intersection poset** of $\mathcal{A}$. Note that in $L(\mathcal{A})$, we have $\hat{0} = V$. We will denote $\hat{1}$, the intersection of all of the hyperplanes in $\mathcal{A}$, by $T(\mathcal{A})$. This element is called the **center** of $\mathcal{A}$.

For any poset P, its **Hasse diagram** is a diagram in which each element in P is represented by a node, and every covering relation $x \lessdot y$ is represented by a line segment that goes upward from x to y. Now consider the poset $L(\mathcal{A})$, where $\mathcal{A}$ is an arrangement. Each of its elements is an affine subspace of $\mathbb{R}^n$. Thus every element x of $L(\mathcal{A})$ has a well-defined dimension, denoted $\dim(x)$. Furthermore, if $x, y \in L(\mathcal{A})$, then $x \lessdot y$ if and only if $x \supset y$ and $\dim(x) + 1 = \dim(y)$. Thus, when drawing its Hasse diagram, we can separate the elements of $L(\mathcal{A})$ into $n+1$ bands, with $\hat{0}$ at the bottom and $\hat{1}$ at the top, such that the elements in band i have the same dimension $n - i$ for $i = 0, ..., n$ (band 0 is at the bottom of the diagram, and band n is at the top).

Example 2.1.1. The following is a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^2$.



Here is the Hasse diagram of its intersection poset.



As expected, the Hasse diagram has 3 bands.

A poset is called **locally finite** if every interval $[x, y]$ is a finite set. Let $\mathrm{Int}(P)$ denote the set of all closed intervals of P. If P is a locally finite poset, then the function $\mu : \text{Int}(P) \to \mathbb{Z}$ defined by

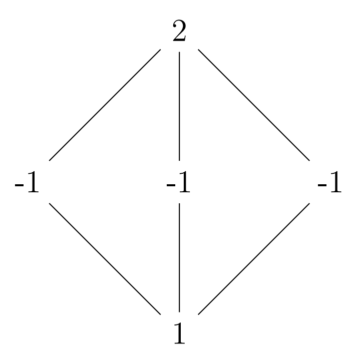
$$\begin{aligned} \mu(x,x) &= 1 \text{ for all } x \in P \\ \mu(x,y) &= -\sum_{x \le z < y} \mu(x,z), \text{ for all } x < y \text{ in } P \end{aligned} \tag{2.1}$$

is called the **Möbius function** of P. If P has a $\hat{0}$, then define $\mu(x) := \mu(\hat{0}, x)$.

Furthermore, define the **characteristic polynomial** $\chi(\mathcal{A}, t)$ of the arrangement $\mathcal{A}$ to be

$$\chi(\mathcal{A}, t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim(x)}.$$

Example 2.1.2. Consider the arrangement of Example 2.1.1. By the definition of μ, we have $\mu(\hat{0}) = 1$, and for $i = 1, 2, 3$, we have $\mu(v_i) = -\mu(\hat{0}) = -1$. Lastly, $\mu(\hat{1}) = -(\mu(v_1) + \mu(v_2) + \mu(v_3) + \mu(\hat{0})) = -(-1 - 1 - 1 + 1) = 2$. Thus, if we fill in the nodes of the Hasse diagram with the Möbius function values of the poset elements that they represent, we get



Hence the characteristic polynomial is $t^2 + (-1 - 1 - 1)t + 2 = t^2 - 3t + 2$.

If $\mathcal{A}$ is an arrangement, then a **subarrangement** of $\mathcal{A}$ is a subset $\mathcal{B} \subseteq \mathcal{A}$. If $x \in L(\mathcal{A})$, define the subarrangement $\mathcal{A}_x$ as

$$\mathcal{A}_x := \{H \in \mathcal{A} : x \subseteq H\}.$$

and define an arrangement $\mathcal{A}^x$ as

$$\mathcal{A}^x := \{x \cap H : H \in \mathcal{A} - \mathcal{A}_x, H \neq \emptyset\}.$$

In other words, $\mathcal{A}_x$ contains all of the hyperplanes that contain x, while $\mathcal{A}^x$ is the intersection of x with the set of hyperplanes that intersect x but do not contain it. If $H \in \mathcal{A}$, $\mathcal{A}' = \mathcal{A} - \{H\}$, and $\mathcal{A}'' = \mathcal{A}^H$, then we call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a **triple of arrangements with distinguished hyperplane** H.

Lemma 2.1.3. *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple of arrangements with distinguished hyperplane* H*. Then*

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'').$$

Proof. Note that $r(\mathcal{A})$ is $r(\mathcal{A}')$ plus the number of regions of $\mathcal{A}'$ that H cuts up, so it suffices establish a bijection between these regions and regions of $\mathcal{A}''$. So let R' be a region of $\mathcal{A}'$ that H intersects. Then $R' \cap H \in \mathcal{R}(\mathcal{A}'')$. Conversely, if $R'' \in \mathcal{R}(\mathcal{A}'')$ then points near R'' on either side of H must be in the same region $R' \in \mathcal{R}(\mathcal{A}')$, since any $H' \in \mathcal{A}'$ separating them would intersect R''. Thus R' is cut in two by H, and we have established a bijection between regions of $\mathcal{A}'$ cut into two by H and regions of $\mathcal{A}''$. □

If $\mathcal{A}$ be an arrangement and $X, Y \in L(\mathcal{A})$ satisfy $X \leq Y$, then let $S(X,Y)$ denote the set of subarrangements $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{A}_X \subseteq \mathcal{B}$ and $T(\mathcal{B}) = Y$.

Lemma 2.1.4. *Let* $\mathcal{A}$ *be an arrangement. Then*

$$\mu(X,Y) = \sum_{\mathcal{B} \in S(X,Y)} (-1)^{|\mathcal{B} - \mathcal{A}_X|}$$

Proof. Let $\nu(X,Y)$ denote the right side of the equation. Now note that

$$\bigcup_{X \leq Z \leq Y} S(X,Z) = \{\mathcal{B} \subseteq \mathcal{A} | \mathcal{A}_X \subseteq \mathcal{B} \subseteq \mathcal{A}_Y\},$$

where the union is disjoint. Thus

$$\sum_{X \le Z \le Y} \nu(X, Z) = \sum_{\mathcal{A}_X \subseteq \mathcal{B} \subseteq \mathcal{A}_Y} (-1)^{|\mathcal{B} - \mathcal{A}_X|} = \sum_{\mathcal{C} \subseteq \mathcal{A}_Y - \mathcal{A}_X} (-1)^{|\mathcal{C}|}.$$

If $X = Y$ the sum is 1; if $X < Y$ then $\mathcal{A}_X \subsetneq \mathcal{A}_Y$, so the sum is zero. Thus ν satisfies the definition of μ. □

Lemma 2.1.5. *Let $\mathcal{A}$ be an arrangement. Then*

$$\chi(\mathcal{A}, t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})}.$$

Proof. Let $S(X) = S(\hat{0}, X)$ and $L = L(\mathcal{A})$. From Lemma 2.1.4 we have

$$\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) t^{\dim X} = \sum_{X \in L} \left(\sum_{\mathcal{B} \in S(X)} (-1)^{|\mathcal{B}|} t^{\dim X} \right)$$

If $\mathcal{B} \in S(X)$ then $T(\mathcal{B}) = X$, so $\dim T(\mathcal{B}) = \dim X$. Since every $\mathcal{B} \subseteq \mathcal{A}$ occurs in a unique $S(X)$, the result follows. □

The following is known as the **Deletion-Restriction Theorem**.

Theorem 2.1.6. *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane H. Then*

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t).$$

Proof. Let

$$R' = \sum_{\mathcal{B} \subseteq \mathcal{A}, H \notin \mathcal{B}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})}.$$

and

$$R'' = \sum_{\mathcal{B} \subseteq \mathcal{A}, H \in \mathcal{B}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})}, \tag{2.2}$$

so that, by Lemma 2.1.5, $\chi(\mathcal{A}, t) = R' + R''$. Also, by Lemma 2.1.5,

$$R' = \chi(\mathcal{A}', t),$$

so it suffices to show $R'' = -\chi(\mathcal{A}'', t)$. So consider equation 2.2. Since $H \in \mathcal{B}, \mathcal{A}_H = \{H\} \subseteq \mathcal{B}$. Thus if $T(\mathcal{B}) = Y$, then $\mathcal{B} \in S(H, Y)$. Let $L'' = L(\mathcal{A}'')$. Then

$$\begin{aligned} R'' &= \sum_{H \in \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})} \\ &= \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H,Y)} (-1)^{|\mathcal{B}|} t^{\dim Y} \\ &= -\sum_{Y \in L''} \sum_{\mathcal{B} \in S(H,Y)} (-1)^{|\mathcal{B} - \mathcal{A}_H|} t^{\dim Y} \\ &= -\sum_{Y \in L''} \mu(H, Y) t^{\dim Y} \\ &= -\chi(\mathcal{A}'', t). \end{aligned}$$

The penultimate equality follows from Lemma 2.1.4 and the fact that the Möbius function $\mu_{\mathcal{A}''}$ of L'' is the restriction of μ to L'', so that $\mu_{\mathcal{A}''}(Y) = \mu(H, Y)$. □

We now come to Zaslavsky's Theorem.

