SCHUBERT VARIETIES, SUBWORD COMPLEXES, FROBENIUS SPLITTING, AND BRUHAT ATLASES

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ABSTRACT. We consider stratifications of schemes by subvarieties, and introduce the notion of a stratification generated by a collection of (reduced) subschemes. We recall some basics of Gröbner degenerations (just for lex order) and Stanley-Reisner theory.

We then state the main theorem, that the lex init of a Kazhdan-Lusztig variety in Bott-Samelson coördinates is the Stanley-Reisner scheme of a subword complex, after defining all those terms. Rather than getting into the combinatorial details, we focus on the principal ingredient in the proof, which is Frobenius splitting.

Now that we have an interesting class of vector spaces with stratifications, we glue them together in atlases to make some famous stratified spaces, such as flag manifolds and wonderful compactifications of groups.

These are notes for the Dijon summer school TLAG 2017, largely compiled from [Kn09] and [KnMi04]. Remerciements to the organizers for the invitation.

CONTENTS

1. HOUR 1: STRATIFICATIONS, AND DEGENERATIONS TO STANLEY-REISNER SCHEMES

Most people's definition of a **stratification $\mathcal{Y}^\circ$ of a variety** X is a disjoint decomposition $X = \coprod_{Y^\circ \in \mathcal{Y}^\circ} Y^\circ$ into locally closed algebraic subsets called **open strata**, with the condition

> For each $Y^\circ \in \mathcal{Y}^\circ$, its closure $\overline{Y^\circ}$ is a union $\coprod \left\{ Z^\circ \in \mathcal{Y}^\circ \,:\, Z^\circ \cap \overline{Y^\circ} \neq \emptyset \right\}$ of open strata.

Date: July 23, 2017.

(Warning: being "open strata" doesn't mean they're open in X – we'll reserve the term **maximal open strata** for this.) In particular $\mathcal{Y}$ is a poset, defining[1] $Z^\circ \leq Y^\circ$ for $Z^\circ \subseteq \overline{Y^\circ}$. One of the obvious sources of such decompositions is the orbits of a group action. In all our interesting examples, $\mathcal{Y}$ will be finite.

The definition I prefer is in terms of the closed subsets $\mathcal{Y} := \{\overline{Y^\circ} \ : \ Y^\circ \in \mathcal{Y}^\circ\}$ I'll call **closed strata**, with the new axiomatization

> For any collection $S \subseteq \mathcal{Y}$ of closed strata, their intersection $\cap S$ is a union of closed strata, namely the ones contained in every $Y \in S$.

(Warning redux: when people who think strata are disjoint speak of "closed strata", they mean what we'll call **minimal strata**.) The poset structure is now simply containment. There are two versions of this axiom, depending on whether the intersection is meant set- or scheme-theoretically. For the moment, we ignore any issues of scheme structure, and only use the word "scheme" instead of "variety" to indicate possible reducibility.

In either axiomatization, we'll always assume that **our strata are irreducible** (though the ambient scheme may not be). This is actually easiest to describe under a *third* axiomatization of "stratifications $\mathcal{Y}$ by (irreducible) varieties", actually an axiomatization of the poset $J(\mathcal{Y})$ of order ideals:

> $J(\mathcal{Y})$ should be a collection of subschemes closed under union, under (reduction of) intersection, and under taking irreducible components.

It's not exactly the same object as in the second version: if we take $\mathcal{Y} := \{Y \in J(\mathcal{Y}) \ : \ Y \text{ is irreducible}\}$ then we get a stratification by closed subvarieties (the second version), and these are equivalent data.

This $J(\mathcal{Y})$ is now nearly an algebra of some sort, modulo the fact that taking irreducible components is multivalued. So given a collection $\mathcal{C}$ of reduced subschemes of X, we can define the **stratification** $J(\mathcal{Y})$ **generated by** $\mathcal{C}$ as the smallest collection containing $\mathcal{C}$ and forming a stratification (in this third sense). It's clear how to compute it: take the closure of $\mathcal{C}$ under the (multivalued) operations intersect and decompose. (Theorem: one can delay taking unions until the end.)

Example. Consider 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\mathcal{C} := \{a = 0, ad = bc, d = 0\}$. Then we get $a = bc = 0$ as an intersection, with components $a = b = 0$, $a = c = 0$. In all, $\mathcal{Y}$ contains

$$\begin{array}{ccccc}
 & & \underline{M_2} & & \\
 & \underline{\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}} & \underline{\det = 0} & \underline{\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}} & \\
\underline{\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}} & \underline{\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}} & \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} & \underline{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}} & \underline{\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}} \\
\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} & & \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}
\end{array}$$

[1] N.B. One of the most important examples (coming later) is Bruhat order, which by convention is the reverse.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So it's missing three of the coordinate spaces, $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \cap \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, which (it will turn out) are somehow glued together into the $\{\det = 0\}$ stratum. More about this anon.

1.1. **Basic elements.**

Call an element $p \in P$ of a finite-height poset $(P, \subseteq)$ **basic** if p is *not* the unique lower bound of $\{q \in P \,:\, q \supset p\}$ [LasSchü96]. Then it is nigh tautological that p *is* the unique lower bound of $\{q \in P \,:\, q \supseteq p\}$ (the proof uses induction on the height).

If $\mathcal{Y}$ is a stratification by closed subvarieties, then $\forall Y \in \mathcal{Y}$,

$$Y = \bigcap \{Z \in \mathcal{Y} \,:\, Z \supseteq Y,\ Z \text{ basic in the poset } \mathcal{Y}\}$$

set-theoretically, not necessarily scheme-theoretically.

In the poset in the example above, the underlined elements are the basic ones, that can't be constructed as intersections of other strata (though sometimes as *components* of those intersections).

1.2. **Gröbner degenerations and Stanley-Reisner theory.**

Given an ideal $I \leq S[z_1, \ldots, z_m]$, define the **lex initial ideal** `init` I as the ideal S-linearly spanned by the lex leading terms of the (uncountably many?) elements of I. (This is an example of a "Gröbner degeneration" of an ideal.) It is necessarily a monomial ideal, thereby defining a schemy union of coordinate spaces; $\sqrt{I}$ is the intersection of some coördinate ideals $C_F := \{z_f = 0 \,:\, f \notin F\}$, $F \subseteq [0, n]$.

