

# Schubert puzzles and invariant trilinear forms

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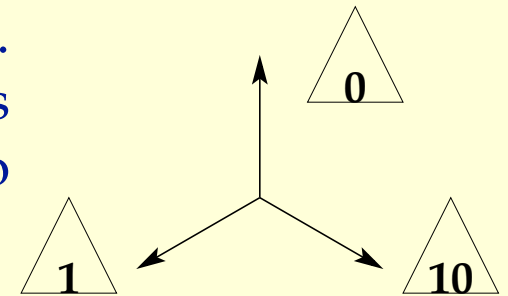
## Abstract

The label content on one side of a Schubert calculus puzzle indicates what flag manifold it is for, so that content should match the label content on the other two sides. Asking that this matching should stem from a *local* conservation law leads us to collections of vectors we recognize as the weights of some minuscule representations. The R-matrices of those representations (which, for 2-step flag manifolds, involve triality of  $D_4$ ) degenerate to give us puzzle formulæ for two previously unsolved Schubert calculus problems:  $K_{\top}(2\text{-step flag manifolds})$  and  $K(3\text{-step flag manifolds})$ . The  $K(3\text{-step flag manifolds})$  formula, which involves 151 new puzzle pieces, implies Buch's correction to the first author's 1999 conjecture for  $H^*(3\text{-step flag manifolds})$ .

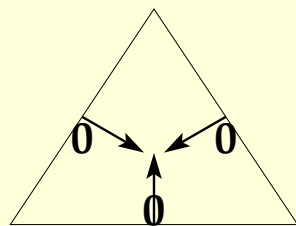
Via computer, we've proved that (under certain assumptions) there will be *no* puzzle-based combinatorial formula for  $H_{\top}^*(3\text{-step})$  nor for  $H^*(4\text{-step})$ .

# Grassmannian puzzles and their cohomology algebras.

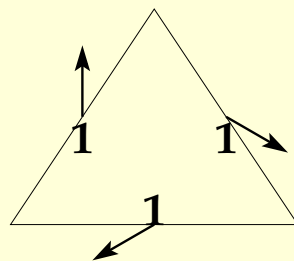
Index Schubert classes on  $Gr_k(\mathbb{C}^n)$  by bit-strings  $0^k 1^{n-k}$ . Introduce a third **edge label** (10) with which to adorn edges of triangles. Associate a vector, rotation-equivariantly, to each edge label on a puzzle piece:



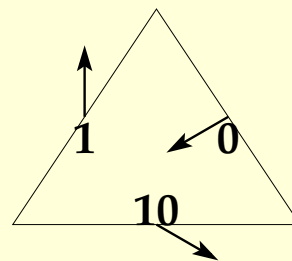
Then there are four ways, up to rotation, to label a triangle such that the three associated vectors cancel each other. We'll assign each little triangle a **fugacity**, and we'll multiply these fugacities when we glue pieces together into a **puzzle**.



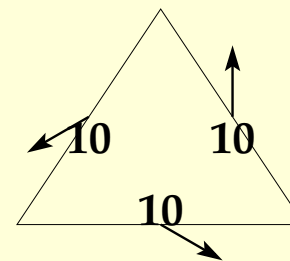
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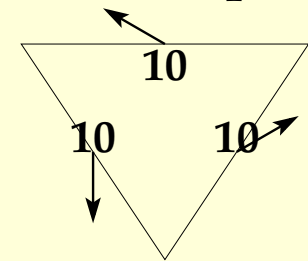
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**$\beta$**



**$\gamma$**

Define  $c_{\lambda\mu}^{\nu}(\beta, \gamma) := \sum \left\{ \text{fugacity}(P) : P \text{ a puzzle of shape } n\Delta, \text{ with } \partial P = \begin{smallmatrix} \lambda & \Delta & \mu \\ & \nu & \end{smallmatrix} \right\}$ .