Theorem 2.1.7. *Let $\mathcal{A}$ be an arrangement in an n-dimensional real vector space. Then*

$$r(\mathcal{A}) = (-1)^n \chi(\mathcal{A}, -1).$$

Proof. The equation holds for $\mathcal{A} = \emptyset$, since $r(\emptyset) = 1$ and $\chi_\emptyset(t) = t^n$. Now by Lemma 2.1.3 and Theorem 2.1.6, both $r(\mathcal{A})$ and $(-1)^n \chi_{\mathcal{A}}(-1)$ satisfy the same recurrence, so the result follows. □

Theorem 2.1.8.

$$r'(\mathcal{A}_n^0) = (-1)^{n-1} \chi(\mathcal{A}_n^0, -1)$$

Proof.

$$r'(\mathcal{A}_n^0) = r(\mathcal{A}_{n-1}) = (-1)^{n-1} \chi(\mathcal{A}_{n-1}, -1) = (-1)^{n-1} \chi(\mathcal{A}_n^0, -1),$$

where the first and third equalitiies come from Theorem 1.2.8 and the second comes from Theorem 2.1.7. □

Now recall that E_n is the number of maximal unbalanced families of subsets of $[n]$. By Theorem 1.2.6, this is equal to $r'(\mathcal{A}_n^0)$, and by Theorem 2.1.8, this is equal to $(-1)^{n-1}\chi(\mathcal{A}_n^0, -1)$. We thus have three ways to calculate E_n. We can either

1. directly count the maximal unbalanced families,
2. directly count the number of regions of the arrangement A_n^0 in H_n, or
3. compute $\chi(A_n^0, t)$ and apply Theorem 2.1.8.

We will now demonstrate these three methods for various n, where feasible.

2.2 $n = 2$

1. There are 2 maximal unbalanced families, each with $2^{2-1} - 1 = 1$ element: $\{\{1\}\}$ and $\{\{2\}\}$.
2. H_2 consists of the line $x_1 + x_2 = 0$, and the arrangement $\mathcal{A}_2^0$ consists of $x_1 = 0$ and $x_2 = 0$ projected onto this line. Both of them project onto the point $(0, 0)$, which splits the line H_2 into two regions.
3. $L(\mathcal{A}_2^0)$ consists of $\hat{0} = H_2$ and the point $\hat{1} = (0, 0)$. Since $\hat{0} \lessdot \hat{1}$, we have $\mu(\hat{0}) = 1$ and $\mu(\hat{1}) = -\mu(\hat{0}) = -1$. Thus the Hasse diagram of $\mathcal{A}_2^0$, where the nodes contain the Möbius function values of the poset elements they represent, is

$\mu(\hat{1}) = -1$

$\mu(\hat{0}) = 1$

Hence $\chi(\mathcal{A}_2^0, t) = t - 1$, and $(-1)^1\chi(\mathcal{A}_2^0, -1) = 2$.

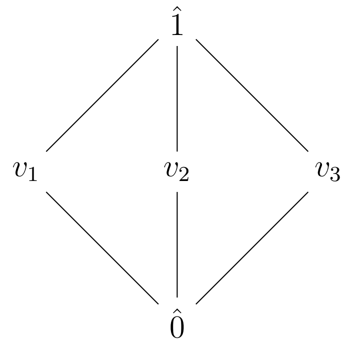
2.3 $n = 3$

1. There are 6 maximal unbalanced families, each with $2^{3-1} - 1 = 3$ elements:

$$\{\{1\}, \{1, 2\}, \{1, 3\}\}$$
$$\{\{2\}, \{1, 2\}, \{2, 3\}\}$$
$$\{\{3\}, \{1, 3\}, \{2, 3\}\}$$
$$\{\{2, 3\}, \{3\}, \{2\}\}$$
$$\{\{1, 3\}, \{3\}, \{1\}\}$$
$$\{\{1, 2\}, \{2\}, \{1\}\}$$

2. H_3 is the plane $x_1 + x_2 + x_3 = 0$. $\mathcal{A}_3^0$ consists of the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_1 + x_2 = 0$, $x_1 + x_3 = 0$, and $x_2 + x_3 = 0$. Of these six planes, the first and fourth, second and fifth, and third and sixth intersect H_3 on the same lines. Thus we only have three lines in H_3, defined by $x_1 = 0, x_2 = 0$, and $x_3 = 0$. Since all three lines pass through $(0, 0, 0) \in H_3$, they divide H_3 into 6 regions.

3. The elements in $L(\mathcal{A}_3^0)$ are $\hat{0} = H_3$, the lines v_i in H_3 defined by $x_i = 0$ for $i = 1, 2, 3$, and $\hat{1} = (0, 0, 0)$. Here is the Hasse diagram:



Note that this is exactly the same Hasse diagram as those in Examples 2.1.1 and 2.1.2. From those examples, we see that $\chi(\mathcal{A}_3^0, t) = t^2 - 3t + 2$ and $(-1)^2\chi(\mathcal{A}_3^0, -1) = 1 - 3(-1) + 2(1) = 6$.

2.4 $n = 4$

1. The maximal unbalanced families each have $2^{4-1} - 1 = 7$ elements and come in four classes:

   (a) $\{\{i\}\{i,j\}\{i,k\}\{i,l\}\{i,j,k\}\{i,j,l\}\{i,k,l\}\}$

   (b) $\{\{j,k,l\}\{k,l\}\{j,l\}\{j,k\}\{l\}\{k\}\{j\}\}$

   (c) $\{\{i\}\{j\}\{i,j\}\{i,k\}\{i,l\}\{i,j,k\}\{i,j,l\}\}$

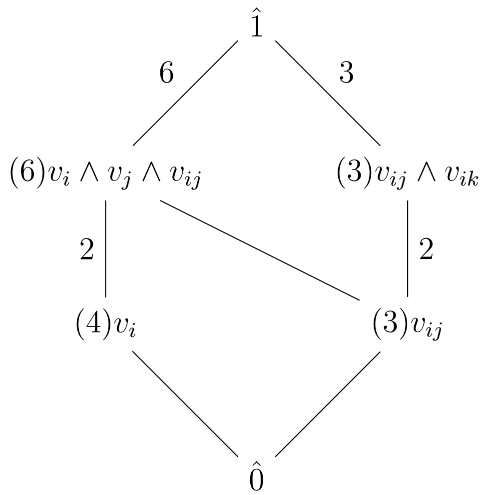
   (d) $\{\{j,k,l\}\{i,k,l\}\{k,l\}\{j,l\}\{j,k\}\{l\}\{k\}\}$

   where (i,j,k,l) is any permutation of $[4]$. Families in the first two classes are uniquely determined by the choice of $i \in [4]$, while families in the latter two classes are uniquely determined by the choice of (i,j). There are thus 4 families in the first class, 4 in the second, $4 \times 3 = 12$ in the third, and 12 in the fourth, for a total of 32 families.

2. H_4 is $x_1 + x_2 + x_3 = 0$. Under the restriction to H_4, the hyperplanes H_S and

H_T are the same if S and T partition $[4]$ (i.e., $S \cup T = [4]$ and $S \cap T = \emptyset$). Thus the only hyperplanes in $\mathcal{A}_4^0$ that we have to consider are $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_1 + x_2 = 0$, $x_1 + x_3 = 0$, and $x_1 + x_4 = 0$. Unfortunately, this arrangement is too complicated for us to compute $r(\mathcal{A}_4^0)$ directly.

3. The Hasse diagram of $L(\mathcal{A}_4^0)$ is too complex to be drawn here, but the structure is ($H_4 = \hat{0}, (0,0,0,0) = \hat{1}$):



Here, each node that appears in the diagram represents a particular form of an element in $L(\mathcal{A}_4^0)$. v_i represents the intersection of H_4 with the hyperplane $x_i = 0$, v_{ij} represents the intersection of H_4 with the hyperplane $x_i + x_j = 0$, and $x \wedge y$ represents the intersection of x and y. Note that we do not need to worry about hyperplanes of the form $x_i + x_j + x_k = 0$ because when restricted to H_4 such hyperplanes are equivalent to $x_l = 0$, where (i, j, k, l) is a permutation of $[4]$.

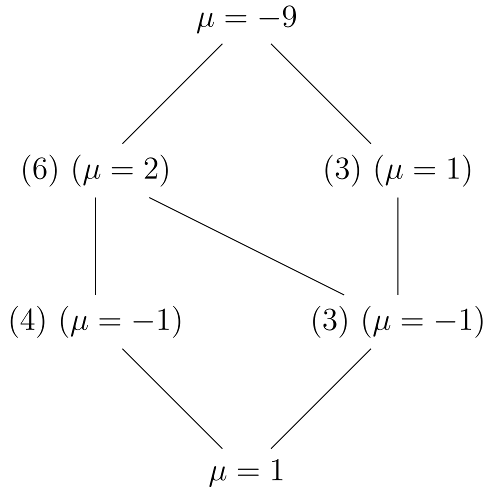
To the left of each form, in parentheses, is the number of times an element of that form actually appears in the poset. v_i, for example, appears 4 times: as v_1, v_2, v_3, and v_4. v_{ij} appears three times: as $v_{12} = v_{34}$, $v_{13} = v_{24}$, and $v_{14} = v_{23}$. $v_i \wedge v_j \wedge v_{ij}$ appears six times: we can have $(i, j) = (1,2), (1,3), (1,4), (2,3), (2,4)$, or $(3,4)$. Lastly, $v_{ij} \wedge v_{ik}$ appears three times: we have four choices for i and $\binom{3}{2} = 3$ choices for (j, k), but $v_{ij} \wedge v_{ik} =$, where (i, j, k, l) is a permutation of [4], so we must divide 4×3 by 4.

