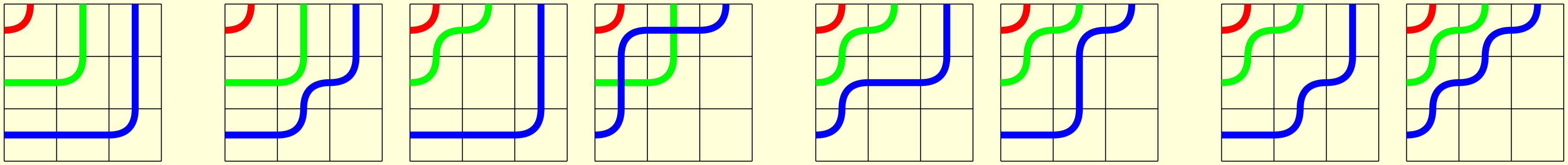
The commuting variety and generic pipe dreams

Allen Knutson* (Cornell) and Paul Zinn-Justin (Melbourne)

Combinatorial Algebra meets Algebraic Combinatorics (CAAC) 2022

Abstract

Nobody knows whether the scheme "commuting pairs of $n \times n$ matrices" is reduced. I'll show a relation of this scheme to matrix Schubert varieties, and give a formula for its equivariant cohomology class (and that of many other varieties) using "generic pipe dreams" that I'll introduce. These interpolate between ordinary and bumpless pipe dreams. I'll rederive both formulæ (ordinary and bumpless) for double Schubert polynomials, as limits. This work is joint with Paul Zinn-Justin.



$$\deg\left(\{(X, Y) \in Mat_3 \colon XY = YX\}\right) = 1 + 2 + 2 + 2 + 4 + 4 + 8 + 8 = 31$$

These slides are available at `https://pi.math.cornell.edu/~allenk/`

Why do matrices commute?

Consider the space $\{M \in \mathrm{Mat}_n : M^2 = 0\}$, defined by n^2 quadratic equations. Such M are nilpotent, hence have trace 0, but this linear equation $\mathrm{Tr}(M) = 0$ can't be algebraically derived from those homogeneous quadratics. (According to Hilbert's Nullstellensatz, any such secret equation *can* be derived using $\sqrt{\ }$; consider the $n = 1$ case.)

Does the scheme $\{(X, Y) \in \mathrm{Mat}_n{}^2 : XY = YX\}$ satisfy secret equations?

Hochster conjectured in 1984 that it does not. (For $n \leq 4$ it sure doesn't.)

Idea: if we "degenerate" this **commuting scheme** C to another one C', and can show that C' is **reduced** (no secret equations), then C likewise is reduced.

I tried this: let $\check{\rho}(t) := \mathrm{diag}(t^1, t^2, t^3, \ldots, t^n)$, and $X' := X\check{\rho}(t)$, $Y' := \check{\rho}(t^{-1})Y$, a linear change of coördinates. Then the scheme becomes

$$\{(X', Y') : X'Y' = (\mathrm{Ad}\, \check{\rho}(t)) \cdot (Y'X')\}, \qquad \text{and these equations at } t \to 0 \text{ become}$$

$$\{(X', Y') : X'Y' \text{ lower triangular}, Y'X' \text{ upper triangular}, \mathrm{diag}(X'Y') = \mathrm{diag}(Y'X')\}.$$

Coming next: What if were to leave out these last n equations?

The lower-upper scheme and its components.

In [K05] I introduced the **(rectangular) lower-upper scheme**

$$E := \{(X, Y) \in M_{k\times n} \times M_{n\times k} \ : \ XY \text{ lower triangular, } YX \text{ upper triangular}\}$$

which bears an action of $B_-(k) \times B_+(n)$, the lower and upper triangular groups:

$$(b_-, b_+) \cdot (X, Y) := \left(b_- X\, {b_+}^{-1}, b_+ Y\, {b_-}^{-1}\right)$$

On any component, we can use it to reduce X to a partial permutation matrix.

Theorem (proved for $k = n$ in [K05]). The components are distinguished by these partial permutations π, so we call them E_π. If $k \leq n$, assumed hereafter, these π are the $k!\binom{n}{k}$ injections $[k] \hookrightarrow [n]$. In particular, E_π satisfies the equations

$$(XY)_{ii} = (YX)_{\pi(i),\pi(i)}\ \forall i \in [k], \quad X \in \overline{X}_\pi := \overline{B_-(k)\pi B_+(n)}, \quad Y \in \overline{B_+(n)\pi^{-1}B_-(k)}$$

For $k = n$, E is a complete intersection (of quadrics), so $\sum_{\pi\in S_n} \deg E_\pi = 2^{n^2-n}$.

(Much more about "degrees" in a couple of slides.)