Define the n**-simplex** Δ^n as $\{(z_0, \ldots, z_n) \,:\, z_i \in \mathbb{R}_+, \sum_i |z_i| = 1\}$. For each $F \subseteq [0, n]$, the subset $\Delta^n \cap \{z_f = 0\ \forall f \notin F\}$ is linearly isomorphic to $\Delta^{|F|-1}$, and we call it a **face** of Δ^n.

Given an ideal $I \leq \mathbb{C}[z_0, \ldots, z_n]$, we get a subset $\Delta^n \cap \operatorname{Spec} I$ of the simplex. Of course this set only depends on $\sqrt{I} = \bigcap_P P$ and the minimal prime ideals P containing it; if I is monomial then these are coördinate ideals, and the subset is a (closed) union of faces, a **simplicial complex on the vertex set** $[0, n]$. ("Stanley-Reisner theory" concerns the passage back and forth between simplicial complexes and squarefree monomial ideals.)

In the poset example above, the only *non*-monomial ideal is the one $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$. If we order the variables so that $-bc$ is the leading term, then this subvariety breaks into the union $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \bigcup_{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. (Remember these were the missing coördinate subspaces?)

We can now picture the original poset as a stratification of the simplex by "open subcomplexes", with the empty face $\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$, the four vertices, but only five of the six open edges, because the sixth $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ is the glue between two of the open triangles.

2. HOUR 2: SCHUBERT VARIETIES

2.1. **Warmup: matrix Schubert varieties.** Let $\mathcal{C}$ be the $n-1$ hypersurfaces in M_n (the space of $n \times n$ matrices) given by $\det(\text{Northwest } i \times i \text{ submatrix}) = 0$. Each of these is $B_- \times B_+$-invariant, hence the stratification $\mathcal{Y}$ they generate is too. In fact its elements are all of the form

$$\overline{X}_\pi := \overline{B_- \pi B_+} \subseteq M_n, \qquad \pi \in S_n$$

which we will call **matrix Schubert varieties**. They were first considered in [Fu92].

This *reverse* of this poset, the **Bruhat order** on S_n is well-studied, e.g. we have a combinatorial description $w \lessdot v$ iff $w = v(i \leftrightarrow j)$ where the numbers in v physically between $i < j$ are not also numerically in between. It is ranked (by codimension), and $\operatorname{codim} \overline{X}_\pi = \{(i<j) \,:\, \pi(i) > \pi(j)\} =: \ell(\pi)$.

The basic elements in (the reverse of!) S_n Bruhat order are the **biGrassmannian** permutations $w \in S_n$, those where both w, w^{-1} have at most one descent. These are of the form

$$w = 1 \ldots r \quad r+b+1 \ldots r+b+a \quad r+1 \ldots r+a \quad r+b+a+1 \ldots n$$

and the corresponding $\overline{X}_w$ is defined by saying that the NW $(r+a) \times (r+b)$ rectangle should have rank at most $\leq r$. (Such a rank condition is equivalent to the vanishing of all the $(r+1) \times (r+1)$ determinants in that rectangle, and those have been known for a long time to generate a prime ideal.)

Fulton's "essential set" $ess(w)$ construction is a way of computing the maximal biGrassmannian $w' \leq w$, and he shows

Theorem 2.1. [Fu92] $\overline{X}_w = \bigcap_{w' \in ess(w)} \overline{X}_{w'}$ *as schemes, i.e.* $I_w = \sum_{w' \in ess(w)} I_{w'}$ *as ideals.*

Following [KnMi05], we lex-order the coördinates of M_n starting from the NE and coming SW in any order – the point being that the leading term of any determinant is the product along the *anti*diagonal. The equations defining $\overline{X}_{w'}$ form a Gröbner basis[2] [St90].

Theorem 2.2.
- [Kn09] *For general ideals* $\texttt{init}\,(I \cap J) \leq \texttt{init}\, I \cap \texttt{init}\, J$, *and* $\texttt{init}\,(I + J) \geq \texttt{init}\, I + \texttt{init}\, J$. *But if* $I = I_w$ *and* $J = J_v$ *then these are equalities.*
- [KnMi05] *Therefore, if we concatenate the Gröbner bases for the* $\overline{X}_{w'}$, *we get one for* $\overline{X}_w = \bigcap_{w' \in ess(w)} \overline{X}_{w'}$. *Without the* `init` *step, this gives theorem 2.1.*
- [KnMi05] *The simplicial complex corresponding to* $\texttt{init}\, I_w$ *is homeomorphic to a ball, and all its links are spheres (in the interior) and hemispheres (on the boundary). Hence its Möbius function is* ± 1 *on interior faces,* 0 *on boundary faces, with which we can compute its Hilbert series, and from there that of* I_w *itself. This reproduces the formula from* [FoKi96] *for double Grothendieck polynomials.*

We'll have the right technology to prove these later; this is just to give an idea of what properties one might hope for about a stratification.

[2] A generating set (p_i) for I, large enough that $(\texttt{init}\, p_i)$ also generates $\texttt{init}\, I$, is a **Gröbner basis**.

Example. Let $w = 1423 = \begin{pmatrix} 1 & . & . & 0 \\ . & . & . & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (biGrassmannian), where the 2×3 dotted region in the NW bears the only nontrivial rank condition: to say that its rank is at most 1, the three 2×2 determinants must vanish,

$$m_{11}m_{22} - \underline{m_{21}m_{12}} = m_{11}m_{23} - \underline{m_{21}m_{13}} = m_{12}m_{23} - \underline{m_{22}m_{23}} = 0$$

where in each polynomial we've underlined the lex first term, the *antidiagonal* one.

The initial scheme, defined by $m_{21}m_{12} = m_{21}m_{13} = m_{22}m_{23} = 0$, has three components

$$\left\{ \begin{pmatrix} * & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & * \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \end{pmatrix} \right\}$$

(of course these are sitting inside the original 4×4 matrices, where the remaining 10 entries are also $*$s, meaning "free"). So the simplicial complex corresponding to this squarefree monomial ideal has three facets (maximal faces), but unfortunately it's 13-dimensional inside Δ^{15}.