**Theorem. 1.** [K-ZJ] These define a commutative associative algebra.

2. [K-Tao '03] For  $\beta = \gamma = 0$ , it's cohomology.

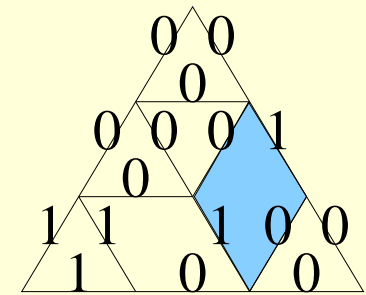
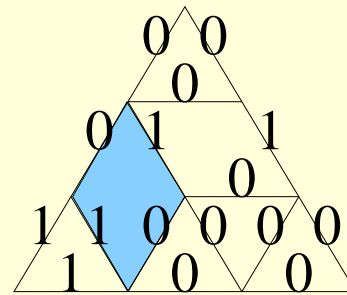
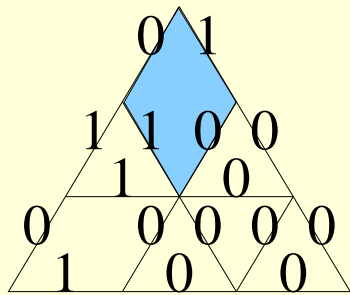
3. [Tao/Buch/Vakil '06] For  $\beta = -1, \gamma = 0$ , it's K-theory.

4. [Wheeler-ZJ '16] For  $\beta = 0, \gamma = -1$ , it's K-theory, in the dual basis.

# Equivariant cohomology, and Green's theorem.

Hereafter our puzzles are always of shape  $n\Delta$ .

For T-equivariant cohomology and K-theory we need to break  $Z_3$ -invariance, which we did in [K-Tao '03] by introducing the **equivariant rhombus**, whose fugacity (in cohomology) is  $y_i - y_j$  depending on its location.



These compute  $c_{010,100}^{100} = c_{100,010}^{100}$  respectively, as  $y_1 - y_3 = (y_1 - y_2) + (y_2 - y_3)$ .

**Theorem** [Pechenik-Yong '15]. There exists a puzzle formula for T-equivariant K-theory of Grassmannians (and another in the dual basis [Wheeler-ZJ '16]), with fugacities in  $K_T = \mathbb{Z}[\exp(\pm y_1), \dots, \exp(\pm y_n)]$ .

**Theorem.** Let  $P$  be a puzzle made of the puzzle pieces just listed, with no (10)s on the boundary. Then the number of 1s on each side is the same.

*Proof:* Sum the vectors over all edges of all little triangles and rhombi. Inside each piece they cancel, so the total vanishes. Likewise, on each internal edge the two summands cancel, leaving only the external edges. After that it's easy.

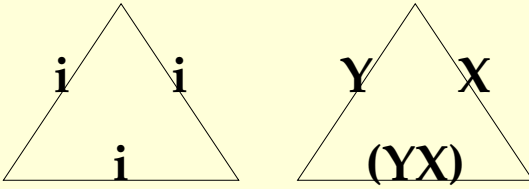
## Edge labels for 2- and 3-step flag manifolds.

A **d-step flag manifold**  $\{(0 \leq V_1 \leq \dots \leq V_d \leq \mathbb{C}^n)\}$  is one with  $\leq d$  interesting subspaces. My puzzle conjecture from '99 was correct for 2-step flag manifolds [Buch-Kresch-Purbhoo-Tamvakis '14], with eight **multinumber** labels

$$0, 1, 2, \quad 20, 21, 10, \quad 2(10), (21)0$$

and while incorrect for 3-step, was conjecturally fixed up by Buch to have twenty-seven labels

$$\{X, 3X : X \text{ above}\}, 3, (32)0, (32)1, (31)0, (32)(10), ((32)1)0, (3(21))0, \\ \text{and Buch's } (3(21))(10), (32)((21)0), 3(((32)1)0), (3(2(10)))0$$

where the puzzle pieces are of the forms  up to rotation.

