Three tales of Hamiltonian geometry

Allen Knutson

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Abstract

I. *Abelian and nonabelian Duistermaat-Heckman measures.*

We recall the multiple ways to define and compute these, and relate nonabelian to abelian.

II. *Pictorial calculations in equivariant cohomology.*

We draw pictures in $\mathfrak{t}^*$, exploiting its double role as (i) target of the moment map Φ_T, and (ii) the generating space for $H^*_T(pt) \cong Sym(\mathfrak{t}^*)$.

III. NEW! *Microlocal geometry of the terms in the Duistermaat-Heckman theorem.*

Given a locally closed submanifold $A \subseteq M$ (e.g. a Morse stratum), one can associate a $\mathcal{D}_M$-module and characteristic cycle in T^*M. The corresponding term in the Duistermaat-Heckman theorem is *itself* the DH measure of this characteristic cycle.

These slides are available at `http://pi.math.cornell.edu/~allenk/`

I. Four approaches to abelian Duistermaat-Heckman measures.

Let $T \circlearrowright (M^{2n}, \omega)$ be a Hamiltonian action, with moment map $\Phi_T : M \to \mathfrak{t}^*$. For convenience assume the generic stabilizer to be finite. Then there are three or four ways to define the associated *Duistermaat-Heckman measure* on $\mathfrak{t}^*$:

1. Push forward M's Liouville measure $[\omega^n]/n!$ along Φ_T, giving a smeared-out version of M's Liouville volume.

2. Define a function on $\mathfrak{t}^*$, taking $\lambda \mapsto vol(M//_\lambda T)$, and multiply by Lebesgue measure. This is maybe the best, as it suggests one look at how $M//_\lambda T$ changes, not just its volume. (DH's paper title gives it away: the symplectic form changes *linearly,* within regions of regular values.)

3. Compute the Fourier transform of $\int_M \exp(\tilde{\omega})$, where $\tilde{\omega} = \omega - \Phi_T$ is the *equivariantly closed extension* of the symplectic form, in Cartan's de Rham model of equivariant cohomology. This is typically the most fruitful for computational purposes (and what we will use in the third tale).

4. (When M is complex projective and $[\omega] = c_1(\mathcal{O}(1))$.) For each $n > 0$, define a Dirac measure $\frac{1}{n^{\dim M}} \sum_{\lambda \in T^*} \dim(\lambda\text{-weight space in } \Gamma(M; \mathcal{O}(n)))\ \delta_{\lambda/n}$ and consider the weak limit $n \to \infty$. (Here $T^* \leq \mathfrak{t}^*$ is the weight lattice.)

A nonabelian analogue.

Now let $K \circlearrowright (M^{2n}, \omega)$ Hamiltonianly, with $T = T_K$ a maximal torus of the compact connected group K. So $\Phi_T = \iota^T \circ \Phi_K$, where $\iota : \mathfrak{t} \hookrightarrow \mathfrak{k}$ is the inclusion. Let $\pi : \mathfrak{k}^* \xrightarrow{/K} \mathfrak{t}^*_+$ be the quotient by the coadjoint action.

One *could* define a measure on $\mathfrak{k}^*$ by $(\Phi_K)_*([\omega^n]/n!)$. But it wouldn't be supported on a polytope, so that's less satisfying.

One *could* take that measure and attempt to "intersect" with $\mathfrak{t}^*_+ \subseteq \mathfrak{k}^*$. But that's not a natural thing to do with measures.

One *could* define a measure on $\mathfrak{t}^*_+$ by $(\pi \circ \Phi_K)_*([\omega^n]/n!)$. This is pretty good, but when M is a coadjoint orbit $K \cdot \lambda$, it gives δ_λ times $vol(K \cdot \lambda)$.

What we *will* do is take that last one and divide by the Vandermonde polynomial $\lambda \mapsto \mathrm{vol}(K \cdot \lambda)$. Call the result the **nonabelian DH measure** $DH_K(M)$.

Theorem. (1) If M is complex projective, and we define a Dirac measure $\frac{1}{n^{\dim M}} \sum_{\lambda \in T^*_+} \dim(\lambda\text{-multiplicity space in } \Gamma(M; \mathcal{O}(n)))\ \delta_{\lambda/n}$, then its weak limit as $n \to \infty$ is also $DH_K(M)$. (2) The function $\lambda \mapsto vol(M//_\lambda K)$ on $\mathfrak{t}^*_+$, times Lebesgue measure, also gives $DH_K(M)$.

