Bruhat cells, subword complexes, and stratified atlases

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MIT Infinite Dimensional Algebras Seminar, March 2021

Abstract

Each finite-dimensional Bruhat cell in Kac-Moody flag varieties is stratified by its intersections with the finite-codimensional Schubert varieties. This stratification has many excellent combinatorial and geometric qualities: it is generated (in a precise sense) by the hypersurface complementary to the dense stratum, and every stratum is normal with anticanonical boundary. I'll trace this to the fact that the hypersurface has leading term = the product of the variables. Using Gröbner bases I'll rederive standard results about Bruhat order, and deeper ones like the fact that subword complexes are balls or spheres. With Bruhat cells firmly in place, I'll use them to put "Bruhat atlases" on famous stratified spaces. This latter work is joint with X. He and J.-H. Lu, with results by Snider, Huang, K-Woo-Yong, Elek, Bao-He, and Galashin-Karp-Lam.

These slides are available at `http://math.cornell.edu/~allenk/`

The objects in play, and outline of the talk.

Let $G, B_\pm, T, W$ be a pinning of a Kac-Moody <u>group</u>, with $X^w_\circ := BwB/B$ the $\ell(w)$-dimensional **Bruhat cell** and and $X_v := \overline{B_- vB}/B$ the $\ell(v)$-codimensional **opposite Schubert variety**. Then $X^w_\circ$ is isomorphic to a finite-dimensional vector space, and is stratified by its intersections with $\{X_v \;:\; v \leq w\}$, each T-invariant.

Call this the **Bruhat decomposition** of $X^w_\circ$.

The plan:

1. The algebra of a stratification.
2. Many excellent properties of the Bruhat decomposition of $X^w_\circ$.
3. Interlude: Frobenius splitting and LMP polynomials.
4. Application to open Bott-Samelson manifolds, and thereby, proofs of the properties via subword complexes.
5. Bruhat atlases (joint with Xuhua He and Jiang-Hua Lu).

The algebra of a stratification.

In the usual axiomatization, a stratification $\mathcal{Y}^\circ$ of a scheme X by irreducible subvarieties is a finite decomposition $X = \coprod_{Y^\circ \in \mathcal{Y}} Y^\circ$ where each Y is locally closed and irreducible, such that $\forall Y^\circ \in \mathcal{Y}$, its closure $\overline{Y^\circ}$ is $\coprod_{Z^\circ \in \mathcal{Y}^\circ, Z^\circ \subseteq \overline{Y^\circ}} Z^\circ$.

We will prefer to take $\mathcal{Y}$ as a collection of *closed* subsets, with $\mathcal{Y}_{\mathrm{irr}} \subseteq \mathcal{Y}$ the irreducible ones. Then $\mathcal{Y}$ has three operations: union, intersection, and (a multivalued one) "take irreducible components".

Upsides: we only deal with closed subsets, and can talk about "generating" a stratification.

Downside (?): if an intersection is nonreduced, when we take the union of its components, we get two elements of $\mathcal{Y}$ with the same support. Maybe we want to insist all intersections are reduced?

Example: let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a(ad - bc) = 0 \right\}$. This hypersurface first generates its two components, $a = 0$ and $ad - bc = 0$, which then generate their intersection $a = bc = 0$, which then generates *its* components $a = b = 0$ and $a = c = 0$, finally giving us the codimension 3 intersection $a = b = c = 0$.

Similarly, the NW determinants in an $n \times n$ matrix generate a poset $\cong S_{n+1}$.

The (excellent) Bruhat decomposition of $X^w_\circ$.

Here are some of the fine properties of $\mathcal{Y}(X^w_\circ)$:

1. Let $Y_1, Y_2 \in \mathcal{Y}$ be unions of strata. Then $Y_1 \cap Y_2$ is reduced.

2. The boundary $\partial Y := \bigcup\{Z : Z \in \mathcal{Y}_{\mathrm{irr}}, Z \subsetneq Y\}$ of a stratum Y is defined by a T-invariant section of the anticanonical bundle (on the regular locus in Y).

