CORNELL UNIVERSITY MATHEMATICS DEPARTMENT SENIOR THESIS

Finite Groups Acting on Surfaces as Rigid Motions of Euclidean Space

A THESIS PRESENTED IN PARTIAL FULFILLMENT
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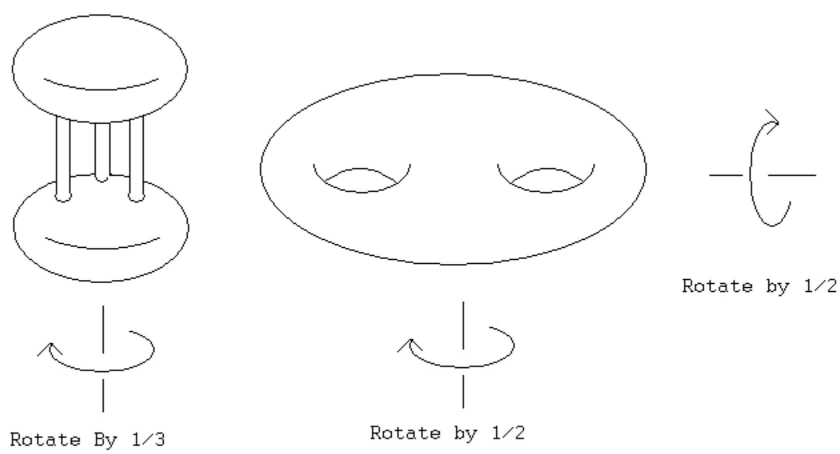
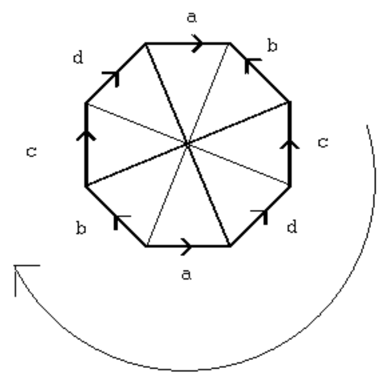
1 Introduction

The first examples normally given of finite groups acting on surfaces via diffeomorphisms involve a fairly symmetric embedding of the surface into euclidian 3-space and the group acting via rotations. All actions of finite groups on the sphere arise this way. Also, a few actions by small groups on surfaces of higher genus arise this way. Any cyclic group acts on the standard embedding of the torus in $\mathbb{R}^3$ by rotation. $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts via rotations on the standard way of drawing a surface of genus 2 in $\mathbb{R}^3$. If you embed surfaces in different ways, sometimes you get other actions. For example, if you view a surface of genus 2 as 2 spheres with 3 symmetrically placed tubes connecting them, then $\mathbb{Z}_3$ also acts via rotation while it does not act on the standard way of embedding the surface in $\mathbb{R}^3$.



It is fairly easy to see that some actions of finite groups on surfaces do not arise in this fashion. All finite groups act on surfaces but only a finite number of non-cyclic or dihedral groups embed in $SO_3(\mathbb{R})$. Even when some group acts on a surface and that group also embeds in $SO_3(\mathbb{R})$, the group's action on the surface need not arise from rigid motions. For example, one can construct an action of $\mathbb{Z}_8 =< g >$ on a surface of genus 2. To construct this action, first observe that $\mathbb{Z}_8 =< g >$ acts by rotation on an octagon with opposite sides identified as in the following picture. By inspection, we see that an octagon with opposite sides identified is a manifold. Its Euler

Characteristic is 1 - 4 + 1 = 2. Thus it is a surface of genus 2. So $\mathbb{Z}_8 =< g >$ acts on a surface of genus 2. $\mathbb{Z}_8$ is cyclic so it embeds in $SO_3(\mathbb{R})$ with g being a rotation by $k\pi/4$ with $(k, 4) = 1$. Acting via rotations, g^4 cannot fix any points not fixed by g. However, this action of $\mathbb{Z}_8$ on a surface of genus 2 has the property that g^4 fixes points which are not fixed by g and thus cannot arise from euclidean rotations of $\mathbb{R}^3$.