If we only consider the nontrivial entries $m_{12}, m_{13}, m_{21}, m_{22}$, then this complex is 1-dimensional inside Δ^3, and easily pictured; it's the 1-ball

$$\begin{array}{ccccc} & & \boxed{m_{12}} & & \\ \boxed{m_{13}} & / & | & / & \boxed{m_{21}} \\ & & \boxed{m_{22}} & & \end{array}$$

inside the 1-skeleton of Δ^3. (The three edges in this 1-ball correspond to the three components above of the initial scheme, in order.) Putting back in the other 12 vertices just takes cones on this complex, making it a 13-ball.

2.2. **Schubert varieties.** Let $B, B_- \leq GL_n$ be the upper/lower triangular matrices. Let $X = GL_n/B = \coprod_{w \in W} BwB/B$, the **Bruhat stratification**.

Theorem. *The Bruhat stratification is generated by* $X \setminus (Bw_0B/B) = \bigcup_\alpha \overline{Bw_0r_\alpha B}/B$, *the complement of the maximal stratum (here* $w_0(i) := n + 1 - i$ *).*

Proof. $\mathcal{Y}$ contains X itself, as the empty intersection, and all the codimension 1 Schubert varieties $\overline{BwB}/B$, the components of $\{X \setminus (Bw_0B/B)\}$.

We first prove that every $w \in S_n$ other than w_0 and $\{w_0r_\alpha\}$ is covered by at least two elements $w_1, w_2 \gtrdot w$. If w_0w is not Grassmannian, then (by definition) there are two $wr_\alpha \gtrdot w$, and w is their greatest lower bound. If $w^{-1}w_0$ is not Grassmannian, then there are two $r_\alpha w \gtrdot w$. If w_0w is biGrassmannian and not r_α, then it takes longer to find two covers but this is still easy and we leave it to the reader.

Then, the intersection of $\overline{Bw_1B}/B \cap \overline{Bw_2B}/B$ has codimension at least 1 in each, but contains $\overline{BwB}/B$, so that must be a component.

Now we claim that $\overline{BwB}/B$ arises in the stratification, by induction on codimension, where we handled $0, 1$ as base cases. The stratification can't be any finer, because of the B-invariance. $\square$

Define the **Schubert varieties** $X_w := \overline{B_- wB}/B$ as the closures of the B_- orbits, and the **opposite Bruhat cells** $X^v_\circ := BwB/B$ as the B-orbits. Then each $X^v_\circ$ is stratified by its (transverse) intersections with the $\{X_w\}$. This is not itself an orbit stratification, despite being derived from one, and we will call it the **Kazhdan-Lusztig stratification**.

2.3. **Bott-Samelson coördinates.** This space $X^v_\circ$ is actually isomorphic to a vector space of dimension $\ell(w)$, and we now put coördinates on it. Given a list $Q = (v_1, \ldots, v_m)$ in W, define the **Bott-Samelson variety**

$$BS^Q := \overline{Bv_1B} \times^B \cdots \times^B \overline{Bv_mB}/B$$

where $\times^B$ means "divide by the diagonal B-action". Forgetting the last factor, we see that this is an (opposite) Schubert variety bundle over a smaller Bott-Samelson, etc., so it is a tower of Schubert varieties (hence a projective variety).

The left-B-equivariant **Bott-Samelson map** $BS^Q \to G/B$ takes

$$[g_1, \ldots, g_m] \quad \mapsto \quad g_1 \cdots g_m B/B$$

The image is closed, irreducible, and B-invariant, therefore must be of the form $\overline{BvB}/B$ for some v. Denote this v by $\mathrm{Dem}(Q)$, the **Demazure/nilHecke product of** Q. Call Q **reduced** if $\sum \ell(v_i) = \ell(\mathrm{Dem}(Q))$, i.e. if the Bott-Samelson map doesn't drop dimension. (Alas, this has nothing to do with reducedness of schemes.) In the reduced case we'll just write $\prod Q$ instead of $\mathrm{Dem}(Q)$.

(More generally, we can replace any adjacent elements v_i, v_{i+1} in Q by $\mathrm{Dem}(v_i, v_{i+1})$ to make Q', and get a natural map $BS^Q \twoheadrightarrow BS^{Q'}$. But anyway...)

Proposition 2.3. *The general fiber of a Bott-Samelson map is connected. (Moreover, the fiber over* $\mathrm{Dem}(Q)B/B$ *is general.) Hence if* Q *is reduced, the Bott-Samelson map is an isomorphism over* $B \prod QB/B$.

Proof. Let $v := \mathrm{Dem}(Q)$. Let $N := B'$ and $N_- := B'_-$, and define $N_v := N \cap vN_-v^{-1}$. It turns out that N_v acts simply transitively on BvB/B. Since the B-S map is N_v-equivariant, it is a bundle over the open set BvB/B (and all fibers there are general).

Let F be the fiber over vB/B. Then $N_v \times F \to BS^Q$ is an open inclusion, hence if F were disconnected then BS^Q would be reducible, contradiction. ∎

3. HOUR 3: `init` IN THE BOTT-SAMELSON COÖRDINATES

If each v_i is a simple reflection, then the Schubert varieties of which BS^Q is a twisted product are all $\mathbb{P}^1$s, hence BS^Q is smooth. (So if Q is a reduced word, then $BS^Q \to X^{\prod Q}$ is a resolution of singularities.) Each $Br_\alpha B/B$ has an easy parametrization

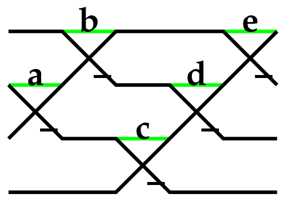
$$z \quad \mapsto \quad \widetilde{r_\alpha}\, e_\alpha(z) \quad = \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& z & 1 &&& \\ &&& -1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \text{ in the } GL_n \text{ case}$$

and putting these together we get a parametrization

$$\begin{array}{ccccc} \mathbb{A}^n & \stackrel{m}{\to} & BvB & \to & X_\circ^v \\ (z_1,\ldots,z_n) & \mapsto & \prod_{i=1}^n \widetilde{r_{\alpha_i}}\, e_\alpha(z_i) & \mapsto & \prod_{i=1}^n \widetilde{r_{\alpha_i}}\, e_\alpha(z_i)B/B \end{array}$$

where the composite is an isomorphism.