3. The stratification is generated (as in the example just given) by the hypersurface $\partial X^w_\circ$.

4. Each (closed) stratum is normal and Cohen-Macaulay.

5. (Not relevant for this talk) There is a T-invariant Poisson structure π on $X^w_\circ$, such that the open strata $Y \setminus \partial Y$ are exactly the T·leaves. (This essentially implies #2.)

6. There is a *Frobenius splitting* on $X^w_\circ$ with respect to which $\mathcal{Y}$ is exactly the compatibly split subschemes.

Frobenius splitting of rings, the definition.

A commutative ring R is **reduced** if for all (or any) $n > 1$, $r^n = 0 \implies r = 0$ for all $r \in R$. It would be nice to phrase this as "$\ker(r \mapsto r^n) = 0$", but the nth power map isn't linear... or is it?

Assume $R \geq \mathbb{F}_p$, and $n = p$, so we can use the Freshman's Dream. If $\ker(F) = 0$, we can ask for a left inverse $\varphi : R \to R$, and hope for the following properties:

1. $\varphi(a + b) = \varphi(a) + \varphi(b)$
2. $\varphi(a^p b) = a\varphi(b)$
3. $\varphi(1) = 1$ this is the hard one to satisfy

Example. Let $\sqrt[p]{\ } : \mathbb{F}_p[z_1, \ldots, z_n] \to \mathbb{F}_p[z_1, \ldots, z_n]$ take a monomial to its pth root (if it exists) or 0 (if not), and extend additively, and call it the **standard Frobenius splitting** of $\mathbb{F}_p[z_1, \ldots, z_n]$. Easy extension to $\mathbb{F}$ a perfect field over $\mathbb{F}_p$.

If φ only satisfies #1 and #2 (e.g. $\varphi = 0$) call it a **near-splitting**.
The basic example is $\mathrm{Tr}(f) := \sqrt[p]{\prod_i z_i f} / \prod_i z_i$, which at $n = 1$ has the residue-like property that $\mathrm{Tr}(f) = 0 \iff f = \frac{d}{dz_1} g$ for some g.

Frobenius splittings of affine space, and compatibly split ideals.

Easy theorem (see e.g. [Brion-Kumar, §1.3]): every near-splitting of $\mathbb{F}[z_1, \ldots, z_n]$ is of the form $h \mapsto \mathrm{Tr}(gh)$, for some unique $g \in \mathbb{F}[z_1, \ldots, z_n]$.

An ideal $I \leq R$ is **compatibly split** w.r.t $\varphi : R \to R$ if $\varphi(I) \leq I$, i.e. iff φ descends to R/I.

Theorem. Let φ be a splitting of R, and $I, J \leq R$ be compatibly split.

1. R is reduced.
2. I is radical.
3. $I \cap J$ is compatibly split.
4. $I + J$ is compatibly split. (Trivial to prove but geometrically striking!)
5. $I : K$ is compatibly split. (Proof: $r \in I : K \iff \exists k, kr \in I \implies \exists k, k^p r \in I \implies \exists k, \varphi(k^p r) \in I \implies \exists k, k\varphi(r) \in I \iff \varphi(r) \in I : K$.)
6. Each prime component of I is compatibly split. (Follows from #5.)

Three-page theorem [Kumar-Mehta '09]. There are only finitely many such I.

Consequently, $\mathcal{Y} := \{$compatibly split subschemes$\}$ forms a stratification in our strong sense, that all intersections of unions are reduced.

Example: the only compatibly split subschemes w.r.t. the standard splitting $\sqrt[p]{}$ are the **Stanley-Reisner subschemes**, the unions of coördinate subspaces.

Three criteria for splitting hypersurfaces.

Let $f \in \mathbb{F}_p[z_1, \ldots, z_n]$, and define $\varphi_f := \mathrm{Tr}(f^{p-1} \bullet)$ a near-splitting.