*Question.* Can we assign a *unit* vector  $\vec{f}_X$  to each valid multinumber  $X$ , with which to prove a similar Green's theorem?

## The “weight” $\vec{f}_X$ of a multinumber $X$ .

Let  $\Lambda$  denote the triangular lattice with  $120^\circ$  symmetry  $\tau$ , and basis  $\vec{f}, \tau\vec{f}$  with Gram matrix  $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ . To each  $d$ -step multinumber, we assign a vector in  $\Lambda^{1+d}$  by

$$\vec{f}_i := \text{the } i\text{th basis vector,} \quad \vec{f}_{YX} = -\tau\vec{f}_Y - \tau^2\vec{f}_X$$

and insist that each valid multinumber  $X$  (for which we only have definitions when  $d \leq 3$ ) enjoy  $|\vec{f}_X|^2 = 2$ . In particular the direct sum  $\Lambda^{1+d}$  is *not* orthogonal.

$d = 1$ . The multinumber  $10$ , and the  $\tau$ -symmetry, force the Gram matrix of  $\vec{f}_0, \tau\vec{f}_0, \vec{f}_1, \tau\vec{f}_1$  to be

$$\begin{bmatrix} 2 & -1 & 2-a & a-1 \\ -1 & 2 & -1 & 2-a \\ 2-a & -1 & 2 & -1 \\ a-1 & 2-a & -1 & 2 \end{bmatrix}$$

$d = 2$ . The multinumbers  $20, 21, 20$  force all the  $2 \times 2$  blocks to match those in the  $d = 1$  case, but with three parameters  $a_{20}, a_{21}, a_{10}$ . Then the multinumbers  $2(10), (21)0$  force  $a_{20} = a_{21} = a_{10}$ . So there is only one parameter,  $a$ .

$d = 3$ . Each multinumber  $X$  already enjoys  $|\vec{f}_X|^2 = 2$ ; no extra condition on  $a$ .

## The vectors of norm-square 2, for $d \leq 3$ .

### Theorem.

1. For  $d \leq 4$ , and  $\alpha \in (0, 1)$ , this Gram matrix is positive definite. For  $d \geq 5$  there is no value of  $\alpha$  giving even semidefiniteness.
2. For  $d = 1, 2, 3$ , the lattice vectors in  $\Lambda^{1+d}$  with norm-square 2 (independent of  $\alpha$ ) are exactly those of the form  $\pm \tau^i \vec{f}_\chi$ , i.e. those vectors that we assign to puzzle-piece edges.  
At  $\alpha = 1$ , the vector configuration  $\{\vec{f}_\chi\}$  is isometric to the weights in the standard rep of  $A_2$  (for  $d = 1$ ), the vector rep of  $D_4$  (for  $d = 2$ ), and the 27-dim rep of  $E_6$  (for  $d = 3$ ). It's  $\tau$ -invariant for  $d = 1, 3$  but not  $d = 2$ , where  $\tau$  acts by  $D_4$  triality, e.g.  $\{\tau \vec{f}_\chi\}$  is the weights of the  $\text{spin}_+$  representation.
3. For  $d = 4$ , the Gram matrix is semidefinite at  $\alpha = 1$ . Modulo a 2-dim kernel, the lattice is  $E_8$ . (Hence its norm-square 2 vectors are the 240 roots.)

The cluster varieties of Dynkin type  $A_2, D_4, E_6, E_8$  are the Grassmannians  $\text{Gr}(3, n)$  for  $n = 5, 6, 7, 8$ . After that, 3-Grassmannians are still cluster but infinite type. We don't know how to work the irreps into the cluster story.

