Bruhat atlases on wonderful compactifications and elsewhere

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In memory of my friend and inspiration, Andrei Zelevinsky

Abstract

In 1985, Andrei identified spaces of representations of the equioriented A_n quiver with certain intersections of Schubert varieties and opposite Bruhat cells, as stratified spaces. This suggests that we might study other stratified spaces using (stratified) atlases of Bruhat cells, or of these intersections.

I'll explain how to recognize that a space might be amenable to this, and the spaces where we've constructed such atlases: G/P, and wonderful compactifications. This is joint work with Xuhua He and Jiang-Hua Lu.

Cluster varieties have suitable stratifications (described by Zwicknagl), and I'll identify the A_n cluster variety with an opposite Bruhat cell intersected with a Schubert variety, much as Andrei did.

These transparencies are available at `http://math.cornell.edu/~allenk/`

Andrei's study of A_n quiver representations.

Fix dimensions $r_0, \ldots, r_n$, and consider linear maps $V_0 \stackrel{\phi_1}{\to} V_1 \stackrel{\phi_2}{\to} \cdots \stackrel{\phi_n}{\to} V_n$, where $\dim V_i = r_i$. Let $\mathrm{Hom} := \left\{ \vec{\phi} = (\phi_0, \ldots, \phi_n) \right\} \cong \mathbb{A}^{\sum_{i=1}^n r_{i-1} r_i}$.

Then Hom is stratified by the discrete invariants $\vec{\phi} \mapsto (\mathrm{rank}(\phi_i \circ \cdots \circ \phi_j))_{i \leq j}$, which by Gabriel's theorem index the orbits of the gauge group $\prod_{i=0}^n GL(V_i)$.

In 1985, Andrei Zelevinskii (!) defined the map

$$(\phi_0, \ldots, \phi_n) \mapsto \begin{bmatrix} 0 & \cdots & & & \phi_n & I_{r_n} \\ \vdots & & & \phi_{n-1} & I_{r_{n-1}} & 0 \\ & & \iddots & \iddots & & \\ & \phi_2 & I_{r_2} & & & \\ \phi_1 & I_{r_1} & & & & \\ I_{r_0} & & & & & \end{bmatrix} B/B \quad \in GL\left(\sum r_i\right)/B,$$

an isomorphism of Hom with $Bw_0w_0^P B/B \cap \overline{B_- \pi_Z B}/B$, additionally identifying each orbit closure Ω_r with $Bw_0w_0^P B/B \cap \overline{B_- \pi_r B}/B$. [Lakshmibai-Magyar '98] used this to study the singularities of these orbit closures, and [K-Miller-Shimozono '06] used it to compute their equivariant cohomology classes.

The stratification of opposite Bruhat cells.

Let $(G, B, B_{,}T = B \cap B_-, W)$ be a pinning of a Kac-Moody group.

Then each **opposite Bruhat cell** $X_\circ^v := B_- vB/B \cong \mathbb{A}^{\ell(v)}$ is stratified by its intersections with **Schubert varieties** $X_w := \overline{BwB}/B$. The strata are nice:

- Each open stratum is smooth.
- Each closed stratum is normal, Cohen-Macaulay, with rational singularities.
- There is a Frobenius splitting, and these are the compatibly split subvarieties.
- There is a Poisson structure, for which these are the T·leaves.

But even moreso, the stratification considered *as a whole* is nice:

- It is the coarsest stratification by varieties with this given open stratum.
- The complement of the open stratum is an anticanonical divisor, and by repeated adjunction, the boundary of every stratum is anticanonical.
- The poset $[1, v]$ of strata is ranked and EL-shellable.

Note that these good properties *do not* hold for the rank stratification of Hom – the finer stratification on $X_\circ^{w_0 w_0^P} \cap X_Z$ is simpler!

Bruhat atlases.

Let M be a manifold, with a stratification $\mathcal{Y}$. Define a **Bruhat atlas** on M to be

- a choice G of Kac-Moody group
- a map $\nu : \mathcal{Y}^{op} \to W_G$, identifying $\mathcal{Y}^{op}$ with an order ideal in the Bruhat order
- an open cover $\{U_f\}_{f \in \mathcal{Y}_{min}}$ of M
- stratified isomorphisms $X_\circ^{\nu(f)} \cong U_f$, $f \in \mathcal{Y}_{min}$.