Theorem. If any of the following hold,

1. f is of degree n, and $z_1 z_2 \cdots z_n$ is a leading term w.r.t. some term order
2. f is of degree n, and the number of $\mathbb{F}_p$-points in $\{f \neq 0\}$ is **not** a multiple of p
3. [LMP := Lakshmibai-Mehta-Parameswaran '98] f's lexicographically leading term is $z_1 z_2 \cdots z_n$

then φ_f is a splitting (or a scalar multiple of one, in case #2), and compatibly splits $\langle f \rangle$ (and hence all the strata it generates). Moreover, #1 $\implies$ #2.

Example: consider $n \times n$ matrices, and let f be the (homogeneous) product of all NW and SE principal minors. If we lex-order the variables from NE to SW, then f satisfies #3, hence #1 and #2.

Example: if f is homogeneous and $n = 3$, then f defines an elliptic curve. #2 says it's not supersingular. #1 implies it's nodal, and generically has only one node. Note that the node will be compatibly split, but is *not* in the stratification (naïvely) generated by $\{f = 0\}$.

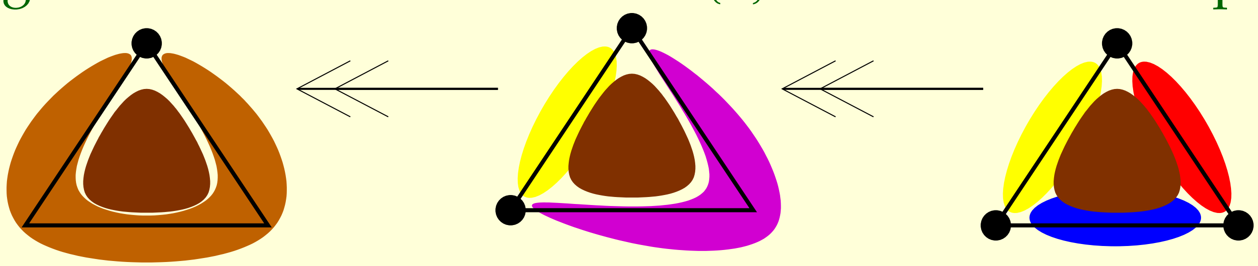
Degeneration and simplicial complexes.

If $I \leq \mathbb{F}[z_1, \ldots, z_n]$ is an ideal, let $\mathrm{lex}(I)$ be a specific *Gröbner degeneration* of I, the ideal spanned over $\mathbb{F}$ by the lex-leading monomials of the elements of I.

Theorem [K]. If f is an LMP polynomial, and I, J are compatibly split w.r.t. φ_f, then $\mathrm{lex}(I), \mathrm{lex}(J)$ are Stanley-Reisner, and $\mathrm{lex}(I \cap J) = \mathrm{lex}(I) \cap \mathrm{lex}(J)$.

Proof sketch. Consider the Gröbner family (f_t) taking $f_1 = f$ to $f_0 = \prod_i z_i$, which simultaneously takes $I_1 = I$ to $I_0 = \mathrm{lex}(I)$. Being compatible is a *closed* condition, i.e. I_t is compatible with φ_{f_t}, so $\mathrm{lex}(I)$ is with $\varphi_{\prod_i z_i}$, hence is S-R.

Let $\mathcal{Y}$ be the stratification generated by $\{f = 0\}$. Define a map $D_f : 2^{[n]} \to \mathcal{Y}_{irr}$ of posets taking $S \mapsto \min\{Y \in \mathcal{Y}_{irr} \; : \; \mathrm{lex}(Y) \supseteq \mathbb{A}^S\}$. (This requires the second half of the theorem.) Interpret it as giving a decomposition of the $(n-1)$-simplex. Taking leading forms in between f and $\mathrm{lex}(f)$ refines the decomposition.