## The puzzle rules, in view of this representation theory.

To each edge of  $\Delta$  or  $\nabla$ , we have a minuscule irrep  $V$  of some group ( $A_2$ ,  $D_4$ , or  $E_6$ ). Each  $\Delta$  piece gives us basis vectors in  $V_S, V_{NW}, V_{NE}$ ; the Green's theorem vanishing condition says that their tensor product is  $T$ -invariant.

Summing those fundamental tensors times the corresponding fugacities, we get tensors  $\vec{\Delta} \in V_S \otimes V_{NW} \otimes V_{NE}$  and  $\vec{\nabla} \in V_N \otimes V_{SE} \otimes V_{NW} \cong V_S^* \otimes V_{NW}^* \otimes V_{NE}^*$  (within which we use the dual basis).

Then  $\vec{\Delta}^{\otimes \binom{n+1}{2}} \otimes \vec{\nabla}^{\otimes \binom{n}{2}}$  contracts, across each interior edge of  $n\Delta$ , to give a tensor in  $V_S^{\otimes n} \otimes V_{NW}^{\otimes n} \otimes V_{NE}^{\otimes n}$ . Expanded in its basis of fundamental tensors, the resulting coefficient on  $\nu \otimes \lambda \otimes \mu$  is exactly the puzzle sum, of products of fugacities. The edge-matching of puzzles is due to the use of dual bases of the two dual minuscule representations associated to an edge, so contractions give 0 or 1.

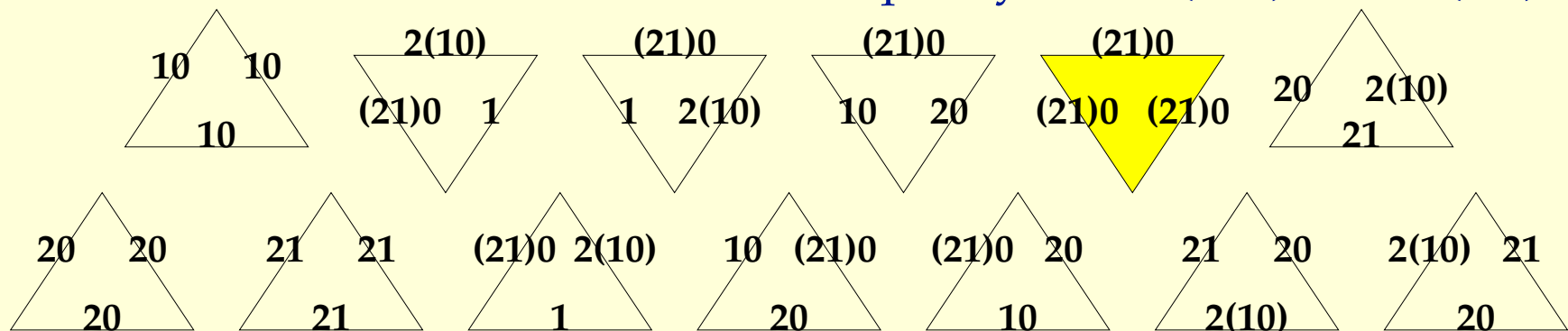
If we want to include the equivariant rhombi, we need to work with a tensor  $R(q) \in V_{SE} \otimes V_{SW} \otimes V_{NE} \otimes V_{NW} \cong \text{End}(V_{SE} \otimes V_{SW})$  including a parameter  $q$  that we specialize to  $y_i - y_j$  for  $H_T^*$  or  $\exp(y_i - y_j)$  for  $K_T$ .

**Theorem.** For  $d = 1, 2, 3$ , the Jimbo  $R$ -matrix of the tensor product  $V_{SE} \otimes V_{SW}$  factors as  $V_{SE} \otimes V_{SW} \rightarrow V_N \rightarrow V_{SE} \otimes V_{SW}$  at  $q = c/3$ , where  $c$  is the dual Coxeter number 3, 6, 12. For  $d = 4$ , we need to take each  $V_* = \mathfrak{e}_8 \oplus 1$  to get a Jimbo  $R$ -matrix, and that  $R$ -matrix again factorizes at  $q = c/3$ , where now  $c = 30$ .