Also, a number m next to a line that goes upward from x to y means that y covers m elements of the form x. If no number is visible, then $m = 1$. v_i and v_{ij} just cover $\hat{0}$. $v_i \wedge v_j \wedge v_{ij}$ cover two elements of the form v_i (namely v_i and v_j) and $v_i j$. $v_{ij} \wedge v_{ik}$ covers two elements of the form v_{ij} (namely v_{ij} and v_{ik}). $\hat{1}$ covers all six elements of the form $v_i \wedge v_j \wedge v_{ij}$ and all three elements of the form $v_{ij} \wedge v_{ik}$.

We can now calculate the Möbius value for the flats and the number of flats of each form:

- $\mu(\hat{0}) = 1$
- $\mu(v_i) = -\mu(\hat{0}) = -1$
- $\mu(v_{ij}) = -\mu(\hat{0}) = -1$
- $\mu(v_i \wedge v_j \wedge v_{ij}) = -(2\mu(v_i) + \mu(v_{ij}) + \mu(\hat{0})) = -(2(-1) - 1 + 1) = 2$
- $\mu(v_{ij} \wedge v_{ik}) = -(2\mu(v_{ij}) + \mu(\hat{0})) = -(2(-1) - 1 + 1) = 1.$
-
$$\begin{aligned}\mu(\hat{1}) &= -(6\mu(v_i \wedge v_j \wedge v_{ij}) + 3\mu(v_{ij} \wedge v_{ik}) + 4\mu(v_i) + 3\mu(v_{ij}) + \mu(\hat{0})) \\ &= -(6(2) + 3(1) + 4(-1) + 3(-1) + 1) \\ &= -9\end{aligned}$$

Thus if we replace each node in the Hasse diagram with the number of elements it represents on the left and its Möbius function value on the right.



The characteristic polynomial is thus

$$\chi(\mathcal{A}_4^0, t) = t^3 + (4(-1) + 3(-1))t^2 + (6(2) + 3(1))t - 9 = t^3 - 7t^2 + 15t - 9.$$

We may now apply Theorem 2.1.8 to see that the number of regions is

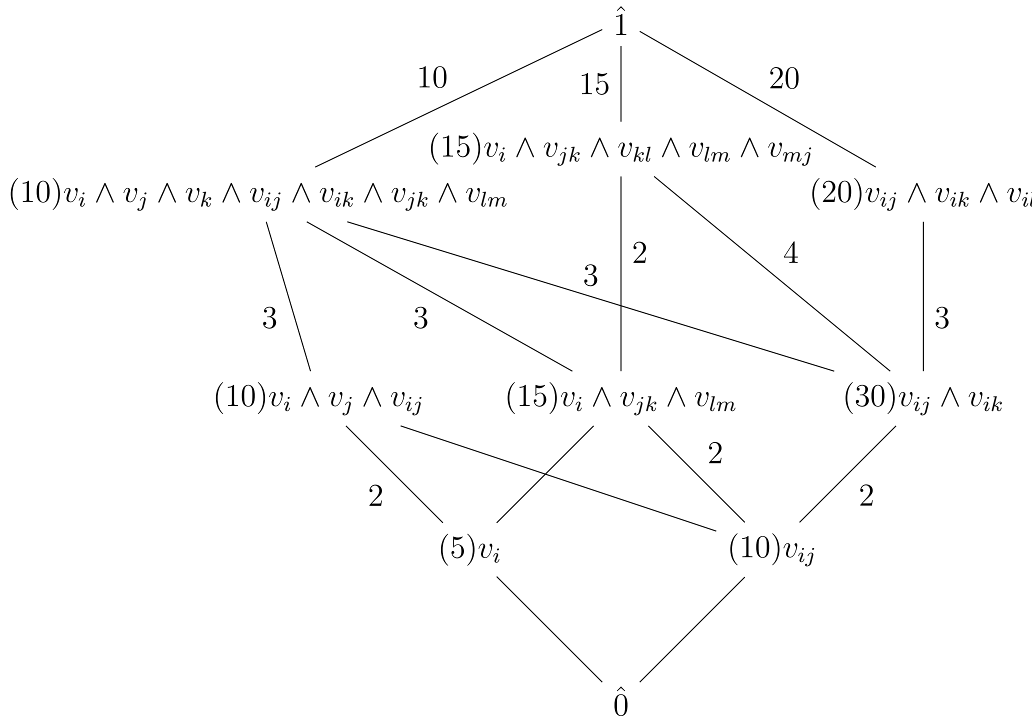
$$(-1)^3\chi(\mathcal{A}_4^0, -1) = -(-1 - 7 - 15 - 9) = 32.$$

2.5 $n = 5$

1. See Section 6.2.

2. H_5 is $x_1 + x_2 + x_3 + x_4 = 0$. Under the restriction to H_5, the hyperplanes H_S and H_T are the same if S and T partition [5]. Thus the only hyperplanes in $\mathcal{A}_5^0$ that we have to consider are $x_i = 0$ and $x_i + x_j = 0$, where $i, j \in [5]$ and $i \neq j$. This is an arrangement of $5 + 10 = 15$ hyperplanes. Unfortunately, this arrangement is too complicated for us to compute $r(\mathcal{A}_5^0)$ directly.

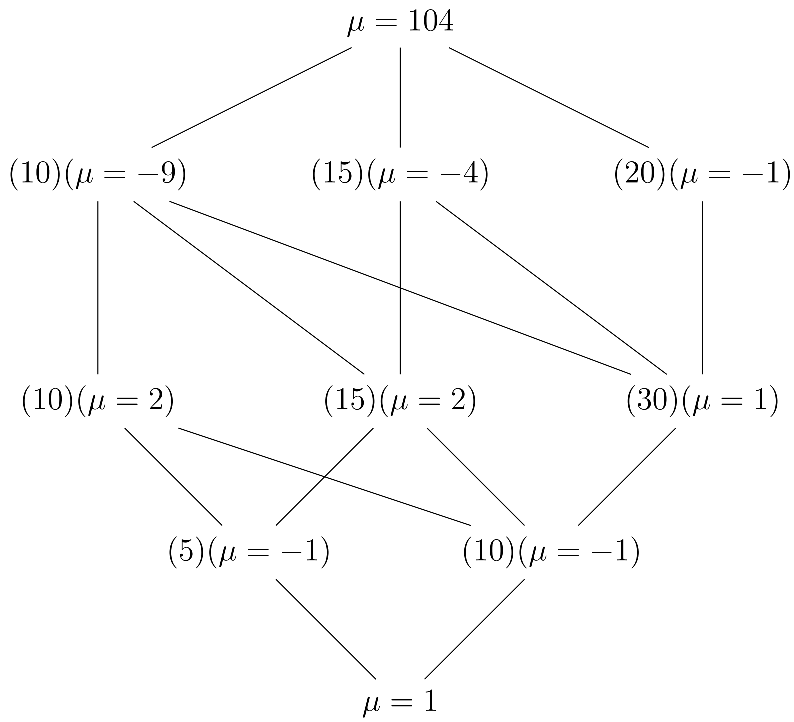
3. As in the case $n = 4$, the Hasse Diagram is too complicated to draw here. Using the notation from the previous case, the abbreviated diagram is



Note that we do not need to worry about hyperplanes of the form $x_i + x_j + x_k = 0$ or $x_i + x_j + x_k + x_l = 0$, because under the restriction to H_5 these hyperplanes are equivalent to $x_l + x_m = 0$ and $x_m = 0$, where (i, j, k, l, m) is a permutation of $[5]$.

v_i and v_{ij} each cover just $\hat{0}$. $v_i \wedge v_j \wedge v_{ij}$ covers v_i, v_j, and v_{ij}; $v_i \wedge v_{jk} \wedge v_{lm}$ covers v_i, v_{jk}, and v_{lm}; and $v_{ij} \wedge v_{ik}$ covers v_{ij} and v_{ik}. $v_i \wedge v_j \wedge v_k \wedge v_{ij} \wedge v_{ik} \wedge v_{ji} \wedge v_{lm}$ covers $v_a \wedge v_b \wedge v_{ab}$ for $(a, b) = (i, j), (i, k), (j, k)$, covers $v_a \wedge v_{bc} \wedge v_{lm}$ for $(a, b, c) = (i, j, k), (j, i, k), (k, i, j)$, and covers $v_{ia} \wedge v_{ib}$ for $(a, b, c) = (i, j, k), (j, i, k), (k, i, j)$. $v_i \wedge v_{jk} \wedge v_{kl} \wedge v_{lm} \wedge v_{mj}$ covers $v_i \wedge v_{ab} \wedge v_{cd}$ for $(a, b, c, d) = (i, j, k, l), (i, k, j, l)$ and covers $v_{ab} \wedge v_{ac}$ for $(a, b, c) = (j, k, m), (k, j, l), (l, k, m), (m, l, j)$. $v_{ij} \wedge v_{ik} \wedge v_{il}$ covers $v_{ia} \wedge v_{ib}$ for $(a, b) = (j, k), (j, l), (k, l)$. Lastly, $\hat{1}$ covers all $\binom{5}{3} = 10$ elements of the form $v_i \wedge v_j \wedge v_k \wedge v_{ij} \wedge v_{ik} \wedge v_{ji} \wedge v_{lm}$, all $\frac{5 \times 4!}{4 \times 2} = 15$ elements of the form $v_i \wedge v_{jk} \wedge v_{kl} \wedge v_{lm} \wedge v_{mj}$, and all 5×4 elements of the form $v_{ij} \wedge v_{ik} \wedge v_{il}$.