$f = xyz + y^3 + z^3$ $\qquad$ $f = y(xz + y^2)$ $\qquad$ $f = xyz$

Given a weighting $w : z_i \mapsto z_i z_{n+1}^{w_i}$ of the variables, if $\prod_i z_i$ gets the lowest power P, then $f_{Rees} := w(f)/z_{n+1}^{P-1}$ is again LMP and melds the decompositions.

Bott-Samelson coördinates on $X^w_\circ$.

To apply the LMP criterion we need to coördinatize $X^w_\circ$, which we do by picking a reduced word Q in W's generators, with $\prod Q = w$.

To *any word* Q of length n, define the **Bott-Samelson manifold** and **map**

$$\begin{array}{rcl} BS^Q := P_{q_1} \times^B P_{q_2} \times^B \cdots \times^B P_{q_n} \times^B B \big/ B & \to & G/B \\ {[p_1, \ldots, p_n]} & \mapsto & p_1 \cdots p_n B/B \end{array}$$

and the parametrization of the open set $BS^Q_\circ := BS^Q \setminus \bigcup_{q \in Q} BS^{Q \setminus q}$

$$\mathbb{A}^n \to BS^Q_\circ, \qquad (z_1, \ldots, z_n) \quad \mapsto \quad \left[e_{q_1}(z_1)\dot{s}_{q_1}, \ \ldots, \ e_{q_n}(z_n)\dot{s}_{q_n} \right]$$

Let $m_Q : \mathbb{A}^n \to BS^Q_\circ \hookrightarrow BS^Q \to G/B$ be the composite.

Theorem [K]. For any Q, the hypersurface $m_Q^{-1}\left(\bigcup_\alpha X_{r_\alpha}\right) \subseteq \mathbb{A}^n$ is defined by an LMP polynomial f_Q. The stratification it generates has $\mathcal{Y}_{irr} = \left\{ m_Q^{-1}(X_w) \right\}$, and these are exactly the subvarieties compatibly split w.r.t. φ_{f_Q}.

More specifically, the lex-leading term of $m_Q^{-1}(X_{r_\alpha})$'s equation is $\prod_{i:\, q_i = \alpha} z_i$.

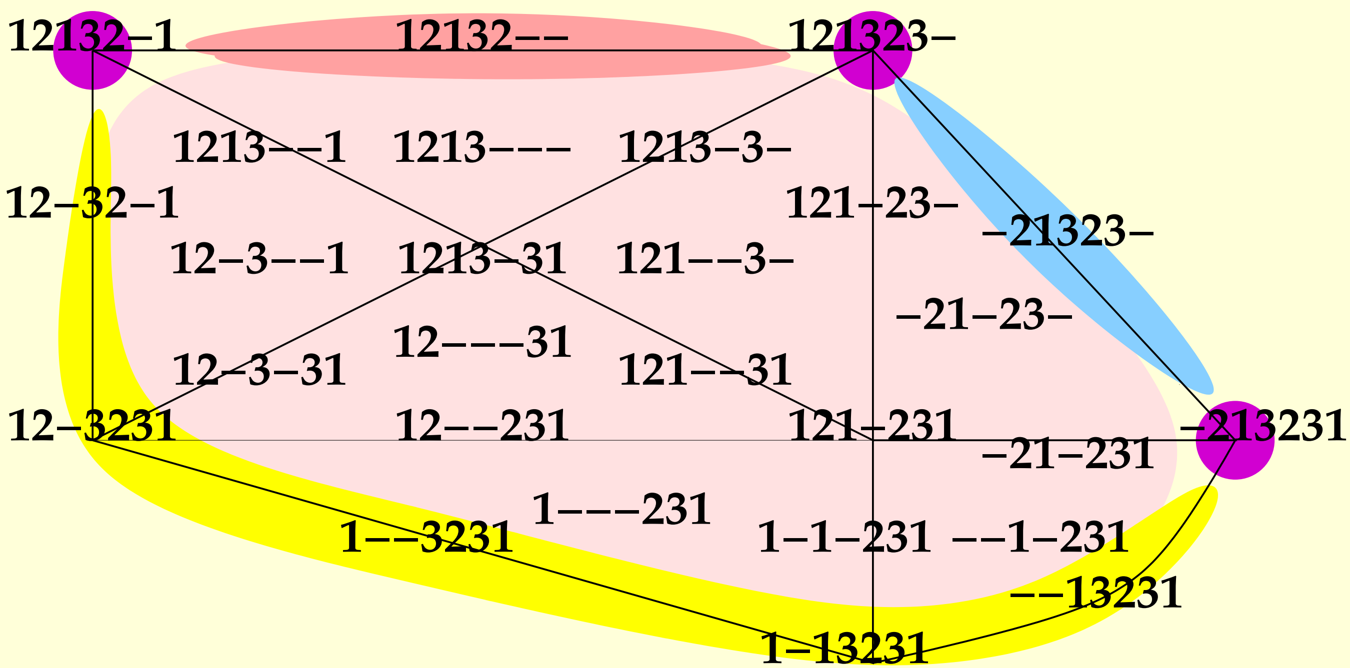
The map $D : 2^Q \to W$ takes $S \subseteq Q$ to the nil Hecke product of its complement.