Example [Snider '10] Let $M = Gr_k(\mathbb{A}^n)$.

Let $\mathcal{Y}$ be the common refinement of the n cyclic shifts of the Bruhat decomposition, the **positroid stratification** considered by Lusztig, Postnikov, Rietsch, Williams, K-Lam-Speyer, …

Let $G = \widehat{GL(n)}$, so $W_G \cong \{f \in Sym(\mathbb{Z}) : f(i+n) = f(i) + n \ \forall i\}$.

Then ν takes $rowspan[\vec{v}_1 \cdots \vec{v}_n] \in Gr_k(\mathbb{A}^n)$ to its "bounded juggling pattern" $f(i) := \min\{j \geq i : \vec{v}_{i \bmod n} \in span(\vec{v}_{i+1 \bmod n}, \ldots, \vec{v}_{j \bmod n})\}$. By [KLS] this is an identification of $\mathcal{Y}^{op}$ with an order ideal in W_G.

Let U_f be the evident permuted big cell, of which there are $\binom{n}{k}$. Then Snider defines an isomorphism $X_\circ^{\nu(f)} \cong U_f$, and checks that it corresponds the anticanonical divisors, which is all that's necessary.

The Coxeter diagram of a stratified manifold.

If $(M, \mathcal{Y})$ is to have a Bruhat atlas, then it needs a choice of G. The map ν is to correspond the divisors in M with length 1 elements of W_G. So attempt to construct a Coxeter diagram $D(M)$:

- The vertices of $D(M)$ are the divisors in $\mathcal{Y}$.
- Given two divisors D_1, D_2, intersect them, decompose that, intersect, decompose, ..., generating a poset. If that doesn't fit in a rank 2 Bruhat order, give up. Otherwise take the smallest such and connect the vertices appropriately.

In particular, if $D_1 \cap D_2$ is irreducible, the poset is $M \supset {D_1 \atop D_2} \supset D_1 \cap D_2$, the $A_1 \times A_1$ Bruhat order. So the vertices don't get connected.

Example. In $Gr_k(\mathbb{A}^n)$, we have the n cyclic shifts D_i of the Schubert divisor.

If $i \neq j \pm 1 \bmod n$, then $D_i \cap D_j$ is irreducible.

If $i = j \pm 1 \bmod n$, then they generate an A_2 poset (except for $k = 1, n-1$).

So the Coxeter diagram is $\widehat{A_{n-1}}$, as in Snider's result.

(If $k = 1, n-1$, then $Gr_k(\mathbb{A}^n)$ is projective space, $\mathcal{Y}$ is the coordinate subspace stratification, and the diagram is completely disconnected.)

H/B_H and its Richardson stratification.

Let $M = H/B_H$ for H finite-dim, and define its open strata to be the nonempty intersections of B_H-orbits and B_H^--orbits, the **open Richardson varieties**. Then

$$\mathcal{Y} \cong \{(a, b) \in W_H : a \leq b\}.$$

The divisors are the $\{X_{r_\alpha}\}$ and $\{X^{r_\alpha}\}$. The Coxeter diagram of M is two copies of H's diagram, not connected to one another: the $\{X_{r_\alpha}\}$ give one copy, the $\{X^{r_\alpha}\}$ the other, and as each $X_{r_\alpha} \cap X^{r_\beta}$ is irreducible there are no connections.

This suggests that we take $G = H \times H$. The combinatorics is easy:

$$\begin{aligned} (a, b) &\mapsto (a, w_0 b) \\ \mathcal{Y}^{op} &\cong \{(a, c) : a \leq w_0 c\} = \bigcup_w [(1, 1), (w, w_0 w)] \end{aligned}$$

Theorem [Kazhdan-Lusztig '79 half the stratification, K-Woo-Yong '13 full]
Let $U_w = w B_H^- B_H / B_H$, $w \in W_H$. Then $U_w \cong X_w^\circ \times X_\circ^w$ as a stratified T-space.

$\mathsf{H/P}$ and its projected Richardson stratification.

Under the natural projection $\mathsf{H}/\mathsf{B}_\mathsf{H} \twoheadrightarrow \mathsf{H}/\mathsf{P}$, define the **projected Richardson stratification** of H/P as the images of the strata in $\mathsf{H}/\mathsf{B}_\mathsf{H}$. These have basically all the good properties of Richardson varieties ([KLS '13], [Billey-Coşkun '13]).