Using Equation 2.1 and the fact that $\mu(\hat{0}) = 1$, we can thus calculate $\mu(x)$ for all $x \in \mathcal{A}_5^0$. Replacing all of the x with $\mu(x)$ in the previous diagram, we get:



The characteristic polynomial is thus

$$\begin{aligned}\chi(\mathcal{A}_5^0, t) &= t^4 + (5(-1) + 10(-1))t^3 + (10(2) + 15(2) + 30(1))t^2 \\ &\quad + (10(-9) + 15(-4) + 20(-1))t + 104 \\ &= t^4 - 15t^3 + 80t^2 - 170t + 104\end{aligned}$$

Using Theorem 2.1.8, the number of regions is

$$(-1)^4\chi(\mathcal{A}_5^0, -1) = 1 + 15 + 80 + 170 + 104 = 370$$

2.6 $n > 5$

For values of n greater than 5, the method we have been using becomes infeasible by hand. The values for $n = 6, 7, 8, 9$ have, however, been calculated by T.S. Evans [4]. They are

$$E_6 = 11292, E_7 = 1066044, E_8 = 347326352, E_9 = 419172756930.$$

This is sequence A034997 in the Online Encyclopedia of Integer Sequences [3]. For $n > 9$, though, E_n is unknown.

CHAPTER 3

THE LOWER BOUND

Maximal unbalanced families correspond to the regions of the hyperplane arrangement defined by the linear forms

$$\sum_{i \in F} y_i \qquad (F \subseteq [n], F \neq \emptyset, [n]) \tag{3.1}$$

where y_i denotes the i-th coordinate of $y \in H_n$. To derive a lower bound on the number of maximal unbalanced families, and thus on the number of regions of the arrangement in (3.1), we will need a combinatorially equivalent form of (3.1) given by the linear forms

$$\sum_{i \in F} y_i \qquad (F \subseteq [n-1], F \neq \emptyset) \tag{3.2}$$

in $\mathbb{R}^{n-1}$. In other words, we consider the arrangement $\mathcal{A}_n$ in $\mathbb{R}^{n-1}$ consisting of the $2^{n-1} - 1$ hyperplanes having as normals all nonzero 0-1 vectors in $\mathbb{R}^{n-1}$. In order to count the number of components of $\mathbb{R}^{n-1} \setminus \mathcal{A}_n$, we will apply Zaslavsky's Theorem (Theorem 2.1.7).

Define the **lattice of flats** L_n of the arrangement $\mathcal{A}_n$ to be the family of all subspaces spanned over $\mathbb{Q}$ by subsets of the set of nonzero 0-1 vectors in $\mathbb{R}^{n-1}$, ordered by inclusion. The rank of L_n is $n-1$. The characteristic polynomial of $\mathcal{A}_n$ is

$$\chi(\mathcal{A}_n, t) = \sum_{x \in L_n} \mu(0, x)\ t^{\operatorname{rank}(L_n) - \operatorname{rank}(x)} = \sum_{k=0}^{n-1} w_k(L_n)\ t^{n-1-k}, \tag{3.3}$$

where μ is the Möbius function of L_n. The quantities $w_k(L_n)$ in (3.3) are called the **Whitney numbers of the first kind**.

The result of Zaslavsky [14] is that the number of chambers of $\mathcal{A}_n$ is

$$(-1)^{n-1}\chi(\mathcal{A}_n, -1) = \sum_{x \in L_n} |\mu(0, x)| = \sum_{k=0}^{n-1} |w_k(L_n)|. \tag{3.4}$$

Unfortunately, we do not have an explicit formula for the polynomial $\chi(\mathcal{A}_n, t)$.

To give a lower bound for the number of chambers of $\mathcal{A}_n$, we consider the linear matroid of all subspaces spanned over the 2-element field $\mathbb{F}_2$ by these same 0-1 vectors, now considered to be the set $\mathbb{F}_2^{n-1} \setminus \{(0,0,\ldots,0)\}$. By abuse of notation, we will denote this matroid by $\mathcal{A}_n^{(2)}$ and its lattice of flats by $L_n^{(2)}$. The rank of $L_n^{(2)}$ is again $n-1$.

Since independence over $\mathbb{F}_2$ implies independence over $\mathbb{Q}$, we have that the map $\mathcal{A}_n \to \mathcal{A}_n^{(2)}$ is a rank-preserving weak map, and so, by a theorem of Lucas [8, Proposition 7.4] (see also [7, Corollary 9.3.7]), we obtain

$$|w_k(\mathcal{A}_n)| \geq |w_k(\mathcal{A}_n^{(2)})|$$

for each k, and so we conclude

$$(-1)^{n-1}\chi(\mathcal{A}_n, -1) \geq (-1)^{n-1}\chi(\mathcal{A}_n^{(2)}, -1). \tag{3.5}$$

To complete the bound, we observe that $\mathcal{A}_n^{(2)}$ is the $(n-1)$-dimensional projective geometry over $\mathbb{F}_2$, and so its characteristic polynomial (see, for example, [2, Example 3.6(3)]) is

$$\chi(\mathcal{A}_n^{(2)}, t) = \prod_{i=0}^{n-2}(t - 2^i). \tag{3.6}$$

Together, (3.4), (3.5) and (3.6) give us a lower bound:

Theorem 3.0.1. *The number of maximal unbalanced families in $[n]$, equivalently, the number of chambers of the arrangement $\mathcal{A}_n$, is at least $\prod_{i=0}^{n-2}(2^i+1)$. Thus*

$$E_n > \prod_{i=0}^{n-2} 2^i = 2^{\frac{(n-1)(n-2)}{2}}.$$

Below is a table giving the lower bound and the actual number of regions for $2 \leq n \leq 9$.

n	Actual Number of Regions	Lower Bound	Ratio
2	2	2	1
3	6	6	1
4	32	30	1.06
5	370	270	1.37
6	11,292	4,590	2.46
7	1,066,044	151,470	7.04
8	347,326,352	9,845,550	35.3
9	419,172,756,930	1,270,075,950	330

CHAPTER 4
THE UPPER BOUND

Let $\mathfrak{M}_n$ denote the set of all maximally unbalanced sets of subsets of $[n]$. Recall that if $\mathcal{F}$ is in $\mathfrak{M}_n$, then $\mathcal{F}$ contains neither $\emptyset$ nor $[n]$. On the other hand, if $F \subseteq [n]$ is neither $\emptyset$ nor $[n]$, then exactly one of F and $[n] \setminus F$ is in $\mathcal{F}$ (Theorems 1.1.3 and 1.2.4). In other words, if $\mathcal{F}$ is maximal, it selects between every nontrivial subset of $[n]$ and is complement. Let $\mathfrak{S}_n$ denote the collection of all families $\mathcal{A}$ of subsets of $[n]$ such that:

- neither $\emptyset$ nor $[n]$ are in $\mathcal{A}$;
- if A is a proper nonempty subset of $[n]$, then exactly one of A and $[n] \setminus A$ are in $\mathcal{A}$.

Note that $\mathfrak{M}_n \subset \mathfrak{S}_n$, and the inclusion is strict for $n \geq 3$ [1].

The collection $\mathfrak{M}_n$ is equipped with a natural notion of adjacency: $\mathcal{F}$ and $\mathcal{G}$ are adjacent if $|\mathcal{F} \setminus \mathcal{G}| = 1$. By Theorem 1.2.6, we can view elements of $\mathfrak{M}_n$ as regions in the hyperplane arrangement $\mathcal{A}_n^0$. If we do this, then this notion of adjacency for families coincides with the notion of adjacency for regions. This is because if two adjacent regions are separated by a single hyperplane H_S, then crossing that hyperplane from the side containing χ_S to the other side corresponds to swapping out S for $[n]\setminus S$, while crossing in the other direction corresponds to swapping $[n]\setminus S$ for S. Thus $\mathcal{F}$ and $\mathcal{G}$ are adjacent families if and only if they differ by a single element, which happens if and only if they are separated by a single hyperplane, which happens if and only if they are adjacent regions.

Theorem 4.0.2. *The adjacency graph on $\mathfrak{M}_n$ is connected.*

Proof. Given any two families $\mathcal{F}, \mathcal{G} \in \mathfrak{M}_n$, we can view them as regions of $\mathcal{A}_n^0$ in H_n. Since H_n is connected, we can get from region $\mathcal{F}$ to $\mathcal{G}$ in $\mathcal{R}(\mathcal{A}_n^0)$ by crossing a series of hyperplanes. This is equivalent to saying we can start with $\mathcal{F}$ and swap out subsets of $[n]$ one at a time for their complements and end with $\mathcal{G}$. In this way, we get a sequence of adjacent unbalanced families starting with $\mathcal{F}$ and ending in $\mathcal{G}$. □

If $\mathcal{F}$ is in $\mathfrak{S}_n$ for some n, define the **signature** of $\mathcal{F}$ – denoted $\operatorname{sig}(\mathcal{F})$ – to be the sequence s of length n such that its i-th term is

$$s_i = |\{F \in \mathcal{F} : i \in F\}| \,.$$

We can view the signature as a map from $\mathfrak{S}$ to $\mathbb{Z}^n$. The goal of this section is to prove that the signature map is injective on $\mathfrak{M}_n$ for each n. Moreover we will show that the signature of an element of $\mathfrak{M}_n$ can never coincide with the signature of an element of $\mathfrak{S}_n \setminus \mathfrak{M}_n$. We will also show that the parities of the entries of $\operatorname{sig}(\mathcal{F})$ are always the same.