Rotate by 1/8

A natural question is: if we allow a higher number of dimensions, can we view all actions of finite groups on surfaces as arising from rigid motions of some higher dimensional euclidean space? This paper will answer this question in the affirmative. After discovering this proof, we found a paper of George Mostow in which he proved a more general result. This theorem is valid for spaces of arbitrary finite dimension which are not necessarily manifolds and for groups which are arbitrary compact Lie Groups. John Douglas Moore and Roger Schlafly later showed that in the case where the space is a Riemannian manifold, this embedding can be taken to be a real-analytic isometry. Our theorem is thus a special case of these general theorems. However, our proof is more elementary and more explicit.

2 Preliminary Remarks about Groups Acting Smoothly on Surfaces

Theorem 1. *Let Σ be a compact orientable surface, let G be a finite group, and let $\phi : G \mapsto Diffeo^+(\Sigma)$ be a faithful action of G on Σ. Then for some k,* *there is a smooth embedding f of Σ into $\mathbb{R}^k$ and an injective homomorphism $\varphi : G \mapsto SO_k(\mathbb{R})$ such that $f^{-1}(\varphi(g) * f(x)) = (\phi(g))(x)$. Twice the order of G plus the number points in Σ fixed by nonidentity elements in G plus 1 suffices for k.*

This means that we can view all smooth orientation-preserving finite group actions on surfaces as arising from rigid motions of the surface embedded in some high dimensional euclidean space.

Proof. Throughout, let Σ be a compact surface without boundary and G be a finite group which acts on Σ faithfully via diffeomorphisms.

Lemma 1. *There is a Riemannian metric on Σ such that this action of G on Σ is via isometries.*

Proof. Let $< \cdot, \cdot >$ be any Riemannian metric on Σ. It is well know that this always exists. Let $< \cdot, \cdot >'$ be defined by the formula $< v_1, v_2 >'_p = \sum_{g \in G} < g_* v_1, g_* v_2 >'_{g(p)}$. It is easy to check that this is g invariant and is, in fact, a Riemannian metric. □

Lemma 2. *If G acts on Σ via isometries, then the stabilizer of a point in Σ is cyclic and the set of fixed points of any nonidentity element is finite.*

Proof. Let $p \in \Sigma$ be fixed by a $g \neq 1$. In a neighborhood of p, g is equivalent to an orientation-preserving euclidean isometry with a fixed point. Thus, g acts like a euclidean rotation locally around p and thus does not fix any other points in a neighborhood of p. So, the set of points fixed by g is discrete. Since Σ is compact, that set is finite. Since G is finite, the set of points fixed by nonidentity elements of G is finite.

Let $p \in \Sigma$ and let G_p denote its stabilizer. G_p acts via orientation preserving orthogonal matrices on the tangent space of p (in a basis orthogonal with respect to the metric) since G acts by orientation preserving isometries. The only finite subgroups of $SO_2(\mathbb{R})$ are cyclic so G_p's action on the tangent space of p is cyclic. The action of G_p on the tangent space at p is known to be faithful so G_p is cyclic.

□

Throughout, let $P = \{p \in \Sigma$ such that there exists $g \in G - \{1\}$ with $g(p) = p\}$ and let G_p denote the stabilizer of some point p. If we ever talk about a metric, it will mean some fixed metric with respect to which G acts via isometries.

First we will select a particular injection of G into $SO_k(\mathbb{R})$. Our strategy for embedding Σ equivariantly into $\mathbb{R}^k$ will be broken up into three parts. We will start by equivariantly embedding P into Σ, then neighborhoods of P, and finally all of Σ.

3 Embedding P

If G acts on a set S, then G has a representation in $O_{|S|}(\mathbb{R})$. This is achieved by identifying the standard basis of $\mathbb{R}^{|S|}$ with S and having some element of $g \in G$ correspond to the permutation matrix which sends s to gs for all $s \in S$. Let $\Phi : G \to O_{|G|}(\mathbb{R})$ be the representation of G arising from the action of G on itself via left multiplication. This representation is faithful since the action is faithful. Let $\Psi : G \to O_{|P|}(\mathbb{R})$ be the representation of G arising from the action of G on P. Consider the following representation:

$$\Upsilon(g) = \begin{pmatrix} \Psi(g) & 0 & 0 & 0 \\ 0 & \Phi(g) & 0 & 0 \\ 0 & 0 & \Phi(g) & 0 \\ 0 & 0 & 0 & \det(\Psi(g)) \end{pmatrix}$$

Υ is faithful since the product of faithful representations with any representation is faithful. It is orientation preserving since the last component insures that all matrices have determinant 1. So Υ is an injection of G into $SO_k(\mathbb{R})$ with $k = 2|G| + |P| + 1$.

