Pipe dreams and conormal varieties

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Abstract

The Bergeron-Billey pipe dream formula for the Schubert polynomial S_π reflects a degeneration of Fulton's matrix Schubert variety $\overline{X}_\pi \subseteq M_{n\times n}$ to a multiplicity-free union of coordinate subspaces: one component for each ╯╭ / + *pipe dream*, or equivalently for each subword with product ∂_π of the triangular word for ∂_{w_0} in the algebra of divided difference operators [K-Miller '05]. This has a minor enhancement to the rectangular word for $\partial_{k+1\ k+2\ \ldots\ k+n\ 1\ 2\ \ldots\ k}$, $\pi \in S_{k+n}$, and some varieties $\overline{X}_\pi^{k\times n} \subseteq M_{k\times n}$.

Here we extend this to the *conormal variety* $C\overline{X}_\pi^{k\times n} \subseteq T^*M_{k\times n}$ of a matrix Schubert variety. The result is now a union of Lagrangian coordinate spaces, but *with* multiplicities. When π has a well-defined associated Temperley-Lieb element $TL(\pi)$ (i.e. is fully commutative, $\Leftrightarrow$ 321-avoiding), then these components and multiplicities are controlled by subwords of the rectangular word in the Temperley-Lieb algebra.

These transparencies are available at `http://math.cornell.edu/~allenk/`

Mildly generalized matrix Schubert varieties.

Fix k and n. Given $\pi \in S_{k+n}$, define

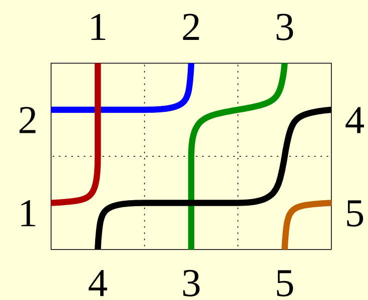
$$\overline{X}_\pi^{k\times n} := \left\{ M \in M_{k\times n} \;:\; \begin{bmatrix} M & 1_k \\ 1_n & 0 \end{bmatrix} \in \overline{B_-\pi B_+} \;\subseteq M_{(k+n)\times(n+k)} \right\}$$

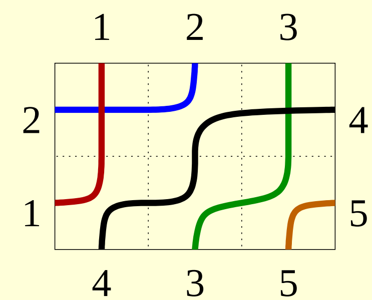
which if $k = n$ and $\pi \in S_n \leq S_{n+n}$, gives the usual matrix Schubert variety $\overline{X}_\pi$.

Generalizations of these hold: (1) [Fulton '92] $\overline{X}_\pi$ is defined as a scheme by the rank inequalities on the NW rectangles (of the $(k+n)\times(n+k)$ matrix).
(2) [K-Miller '05] Fulton's determinants are a Gröbner basis w.r.t. any term order picking out their *antidiagonal* terms (which are squarefree monomials).
(3) [K-Miller '05] The components of the resulting Stanley-Reisner scheme correspond naturally to the pipe dreams of [Bergeron-Billey '93].

Example: $\overline{X}_{21435}^{2\times 3} = \left\{ \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \;:\; \underline{m_{11}} = 0,\, m_{12}m_{23} - \underline{m_{13}m_{22}} = 0 \right\}$

So init $\overline{X}_{21435}^{2\times 3}$ has two components:





Conormal varieties, Arnol′d's lemma, and projective duality.

Given $A \subseteq B$ manifolds (A locally closed), define the **conormal bundle**

$$CA := \left\{(b, \vec{v}) \in T^*B \;:\; b \in A,\ \vec{v} \perp T_b A\right\}$$

which is always Lagrangian and **conical**, meaning invariant under scaling the cotangent fibers. If A is only a subvariety (but B still smooth), define the **conormal variety** $CA := \overline{CA_{reg}}$.