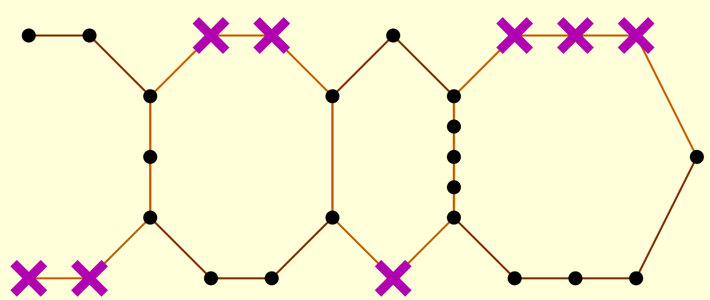
It keeps the w_0 symmetry, and for H/P cominuscule, it is also invariant under

$$\{w \in W_\mathsf{H} : w \cdot (\text{the Weyl alcove}) = (\text{a translate})\} \qquad \cong Z(\mathsf{H}) \text{ [Yakimov '10].}$$

Theorem [He-K-Lu]. The Coxeter diagram of $\mathsf{H/P}$ is two copies of H's diagram, glued together along P's diagram, using the duality involution thereon.

Taking that for our G, we can construct a Bruhat atlas on $\mathsf{H/P}$.

(To be precise: for tiny $\mathsf{H/P}$ we could connect up less. But this G works.)



H is the black dots,
P uses the dots on
the vertical lines

If $\mathsf{H/P}$ is cominuscule, but not $\mathbb{P}^n$, this diagram is just the affine diagram!

The wonderful compactification of a group.

Let H be an adjoint group, and $\overline{H}$ its wonderful compactification.
Define the open strata in the **double Bruhat decomposition** of $\overline{H}$ to be the intersections of $(B_H \times B_H)$-orbits with $(B_H^- \times B_H^-)$-orbits.
Inside H, these are the double Bruhat cells of [Fomin-Zelevinsky '99].

There are three kinds of divisors: the ones in $\overline{H} \setminus H$, which are $H \times H$-invariant and generate a boolean lattice [de Concini-Procesi '83], the $(B_H \times B_H)$-invariant divisors inside H, and the $(B_H^- \times B_H^-)$-divisors inside H.

Theorem [He-K-Lu]. The Coxeter diagram of $\overline{H}$ is two copies of H's diagram, each vertex glued to that in a third copy, which is completely disconnected.

Taking that for our G, we can construct a Bruhat atlas.

Springer studied the $(B_H \times B_H)$-orbits on $\overline{H}$, which amounts to looking at the stratification in the attracting neighborhood of the $(B_H \times B_H)$-fixed point, which rips out the $(B_H^- \times B_H^-)$-divisors.

For that smaller space, the Coxeter diagram is just the Nakajima doubling of H's diagram, and we recover the [Chen-Dyer '03] description of Springer's poset.

Why is the Nakajima doubling showing up here???

The full diagram is a sort of fiber product of two, and appeared also in [Li '10].

The A_n cluster variety.

Zwicknagl showed that on a cluster variety, there are finitely many T-invariant Poisson subvarieties, the $T\cdot$**leaves**. Since these are singular, we shouldn't hope to cover them with Bruhat cells $X^v_\circ$, but perhaps with $X^v_\circ \cap X_w$, as in [Zelevinskii].

The A_{n-3} cluster variety is the cone $\widehat{Gr_2(\mathbb{A}^n)}$, which Plücker embeds into $\Lambda^2(\mathbb{A}^n)$. So we could additionally ask that that serve as $X^v_\circ$.

Theorem. Embed the Dynkin diagram D_{n-2} into D_n in the obvious way, and choose a particular antler to amputate producing A_{n-2}, A_n.

Let $v(k) := w_0(D_k) w_0(A_k) \in W_{D_n}$, for $k = n, n-2$.

Then $X^{v(n)}_\circ \cong \Lambda^2(\mathbb{A}^n)$, and $X^{v(n)}_\circ \cap X_{v(n-2)} \cong \widehat{Gr_2(\mathbb{A}^n)}$, as stratified T-spaces.

It doesn't seem likely that there will be a similar description of other cones over Grassmannians. But perhaps of other finite-type cluster varieties?