Map P into the first $|P|$ standard basis vectors. Call this set P'. G acts on P' via this representation in a way identical to the way G acts on P via its action on Σ.

At this point, everything should seem well motivated except why we take two copies of Φ. This will be needed when we embed all of Σ into $\mathbb{R}^k$ to provide extra "wiggle room."

4 Embedding Neighborhoods of P

For convenience, rescale the Riemannian metric on Σ so that the minimum distance between points in P is greater than 1 and so that points in P have a closed disk of radius 1/2 diffeomorphic to a disk in the plane that are mutually disjoint. Before we proceed, we will need the following lemma.

Lemma 3. *$\Phi(g)$ has $e^{2\pi i/|g|}$ as an eigenvalue.*

Proof. g acts on $\{1\ g\ g^2\ \ldots\ g^{|n-1|}\}$ by cyclic permutation. Thus, after reordering the basis of $\mathbb{R}^{|G|}$, $\Phi(g)$ will be block diagonal with a $|g| \times |g|$ block equaling:

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Call this matrix A. $char(A) = \det(\lambda I - A) = \lambda^{|g|} - (-1)^{|g|}$ which has $e^{2\pi i/|g|}$ as a root. □

For some $p \in P$, take some neighborhood of p of radius $1/2$. We will embed neighborhoods around the fixed points of radius $1/2$ into $\mathbb{R}^k$. The $1/2$ is chosen since all the points in P are at least 1 unit apart so we will not have to worry about these neighborhoods intersecting. Let g be a generator of G_p and let $M = \Upsilon(g)$, the matrix corresponding to g. Use coordinates so that M is block diagonal with the blocks being 2×2 rotation matrices or 1. Let V_j be an eigenspace of M corresponding to $\lambda = \exp(2\pi i/j)$. Let D be the disk of radius $1/2$ around $p \in \Sigma$. Let n be the order of g. Let p' be the image of p in $\mathbb{R}^k$. We will define the embedding f component-wise on all of the V_j. Use polar coordinates (r, θ) to parameterize D and the V_j's. Map D into V_j by the map $(r, \theta) \to (r^{n/j}, n\,\theta/j)$. p' is not 0 and is fixed by M so it is in the eigenspace corresponding to eigenvalue 1. Map D into this eigenspace by sending all points of D to p'.

By the lemma, one of the j's is n since M has $\Phi(g)$ as submatrix and $\Phi(g)$ has $e^{2\pi i/|g|}$ as an eigenvalue. This means that f is injective even though its projection into any given V_j is normally not injective. This is also clearly differentiable since if we use complex coordinates, it is sending z to $z^{n/j}$ and n/j is an integer. This map is a smooth embedding since one j is n so, one of these maps is $z \to z$ which has nonzero derivative. Let D' be the image of D in $\mathbb{R}^k$. One can check directly that the D' is mapped to itself by M. Since $< g >= G_p$, any matrix corresponding to an element of G which fixes p maps D' to itself.

To extend the map to points in neighborhoods of points in p's orbit, use the following procedure. Assume $h(p) = q$, $h \in G$ and h corresponds to the matrix L. Let q' be the image of q in $\mathbb{R}^k$. $h(D)$ is a neighborhood of q with the same properties as D. Map $h(D)$ into $\mathbb{R}^k$ in the analogous way as was done before except replacing D with $h(D)$, p' with q' and M with LML^{-1}. $L(D')$ is the image of $h(D)$ since L maps eigenspaces of M to eigenspaces of LML^{-1} and so on. Now repeat this construction for all points in $p's$ orbit and then all points in P one orbit at a time. Assume A maps p' to q' with A being a matrix corresponding to an element of G. Then $L^{-1}AL$ fixes p' so it is a power of M. So, $L^{-1}AL$ maps D' to D'. So A maps D' to $L(D)$. This shows that the embedded disks are mapped among themselves by G. Call the map of disks in Σ into $\mathbb{R}^k$ f. At this point, we have succeeded in embedding a neighborhood of P into $\mathbb{R}^k$ smoothly in a way such that the two actions agree.