Arnol′d's lemma. If $L \subseteq T^*M$ is Lagrangian, closed, and conical, then each component $X \subseteq L$ is a conormal variety: specifically, $X = C\left(\pi_{T^*M \twoheadrightarrow M}(X)\right)$.

So if $X \subseteq V$ is an affine variety, then $CX \subseteq T^*V \cong V \times V^* \cong T^*(V^*)$, but Arnol′d's lemma only applies on the V^* side if X itself was conical, the cone over a projective variety $\mathbb{P}X$. In that case, there exists a unique variety $Y \subseteq V^*$ such that $C(X \subseteq V) = C(Y \subseteq V^*)$, and $\mathbb{P}X, \mathbb{P}Y$ are called **projectively dual**.

The easy case is $X \leq V$ a linear subspace, in which case $Y = X^\perp \leq V^*$.

But projective duality is **very** strange – for example, the orbits of a group G on a rep V and its dual V^* are in correspondence, but the posets of orbit closures can be completely different.

Gröbner degeneration of conormal varieties.

Let $X \subseteq V$ be an affine variety, with a Gröbner basis (g_i) w.r.t. a term order given by some integral weighting of the variables (i.e. by a circle subgroup S of the diagonal matrices $T \leq GL(V)$). Then $\operatorname{init} X$ is T-invariant, a schemy union of coordinate subspaces of V.

Extend the action of S on V to a symplectic action on T^*V. Then $\operatorname{init} CX$ is Lagrangian, conical, and T-invariant, so by Arnol′d's lemma must supported on a union of conormal bundles to coordinate subspaces of V.

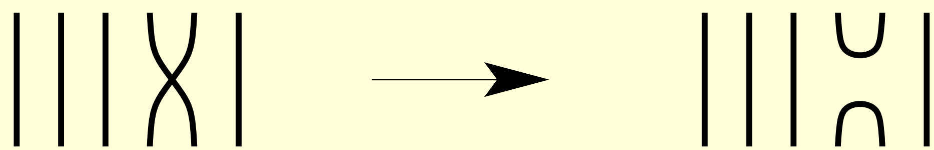
Example: Let X be the hyperbola $xy = t$ degenerating at $t = 0$ to X', the two axes. Then $CX = \{(x, y, a, b) \ : \ xy = t, \ ax = yb\}$. At $t \to 0$ it contains $C(X')$, but also contains the conormal variety to the origin, with multiplicity 2.

Easy theorems:
(1) $\operatorname{init} CX \subseteq \operatorname{init} X \times V^*$.
(2) $\operatorname{init} CX \supseteq C(\operatorname{init} X)$.
(3) If X is conical, with projective dual Y, then $\operatorname{init} CX \supseteq C(\operatorname{init} X) \cup C(\operatorname{init} Y)$.

As the example above shows, extra components of $\operatorname{init} CX$ can develop where $\operatorname{init} X$ develops singularities, and they need not be multiplicity 1.

Combinatorial interlude: the Temperley-Lieb algebra.

Consider replacing each divided difference operator ∂_i by the corresponding *Temperley-Lieb* generator e_i:



These pictorially satisfy $[e_i, e_j] = 0$ for $|i - j| > 1$, and $e_i e_{i+1} e_i = e_i$; in particular, they do not braid. So the rule above only extends to those $w \in S_n$ for which no braid moves are required, called **fully commutative** elements.

In order for them to define a closed algebra, we have to give a value for e_i^2, and we will use one of the standard choices, $2e_i$.

Theorem. The fully commutative permutations w are the 321-avoiding ones. There are Catalan many of them, and $\{TL(w)\}$ give a basis of the Temperley-Lieb algebra on n strands.