These observations are already enough to yield an upper bound. To see this, observe that there are $(2^{n-1})^n$ possible signatures of a family with $2^{n-1}-1$ elements, since each entry in the signature can only range from 0 to $2^{n-1}-1$. Furthermore, if we require that all entries are even or all are odd, there are fewer than $(2^{n-1})^n/2^{n-1} = 2^{(n-1)^2}$ such signatures, which is an upper bound.

To verify the above claims, define $\delta_F \in \{-1, 1\}^n$, for F a nonempty proper subset of $[n]$ and $i < n$, by

$$\delta_F(i) = \begin{cases} 1 & \text{if } i \in F \\ -1 & \text{if } i \notin F. \end{cases}$$

If $\mathcal{F}$ is a family of nonempty proper subsets of n, define

$$\delta_{\mathcal{F}} = \sum_{F \in \mathcal{F}} \delta_F.$$

Lemma 4.0.3. *If $\mathcal{F}$ is a nonempty unbalanced family of subsets of $[n]$, then $\delta_{\mathcal{F}}$ is not constant.*

Proof. Suppose that this is not the case and notice that the cardinalities of

$$\{F \in \mathcal{F} : i \in F\}$$

and

$$\{F \in \mathcal{F} : i \notin F\}$$

do not depend on i; this is because their difference is the constant entry of $\delta_{\mathcal{F}}$ and their sum is the cardinality of $\mathcal{F}$. This means, however, that the signature of $\mathcal{F}$ is constant and, in particular, that the uniform probability measure on $\mathcal{F}$ witnesses that $\mathcal{F}$ is balanced, a contradiction. □

Theorem 4.0.4. *The function which takes an element of $\mathfrak{S}_n$ to its signature is one-to-one on $\mathfrak{M}_n$. Furthermore, the signature of an element of $\mathfrak{M}_n$ can never coincide with the signature of a balanced family in $\mathfrak{S}_n$.*

Proof. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are distinct elements of $\mathfrak{S}_n$ and that $\mathcal{B}$ is unbalanced. Observe that $\mathcal{A}$ and $\mathcal{B}$ have the same signature if and only if $\delta_{\mathcal{B} \setminus \mathcal{A}} = (0, \ldots, 0)$: as $\mathcal{A}$ and $\mathcal{B}$ differ by a series of swaps, $\mathcal{B} \setminus \mathcal{A}$ is the family of swapped sets. Hence, $\mathrm{sig}(\mathcal{B}) = \mathrm{sig}(\mathcal{A}) + \delta_{\mathcal{B} \setminus \mathcal{A}}$. Since $\mathcal{B}$ is unbalanced, so is $\mathcal{B} \setminus \mathcal{A}$, so Lemma 4.0.3 implies that $\delta_{\mathcal{B} \setminus \mathcal{A}}$ is not constant and, in particular, is not identically 0. Consequently, $\mathcal{A}$ and $\mathcal{B}$ have distinct signatures. □

Proposition 4.0.5. *If $\mathcal{F}$ is an element of $\mathfrak{M}_n$, then either all entries of* $\mathrm{sig}(\mathcal{F})$ *are even or all entries are odd.*

Proof. If $\mathcal{F}$ is the family of all nonempty subsets of $[n]$ that do not contain 1, then $\mathrm{sig}(\mathcal{F})$ is the sequence $(0, 2^{n-2}, \ldots, 2^{n-2})$. In particular, the conclusion of the proposition holds for $\mathcal{F}$. Next observe that if $\mathcal{G}_0$ and $\mathcal{G}_1$ are adjacent elements of $\mathfrak{M}_n$, then every coordinate of $\mathrm{sig}(\mathcal{G}_0)$ differs by ± 1 from the corresponding coordinate of $\mathrm{sig}(\mathcal{G}_1)$. The proposition now follows from the connectedness of the adjacency graph on $\mathfrak{M}_n$. □

The above proof actually shows that the adjacency graph on $\mathfrak{M}_n$ is bipartite: families with odd signature entries can only adjacent to families with even signature entries, and families with even signature entries can only be adjacent to families with odd signature entries.

We have thus proved the upper bound:

Theorem 4.0.6. $E_n \leq 2^{(n-1)^2}$.

Again we can create a table giving the upper bound and the actual number of regions for $2 \leq n \leq 9$.

n	Actual Number of Regions	Upper Bound	Ratio
2	2	2	1
3	6	16	2.7
4	32	512	16
5	370	65536	177
6	11,292	33,554,432	2972
7	1,066,044	68,719,476,736	64462
8	347,326,352	5.63×10^{14}	1,620,810
9	419,172,756,930	1.84×10^{19}	44,007,498

CHAPTER 5

THE SWAP DISTANCE FUNCTION

In this chapter, we develop a notion of distance between maximal unbalanced families.

Let $\mathrm{sw}_F : \mathfrak{M}_n \to \mathfrak{M}_n$ be the swap function, defined by $\mathrm{sw}_F(\mathcal{F}) = (\mathcal{F} \setminus F) \cup \{[n] \setminus F\}$ if $F \in \mathcal{F}$ and be undefined otherwise.

Furthermore, let $d : \mathfrak{M}_n^2 \to \mathbb{Z}$ be the swap distance within $\mathfrak{M}_n$. That is, for all $\mathcal{F}, \mathcal{G} \in \mathfrak{M}_n$, we define $d(\mathcal{F}, \mathcal{G})$ to be the smallest δ such that there is a sequence $\mathcal{F}_0, \ldots, \mathcal{F}_\delta$, where $\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_\delta = \mathcal{G}$, and for all $i < \delta$, there is some nonempty $F \subsetneq [n]$ such that $\mathcal{F}_{i+1} = \mathrm{sw}_F(\mathcal{F}_i)$; in addition, all of the $\mathcal{F}_i$ are in $\mathfrak{M}_n$. In other words, $d(\mathcal{F}, \mathcal{G})$ is the minimum number of swaps needed to get from $\mathcal{F}$ to $\mathcal{G}$ while staying in $\mathfrak{M}_n$. Note that $d(\mathcal{F}, \mathcal{G}) = d(\mathcal{G}, \mathcal{F})$, since we can flip the sequence $\mathcal{F}_0, ..., \mathcal{F}_\delta$ to get a sequence $\mathcal{F}_\delta, ..., \mathcal{F}_0$ starting at $\mathcal{G}$ and ending at $\mathcal{F}$ after a series of δ swaps.

Theorem 5.0.7. *If $\mathcal{F}, \mathcal{G} \in \mathfrak{M}$, then there is a sequence $\mathcal{F}_0, \ldots, \mathcal{F}_\delta$ satisfying the following properties:*

1. *$\mathcal{F}_0 = \mathcal{F}$.*
2. *$\mathcal{F}_\delta = \mathcal{G}$.*
3. *For all $i < \delta$, there is some nonempty $F \subsetneq [n]$ such that $\mathcal{F}_{i+1} = sw_F(\mathcal{F}_i)$.*
4. *$\mathcal{F}_i \in \mathfrak{M}$ for all i.*
5. *$\mathcal{G}$ is obtained from $\mathcal{F}$ without swapping the same set twice. That is, if F_i is the unique set such that $sw_F(\mathcal{F}_i) = \mathcal{F}_{i+1}$, then $i_1 \neq i_2$ implies that $F_1 \neq F_2$ and $F_1 \neq [n] \setminus F_2$.*

Proof. By Theorem 1.2.6, we may view $\mathcal{F}$ and $\mathcal{G}$ as regions in $\mathcal{A}_n^0$. Let $\mathbf{x} \in \mathcal{F}$ and $\mathbf{y} \in \mathcal{G}$ be points in the regions such that the straight line l connecting $\mathbf{x}$ to $\mathbf{y}$ does not cross any places where hyperplanes of $\mathcal{A}_n^0$ intersect.

Parametrize l so that $l(0) = \mathbf{x}$ and $l(1) = \mathbf{y}$. As t increases from 0 to 1, $l(t)$ passes from $\mathcal{F}$ to $\mathcal{G}$; every time it crosses a hyperplane, it only crosses one such plane, so as such a crossing occurs, a set in $\mathcal{F}_i$ gets swapped for its complement. Also, $\mathcal{F}_i$ is always maximally unbalanced. This means that $\mathcal{F}$ is transformed into $\mathcal{G}$ by swapping sets in $\mathcal{F}$ for their complements one at a time while staying in $\mathfrak{M}$. The sequence of families that $\mathcal{F}$ is turned into by this series of swaps is a sequence $\mathcal{F}_0, \ldots, \mathcal{F}_\delta$ that satisfies (1), (2), (3), and (4).