5 Embedding the Rest of Σ

We have already embedded disks of radius 1/2 around all points in P into $\mathbb{R}^k$. Let $D(r)_p$ denote a closed disk of radius r around a point p. Let $S = \{x \in \mathbb{R}^k : g(x) = x \text{ for some } g \in G - 1\}$. Our strategy will be first to map $\Sigma - \bigcup_{p \in P} D(1/4)_p$ into $\mathbb{R}^k - S$ equivariantly. This will induce a map from $(\Sigma - \bigcup_{p \in P} D(1/4)_p)/G$ into $(\mathbb{R}^k - S)/G$. We will then invoke the Relative Whitney Embedding Theorem to approximate this map with an embedding. Then we will lift to an equivariant embedding of Σ into $\mathbb{R}^k$.

Put a cell structure on Σ using the following procedure. Consider the manifold with boundary $(\Sigma - \bigcup_{p \in P} D(1/4)_p)/G$. Since the disks were constructed to respect G's action, this is well defined. It is a manifold not an orbifold since we have removed all of the fixed points. Use the boundaries of the images of $D(1/4)_p's$ and $D(1/2)_p's$ as edges in a cell structure. Add vertices to the boundaries of the disks and add addition vertices and edges until we have a cell structure. Now lift this cell structure to a cell structure on Σ. Since G acts on Σ/G trivially, the cell structure on Σ is respected by G, i.e. G sends vertices to vertices, edges to edges, and faces to faces.

G acts freely on cell structure of $\Sigma - \bigcup_{p \in P} D(1/4)_p$ in the sense that no nonidentity element maps a cell to itself. The projection of $\Sigma - \bigcup_{p \in P} D(1/4)_p$ onto $(\Sigma - \bigcup_{p \in P} D(1/4)_p)/G$ is a $|G|$-fold covering since the action is free. By standard unique lifting properties from elementary covering space theory, we have that the inverse image of a cell is a disjoint collection of $|G|$ cells. The orbit of a cell in $\Sigma - \bigcup_{p \in P} D(1/4)_p$ under the action of G is the same as the inverse image of the projection of that cell. Thus, the size of the orbit of a cell under the action of G is $|G|$ so G acts freely on the cell structure.

We will map $\Sigma - \bigcup_{p\in P} D(1/4)_p$ into $\mathbb{R}^k - S$ by first mapping in the 0-cells then 1-cells and then 2-cells. All cells contained in $\bigcup_{p\in P} D(1/4)_p$ have already been mapped in by the previous sections. Select a 0-cell $q \in \Sigma - \bigcup_{p\in P} D(1/4)_p$. Map q to a point $q' \in \mathbb{R}^k - S$ arbitrarily. Map $g(q)$ to $g(q')$ for all $g \in G$. $g(q') \notin S$ since S is G invariant. q is not in P so the orbit of q has size $|G|$. So a point $g(q)$ is uniquely determined by g so this procedure is well defined. Select points in each of the orbits of 0-cells and repeat this procedure.

Let N be a 1-cell with boundary points a and b. Assume N is not contained in the disks that we have already embedded (a and b are allowed to be in those disks however). Let a' and b' be the images of a and b in $\mathbb{R}^k - S$ respectively. $\mathbb{R}^k$ is path connected so map N continuously to any path connecting a' and b'. This may intersect S. Let this path be denoted by N' as set and f_0 as a map. Let $S_g \subseteq \mathbb{R}^k$ be the subspace fixed by g. Fix $g \in G$ not equal to 1. Because of the way we embedded the 0-cells, the boundary of N' does not intersect S and thus does not intersect S_g. We showed earlier that the matrix corresponding to g had 2 2-dimensional eigenspaces with non identity eigenvalue so S_g has codimension 4 or higher in the manifold $\mathbb{R}^k$. Since N' has dimension 1 and $1 < 4$, we can perturb f_0 to make N' miss S_g without changing the value of f_0 on the boundary of N. Call this new map f_1 and view this as a map into the manifold $\mathbb{R}^k - S_g$. Let h be another nonidentity group element. $S_h - S_h \bigcap S_g$ is a manifold of codimension at least 4 in the manifold $\mathbb{R}^k - S_g$. N' is 1-dimensional so f_1 can be perturbed slightly to a map f_2 into $\mathbb{R}^k - S_g$ so that N' avoids S_h. Repeat this for all nonidentity group elements. Call the final map f. f is a map of N into $\mathbb{R}^k - S$. Map the 1-cell $g(N)$ to $g(N')$. Since S is G invariant, $g(N')$ will not intersect S. As before, do this for each orbit of 1-cells. So, we have an equivariant map f from the 1-skeleton of $\Sigma - \bigcup_{p\in P} D(1/4)_p$ into $\mathbb{R}^k - S$ that agrees with our original map on the 0-skeleton. At this point, we are only requiring that this map be continuous since the Whitney Embedding Theorem will provide the smoothness.