(Had we mistakenly measured "$\dim(\lambda$-isotypic component$)$" there, we'd've been off by that same $vol(K \cdot \lambda)$ factor.)

Computing one from the other.

Let $P_T = \Phi_T(M) \subseteq \mathfrak{t}^*$ and $P_K = (\pi \circ \Phi_K)(M) \subseteq \mathfrak{t}^*_+$, the abelian and nonabelian moment polytopes.

Theorem. We can compute P_T from P_K as $conv(W \cdot P_K)$, the convex hull of the Weyl group translates.

The reverse isn't possible, e.g. for $K = SU(V \cong \mathbb{C}^2)$ the nonabelian moment polytopes of $\mathbb{P}(V)$ and $\mathbb{P}(V \oplus \mathbb{C})$ are a point and an interval.

Perhaps surprisingly, one can go *both* directions if one keeps track of the DH measures, not just their supports.

Theorem.

1. Given $DH_T(M)$, apply the differentiation operators $\prod_{\beta \in \Delta_+} \partial_\beta$, and restrict the result to $\mathfrak{t}^*_+$ to get $DH_K(M)$.

2. Given $DH_K(M)$, apply to $\sum_{w \in W} (-1)^w (w \cdot DH_K(M))$ the integration operators $\prod_{\beta \in \Delta_+} int_\beta$ to get $DH_T(M)$.

This two-way result is maybe less surprising in K-theory: (1) K-reps are determined by their characters and (2) T meets every K-conjugacy class. To actually get a hold of it one uses the Weyl character formula.

When is $\mathrm{DH}_K(M) = \mathrm{DH}_T(N)$?

Let $K \circlearrowright (M,\omega)$. The **imploded cross-section** $M_{cs} := \Phi_K^{-1}(\mathfrak{t}_+^*)/\sim$ of [Guillemin-Jeffrey-Sjamaar '02] is the rather nasty space made by modding out null directions in the at-best symplectic-manifold-with-corners $\Phi_K^{-1}(\mathfrak{t}_+^*)$.

Theorem [GLS02]. M_{sc} is a Hamiltonian T-space, and

1. $\mathrm{DH}_K(M) = \mathrm{DH}_T(M_{cs})$
2. $M_{cs} \cong (M \times (T^*K)_{cs})//K_\Delta$, i.e. there is a "universal" case to consider
3. $(T^*K)_{cs} \cong G//N$ where $G = K^{\mathbb{C}}$ and $N \leq G$ is maximal unipotent.

If $T \circlearrowright (M,\omega)$, one can make a space $\mathrm{Ind}_T^K(M) := K \times^T M$ with a natural presymplectic form, which will be symplectic when $\Phi_T(M) \subseteq (\mathfrak{t}_+^*)^\circ$.
In that case $\mathrm{DH}_K(\mathrm{Ind}_T^K(M)) = \mathrm{DH}_T(M)$, and there is an alternate construction $\mathrm{Ind}_T^K(M) \cong (M \times (N\backslash\backslash G))//T_\Delta$.

The composite recipe $\mathrm{Ind}_T^K(M_{sc})$ can therefore be realized as $(M \times G_0)//K$, where $G_0 := (G//N \times N\backslash\backslash G)//T_\Delta$ is the **Vinberg semigroup**. Its coördinate ring is the *graded* degeneration of the Peter-Weyl ring $\bigoplus_\lambda V_\lambda \otimes V_\lambda^*$.