Example. Let $Q = 21321$, meaning, $(r_2, r_1, r_3, r_2, r_1)$ in S_4. Picture the product m as



$$\begin{pmatrix} be-d & -b & 1 & 0 \\ ae-c & -a & 0 & 1 \\ e & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(Each terms comes by following a path from left to right, possibly through variables sitting atop crosses.) Now we want to understand the stratification, i.e. $m^{-1}(X_w)$. Since it's generated by the divisors X_{r_α}, we start by pulling back each equation $\det(M_{NW\, i\times i})$ of X_{r_i}:

$$\begin{aligned} \det\begin{pmatrix} be-d \end{pmatrix} &= be-d \\ \det\begin{pmatrix} be-d & -b \\ ae-c & -a \end{pmatrix} &= ad-bc \\ \det\begin{pmatrix} be-d & -b & 1 \\ ae-c & -a & 0 \\ e & -1 & 0 \end{pmatrix} &= c \end{aligned}$$

Proposition 3.1. *Let Q be a list of simple reflections (w_q), and f_α the equation of X_{r_α} inside $X_\circ^{Dem(Q)}$. With respect to the lexicographic term order on variables, the leading term of the polynomial $m^{-1}(f_\alpha)$ is $\prod_{q\in Q,\, w_q=r_\alpha} z_q$. (In the above example those are be (for $2\underline{1}32\underline{1}$), ad (for $\underline{2}13\underline{2}1$), c (for $21\underline{3}21$).)*

What's the corresponding statement for Schubert varieties not of codimension 1?

Theorem 3.2. [Kn09, §7.3] *Let Q be a list of simple reflections (w_q), and $BS^Q \to G/B$ the Bott-Samelson map. Let $m^{-1}(X_w) \subseteq \mathbb{A}^{|Q|}$ be the pullback of the Schubert variety, and denote its ideal by I_w.*

Then $J_w := \texttt{init}\ I_w$ is a squarefree monomial ideal (it is reduced), and its prime components are coördinate ideals C_F where the complement $Q \setminus F$ *is a reduced word for w.*

This simplicial complex $\Delta(Q, w)$, called a **subword complex** *in* [KnMi04]*, is homeomorphic to a ball (and "shellable"). Its boundary sphere is $\bigcup_{w'>w} \Delta(Q, w')$.*

For general ideals $\texttt{init}\ (I\cap J) \le \texttt{init}\ I \cap \texttt{init}\ J$, *and* $\texttt{init}\ (I+J) \ge \texttt{init}\ I + \texttt{init}\ J$. *But if $I = I_w$ and $J = I_v$ then these are equalities.*

Most of the proof coming soon!

Corollary 3.3. (1) *Each I_w is Cohen-Macaulay (using the shellability).*
(2) *Each I_w is normal, so one can speak of an anticanonical divisor, and $\bigcap_{w'>w} I_{w'}$ defines an anticanonical divisor.*

(3) $I_w = \bigcap_{w' \le w,\ \text{biGrassmannian}} I_{w'}$. *Moreover, one can get a Gröbner basis for* I_w *by concatenating Gröbner bases for the maximal biGrassmannians* $\le w$.

(4) *Consider the poset morphism* $2^Q \to W$, $F \mapsto \mathrm{Dem}(Q \setminus F)$, *we'll also call* m. *Then* $\Delta(Q, w) = m^{-1}([w, w_0])$, *i.e. this* m *defines a "Bruhat decomposition of* Δ^Q*".*

Some proof sketches. (1) C-Mness is about vanishing of certain local cohomology; the shelling lets us use Mayer-Vietoris to prove this inductively.

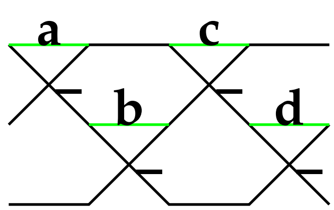
(2) C-M $\implies$ S2, so we need R1 (the other half of Serre's criterion for normality). The only locus where X_w might fail to be R1 is around the $X_{w' > w}$. But in $\mathtt{init}\ I_w$, that's the boundary sphere of the ball, and $\mathtt{init}\ I_w$ is generically smooth there; since R1 is an open condition in families we learn I_w was also R1 there, hence normal. We're not explaining the anticanonical comment here, but see §4.1.

(3) The sum of radical monomial ideals is radical; now use the commuting-with-`init` statement.

(4) This is just a rewriting of the definition of subword complex.

□

Example. $Q = 1212$.



$$\begin{pmatrix} ac - b & 1 - ad & a \\ c & -d & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

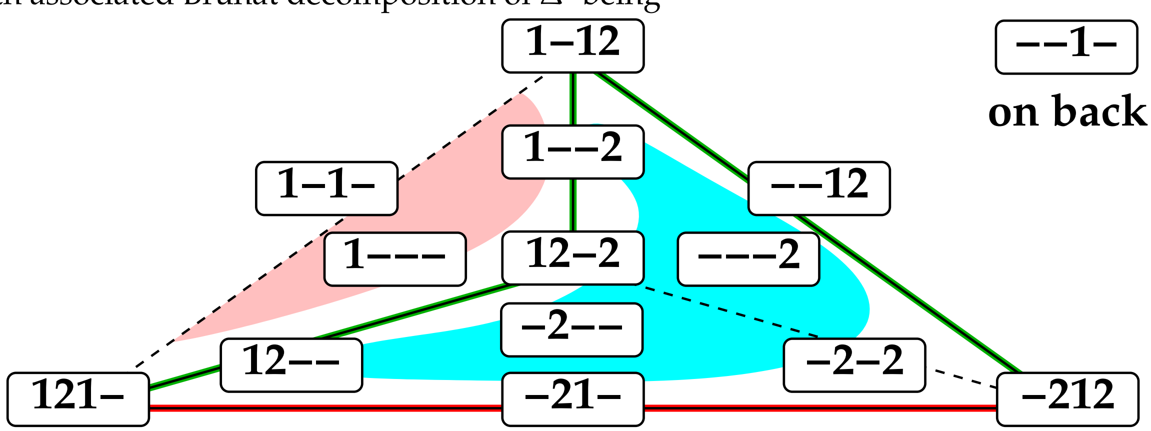
The ideals, with lex leading terms[3] underlined are

