Math Explorers' Club - Graph Theory Session 1

Hugo Mainguy

1 §The Seven Bridges of Königsberg

In mathematics, many great problems start with a question following a simple observation. In particular, Leonhard Euler, while in the city, wondered whether it was possible to take each bridge exactly once. Take some time and try to see if you can do it!

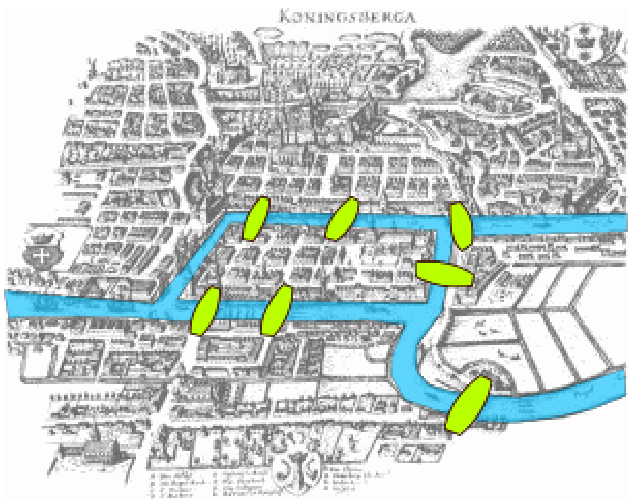

Figure 1: The seven bridges of Königsberg (today Kaliningrad, Russia) as existing in 1734

What is the issue? Would adding a bridge fix the problem, and if so, where should it be put? If not, how about two?

In mathematics, it is often useful to try to have some kind of mathematical model for the phenomenon we are observing - that is, how can we represent this problem more abstractly? In particular, we have pieces of land, and bridges that represent our only way of going from one place to another.

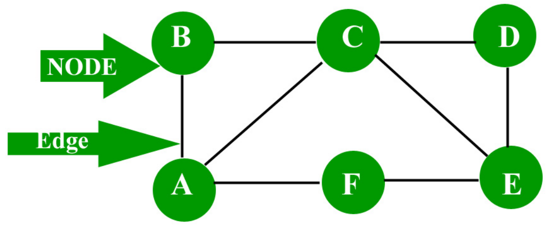

Figure 2: An example of a graph

The figure above represents a mathematical object called a **graph**: it has nodes (also called **vertices**, depending on who you talk to), which are connected by **edges**. In our previous example, we can represent the islands as vertices, and the bridges as edges. We define the **degree** of a vertex as the number of edges it is connected to - the number of bridges reaching an island here. Now, let's find out what a graph version of our problem would look like.

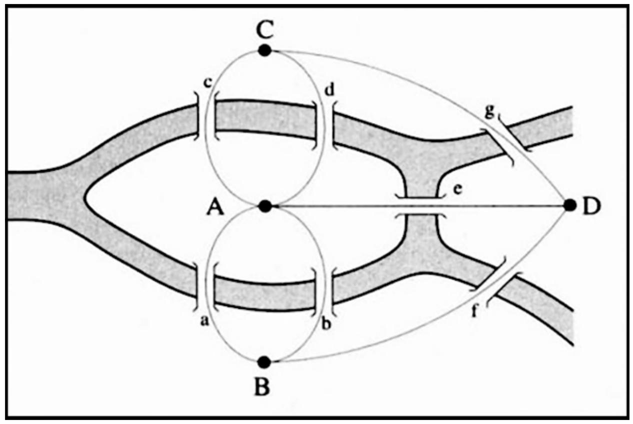

Figure 3: Our original problem, modeled as a graph

Now, let's attempt to figure out why it was impossible to draw a cycle (a full loop) with the seven bridges - perhaps by contrasting with an example where you add two bridges. *Hint: If you visit an island, you take a bridge to get there, and take a bridge to leave.*

Such a loop is called an **Euler cycle** (named after Euler, who essentially founded the field of graph theory). To have an Euler cycle, it is necessary and sufficient that the graph is connected (there is a path between any pair of vertices) and **every vertex has even degree**; that is, there is an even number of edges coming out of every vertex. (If and only if means that no matter the graph, if every vertex has even degree, there is an Euler cycle, otherwise, it does not. So it is actually very easy to check!

Now, an **Euler path** is an Euler cycle, but that doesn't start and end in the same place (you go from point a to b instead of a back to a). What do you think is a necessary and sufficient condition for the existence of an Euler path?

What are some Euler cycle applications? Some involve garbage collection or mail delivery routes (you want to go everywhere exactly once), or travel planning! Can you think of any others?