Hence when M is complex projective, M degenerates to $(M \times G_0)//K$, called the **horospherical degeneration** in case M is spherical (i.e. M_{sc} is toric). Note that $(M \times G_0)//K$ has an extra T-action, an algebraic analogue of the "Thimm trick" that uses $M \xrightarrow{\Phi_K} \mathfrak{k}^* \xrightarrow{\pi} \mathfrak{t}_+^*$ as a "moment map". **The End.**

II. Pictures of equivariant classes.

Let $T \circlearrowright (M, \omega)$ for M compact, with moment map Φ_T, and assume M^T finite for convenience. Then we have two H^*_T-algebra homomorphisms

$$H^*(M) \xleftarrow{/H_T^{>0}(pt)} H^*_T(M) \quad \longrightarrow \quad H^*_T(M^T) = \bigoplus_{f \in M^T} H^*_T$$

$$\alpha \quad \mapsto \quad (\alpha|_f)_{f \in M^T}$$

where the left one is (Kirwan) surjective, and the right one is an injection into a particularly convenient ring. This suggests that to do calculations in $H^*(M)$, it might be more convenient to lift them to $H^*_T(M^T)$. Note that if a closed submanifold $A \subseteq M$ gives our class $\alpha = [A]$ by the Thom isomorphism, then $[A]|_f = \prod(T\text{-weights in } T_f M / T_f A)$ (or 0 if $f \notin A$).

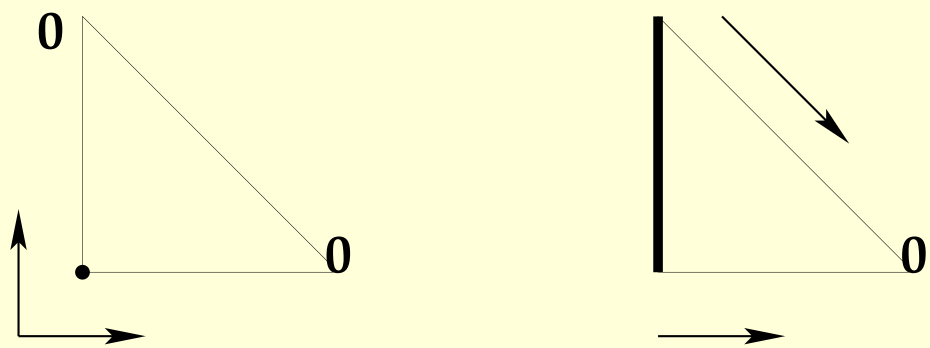
(There is a lot of work on describing the image of the second map, but we won't actually use any of it!)

If Φ_T is injective on M^T, then there's a natural place to list the elements in the tuple $(\alpha|_f)_{f \in M^T} \in \bigoplus_{f \in M^T} H^*_T$; draw $\alpha|_f$ *at the point* $\Phi_T(f) \in \mathfrak{t}^*$.

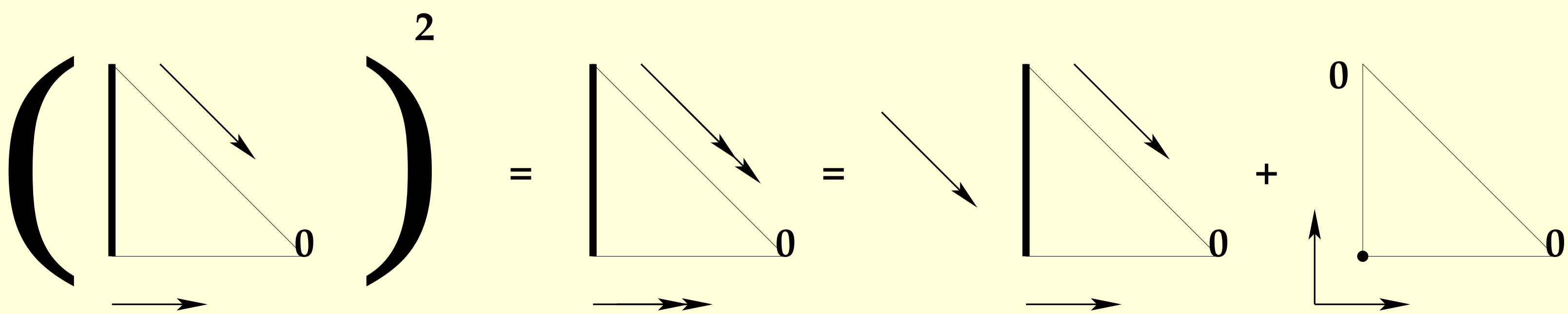
But we can do much better, since $\alpha|_f \in H^*_T \cong Sym(\mathfrak{t}^*)$: draw $\alpha|_f$ as a polynomial *in arrows living in* $\mathfrak{t}^*$.