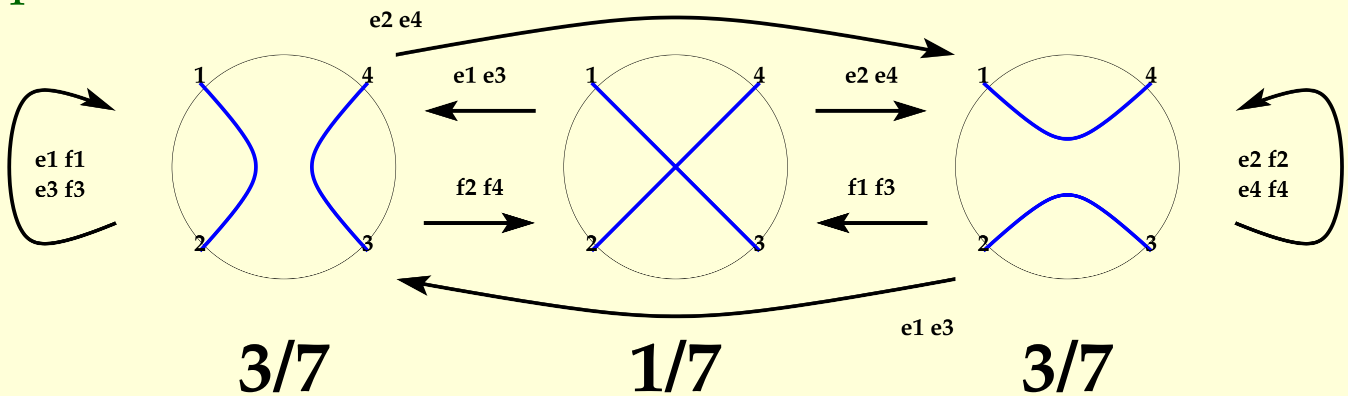
A stranger appears: the de Gier-Nienhuis Markov chain.

Consider a Markov chain, whose states $\rho \in S_{2n}$ are the $(2n-1)!!$ perfect matchings of $1\dots 2n$, and whose transitions are as follows:

Spin a wheel-of-fortune, that stops between some $i, i+1 \bmod 2n$, and separately flip an unfair coin that comes up "e" $\frac{2}{3}$ of the time, "f" $\frac{1}{3}$ of the time.

(This $\frac{2}{3}, \frac{1}{3}$ is very important! 1 and 0 also works well, but $\frac{1}{2}, \frac{1}{2}$ does *not*.)

If e comes up, connect $\rho(i)$ to $\rho(i+1)$ and i to $i+1$.
If f comes up, connect $\rho(i)$ to $i+1$ and i to $\rho(i+1)$.



Theorem [KZJ07] (conjectured by [dGN05] based on OEIS numerology).
Let $P(\rho) :=$ the fraction of time spent in state ρ, and $\mathrm{diam}(i) := i + n \bmod 2n$.
1. $P(\rho)/P(\mathrm{diam}) \in \mathbb{N}$ for each perfect matching ρ.
2. For $\pi \in S_n$, let $\rho(i) := \pi(i) + n \bmod 2n$. Then $P(\rho)/P(\mathrm{diam}) = \deg E_\pi$.

We give a similar geometric interpretation of the other $P(\rho)$ in [KZJ07], using an enlargement of the lower-upper scheme to the "Brauer loop scheme".

Degrees, degenerations, and projective duality.

If V is an n-dim vector space over $\mathbb{C}$, and $X^k \subseteq V$ is defined by the vanishing of some homogeneous polynomials (g_i), then we have many ways to compute the "degree d of the affine cone X":

1. $d = \#(X \cap P^{n-k})$ — P is a random plane of complementary dimension
2. $d = vol(X \cap S^{2n-1})/vol(Q^k \cap S^{2n-1})$ — Q any plane of the same dimension
3. $d = \lim_{D\to\infty} \dim(\mathbb{C}[x_1, \ldots, x_n]/\langle g_1, \ldots\rangle)_D / \binom{D}{k}$ — computed from the ring
4. $[\mathbb{P}X] = d[\mathbb{P}^{k-1}] \in H^{2(n-k)}(\mathbb{P}V) = \mathbb{Z} \cdot [\mathbb{P}^{k-1}]$ — in ordinary cohomology
5. $[X] = d\hbar^{n-k} \in H^*_{\mathbb{C}^\times}(V) \cong \mathbb{Z}[\hbar]$ — in dilation-equivariant cohomology

The degree is invariant under "Gröbner degenerations" of X, of which we will only discuss three special cases. In the first two, we start by splitting V as $H \oplus L$, a Hyperplane plus a Line, and we let $\mathbb{C}^\times \circlearrowright V$ by $z \cdot (h, \ell) := (h, z\ell)$.

Lexing the variable: $X' := \lim_{z\to 0}(z \cdot X)$. Each $g_i \mapsto$ its terms with $\ell^{\max}$.