Subword complexes $\Delta(Q, w)$.

Theorem [K-Miller ’05]. Let Q be a word in W’s generators, f_Q its polynomial, and $D : 2^Q \to W$ the $S \mapsto \mathrm{nilHecke}(S^c)$ just obtained geometrically.

1. The preimage $D^{-1}(w)$, considered as a union of open simplices, is an open ball (or very rarely a sphere).
2. Its closure is the **subword complex** $\Delta(Q, w) := \bigcup_{v \geq w} D^{-1}(v)$, and is a (shellable) closed ball or sphere, whose boundary is $\bigcup_{v > w} D^{-1}(v)$.

This gives a **Bruhat decomposition of the simplex**, from which one can derive typical results about Bruhat order like, $w \geq v$ iff some reduced word for w contains one for v iff every reduced word for w contains one for v.



Geometric and cohomological consequences for Schubert varieties.

1. Shellable balls and spheres have Cohen-Macaulay Stanley-Reisner schemes, and Cohen-Macaulayness is open in families, hence $m_Q^{-1}(X_w)$ is C-M. ([Conca-Varbaro '20] prove a sort of converse to this semicontinuity: if Y is C-M and $lex(Y)$ is Stanley-Reisner, then $lex(Y)$ is still C-M!)
2. The singularities of $m_Q^{-1}(X_w)$ (if any) lie inside $m_Q^{-1}\left(\bigcup_{v>w} X_v\right)$, which degenerate to the spherical boundary of $lex(m_Q^{-1}(X_w))$. But $lex(m_Q^{-1}(X_w))$ is generically regular along that boundary, hence, $lex(m_Q^{-1}(X_w))$ is R1 (again an open condition in families). Cohen-Macaulayness makes it S2, so together, $m_Q^{-1}(X_w)$ is normal.
3. As Q varies over reduced words for $v \geq w$, these affine varieties give local pictures of the singularities on X_w, hence, X_w is normal and Cohen-Macaulay.
4. The degeneration of $m_Q^{-1}(X_w)$ to its lex is T_G-equivariant, so we can compute the class $[X_w]|_{\prod Q} \in K_T(pt)$ as an alternating sum over interior faces of $\Delta(Q, w)$, recovering the (AJS/Billey)/Graham/Willems formula. (The requisite Möbius function calculation $\mu(F) = 1 - \chi(link(F))$ here is easy only because the links are hemispheres or spheres, giving $\mu = 0, \pm 1$.)

Recognizing stratified vector spaces as Bruhat cells.

Given a stratification $\mathcal{Y}$ on $\mathbb{A}^n$, when is that $\mathbb{A}^n$ isomorphic to some $X^w_\circ$?