Theorem: Two lines in the plane intersect in a point.

We do a sample computation in $H^*(\mathbb{CP}^2)$. Let $T \circlearrowright \mathbb{C}^3$ with weights $(0,0)$, $(0,1)$, $(1,0)$ and let $\mathbb{CP}^0 := \{[*,0,0]\}$, $\mathbb{CP}^1 := \{[*,*,0]\}$. The correponding pictures are



and with them we compute $[\mathbb{CP}^1]^2 = (y_1 - y_2)[\mathbb{CP}^1] + [\mathbb{CP}^0]$:



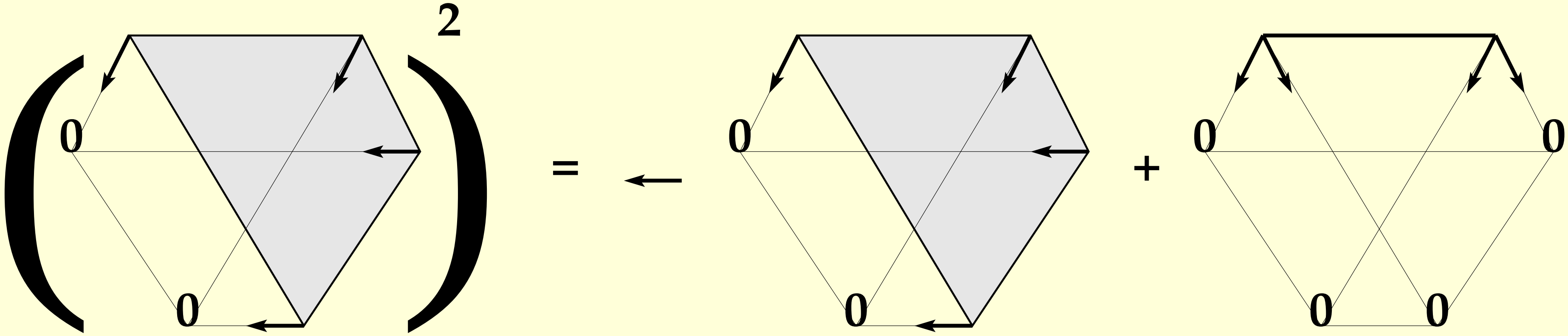
If we then set the $H_T^{>0}(pt)$-coefficients to 0, we get $[\mathbb{CP}^1]^2 = [\mathbb{CP}^0]$ in $H^*(\mathbb{CP}^2)$.

A more complicated Schubert calculus computation.

Consider flags $(V_1 < V_2 < \mathbb{C}^3)$ such that $V_\bullet \in X_{213}(F_\bullet) \cap X_{213}(G_\bullet)$, i.e. $V_1 \leq F_2, G_2$. If $F_2 \neq G_2$ and $H_2 > H_1 := F_2 \cap G_2$, then $X_{213}(F_\bullet) \cap X_{213}(G_\bullet) = X_{312}(H_\bullet)$.

In the equivariant calculation, $F_\bullet = G_\bullet$; the above computes only the nonequivariant terms.

$[X_\pi]|_\rho \neq 0$ only for $\rho \geq \pi$ in Bruhat order. In this tiny example the Schubert varieties are smooth; otherwise, one needs the AJS/Billey formula for point restrictions of Schubert classes.