$$I_{213} = \langle \underline{ac} - b \rangle, \qquad I_{132} = \langle \underline{bd} - c \rangle$$

$$I_{312} = \langle \underline{ac} - b, 1 - \underline{ad},\ \underline{bd} - c \rangle, \qquad I_{231} = \langle -\underline{b}, \underline{c} \rangle$$

$$I_{321} = \langle \underline{c}, \underline{b}, 1 - \underline{ad} \rangle$$

with associated Bruhat decomposition of Δ^3 being



Incidentally, though BS^Q and $\Delta(Q, w)$ are defined using a word Q, there are obvious isomorphisms if we change Q by a commuting move, though not by a braid move. In

[3]Note that the two obvious equations of I_{312} are not a Gröbner basis! We need to include the third one.

S_n the permutation 321 is the smallest one supporting a braid move, and indeed, the condition to have only one reduced word (up to commuting moves) is to be 321-avoiding.

Example: matrix Schubert varieties redux. Let Q be a reduced word for $w_0 w_0^P := (i \mapsto i + n \bmod 2n) \in S_{2n}$, which is visibly 321-avoiding hence the word doesn't matter much and is called "the" **square word**. (Here P is the maximal parabolic of $GL(2n)$ with Levi $GL(n) \times GL(n)$.) Then consider the composite

$$BS_\circ^Q \cong Bw_0 w_0^P B/B \cong Bw_0 w_0^P P/P = Bw_0 P/P = Pw_0 P/P \qquad \hookrightarrow GL(2n)/P \cong Gr(n, 2n)$$

where the second isomorphism depends on $w_0 w_0^P$ being minimal in its W/W_P coset. In particular, the latter orbit is the big cell in the Grassmannian, the image of the $GL(n)^2$-equivariant map $\mathtt{graph}: M_n \to Gr(n, 2n)$.

Fulton observed (in other language) that this isomorphism of cells takes the matrix Schubert variety stratification to *a coarsening of* the Kazhdan-Lusztig stratification on $X_\circ^{w_0 w_0^P}$, which essentially derives from the latter's surprising $GL(n)^2$-action. (Most of the stratification is not an orbit stratification, but this coarsening is.)

3.1. **Frobenius splitting.** While every ideal has a monomial $\mathtt{init}$, it's very special for that monomial ideal to be radical ($\mathtt{init}\, I$ radical $\implies$ I radical, but not the reverse). So let's better understand why an ideal is radical.

Fix $n \in \mathbb{N}, n > 1$. A (commutative) ring R is reduced if $x \mapsto x^n$ only takes $x \mapsto 0$. We'd like to say "if $\ker(x \mapsto x^n) = 0$" but this isn't an additive map... *unless* n is a prime we'll rename p, and $R \geq \mathbb{F}_p$.

Now asking $\ker = 0$ is the same as asking it to have a 1-sided inverse. Define a **near-splitting** φ of R to be a map $\varphi : R \to R$, a sort of pth root operation, such that

- $\varphi(a + b) = \varphi(a) + \varphi(b)$
- $\varphi(a^p b) = a\varphi(b)$

and a **splitting** if it also satisfies

- $\varphi(1) = 1$.

It's now really obvious that if a ring has a splitting, then it's reduced. Where do splittings come from?

To specify a near-splitting on $\mathbb{F}[x_1, \ldots, x_n]$ where $\mathbb{F}$ is a perfect field of characteristic p, it's enough to specify it on monomials m. (Actually, it's enough to specify it on monomials with exponents in $[0, p)$, and one can do so freely.)

Example. On affine space, let $\varphi(m) = \begin{cases} \sqrt[p]{m} & \text{if } \exists m', (m')^p = m \\ 0 & \text{if } \nexists m', (m')^p = m \end{cases}$ and call this the **coördinate splitting**.

There's a basic example of near-splittings on $\mathbb{A}^n$, which we describe on monomials:

$$\mathrm{Tr}(m) := \begin{cases} \sqrt[p]{m \prod x_i} / \prod x_i & \text{if } \exists m', (m')^p = m \prod x_i \\ 0 & \text{if } \nexists m' \end{cases}$$

(What is this thing? It's a characteristic p analogue of "residue" – in one variable, it measures the failure of a Laurent polynomial to be a derivative. It's sometimes more natural to think of it as applying to top forms $f\,dx_1\,dx_2\cdots\,dx_n$ rather than to functions, at which point it multiplies the grading by $\frac{1}{p}$.)

Proposition 3.4. *The map*

$$f \in \mathbb{F}[x_1, \ldots, x_n] \quad \mapsto \quad \mathrm{Tr}(f\bullet)$$

is a bijection to the space of near-splittings.

Proof. $\mathrm{Tr}(f\bullet)$ is trivially checked to be a near-splitting. Each near-splitting is determined by its value on the p^n monomials with exponents in $[0, p)$. From there it's a boring calculation to construct the inverse. □

To get examples of splittings on subvarieties of affine space, define an ideal $I \leq R$ to be **compatibly (near-)split by** φ if $\varphi(I) \leq I$. Then the near-splitting descends to R/I. When φ is a splitting such ideals are automatically radical, since the quotient is split so reduced.

Example. An ideal I is compatibly split w.r.t. the coördinate splitting iff it's Stanley-Reisner, i.e., generated by squarefree monomials. (In what counts as a hard theorem in Frobenius splitting, there's a 4-page paper [KuMe] showing that *every* splitting ϕ has only finitely many compatibly split subvarieties!)