Let f be the equation of the union of divisor strata. If $\mathrm{lex}(f)$ is not squarefree, then these coördinates aren't going to work. If it is squarefree but not all of $\prod_i z_i$, we will need to add some divisors.

Example: let $f \in \mathbb{F}[m_{11}, \ldots, m_{dd}]$ be the product of the coefficients of the characteristic polynomial. If we order the variables NE to SW by columns, the leading term is squarefree but only of degree $\binom{d+1}{2}$. We can throw in the NW $k \times k$ minors, $k = 1, \ldots, d-1$ to fix that.

For f_Q from Q, we have the inequality

$$\text{codim(Newton polytope of } f_Q) \;\geq\; \#\{\text{factors of } f_Q\}$$

those being the dimension of the effective torus action (rank of G) vs. the number of distinct letters in Q (taken from G's Dynkin diagram).
In the example these are $d \not\geq 2d-1$, so that one cannot be $\cong X^w_\circ$.

From a pair $(\mathbb{A}^n, f = \prod f_i)$, we can guess G's Coxeter diagram by looking at the stratification generated by pairs of divisors, and obtain Q from $(\mathrm{lex}(f_i))$.

Degenerations of f_Q: conjectures.

Let D be G's Dynkin diagram, Q a word in D of length n, and $f_Q \in \mathbb{F}[z_1, \ldots, z_n]$ the polynomial associated with (D, Q). We state some conjectures about its leading forms:

1. Around the vertex $(1, 1, \ldots, 1)$ the Newton polytope of f_Q is an orthant, i.e. defined by the intersection of n equations and inequalities.
2. The equations and inequalities are in correspondence with positions in Q. A position in Q gives an equation iff it is the last occurrence of that simple root.
3. The facet of $Newton(f_Q)$ corresponding to position i defines a leading form f_Q^i, which is again LMP. Let D' be D with an extra copy α' of α_i attached (where α_i attaches), and Q' be Q with every **later** occurrence of α_i replaced with α'. Then f_Q^i is the polynomial associated with (D', Q').

Example: $D = A_2$, $Q = 1221$, $f_Q = (z_1 z_4 - z_2 z_3 + 1)(z_2 z_3 - 1)$.
Its LMP initial forms are $z_1 z_4 (z_2 z_3 - 1) = f_{1221'}$ and $(z_1 z_4 - z_2 z_3) z_2 z_3 = f_{122'1}$.

If we partially order LMP polynomials by "is a leading form of", then this conjecture implies that the set of f_Q polynomials forms an order ideal inside this poset of LMP polynomials.

Bruhat atlases: definition.

What about stratified varieties that *aren't* just vector spaces?

Definition [X. He–K–J. H. Lu]. A **Bruhat atlas** on a stratified manifold $(M, \mathcal{Y})$ is the data

1. a Kac-Moody group H (with Bruhat cells $X^v_\circ \subseteq H/B_H$, etc.)
2. a poset injection $v : {\mathcal{Y}_{irr}}^{op} \hookrightarrow W_H$, with image an order ideal $\bigcup_{p \in \mathcal{Y}_{min}} [1, v(p)]$
3. an open cover $\{U_p \ : \ p \in \mathcal{Y}_{min}\}$ with $U_p \ni p$, so, $M = \bigcup_{p \in \mathcal{Y}_{min}} U_p$
4. stratified chart maps $c_p : \ U_p \xrightarrow{\sim} X^{v(p)}_\circ$, i.e. $c_p^{-1}(X_{v(Y)}) = U_p \cap Y$ for each $Y \in \mathcal{Y}_{irr}$.

What $(M, \mathcal{Y})$ could be a candidate for this structure? How to choose H etc.?

Make a Coxeter diagram D whose vertices are the divisors in $\mathcal{Y}$, since they map to length 1 elements of W_H. If divisors Y_1, Y_2 generate a rank 2 Bruhat poset, use that knowledge to connect them in D for H; e.g. finding the poset on the right would tell us to connect the Y_1, Y_2 vertices into an A_2 subdiagram. Then let H be the corresponding Kac-Moody group.