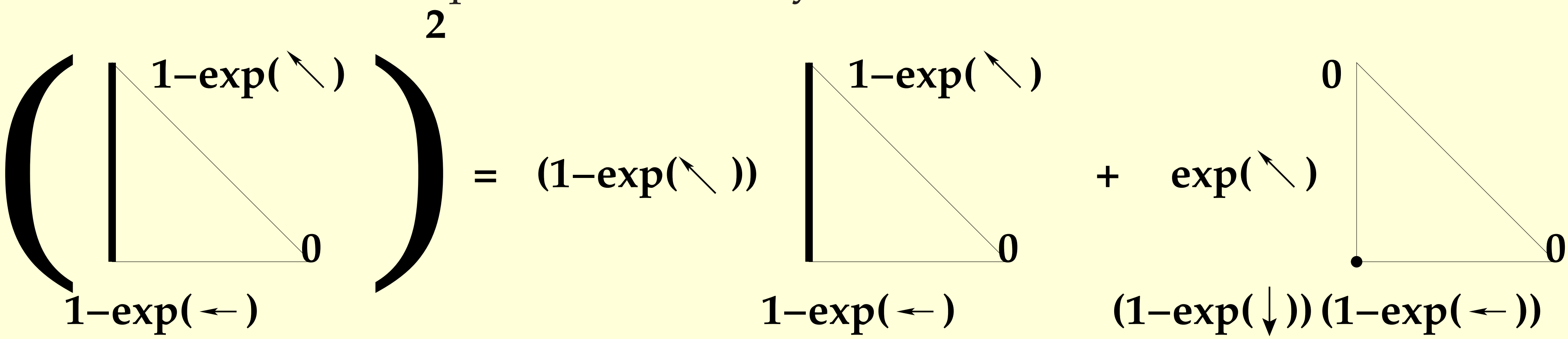
If we define the "support" of α as $\{f \in M^T : \alpha|_f \neq 0\}$, then by these support calculations one can easily prove e.g. $c^\sigma_{\pi\rho} \neq 0 \implies \sigma \geq \pi, \rho$.

Extending to K-theory.

Bott's K-theoretic version of the Thom isomorphism is based on the SES of equivariant sheaves on the T-representation L of weight λ

$$0 \to \mathcal{O}_L \otimes \mathbb{C}_{-\lambda} \xrightarrow{z\cdot} \mathcal{O}_L \to \mathcal{O}_{\{0\}} \to 0$$

giving $[\{0\}]|_0 = 1 - \exp(-\lambda) \in K_T(L)$, for the origin in the 1-d rep $\mathbb{C}_\lambda$ of T. The $[\mathbb{CP}^1]^2$ calculation in equivariant K-theory now becomes



Reducing to nonequivariant K-theory involves setting each $\exp(\vec{v}) \mapsto 1$.

The End.

III. Microlocal geometry of the individual terms in the Duistermaat-Heckman theorem.

Let (M^{2n}, ω) be a compact symplectic manifold. Its **volume** can be computed as $\int_M \frac{\omega^n}{n!}$ or as $\int_M \exp(\omega)$. The latter is more suggestive of Hirzebruch-Riemann-Roch, which is about $\int_M \exp(\omega)\,\mathrm{Td}(M)$, since $\mathrm{Td}(M) = 1+$ higher degree terms. Alternately, one can view this as the pushforward of the Liouville measure $[\omega^n]/n!$ to a point.

Both of these have generalizations when a torus T acts on M (or more generally a compact group K, which we won't discuss here). The first requires extending $\omega \in \Omega^2(M)$ to an *equivariantly closed* symplectic form $\omega - \Phi_T \in \Omega^2_T(M)$, the second requires picking a moment map $\Phi_T \colon M \to \mathfrak{t}^*$. (So, the same data.)

Theorem [Atiyah-Bott '84], elucidating [Duistermaat-Heckman '84]. The **Duistermaat-Heckman measure** $DH_T(M, \omega) := (\Phi_T)_*([\omega^n]/n!)$ on $\mathfrak{t}^*$ is the Fourier transform of

$$\int_M \exp(\omega - \Phi_T) \qquad = \sum_{f \in M^T} \frac{\exp(-\Phi_T(f))}{\prod(T\text{-wts in } T_f M)} \quad \text{if } |M^T| < \infty$$

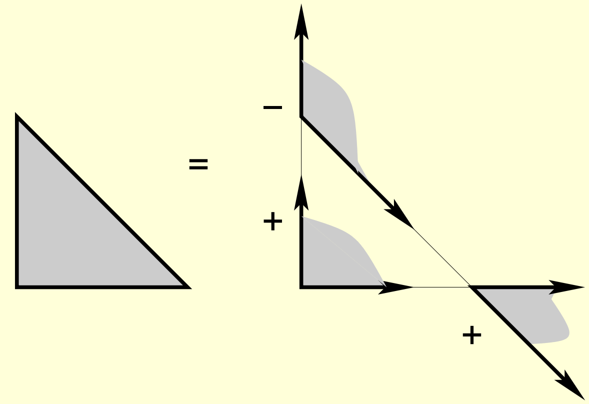
At this point, it would be nice if we could Fourier transform this function on $\mathfrak{t}$ term by term.