Multiplying generators ∂_i of the nil Hecke ring gives basis elements ∂_w, or 0 if strands cross twice. But when we multiply generators e_i of Temperley-Lieb, we get basis elements $TL(w)$ times $2^{\#\text{ of loops}}$.

The antidiagonal Gröbner degeneration of $C\overline{X}_\pi$.

We recapitulate one of the main results of [K-Miller '05]:

$$\text{init } \overline{X}_\pi = \bigcup_{\substack{\text{╯╭},\,\text{┼ pipe dreams} \\ \text{with product } \pi, \\ \text{no two pipes cross twice}}} \left(\mathbb{A}^{\text{╯╭}} := \{ M \in M_{k\times n} \,:\, m_{i,j} = 0 \text{ at ┼s} \} \right) \quad \subseteq M_{k\times n}.$$

So far we know that each component of $\text{init } C\overline{X}_\pi^{k\times n}$ is the conormal bundle to a coordinate subspace of $M_{k\times n}$. To specify a coordinate subspace is the same amount of data as in a pipe dream: one bit for each matrix entry.

But for the conormal varieties, it turns out to be natural to use the tiles ╯╭, ╮╰.

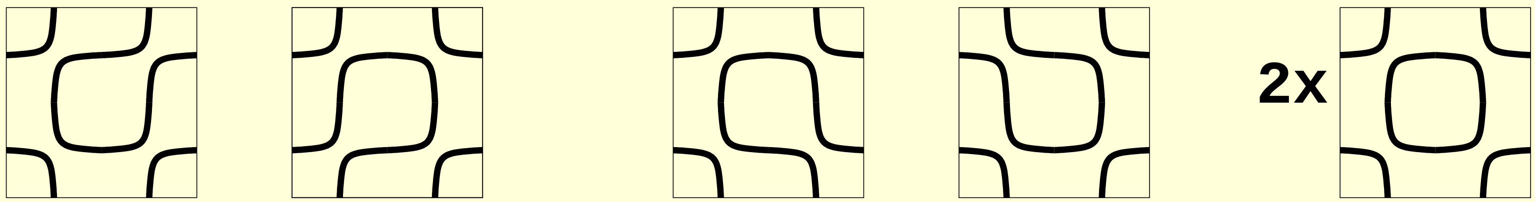
Theorem [K-Zinn-Justin]. If π is fully commutative, and $TL(\pi)$ its corresponding Temperley-Lieb basis element, then as a cycle

$$[\text{init } C\overline{X}_\pi^{k\times n}] = \bigcup_{\substack{\text{╯╭},\,\text{╮╰ pipe dreams} \\ \text{with connectivity } TL(\pi)}} 2^{\#\text{ of loops}} \left[\mathbb{A}^{\text{╯╭}} \times \mathbb{A}^{\text{╯╭}} \right] \quad \subseteq M_{k\times n} \times M^*_{k\times n}.$$

The first nonlinear example.

$$\overline{X}^{2\times 2}_{1324} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \overline{B_- \pi B_+}, \text{ i.e. } \operatorname{rank} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \leq 1 \right\}$$

Then $\overline{X}_\pi$ is isomorphic to its projective dual Y, and $\operatorname{init} C\overline{X}_\pi$ has components



two from $C(\operatorname{init} \overline{X}_\pi)$, two from $C(\operatorname{init} Y)$, and one surprise component.

In particular, their projections to $M_{2\times 2}, M^*_{2\times 2}$ have dimensions $(3,1), (3,1)$, $(1,3), (1,3)$, $(2,2)$; when neither projection has dimension $\dim \overline{X}_\pi$ we see a component that can't be seen from either X or Y.

Theorem [K-ZJ]. Let π be 321-avoiding. Then there is a unique π' such that the connectivity of $TL(\pi)$ and $TL(\pi')$ are left-right mirror, and the projective dual of $\overline{X}_\pi$ is also the left-right mirror of $\overline{X}_{\pi'}$.