Theorem 3.5. *Let* $I, J \leq R$ *be compatibly split w.r.t* φ*, and* K *an ideal. Then*

- $I \cap J$ *is split. (We'll omit "compatibly" without confusion.)*
- $I + J$ *is split.*
- *The* **colon ideal** $I : K$*, i.e.* $\{j \in R \ : \ jK \leq I\}$*, is split.*
- *The prime components of* I *are split.*

If C *is a list of compatibly split subschemes of* $\operatorname{Spec} R$*, then the stratification* $J(\mathcal{Y})$ *they generate consists only of compatibly split subschemes. Its* $\mathcal{Y}$*, the compatibly split varieties, is a stratification in the strong sense of v2 of the definition: each* $Y \cap Y'$ *is a* reduced *union of other strata.*

Proof.
- Universal algebra, i.e. $\phi(I \cap J) \subseteq \phi(I) \subseteq I$, similarly with J.
- Now we have to use the fact that φ is additive.
- $r \in I : K \iff \forall k \in K, kr \in I \implies \forall k \in K, k^p r \in I \implies \forall k \in K, k\varphi(r) = \varphi(k^p r) \in I \iff \varphi(r) \in I : K$.
- Since I is radical, it's $\bigcap_{P \geq I \text{ minimal prime}} P$,
  and then $Q \geq I$ minimal is $I : \bigcap_{P \geq I \text{ minimal prime}, P \neq Q} P$ (these P, Q are prime ideals).

We've seen that the (multiple-valued) operations with which we built $J(\mathcal{Y})$ from $\mathcal{C}$ preserve compatible splitness. □

So how do we get started?

Proposition 3.6. *The near-splitting* $\mathrm{Tr}(f^{p-1}\bullet)$ *compatibly near-splits the ideal* $\langle f \rangle$.

Proof. $r \in \langle f \rangle \iff r = fs \implies \mathrm{Tr}(f^{p-1}r) = \mathrm{Tr}(f^{p-1}fs) = f\,\mathrm{Tr}(s) \in \langle f \rangle$. □

So how do we determine that $\mathrm{Tr}(f^{p-1}\bullet)$ is a splitting, not just a near-splitting?

Proposition 3.7. [LakMeP, essentially] *If* $f \in \mathbb{F}[x_1, \ldots, x_n]$ *has degree* n *and and* $\texttt{init}\, f = \prod_i x_i$*, then* $\mathrm{Tr}(f^{p-1}\bullet)$ *is a splitting.*

Proof. Let P be the Newton polytope of f, of which $\prod_i x_i$ is a vertex. Then $\prod_i x_i^{p-1}$ is a vertex of the Newton polytope of f^{p-1}, and again has coefficient 1. Since the degree of each monomial is $\leq n(p-1)$ all other terms have some exponents $< p-1$, so die under $\mathrm{Tr}()$. $\square$

(It turns out [Kn09] that for f of degree n, one gets a splitting iff the number of $\mathbb{F}_p$-points with $f \neq 0$ is not a multiple of p. Incidentally, the Chevalley-Warning theorem says that this can't happen for f of degree $< n$.)

Example: matrix Schubert varieties reredux. Consider the vector space $M_{k\times n}$ of $k \times n$ matrices and let

$$f = \prod_{i=1}^{k} \det(\text{NW } i \times i) \prod_{i=k+1}^{n-k} \det(k \times k \text{ from column } i) \prod_{i=1}^{k-1} \det(\text{SE } i \times i).$$

Then if we order the variables from NE to SW, the lex init terms are the antidiagonal in each determinant, and proposition 3.7 applies. Part of this stratification is the MSV stratification, so those are all compatibly split.

As a corollary, we get Fulton's theorem over $\mathbb{F}_p$ for each p, from which we can infer it over any field using standard "spreading-out" tricks not described here.

3.1.1. *Interlude: a lemma on flat families.*

Lemma 3.8. *Let A be a discrete valuation ring e.g. $\mathbb{C}[[z]]$, so $S = \operatorname{Spec} A$ has two points $S^\times$, 0.*

Let $X \cup Y \supseteq X, Y$ be three flat families over S, where X, Y are reduced, and crucially, assume $(X \cup Y)_0$ is reduced. Then $X \cap Y$ is the closure of $X^\times \times Y^\times$, i.e. also is flat.

Proof. Consider two gluings of X to Y, along their common subschemes $\overline{X^\times \cap Y^\times} \hookrightarrow X \cap Y$:

$$(X \coprod Y) \Big/ \overline{X^\times \cap Y^\times} \quad \twoheadrightarrow \quad (X \coprod Y) \Big/ (X \cap Y) \quad \cong \quad X \cup Y.$$

Call this map $\pi : Z_1 \twoheadrightarrow Z_2$. It is finite, and an isomorphism away from $t = 0$, and Z_1, Z_2 are reduced, so $\mathrm{Fun}(Z_1)$ is integral inside $\mathrm{Fun}(Z_2)[t^{-1}]$.

If $\overline{X^\times \cap Y^\times} \neq X \cap Y$, so $\mathrm{Fun}(Z_1) \neq \mathrm{Fun}(Z_2)$, then there exists $r \in \mathrm{Fun}(Z_2)$ such that $r/t \in \mathrm{Fun}(Z_1) \setminus \mathrm{Fun}(Z_2)$. By the integrality, r/t satisfies a monic polynomial of degree m with coefficients in $\mathrm{Fun}(Z_2)$. Hence $r^m \equiv 0 \bmod t$, but $r \not\equiv 0 \bmod t$ (since $r/t \notin \mathrm{Fun}(Z_2)$), so $\mathrm{Fun}(Z_2)/\langle t \rangle = \mathrm{Fun}((X \cup Y)_0)$ has nilpotents, contrary to assumption. $\square$

Theorem 3.9. *Let $f \in \mathbb{F}[x_1, \ldots, x_n]$ be of degree n with leading term $\prod_i x_i$, where $\mathbb{F}$ is perfect of characteristic p, and let $\varphi := \mathrm{Tr}(f^{p-1}\bullet)$ be the associated splitting.*