Choices to be made when Fourier transforming term by term.

The easy parts: $\mathrm{FT}\exp(-\lambda) = \delta_\lambda$ (Dirac delta), and $\mathrm{FT}(\lambda f) = D_\lambda \mathrm{FT}(f)$ (directional derivative). So $\mathrm{FT}(f/\lambda)$ involves... integrating? How to choose the constant?

Theorem [e.g. Guillemin-Lerman-Sternberg]. Pick $\vec{X} \in \mathfrak{t}$ such that $\langle \vec{X}, \lambda \rangle \neq 0$ for all isotropy tangent weights. If $(\lambda_i^f)_{i=1}^n$ are $\pm$ the weights at f, sign picked so that $\langle \vec{X}, \lambda_i \rangle > 0\ \forall i$, define $\mathrm{FT}(\exp(-\Phi_T(f))/\prod_i \lambda_i := \left(\prod_i \mathrm{int}_{\lambda_i}\right) \delta_{\Phi_T(f)}$. Then

$$DH_T(M, \omega) = \sum_{f \in M^T} (-1)^{\#\text{ flipped } \lambda_i^f} \left(\prod_i \mathrm{int}_{\lambda_i^f} \right) \delta_{\Phi_T(f)}$$



Here "int_λ" means "integrate in direction λ".

Our only measures are linear combinations of such terms. Call two such measures **Fourier equivalent** if they come from the same function on $\mathfrak{t}$. At most one class representative can be supported inside a fixed pointed cone.

Theorem [Prato-Wu '94]. A similar theorem holds for M noncompact, as long as $\langle \vec{X}, \Phi_T \rangle$ is proper and bounded below. (In particular, a bad choice of $\vec{X}$ leads to measures that are only Fourier equivalent to the right answer, not equal.)

Interlude: characteristic cycles and CSM classes.

Given a locally closed submanifold $\iota\colon A \hookrightarrow M$ of a complex manifold (both algebraic, say) we can associate the $\mathcal{D}_M$-module $\iota_*(\mathcal{O}_A)$ of distributions supported on A.

The algebra $\mathcal{D}_M$ of differential operators has a filtration by degree, with $\operatorname{gr} \mathcal{D}_M \cong \mathcal{O}_{T^*M}$. If we give $\iota_*(\mathcal{O}_A)$ a "good" compatible filtration, then $\operatorname{gr} \iota_*(\mathcal{O}_A)$ defines a sheaf on T^*M supported on a conical Lagrangian $cc(A)$ with multiplicities, the **characteristic cycle** of $\iota_*(\mathcal{O}_A)$.

Examples, where $M = \mathbb{C}$ *so* $\mathcal{D}_M = \mathbb{C}\langle \hat{x}, \frac{d}{dx} \rangle$ *and* $\operatorname{gr} \mathcal{D}_M \cong \mathbb{C}[q, p]$.

$A \subseteq \mathbb{C}$	generator g of $\iota_*(\mathcal{O}_A)$	$\operatorname{ann}(g)$	$cc(A) := \operatorname{supp}(\operatorname{gr} \iota_*(\mathcal{O}_A))$	cartoon
$\mathbb{C} \subseteq \mathbb{C}$	1	d/dx	$\{(q,p) : p = 0\}$	$-$
$\mathbb{C}^\times \subseteq \mathbb{C}$	x^{-1}	$(d/dx)\hat{x}$	$\{(q,p) : pq = 0\}$	$+$
$\{0\} \subseteq \mathbb{C}$	δ_0	$\hat{x}$	$\{(q,p) : q = 0\}$	$\vert$

Theorem [Victor "Ginsburg" '86]. Let $csm(A) := (-1)^{\operatorname{codim}_M A}[cc(A)]$ in $H^*_{\mathbb{C}^\times}(T^*M) \cong H^*(M)[\hbar]$. Then (1) $csm(A) = csm(A \setminus B) + csm(B)$ for B closed in A, and (2) $csm(M)|_{\hbar=-1} = c(TM)$.

In particular this defintion extends well-definedly to constructible functions, giving the **Chern-Schwartz-MacPherson classes**.