An identical procedure works for the 2 cells as well. The map is already defined on the 1-skeleton so it is already defined on the boundaries of 2-cells. Since $\mathbb{R}^k$ is contractible and thus simply connected, we can fill in any loop with a disk. Thus we can map a 2-cells N from the cell structure of $\Sigma - \bigcup_{p\in P} D(1/4)_p$ into a 2-cell with boundary being the image ∂N. Since $2 < 4$, we can push the image of the N off S one subspace at a time without changing the value of the function on ∂N. Then we can use the action of G to extend the map to the orbit of the N. After doing this procedure for each orbit of two cells, we will have defined f on all of Σ. This is the point in the argument where it becomes relevant that we took two copies of Φ. If we had only taken one copy, the codimension of S_g would be 2 so we would be unable to perturb the images of the 2-cells to miss the fixed sets. So we can construct a map $f : \Sigma \to \mathbb{R}^k$ such that f is continuous and equivariant everywhere, a smooth embedding on $\bigcup_{p\in P} D(1/2)_p$ and maps $\Sigma - \bigcup_{p\in P} D(1/4)_p$ into $\mathbb{R}^k - S_g$.

Since f is equivariant, it induces a map $f' : (\Sigma - \bigcup_{p\in P} D(1/4)_p)/G \to (\mathbb{R}^k - S)/G$. G acts freely on $\mathbb{R}^k - S$ so $(\mathbb{R}^k - S)/G$ is a k-manifold. Likewise, G acts freely on $\Sigma - \bigcup_{p\in P})D(1/4)_p$ and thus $(\Sigma - \bigcup_{p\in P} D(1/4)_p)/G$ is a 2-manifold (with boundary). The Relative Whitney Embedding Theorem states that any map from a manifold to a manifold of at least twice the dimension can be approximated by a smooth embedding. If the original map was already a smooth embedding on some closed set, the approximation can be taken so that it agrees with the original map on that set. Since k is at least 5 for G nontrivial, k is greater than twice the dimension of $(\Sigma - \bigcup_{p\in P} D(1/4)_p)/G$. So we can approximate f' by a smooth embedding g' such that g' agrees with f' on the image of the annuli of radii 1/4 to 1/2 around points in P/G. Since the ambient space is taken to be $(\mathbb{R}^k - S)/G$, the image of g' misses S/G. Lift g' to a smooth embedding of g of $\Sigma - \bigcup_{p\in P} D(1/4)_p$ into $\mathbb{R}^k - S$. Define g on $\bigcup_{p\in P} D(1/2)_p$ to agree with f. The result is smooth since g already agreed with f on $\bigcup_{p\in P}(D(1/2)_p - D(1/4)_p)$. Thus, g is a smooth embedding of Σ into $\mathbb{R}^k$ such that the action of G on Σ agrees with the action of the representation of G on the image of Σ.

Throughout, we have been assuming that the action of G had fixed points. If it does not, everything still works and is even easier. The only situation that was not addressed is the case of $G = 1$ since then $k = 3$ which is less than twice the dimension of Σ. However, the standard embedding of a surface works in this case. □

6 Conclusion and Problems

In conclusion, all actions of finite groups on compact orientable surfaces without boundary arise from rigid motions of some high dimensional euclidian space. The bound of $2G + P + 1$ seems rather crude. Trivially, we can reduce it to $2G + P$ - 2 since Φ and Ψ are not irreducible representations and can each be reduced in dimension by 1 without changing any of the properties we used. A possible future project could involve searching for a better or even best upper bound in terms of genus, $|G|$, $|P|$ or any other relevant information.

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8 Bibliography

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