## New Schubert calculus formulæ.

As in the  $d = 1$  case and its two K-pieces, we need to take certain degenerations of these R-matrices to get the actual puzzle formulæ (which spoils the G-equivariance of the R-matrix; this is weird and deserves better understanding).

**Theorem.** Add the following K-pieces to compute Schubert calculus in  $K_T(2\text{-step flags})$ . Some nonequivariant rhombi acquire an extra fugacity factor  $\exp(y_i - y_j)$ , if the sum of lengths of top multi-numbers is greater than that of bottom multi-numbers – and in case of equality, if  $\max(\text{NE}) > \max(\text{SE})$ .



Even for *nonequivariant* K, there was no conjectured rule before.

In the 3-step case, degenerating some terms to zero takes others to infinity, unless we first pass to the nonequivariant situation  $q = c/3$ .

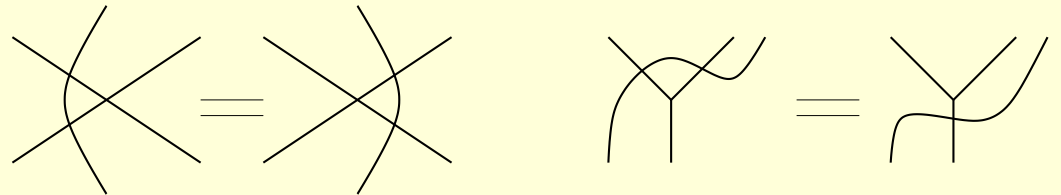
**Theorem.** Buch's 27-label correction to my 1999 conjecture correctly computes Schubert calculus in  $H(3\text{-step flags})$ . One can add 151 K-pieces in order to compute K-theory. Each spoils the inversion count, sometimes by up to 5.



## Ingredients of the proof (for $d = 1, 2$ at least).

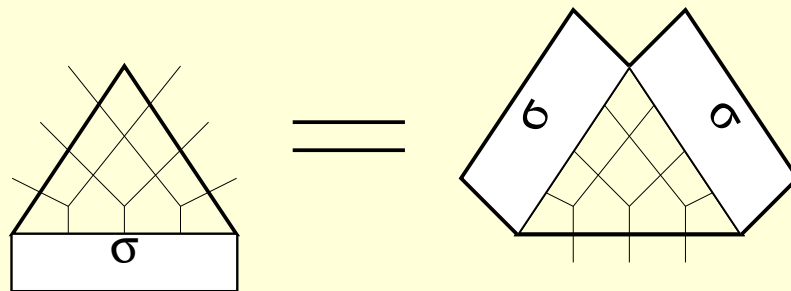
It's convenient and customary among the integrable crowd to work with the dual picture, where the vertical rhombi (into which we'll place R-matrices) and the triangles at the bottom (into which we'll place the invariant trilinear form) as crossing lines and trivalent vertices.

Then the “Yang-Baxter” and “bootstrap” equations become



Given an R-matrix, *define*  $S_{\lambda|\sigma}$  as the matrix coefficient  $\left( \prod_{\text{Id}}^{\sigma} R \right)^{\lambda}$ , i.e. using a wiring diagram for  $\sigma$ . **In a certain limit of R**, this reduces to the restriction of the equivariant Schubert class.

**Proposition.**



**Proof.** Clearly suffices to handle  $\sigma = r_i$ . Use YBE and bootstrap equations repeatedly.

**Corollary.**  $\sum_{\nu} (\sum_{\Delta} \text{fugacity}(\Delta)) S_{\nu|\sigma} = S_{\lambda|\sigma} S_{\mu|\sigma}$ . Once we have that for all  $\sigma$ , we're done.