Exploiting the additivity.

For C a T-invariant cycle in M (i.e. linear combo of subvarieties of M), write $DH_T(C)$ for $(\Phi_T)_*(\text{Liouville measure on } C_{reg}$, weighted with multiplicities).

Let $M = \coprod_{f \in M} M_f^\circ$ be the Morse decomposition w.r.t. a component $\langle \vec{X}, \Phi_T(\bullet) \rangle$ of the moment map. Then

$$\begin{aligned} DH_{T\times\mathbb{C}^\times}(M) &= FT \int_M \exp(\tilde{\omega}) = FT \int_{T^*M} \exp(\tilde{\omega})[M] = FT \int_{T^*M} \exp(\tilde{\omega}) csm(M) \\ &= FT \int_{T^*M} \exp(\tilde{\omega}) \sum_{f \in M} csm(M_f^\circ) = \sum_{f \in M} FT \int_{T^*M} \exp(\tilde{\omega}) csm(M_f^\circ) \\ &= \sum_{f \in M} (-1)^f \, FT \int_{T^*M} \exp(\tilde{\omega})[cc(M_f^\circ)] = \sum_{f \in M} (-1)^f \, DH_{T\times\mathbb{C}^\times}(cc(M_f^\circ)) \end{aligned}$$

Lemma [Weber '12]. If $p \notin A$ then $[cc(A)]|_p \equiv 0 \bmod \hbar$.

Corollary. $DH_{T\times\mathbb{C}^\times}(cc(M_f^\circ))$ is Fourier equivalent to a measure whose $\hbar \mapsto 0$ projection is the corresponding term in the DH formula.

Irritatingly, I haven't been able to remove the "Fourier equivalent to" weasel words, and know examples of non-Morse decompositions where they are definitely necessary – the projection is improper on the support.

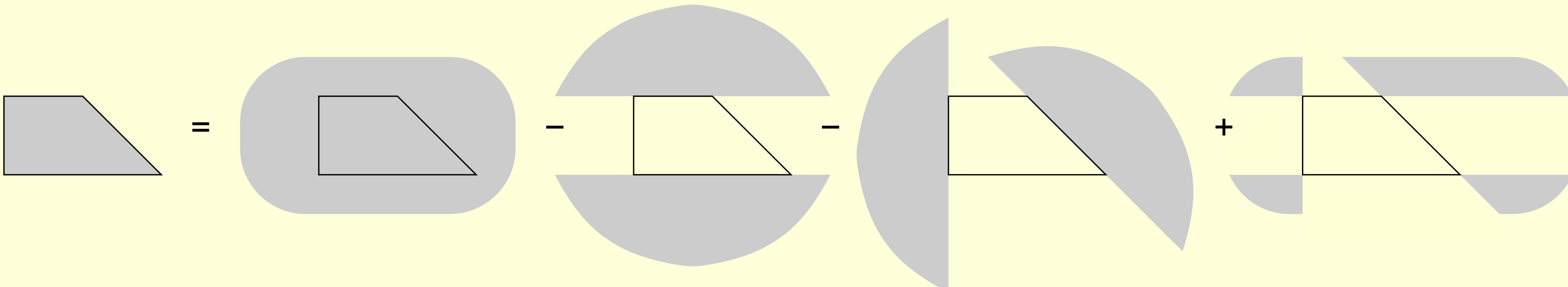
The Brianchon-Gram theorem.

Let $M = TV_P$ be the toric variety of a simple integral polytope, and $M = \coprod_{F \subseteq P} M_F^\circ$ be the decomposition into $T^{\mathbb{C}}$-orbits, indexed by faces F of P.

We can run the same computation as last slide, obtaining

$$DH_T(M) = \sum_{F \in P} (-1)^{\operatorname{codim} F} DH_T(cc(M_F^\circ))$$

and then discovering that $DH_T(cc(M_F^\circ))$ is the term in the outward-pointing *Brianchon-Gram theorem*, whose statement is best done through example.



[Guillemin-Ohsawa-Viktor Ginzburg-Karshon '02] also gave geometric meaning to individual DH terms, and [Harada-Karshon '12] connected that to the Brianchon-Gram theorem.

Really The End.