1. *Let I be a compatibly split ideal w.r.t. φ.*
   Then $\mathtt{init}\ I$ is compatibly split w.r.t. $\mathrm{Tr}((\mathtt{init}\ f)^{p-1}\bullet)$, i.e. is a Stanley-Reisner ideal.
2. *Let $\mathcal{Y}$ be the set of compatibly split subvarieties, containing Y_1, Y_2.*
   Then $\mathtt{init}\ (Y_1 \cap Y_2) = \mathtt{init}\ (Y_1) \cap \mathtt{init}\ (Y_2)$ and $\mathtt{init}\ (Y_1 \cup Y_2) = \mathtt{init}\ (Y_1) \cup \mathtt{init}\ (Y_2)$.
3. *There is a well-defined map $2^n \twoheadrightarrow \mathcal{Y}$, taking $F \mapsto \min\{Y \in \mathcal{Y} : \mathtt{init}\ Y \supseteq \mathbb{A}^F\}$, giving a decomposition of the simplex.*
4. *This is a map of posets, and onto.*

Proof. (1) One does the lex degeneration one variable at a time, introducing a parameter and taking its flat limit to 0. The key point is that being compatibly split is a *closed* condition in families, unlike being reduced with is an open condition.

(2) The first statement is a direct consequence of the lemma (and the reducedness provided by Frobenius splitting). The second is essentially Mayer-Vietoris for the structure sheaves, plus the first statement.

(3) The whole space $\mathbb{A}^n$ is compatibly split, so the $\min$ is not over an empty collection. Say Y_1, Y_2 were two minima, so $\mathbb{A}^F$ is contained in $\texttt{init}\ Y_1, \texttt{init}\ Y_2$. Then it is contained in $\texttt{init}\ Y_1 \cap \texttt{init}\ Y_2 = \texttt{init}\ (Y_1 \cap Y_2)$. If $Y_1 \neq Y_2$, then $\mathbb{A}^F$ is contained in $\texttt{init}$ of some component of $Y_1 \cap Y_2$, contradicting minimality.

(4) Given $Y \in \mathcal{Y}$, pick Z a component of $\texttt{init}\ Y$, hence of the same dimension as Y. Therefore no $Y' \subset Y$ can have $\texttt{init}\ (Y') \supseteq Z$, so Y is the unique minimum, and $Z \mapsto Y$.

□

In fact, considered as finite topological spaces, the target has the quotient topology. In the Kazhdan-Lusztig example this recovers the usual characterization of Bruhat order in terms of subwords.

4. Hour 4: Bruhat atlases [He-Knutson-Lu].

Now that we have such nice stratifications on a vector space $X^v_\circ$, are there other stratified varieties we can study with them?

Recall an **atlas** of a real differentiable manifold M is a covering $M = \bigcup\{U\}$ by open sets U, each with a chart isomorphism $\psi_U : \mathbb{R}^n \to U$. If M carries a stratification $\mathcal{Y}$, then each U gets a stratification

$$\mathcal{Y}(U) := \{Y \cap U \ :\ Y \in \mathcal{Y}, Y \cap U \neq \emptyset\}$$

where the corresponding subposet is an order ideal in the poset $\mathcal{Y}$. It doesn't make sense to ask that ψ_U be an isomorphism of stratified spaces, until we specify a stratification on $\mathbb{R}^n$.

But thanks to the previous days we have a bevy of stratified vector spaces: the opposite Bruhat cells $X^v_\circ$. Each of these has one minimal stratum $X^v_\circ \cap X_v$, a point, so our minimal strata $\mathcal{Y}_{min}$ had better be points and our charts on M should be centered around them.

Note that the indexings $[1, v]$, $[1, v']$ of the strata in two Bruhat cells $X^v_\circ$, $X^{v'}_\circ$ are related, and we will bring this into play also, in our definition of **Bruhat atlas** on $(M, \mathcal{Y})$:

1. A choice of Kac-Moody flag variety H/B, to supply the cells.
2. A usual atlas $M = \bigcup_{f \in \mathcal{Y}_{min}} U_f$, with $f \in U_f$.
3. A ranked poset embedding $v :\ \mathcal{Y}^{op} \hookrightarrow W_H$,
   restricting to isomorphisms $\mathcal{Y}(U_f)^{op} \cong [1, v(f)]$.
4. Chart isomorphisms $\psi_f :\ X^{v(f)}_\circ \to U_f$, inducing commutative diagrams

$$\begin{array}{ccc} X^{v(f)}_\circ & \xrightarrow{\psi_f} & U_f \\ \uparrow & & \uparrow \\ X^{v(f)}_\circ \cap X_{v(Y)} & \to & U_f \cap Y \end{array}$$

for each $Y \in \mathcal{Y}$.

First example: $M = G/B$ with the Bruhat stratification? This has only one minimal stratum, and we can't cover G/B with one chart. So no, this stratification isn't rich enough (in much the same way that we needed to refine the matrix Schubert variety stratification in order to get proposition 3.7 to apply).

Real first example: $M = G/B$ with the **Richardson stratification** $\mathcal{Y} := \{X_a \cap X^b : a \le b\}$, with $\mathcal{Y}_{min} = \{X_a \cap X^a\} \cong W_G$. Here we take $H = G \times G$ (finite-dimensional!), and embed

$$\begin{array}{rcl} \nu : \{(a,b) \,:\, a \le b\} & \hookrightarrow & W_H = W_G \times W_G \\ (a,b) & \mapsto & (w_0 a, b) \end{array}$$

with chart isomorphisms given by the [KnWY] strengthening of the Kazhdan-Lusztig lemma:

$$X_\circ^f \times X_f^\circ \quad \xrightarrow{\sim} \quad U_f := f \cdot B_- B_+ / B_+ \qquad \text{as stratified spaces}$$

Second (historically first) example: Grassmannians $Gr(k,n)$ with the cyclic Bruhat or "positroid" stratification [Sn10]. Here $H = \widehat{GL_n}$.