We still need to show that $\mathcal{F}_0, \ldots, \mathcal{F}_\delta$ satisfies (5). Since l is straight, it will cross each hyperplane in $\mathcal{A}_n^0$ at most once. This means that for all $F \subset [n]$, as t goes from 0 to 1, $\mathcal{F}_i$ cannot switch F for $[n] \setminus F$ and later switch $[n] \setminus F$ for n. So our sequence $\mathcal{F}_0, ..., \mathcal{F}_\delta$ is the desired sequence. □

Theorem 5.0.8. *For all $\mathcal{F}, \mathcal{G} \in \mathfrak{M}$, we have $d(\mathcal{F}, \mathcal{G}) = |\mathcal{G} \setminus \mathcal{F}|$.*

Proof. By Theorem 5.0.7, we can get from $\mathcal{F}$ to $\mathcal{G}$ through a series of swaps such that the family stays in $\mathfrak{M}$ and no set gets swapped twice. Since for every nonempty $F \subsetneq [n]$ both $\mathcal{F}$ and $\mathcal{G}$ contain exactly one of F and $[n] \setminus F$, the only way we can do this is by only swapping out sets in $\mathcal{F} \setminus \mathcal{G}$ for sets in $\mathcal{G} \setminus \mathcal{F}$ one at a time. This takes $|\mathcal{G} \setminus \mathcal{F}|$ swaps, so $d(\mathcal{F}, \mathcal{G}) = |\mathcal{G} \setminus \mathcal{F}|$. □

CHAPTER 6

ORBITS

A **group** $(G, \cdot)$ is a set G with a binary operation $\cdot : G \times G \to G$ such that

- (Closure) For all $a, b \in G$, we have $a \cdot b \in G$.
- (Associativity) For all $a, b, c \in G$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- (Identity) There exists $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$.
- (Inverse) For each $a \in G$, there exists an element $b \in G$ such that $a \cdot b = b \cdot a = e$.

If there is no risk of confusion, we omit the $\cdot$.

Example 6.0.9. The **symmetric group** on a finite set X is the group whose elements are all bijective functions $X \to X$ and whose group operation is function composition. The **symmetric group of degree** n is the symmetric group on $[n]$ and is denoted S_n. We may denote elements f of S_n by strings that are permutations of the elements of $[n]$, such that $f(i)$ is the i-th symbol in the string. For example, if $n = 4$, then 1324 denotes the bijective function $f : [4] \to [4]$ such that $f(1) = 1, f(2) = 3, f(3) = 2$, and $f(4) = 4$.

If G is a group and X is a set, then a **group action** of G on X is a function $\cdot : G \times X \to X$ satisfying

- (Associativity) For all $g, h \in G$ and all $x \in X$, we have $(gh) \cdot x = g \cdot (h \cdot x)$
- (Identity) If e is the identity element in G, then $e \cdot x = x$ for all $x \in X$.

In this case, we say that G **acts on** X.

S_n acts on $\mathfrak{M}_n$ in the following sense. Suppose that $s \in S_n$ and $\mathcal{F} = \{F_1, ..., F_m\} \in \mathfrak{M}_n$. Then define

$$s \cdot \mathcal{F} = \{s \cdot F : F \in \mathcal{F}\}.$$

Example 6.0.10. Suppose $\mathcal{F} = \{\{1\}, \{2\}, \{1, 2\}\}$ and $s = 132$ (i.e., s swaps 2 and 3). Then

$$s \cdot \mathcal{F} = \{\{1\}, \{3\}, \{1, 3\}\}.$$

If G is a group acting on X, then the **orbit** of an element $x \in X$ is the set of elements of X to which x can be moved by the elements of G. In other words, if Gx denotes the orbit of x, then

$$Gx = \{gx | g \in G\}.$$

The orbits of a set X partition X. That is, if $P = \{Gx | x \in X\}$, then

- $\emptyset \notin P$,
- $\bigcup_{p \in P} p = X$, and
- For all $p_1, p_2 \in P$, we have $p_1 \cap p_2 = \emptyset$.

Example 6.0.11. Suppose we have S_3 acting on $\mathfrak{M}_3$. The orbit of $\mathcal{F} := \{\{1, 3\}, \{2, 3\}, \{3\}\}$ consists of three elements:

- $231 \cdot \mathcal{F} = 321 \cdot \mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{1\}\}$,
- $132 \cdot \mathcal{F} = 312 \cdot \mathcal{F} = \{\{1, 2\}, \{2, 3\}, \{2\}\}$, and
- $123 \cdot \mathcal{F} = 213 \cdot \mathcal{F} = \{\{1, 3\}, \{2, 3\}, \{3\}\}$.

The orbit of $\mathcal{G} := \{\{1\}, \{2\}, \{1, 2\}\}$ also consists of three elements:

- $231 \cdot \mathcal{G} = 321 \cdot \mathcal{G} = \{\{2\}, \{3\}, \{2, 3\}\}$,
- $132 \cdot \mathcal{G} = 312 \cdot \mathcal{G} = \{\{1\}, \{3\}, \{1, 3\}\}$, and
- $123 \cdot \mathcal{G} = 213 \cdot \mathcal{G} = \{\{1\}, \{2\}, \{1, 2\}\}$.

Since these orbits are disjoint and $|\mathfrak{M}_3| = E_3 = 6$, we see that these are the only 2 orbits in $\mathfrak{M}_3$.

Let O_n denote the number of orbits of maximal unbalanced families. From Example 6.0.11, we have already seen that $O_3 = 2$. Note that the i-th family in the first orbit consists of all proper subsets of [3] containing i, and the i-th family in the second orbit consists of all proper subsets of [3] that do not contain i. Thus for a particular i, we will denote the set of all proper subsets of $[n]$ containing i as $\underline{i}$, and we will denote the orbit containing all families of the form $\underline{i}$ by $\mathbf{i}$. Similarly, we will denote the set of all proper subsets of $[n]$ not containing i as $\not{\underline{i}}$ and the orbit containing all familes of the form $\not{\underline{i}}$ by $\not{\mathbf{i}}$.

Another thing to note is that for $i = 1, 2, 3$, the elements in the i-th family of the first orbit are the complements of the elements in the i-th family of the second orbit. That is, given an $\mathcal{F} \in \mathfrak{M}_n$, define

$$\bar{\mathcal{F}} := \{[n] \setminus F | F \in \mathcal{F}\}$$

to be the **complement** of $\mathcal{F}$. Then $231 \cdot \mathcal{F}$ and $231 \cdot \mathcal{G}$ are complements, $132 \cdot \mathcal{F}$ and $132 \cdot \mathcal{G}$ are complements, and $\mathcal{F}$ and $\mathcal{G}$ are complements. Similarly, given an orbit $\mathcal{O}$, define its complement $\bar{\mathcal{O}}$ by

$$\bar{\mathcal{O}} := \{\bar{\mathcal{F}} | \mathcal{F} \in \mathcal{O}\}.$$

Thus we see that the two orbits of $\mathfrak{M}_3$ are complements of each other. In general, $\mathbf{i}$ and $\cancel{\mathbf{i}}$ are complements of each other.

Lastly, note that any family in the first orbit of $\mathfrak{M}_3$ is adjacent to some family in the second orbit. For example, $\mathcal{F}$ can be transformed into $231 \cdot \mathcal{G}$ by swapping out $\{1, 3\}$ for $\{2\}$. In general, we will say that two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are **adjacent** if each family in $\mathcal{O}_1$ has a corresponding family in $\mathcal{O}_2$ to which it is adjacent and each family in $\mathcal{O}_2$ has a corresponding family in $\mathcal{O}_1$ to which it is adjacent.

6.1 $n = 4$

The four orbits and their sizes have already been calculated in item (1) under Section 2.4. Note that the first orbit is $\mathbf{i}$ and the second orbit is $\cancel{\mathbf{i}}$, the first orbit's complement. The third orbit is adjacent to the first orbit: it is exactly the same as it except that $\{i, k, l\}$ has been replaced with $\{j\}$; we will denote it by $(\mathbf{i}, ikl \to j)$. The fourth orbit is adjacent to the second orbit: it is exactly the same as the second orbit except that $\{j\}$ has been replaced with $\{i, k, l\}$. It is also adjacent to the third orbit: in the third orbit, swap i with k and j with l to get

$$\{\{k\}\{l\}\{k, l\}\{i, k\}\{k, j\}\{k, l, i\}\{k, l, j\}\}.$$

This is still in the third orbit, but if we swap $\{i, k\}$ for $\{j, l\}$, we get something in the fourth orbit, as desired. Lastly, this orbit can be viewed as the complement of the third orbit, so we will denote it by $\cancel{(\mathbf{i}, ikl \to j)}$.

6.2 $n = 5$

In this case, there are 12 orbits. We will use the same notation as before to denote them: if x is an orbit, then $x, a \to b$, where a and b are strings, denotes the result when the set containing the letters in a is swapped for the set containing the letters in b. The 12 orbits are:

1. $\mathbf{i}$. This orbit has size 5.
2. $\mathbf{i}, ijkl \to m$. This orbit has size 20.
3. $\mathbf{i}, ijkl \to m, ijk \to lm$. This orbit has size 60.
4. $\mathbf{i}, ijkl \to m, ijk \to lm, ijl \to km$. This orbit has size 60.
5. $\mathbf{i}, ijkl \to m, ijk \to lm, ijkm \to l$. This orbit has size 30.
6. $\mathbf{i}, ijkl \to m, ijk \to lm, ijl \to km, ikl \to jm$. This orbit has size 10.

7-12. The complements of orbits 1-6, respectively.