2 §Map Coloring

We now come to a seemingly unrelated topic, which also arose from a natural observation in the nineteenth century. Given this map of the United States (which you can print), try to color all states with as few colors as you can, with the constraint that no neighboring states should be of the same color. (If you have no colored pencils, you can use numbers to represent colors - write 1 instead of red, 2 for blue, etc.)

Figure 4: A blank map of the United States, for you to color!

How many colors did that take you? If you were to color differently, could you use fewer colors? Why or why not?

Normally, you should have been able to color the map with four colors! But why is it related to graph theory? Can we represent this problem with a graph that has vertices and edges?

Now, draw your own map of any place you want (real or imaginary), separate it into regions like the US state map, and color it in the same way. How many colors do you need? After sharing with the other students, can we make a conjecture on the number of colors needed?

Mapmakers and mathematicians soon observed what we just witnessed, but it is not enough to try to color thousands of maps and see that a certain number is always enough - we need to prove it. Early on, a proof was given that six colors were always enough, then five a few years later. Four was much harder, and over more than a century, many proof attempts were made, sometimes to be found wrong twenty years later. Finally, it was proven that four colors is always enough thanks to the first big computer proof result in 1976 - there are about 1955 subcases that need to be considered!

3 §The Art Gallery Problem

The statement of the Art Gallery Problem is as follows: imagine you are responsible for the security of an art gallery, which is represented by a polygon of your choosing (but may not have holes). You are allowed to use ”vertex guards” - that is, guards that stand at a vertex and do not move. How many guards do you need to ”see” the whole art gallery?

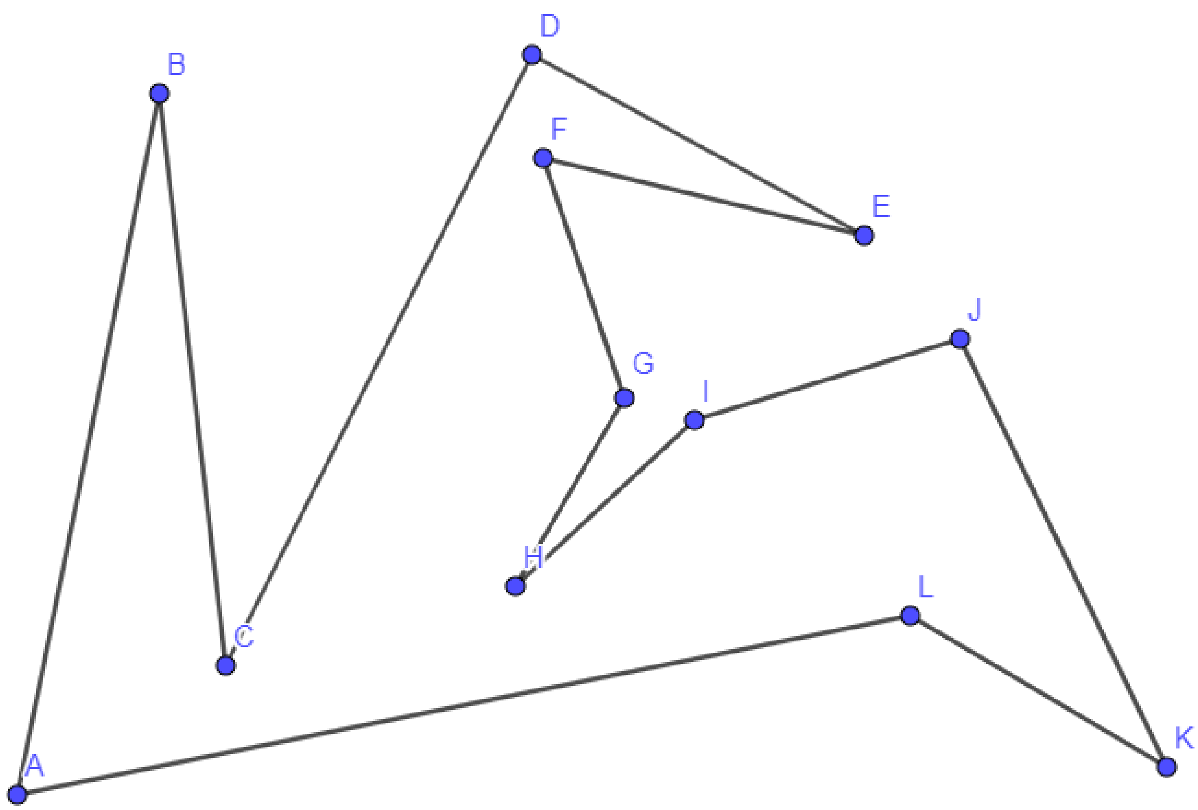

Figure 5: A polygon, representing a (particularly interesting) art gallery.