More generally, G/P has a **projected Richardson stratification** [KnLamSp13], whose H is pretty tricky. Given $(M, \mathcal{Y})$, how can we come up with some H?

Recipe for H. Applying ν to the codimension 1 strata in $\mathcal{Y}$, we get length 1 elements of W_H, so we *define* the Coxeter diagram of H to have vertex set $\{Y \in \mathcal{Y} \,:\, \mathrm{codim}_M Y = 1\}$. Then, to see how Y_1, Y_2 should be connected in this Coxeter diagram (as $A_1 \times A_1, A_2, B_2, G_2$), we consider the stratification generated by Y_1, Y_2 and see if it's a Bruhat order. For example, we don't connect the vertices iff the intersection of the divisors is irreducible (or empty).

Examples:

(1) G/B. The Richardson divisors are $\{X_{r_\alpha}\} \cup \{w_0 X_{r_\alpha} = X^{w_0 r_\alpha}\}$, and the recipe gives the $G \times G$ diagram we used.
(2) $Gr(k,n)$. The positroid divisors are the n cyclic shifts of the Schubert divisor, and the recipe gives the affine Dynkin diagram $\widehat{A_{n-1}}$.
(3) G/P. Let $\pi : G/B \to G/P$ be the projection. The projected Richardson divisors are $\{\pi(X_{r_\alpha})\} \cup \{\pi(X^{w_0 w_0^P r_\alpha} \,:\, -\alpha$ not a root of $P\}$. The recipe gives the union of two copies of G's Dynkin diagram, glued along P's subdiagram, but using the duality involution $-w_0^P$ on P's diagram.
(4) The wonderful compactification $\overline{G}$, with stratification given by the $B_- \times B_-$-orbits intersect the $B \times B$-orbits. Now there are three kinds of divisors:
    - the closures of the $B_+ \times B_+$ divisors inside G, giving a copy $\{\alpha_+\}$ of G's Dynkin diagram,
    - the closures of the $B_- \times B_-$ divisors inside G, giving a copy $\{\alpha_-\}$ of G's Dynkin diagram disconnected from the previous, and
    - the components of $\overline{G} \setminus G$, giving a third copy $\{\alpha_0\}$ except that it's all disconnected,

    where each α_0 is connected to α_+, α_-.

Another family of examples. Let $Q, P \le G$ be standard parabolics, where Q is of finite type. Then the orbits $QxP/P \le G/Q$ (typically noncompact) have projected Richardson stratifications and Bruhat atlases (not specified here). If $Q = G$ then this generalizes example

(3) above. There is another more exciting case, though: $G = F\big(\mathbb{C}((z))\big)$, $P = Q = F\big(\mathbb{C}[[z]]\big)$, and the orbits are called "the $Gr^\lambda_\circ$s inside F's affine Grassmannian".

4.1. **Frobenius splittings of schemes.** Since Frobenius splitting played such a role in studying the stratifications of the charts individually, it'd be nice to extend the technique to the spaces we're gluing together from those charts.

If (R, φ) is a split ring, then φ extends to $R[s^{-1}]$ by $\varphi(r/s^k) = \varphi(rs^j/s^{mp}) := \varphi(rs^j)/s^k$. So we can sheafify the notion, and ask for a Frobenius splitting of a scheme X as a map $\varphi : \mathcal{O}_X \to \mathcal{O}_X$. It's better to write $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$, as this makes it a map of $\mathcal{O}_X$-modules.

On affine space, we used functions and $\mathrm{Tr}()$ to build near-splittings. What object do we use on schemes? There's a good answer (see [BrKu05, §1.3]) when X is regular, which is equivalent (!) to the Frobenius F being flat:

$$\begin{aligned}
\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) &\cong F_*(F^!\mathcal{O}_X) && \text{which uses finiteness of } F\\
&\cong \mathcal{H}om_{\mathcal{O}_X}(F^*\omega_X, \omega_X) && \text{using Serre duality for finite flat morphisms}\\
&\cong \mathcal{H}om_{\mathcal{O}_X}((\omega_X)^p, \omega_X)\\
&\cong \omega_X^{1-p} \cong (\omega_X^{-1})^{p-1}
\end{aligned}$$

So if σ is an anticanonical section, i.e. of ω_x^{-1}, we can take its $(p-1)$st power and build a near-splitting (as in proposition 3.7) that compatibly near-splits the divisor $\sigma = 0$. (As before, one must be lucky to get an actual splitting.)

Where do we get anticanonical sections, i.e. of $\mathrm{Alt}^{top}(TX)$? We could wedge together smaller tensors, e.g. vector fields. This works great on n-dim toric varieties, where we have n (commuting!) independent vector fields whose wedge is an anticanonical section. The corresponding divisor is the complement of the open T-orbit, hence is invariant under the vector fields (as follows from their commuting).

If we want something similar using higher rank tensors, we still want all the constituents to commute in some sense, which will be w.r.t. the Schouten bracket. The simplest case is to have one 2-tensor π and a bunch of commuting vector fields. The condition $[\pi, \pi] = 0$ says that π is a **Poisson tensor**.

It turns out that in all the known examples of spaces with Bruhat atlases, the anticanonical can be built as $\pi^{\wedge top} \wedge \vec{v}_1 \wedge \vec{v}_2 \ldots$ where the $\vec{v}_i$ are vector fields from a torus action preserving π, with the effect that all of the strata built by intersect-decompose are T-invariant and Poisson. Then, that anticanonical generates the Frobenius splitting, so the strata are in addition Frobenius split.

Non-example: the Hilbert scheme of n points in the plane. The Poisson tensor $xy\frac{d}{dx} \wedge \frac{d}{dy}$ on the plane $\mathbb{A}^2$ gives Poisson tensors on $(\mathbb{A}^2)^n$, $(\mathbb{A}^2)^n/S_n$, and its resolution $\mathrm{Hilb}_n(\mathbb{A}^2)$, with which one can build a Frobenius splitting on the Hilbert scheme. From there, we get a stratification by the compatibly split subvarieties, studied in [R12]. This scheme turns out to have infinitely many T^2-invariant Poisson subvarieties, so cannot be given a Bruhat atlas (though an open subset can).

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