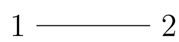
6.3 $n > 5$

We have, by [5],

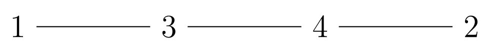
$$O_6 = 56, O_7 = 576, O_8 = 16640.$$

6.4 Adjacency Graphs

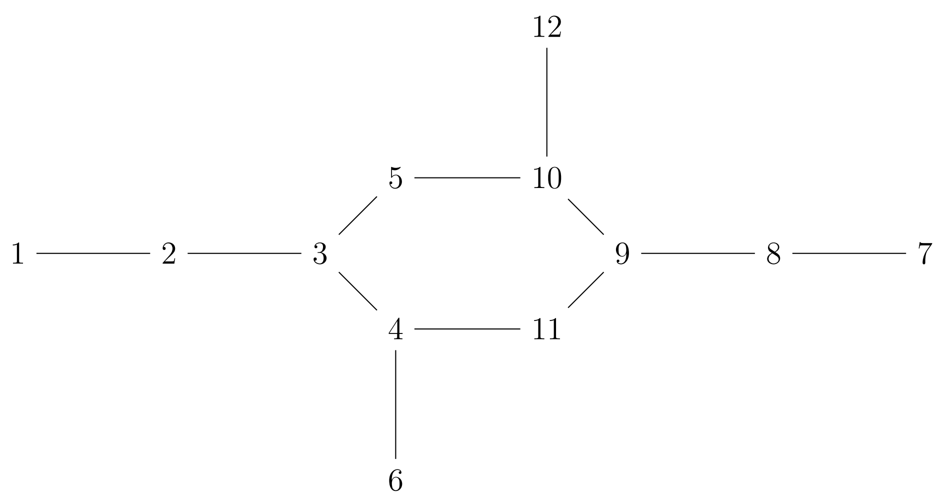
Using our notation of adjacency, we can draw **adjacency graphs** of orbits. In these graphs, the nodes are orbits, and two nodes have an edge between them if the orbits they correspond to are adjacent. In the diagrams that follow, the i-th node will contain i, under the numbering that we have established in the previous section. For $n = 3$, we have two orbits that are adjacent to each other, so the graph looks like:



For $n = 4$, orbit 1 is adjacent to orbit 3, orbit 3 is adjacent to orbit 4, and orbit 4 is adjacent to orbit 2, so the graph looks like:



For $n = 5$, the graph looks like:



6.5 Properties

Theorem 6.5.1. *If $\mathcal{F}$ is a maximal unbalanced family, then so is $\bar{\mathcal{F}}$.*

Proof. Since $\mathcal{F}$ has $2^{n-1} - 1$, so does $\bar{\mathcal{F}}$, so it suffices to show that $\bar{\mathcal{F}}$ is unbalanced. Suppose, for sake of contradiction, that it is balanced. Then there exist $S_1, ..., S_m \in$

$\bar{\mathcal{F}}$ and $\alpha_1, ..., \alpha_m > 0$ such that

$$\begin{aligned}
(1, ..., 1) &= \sum_{i=1}^{m} \alpha_i \chi_{S_i} \\
&= \sum_{i=1}^{m} \alpha_i((1, ..., 1) - \chi_{[n]\setminus S_i}) \\
&= \sum_{i=1}^{m} \alpha_i(1, ..., 1) - \sum_{i=1}^{m} \alpha_i \chi_{[n]\setminus S_i} \\
&= \sum_{i=1}^{m} (\alpha_i, ..., \alpha_i) - \sum_{i=1}^{m} \alpha_i \chi_{[n]\setminus S_i}
\end{aligned}$$

Let $\alpha = \sum_{i=1}^{m} \alpha_i$. Then

$$\begin{aligned}
(\alpha - 1, ..., \alpha - 1) &= \sum_{i=1}^{m} \alpha_i \chi_{[n]\setminus S_i} \\
\implies (1, ..., 1) &= \sum_{i=1}^{m} \frac{\alpha_i}{\alpha - 1} \chi_{[n]\setminus S_i}
\end{aligned}$$

Since $S_1, ..., S_m \in \bar{\mathcal{F}}$, we have $[n] \setminus S_i \in \mathcal{F}$ for all i. Furthermore, the χ_{S_i} are 0-1 vectors whose weighted sum is the all-ones vector, so the sum of the weights α must be greater than 1. Thus the above equation implies that $\mathcal{F}$ is balanced, a contradiction. □

Theorem 6.5.2. *O_n is always even.*

Proof. For each orbit $\mathcal{O}$, $\bar{\mathcal{O}}$ is also an orbit. Thus it suffices to show that no orbit is its own complement. Suppose, for sake of contradiction, that there is an orbit $\mathcal{O}$ that is its own complement. Since the adjacency graph of $\mathfrak{M}_n$ is connected (Theorem 4.0.2), the adjacency graph of orbits is connected. In particular, there is a path from $\mathbf{i}$ to $\mathcal{O}$; suppose it has length m. That is, there is a sequence of orbits $\mathcal{O}_0, ..., \mathcal{O}_m$ such that $\mathcal{O}_0 = \mathbf{i}, \mathcal{O}_m = \mathcal{O}$, and $\mathcal{O}_i$ is adjacent to $\mathcal{O}_{i+1}$ for all $i = 0, ..., m-1$. By symmetry, there is a path from $\bar{\mathbf{i}}$ to $\bar{\mathcal{O}} = \mathcal{O}$, namely $\bar{\mathcal{O}}_0, ..., \bar{\mathcal{O}}_m$.

We can combine these two paths to get a path $\mathcal{O}_0 = \mathbf{i}, ..., \mathcal{O}_m = \bar{\mathcal{O}}_m, ..., \bar{\mathcal{O}}_0 = \bar{\mathbf{i}}$ of length $2m$ from $\mathbf{i}$ to $\bar{\mathbf{i}}$.

From the proof of Theorem 4, we know that families with odd signature entries can only adjacent to families with even signature entries, and families with even signature entries can only be adjacent to families with odd signature entries. For a fixed orbit, the signatures of families in that orbit are all permutations of each other. Thus, in any orbit $\mathcal{O}$, the entries of the families in the orbit are either all even or all odd; furthermore, orbits with odd signature entries can only adjacent to families with even signature entries, and families with even signature entries can only be adjacent to families with odd signature entries. Hence the adjacency graph of the orbits is bipartite: one independent set X consists of the orbits whose families' signatures contain even entries, while the other set Y consists of the orbits whose families' signatures' entries are even. The signatures of the families in $\mathbf{i}$ are some permutation of $(2^{n-1} - 1, 2^{n-2} - 1, ..., 2^{n-2} - 1)$, while the signatures of the families in $\bar{\mathbf{i}}$ are some permutation of $(0, 2^{n-2}, ..., 2^{n-2})$. Thus $\mathbf{i} \in X$ and $\bar{\mathbf{i}} \in Y$. But this means that any path from $\mathbf{i}$ to $\bar{\mathbf{i}}$ must have odd length, which contradicts our earlier result that this length is $2m$. □

Theorem 6.5.3. *Suppose that a given signature* $\mathrm{sig}(\mathcal{F})$ *repeats itself. That is, suppose that there exists some* d *such that* $\mathrm{sig}(\mathcal{F})(i) = \mathrm{sig}(\mathcal{F})((i + d) \bmod n)$ *for all* i. *Then* $\mathcal{F}$ *is not a maximal unbalanced family.*

Proof. Since the signature $\mathrm{sig}(\mathcal{F})$ repeats itself, we can write it as $\mathrm{sig}(\mathcal{F}) = \underbrace{(X, X, ..., X)}_{n/j}$, where X is a sequence and $n > j$. It is not hard to show that, in general, the symmetries of the witness vector of a family are the symmetries of the signature of that family, implying that there must be a witness vector y

corresponding to the signature that is of the form $y = \underbrace{(Y, Y, ..., Y)}_{n/j}$. Suppose $Y = (y_0, ..., y_{k-1})$. Since y lies in H_0, its entries sum to 0, so we have

$$(n/j)(y_0 + ... + y_{k-1}) = 0 \implies y_0 + ... + y_{k-1} = 0.$$

Since $k < n$, the hyperplane $H = x_0 + ... + x_{k-1} = 0$ is in $\mathcal{A}$. Thus y lies on a hyperplane H in the arrangement $\mathcal{A}$. Hence $\mathcal{F} = \mathcal{F}_y$ is not maximal. □

Theorem 6.5.4. *Let $\mathcal{O}$ be an orbit. Then n divides $|\mathcal{O}|$.*

Proof. Let ρ denote the permutation which "shifts" signatures left by one: $\rho(\langle s_0, s_1, \ldots, s_{n-1}\rangle) = \langle s_1, \ldots, s_{n-1}, s_0\rangle$. Let $s_{\mathcal{F}} = \text{sig}(\mathcal{F})$ for all $\mathcal{F} \in \mathcal{O}$. Also, fix a $\mathcal{F}_0 \in \mathcal{O}$ and let $s = \text{sig}(\mathcal{F}_0)$. Note that the sets $\{s_{\mathcal{F}}, \rho(s_{\mathcal{F}}), ..., \rho^{n-1}(s_{\mathcal{F}})\}$ partition the set of signatures of families in $\mathcal{O}$, and recall that the signature function is one-to-one on any $\mathcal{O}$. Thus it suffices to show that for all $\mathcal{F} \in \mathcal{O}$, the signatures $s_{\mathcal{F}}, \rho(s_{\mathcal{F}}), ..., \rho^{n-1}(s_{\mathcal{F}})$ are all distinct. That is, it suffices to show that for all $\pi \in S_n$ and all $i = 1, ..., n-1$, $\pi(s) \neq \rho^i\pi(s)$.