In general, for a polygon with n vertices, how many guards would you expect to be always sufficient, and sometimes necessary? (Meaning that if I have n vertices in my polygon, no matter what the polygon is, this number of guards will be enough, and I will sometimes need this exact number).

We will not give a definite answer to that question yet, but will keep it in mind.

To answer this question, we will start by **triangulating** the polygon. To do this, we connect pairs of vertices and keep going until we only have triangles left at the end, and all these triangles use three vertices of the art gallery (we do not make any new vertices). Note: there are many possible solutions.

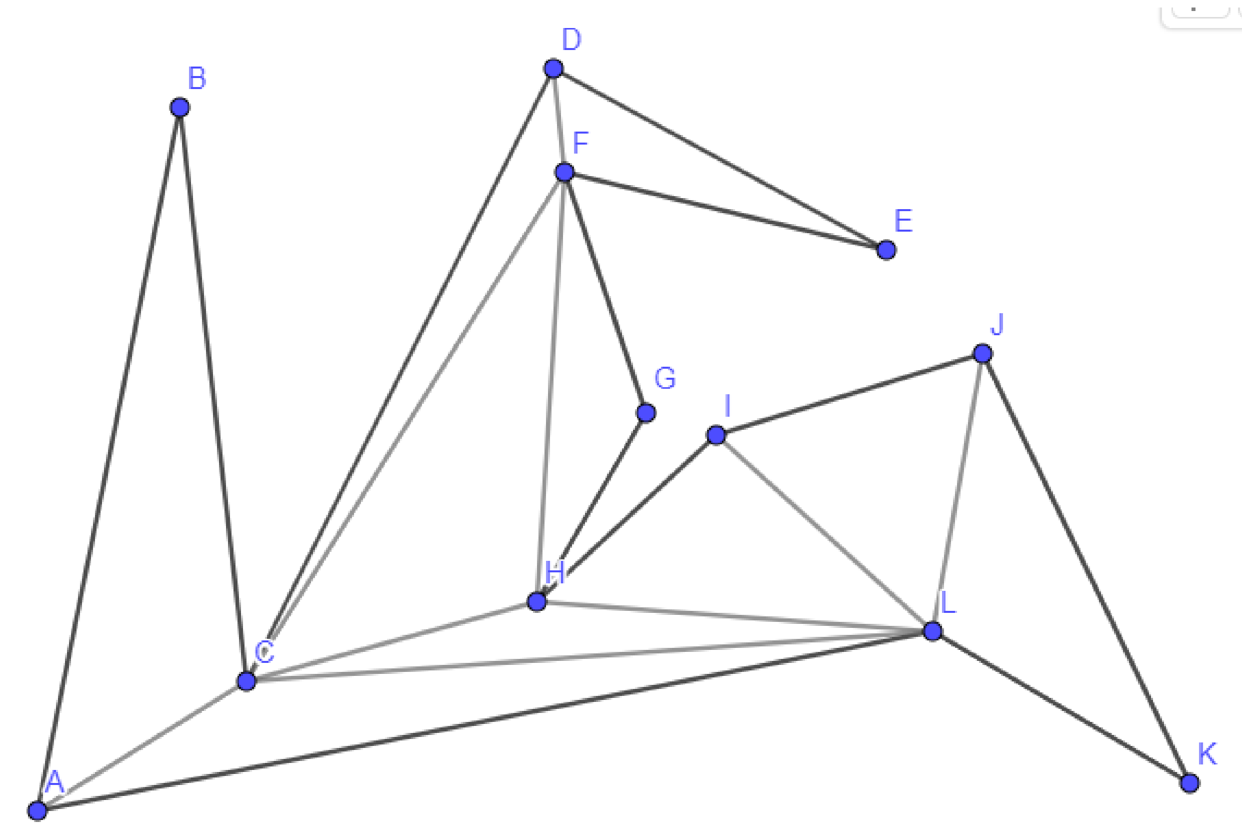

Figure 6: One possible triangulation of the polygon above.

Now, just like in the previous exercise, try to color the vertices of the polygon, while keeping in mind that two vertices connected by an edge must be of different colors. How many colors do you need?

Now let's go back to the original problem. In general, for a polygon with n vertices, how many guards would you expect to be always sufficient, and sometimes necessary? *Hard question: can you find an example of a polygon for which this number is necessary, whether for a specific n, or even better, for all n?*

If you managed to do this, congratulations! You just proved the **Art Gallery Problem**, a problem that remained unsolved for a long time before this elegant solution was found in the 1970s! This problem is part of a graph theory related field called computational geometry, which uses algorithms to solve geometric problems - if you are interested in any of these problems or their extensions, do not hesitate to reach out!