Next is to extend the definition of v into higher codimensions.

By far the hardest part is writing down (c_p).

$$\begin{array}{c} M \\ / \quad \backslash \\ Y_1 \quad Y_2 \\ | \ \times \ | \\ C \quad C' \\ \backslash \quad / \\ C'' \end{array}$$

Bruhat atlases: examples.

Known (H, v, and (c_p)):

1. $M = G/B$ (finite dim) with the Richardson stratification [K-Woo-Yong '12]. $H = G \times G$ and $c_w : X_w^\circ \times X_\circ^w \xrightarrow{\sim} wB_-B/B$, extending [Kazhdan-Lusztig '79].
2. $M = Gr(k, n)$ with the positroid stratification [M. Snider '11]. $H = \widehat{GL(n)}$.
3. $M = \overline{PSO(2n)/SO(2n-1)}$ stratified by B-orbits plus extra [D. Huang '19]. $H = SO(2n+2)$.

Known combinatorially (H and v):

1. $M = G/P$ (finite dim). The diagram D is two copies of G's, glued together along P's, but using $-w_0^P$ to make the identification. [He-K-Lu]
2. $M = \overline{G_{ad}}$. The diagram D is two Nakajima diagrams (G's diagram plus framing vertices) glued together along their framing vertices. [He-K-Lu]

Baseless conjecture (H only):

1. $M = T^*GL_n/B$, $\mathcal{Y}$ defined by [Mehta-van der Kallen]. D is a sort of broom. [K-S. Sam]

Bruhat atlases, and, degenerations of multiplicity-free subvarieties of H/B_H.

If $(M, \mathcal{Y}, H, \nu, (c_p))$ is a stratified manifold with a Bruhat atlas, define $M_0 := \bigcup_{p\in\mathcal{Y}_{min}} X^{\nu(p)}$, a subscheme of H/B_H. What is its relation to M?

Example: $M = G/B$. Let $M_t := (1, \check{\rho}(t)) \cdot (G/B)_\Delta \subseteq (G/B)^2$. Then the flat limit $\lim_{t\to 0} M_t$ is $\bigcup_w (X_w \times X^w)$, isomorphic to M_0.

Example: $M = Gr(k, n)$. Following [Görtz '01], let

$$M_t := \left\{ (V_1, V_2, \ldots, V_n) \in Gr(k,n)^n \,:\, V_i \geq \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ t & & & 0 \end{bmatrix} \cdot V_{i-1 \bmod n} \right\}$$

For $t \neq 0$ the $\geq$ is $=$ and $M_t \cong Gr(k, n)$, but M_0 is $\cong$ the union described above. One can fit this inside Gaitsgory's degeneration of $Gr_G \times G/B$ to $\widehat{G}/\widehat{B}$.

Theorem [Brion, '03]. If $Z \subseteq G/B$ irreducible is homologous to a union M_0 of distinct Schubert varieties, then Z has a flat degeneration F to M_0, and is C-M.

What extra structure lets one extend the open sets $X_\circ^{\nu(p)} \subseteq M_0$ to open sets in F, thereby to $F_1 = Z$?

Kazhdan-Lusztig atlases.

If $(M, \mathcal{Y}, H, v, (c_p))$ is a stratified manifold with a Bruhat atlas, then $Y \in \mathcal{Y}_{irr}$ doesn't naturally inherit one, even if smooth. So we generalize the definition:

Definition [HKL]. A **Kazhdan-Lusztig atlas** on $(Z, \mathcal{Y})$ is again the data H, v, $(U_p)_{p \in \mathcal{Y}_{min}}$, but now

1. the ranked poset injection $v : {\mathcal{Y}_{irr}}^{op} \hookrightarrow W_H$ has image $\bigcup_{p \in \mathcal{Y}_{min}} [v(Z), v(p)]$
2. the chart maps c_p are to $X_\circ^{v(p)} \cap X_{v(Z)}$ i.e. $c_p^{-1}(X_{v(Y)}) = U_p \cap Y$ for each $Y \in \mathcal{Y}_{irr}$.