Let $s' = \pi(s)$, and let j be the least non-negative integer such that $s' = \rho^j(s')$. It suffices to show that $j = n$. If $s' = \rho^j(s')$, then $s' = \rho^{jk}(s')$ for any integer k. Now there exists some k such that $jk \equiv \gcd(j, n) \pmod{n}$. Hence $s' = \rho^{\gcd(j,n)}(s')$. If $j \nmid n$, then $\gcd(j, n) < j$, contradicting the leastness of j, so $j \mid n$.

Therefore,

$$s' = \underbrace{(X, ..., X)}_{n/j}.$$

where X is a sequence of length j. Since s' is a permutation of the signature s of a maximal unbalanced family $\mathcal{F}_0$, it itself is a maximal unbalanced family. By the contrapositive of Theorem 6.5.3, s' does not repeat itself. Thus $n/j = 1$ and $n = j$. □

Since each orbit's size is a multiple of n, we must have $n \mid E_n$. Incidentally, we can also see that $2 \mid E_n$, since the complement of any maximal unbalanced family is also unbalanced.

CHAPTER 7

THRESHOLD FUNCTIONS

Maximal unbalanced familes might be viewed in the context of **threshold families** whose defining weights sum to zero. See, for example [6], where attention is restricted to uniform families of subsets (i.e., all subsets having the same cardinality), and the signature of a family is called its **degree sequence**. Threshold functions arise in electrical engineering [13], switching theory, and artificial intelligence [12].

A Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ is called a **threshold function** if and only if there exist a **weight vector** $W = (w_1, ..., w_n)$ with real components and a **threshold** $T \in \mathbb{R}$ such that for all $X = (x_1, ..., x_n) \in \{0, 1\}^n$,

$$W \cdot X \geq T \Leftrightarrow f(X) = 1,$$

$$W \cdot X < T \Leftrightarrow f(X) = 0.$$

Note that if W and T are given, then a threshold function is determined uniquely. We denote that function by $[W; T]$. Let T_n be the number of threshold functions on n variables. The following table lists T_n for various values of n. It is sequence A000609 in the Online Encyclopedia of Integer Sequences [10].

n	E_n	T_n
1	0	4
2	2	14
3	6	104
4	32	1882
5	370	94572
6	11292	15028134
7	1066044	8378070864
8	347326352	17561539552946
9	419172756930	144130531453121108

It appears that $E_n < T_n$. We will show that this is the case for all n.

Define the n-**dimensional hypercube**, or n-**cube**, as the polytope with vertices $\{0,1\}^n$. Thus we can think of threshold functions as mapping vertices of the n-cube to the set $\{0,1\}$. Given a threshold function $[W;T]$, the hyperplane $H_{W,T}$ such that $W \cdot X = T$ for all $X \in H_{W,T}$ divides the n-cube into two pieces. On one side, we have $W \cdot X \geq T$, and on the other, we have $W \cdot X < T$. Thus the vertices that $[W;T]$ maps to 1 are on one side of $H_{W,T}$, while the vertices that map to 0 are on the other. Hence there is a one-to-one correspondence between threshold functions and ways to divide the vertices of the n-cube using a hyperplane.

A threshold function f that assigns different values to opposite vertices of the n-cube is known as a **symmetric** threshold function. Note that $[W;T]$ is symmetric if and only if $H_{W,T}$ contains $(1/2, ..., 1/2)$. Let T_n^s be the number of symmetric threshold functions on n variables.

Theorem 7.0.5. $T_n = T_{n+1}^s$.

Proof. It suffices to exhibit a bijection between the set of symmetric threshold function f on $n+1$ variables and the set of threshold functions on n variables. In one direction, given a symmetric threshold function f on $n+1$ variables, we can define a threshold function g on n variables by $g(x_1, ..., x_n) = f(x_1, ..., x_n, 0)$. In the other direction, given a threshold function $[W; T]$ on n variables, view $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ and construct the hyperplane $H_{W',T'}$ in $\mathbb{R}^{n+1}$ going through $H_{W,T}$ and $(1/2, ..., 1/2)$; then $[W'; T']$ is a symmetric threshold function on $n+1$ variables. It is not hard to show that these two transformations are onto, thus showing that we really have a bijection. □

Theorem 7.0.6. $T_n^s = r(\mathcal{A}_n^T)$.

Proof. Given a threshold function f, the choice of W and T such that $f = [W; T]$ is not unique. Symmetric threshold functions $[W; T]$ are completely determined by W, since $H_{W,T}$ must contain $(1/2, ..., 1/2)$. Two different W's in $\mathbb{R}^n$ give the same f value when evaluated at a pair of opposite hypercube vertices if and only if the two weight vectors are on the same side of a hyperplane that perpendicularly bisects the two vertices. So, if we group weight vectors into equivalence classes according to whether they produce the same function, these equivalence classes are just the regions of the hyperplane arrangement $\mathcal{A}_n^T$ formed by this collection of perpendicular bisectors. Then the number of symmetric threshold functions will just be the number of regions in this arrangement. □

We now relate maximal unbalanced families to threshold functions. Recall that T_n is equal to the number of ways one can use a hyperplane $H_{W,T}$ to divide the vertices of the n-cube into two sets. Suppose we require that $T = 0$ and $W \cdot (1, ..., 1)$. This means that the dividing hyperplane $H_{W,T}$ must contain the diagonal passing through the origin and $(1, ..., 1)$. Then a vertex $v \in \{0, 1\}^n$ and

W are on the same side of $H_{W,T}$ if and only if $W \cdot v > 0$. Now recall that each maximal unbalanced collection $\mathcal{F}_y$ is specified by a witness vector $y \in H_n$: we have $\mathcal{F}_y = \{S \subsetneq [n] | y \cdot \chi_S > 0, S \neq \emptyset\}$. We know that every vertex v of the unit cube is χ_S for some nonempty $S \subsetneq [n]$, and vice versa. Furthermore, $W \in H_n$. Thus we see that a choice of dividing the vertices of the n-cube by $H_{W,T}$ is a way of picking a maximal unbalanced collection $\mathcal{F}_y$, and vice versa. In other words, the number of threshold functions where $T = 0$ and $W \cdot (1, ..., 1)$ is E_n. Thus we have shown:

Theorem 7.0.7. *$E_n < T_n$.*

Now suppose that we require $T = 0$ but put no restrictions on W. Once again, a vertex $v \in \{0, 1\}^n$ and W are on the same side of $H_{W,T}$ if and only if $W \cdot v > 0$. Also, we can once more construct $\mathcal{F}_y = \{S \subseteq [n] | y \cdot \chi_S > 0, S \neq \emptyset\}$ for any $y \in \mathbb{R}^n$. As y varies, $\mathcal{F}_y$ changes if and only if y passes through a hyperplane H_S, where $S \subseteq [n]$ is nonempty. Thus the number of possible values for $\mathcal{F}_y$ is the number of regions of $\mathcal{A}_n$. As before, every vertex v of the unit cube is χ_S for some nonempty $S \subsetneq [n]$, and vice versa, so a choice of dividing the vertices of the n-cube by $H_{W,T}$ is a way of picking a set $\mathcal{F}_y$. Thus the number of ways one can use a hyperplane $H_{W,T}$ to divide the vertices of the n-cube into two sets is equal to $r(\mathcal{A}_n) = E_{n+1}$. In other words, the number of threshold functions where $T = 0$ is E_{n+1}.

The following bounds for the number of threshold functions are known [16]:

Theorem 7.0.8. *$n^2(1 - 10/\ln n) \leq \log_2 T_n \leq n^2$ as $n \to \infty$.*

Thus $\log_2 T_n \to n^2$ as $n \to \infty$. Compare this to our bounds for E_n, which say that $\log_2 E_n \to Cn^2$ for some $C \in [1/2, 1]$ as $n \to \infty$.

CHAPTER 8

UNANSWERED QUESTIONS

8.1 Relative Sizes of E_n and O_n

Note that

$$\frac{O_4}{E_2} = \frac{4}{2} = 2$$
$$\frac{O_5}{E_3} = \frac{12}{6} = 2$$
$$\frac{O_6}{E_4} = \frac{56}{32} = 1.75$$
$$\frac{O_7}{E_5} = \frac{576}{370} \approx 1.56$$
$$\frac{O_8}{E_6} = \frac{16640}{11292} = 1.47$$

It appears that

$$2 \geq \frac{O_n}{E_{n-2}} \geq 1$$

for all n, and that the ratio O_n/E_{n-2} is decreasing as n increases. We do not know whether or not this is true.

8.2 Finding Witness Vectors

Given a maximal unbalanced family, we want to find a witness to its being unbalanced. That is, if $\mathcal{F} = \{F_1, ..., F_m\}$ is unbalanced, we want an easy way to find y such that $\chi_{F_i} \cdot y > 0$ for all i. Theorem 1.2.3 assures us that such a y exists. One can, of course, find such a y by solving all m inequalities for the n coordinates of y; the question is whether or not a more efficient algorithm exists. Obvious algorithms based on the signature of $\mathcal{F}$ do not seem to work.

8.3 Maxima and Minima of Signature Sums

The signature of the family **1** is $(2^{n-1}-1, 2^{n-2}-1, ..., 2^{n-2}-1)$. If we add the components of this vector, we get its **signature sum**

$$(2^{n-1}-1)+(n-1)(2^{n-2}-1)=(n+1)2^{n-2}-n.$$

We conjecture that the maximum signature sum. Similarly, the signature of the family $\not{1}$ is $(0, 2^{n-2}, ..., 2^{n-2})$, with signature sum $(n-1)2^{n-2}$, and we conjecture that this is the minimum signature sum.

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