Examples:

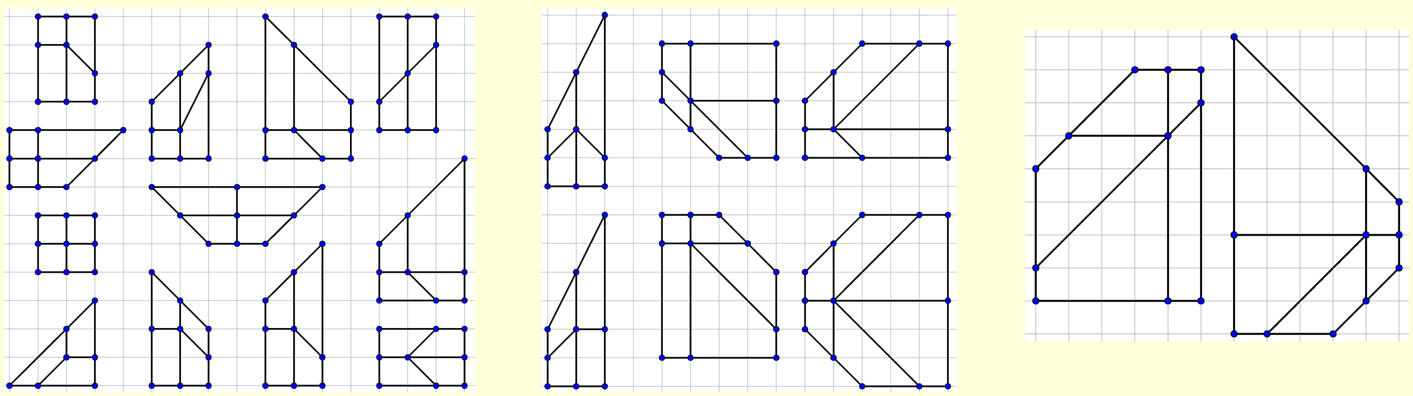
1. The Zelevinsky isomorphism $\mathrm{Rep}(A_n, \vec{d}) \cong X_\circ^{w_0 w_0^P} \cap X_v$ of the (non-moduli) space of quiver representations with (a coarsening of) the RHS. Extended in [Kinser-Rajchgot '15] to the non-equioriented case
2. $Z = G/P$ with the projected Richardson stratification, $H = \widehat{G}$. The combinatorics v is in [He-Lam '15], the maps (c_p) in [Huang '20], [Galashin-Karp-Lam '20] (not quite the same though!).

It seems very hard to guess $H, v(Z) \in W_H$ from $(Z, \mathcal{Y})$. But as these examples show, doing so can allow for the use of much tamer H.

Kazhdan-Lusztig atlases on smooth toric surfaces.

[Elek '16] classified toric surfaces M with simply-laced K-L atlases, under the additional assumption that M has a toric degeneration to M_0. This required

1. classifying Richardson surfaces in all H/B_H, of which there are 10 plus an infinite family, the "pizza slices", with associated elements of $SL_2(\mathbb{Z})$
2. gluing those together into polygons, the "pizzas"
3. making sure the pizzas only wrap around the center once, an $\widetilde{SL_2(\mathbb{Z})}$ calculation, whose abelianization is $\mathbb{Z}/12$. Each pizza slice thereby obtains a "nutritive value" ν in $\frac{1}{12}\mathbb{Z}$, with $\sum \nu = 1$. All $\nu > 0$ except one in G_2.



4. developing a theory of "pizza toppings", one for each simple root of a candidate Kac-Moody H, and finding ranked poset injections $\nu : \mathcal{Y} \to W_H$.