Revlexing the variable: $X' := \lim_{z\to\infty}(z \cdot X)$. Each $g_i \mapsto$ its terms with $\ell^{\min}$.

Let $CX \subseteq V \times V^* \cong T^*V$ be the closure of the space of pairs

$$\{(v \in V, f \in V^*) : v \text{ a smooth point in } X, f \perp T_vX\}.$$

Assuming X is a variety, the projection of the Lagrangian CX to V is X, while the projection to V^* is the 19th century **projective dual** $X^\perp$. (And yes, $X^{\perp\perp} = X$.)
Lexing X is projective dual to revlexing $X^\perp$. (Third, combined, degen to come later)

Warmup: the projections of E_π to X and Y.

How can we use these tricks to compute the degree of E_π? Let's warm up with the images $\overline{X}_\pi$ (a **matrix Schubert variety** [Fulton '92]) and $w_0 \overline{X}_{w_0 \pi^{-1} w_0} w_0$ of the projections $(X, Y) \mapsto X, Y$. (Examples to come on the next slide.)

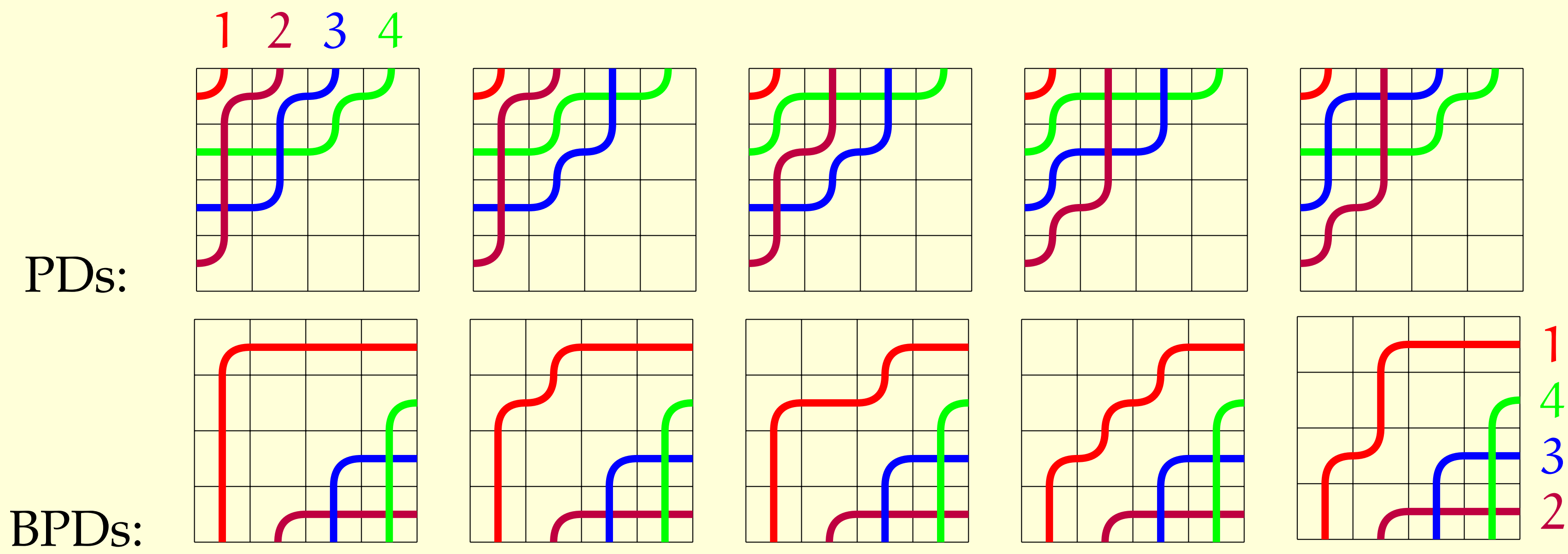
A(n **ordinary) pipe dream** δ [N. Bergeron-Billey '93] for $\pi \in S_n$ is an $n \times n$ square filled with the tiles in the NW triangle, down the antidiagonal, and in the SE triangle. The pipes must connect $1 \ldots n$ on the West to $\pi(1) \ldots \pi(n)$ on the North, no two pipes crossing twice. Write $\delta \vdash \pi$.

Theorem [K-Miller '05]. To a pipe dream $\delta \vdash \pi$, associate a coördinate space $\mathbb{C}^\delta \leq Mat_n$ with $m_{ij} = 0$ if $\delta(i,j) =$. There is an "iterated revlex from NW" degeneration of $\overline{X}_\pi$ to $\bigcup_{\delta \vdash \pi} \mathbb{C}^\delta$, and consequently, $\deg \overline{X}_\pi = \#\{\delta : \delta \vdash \pi\}$.

A **bumpless pipe dream** δ [Lam-Lee-Shimozono '18] for $\pi \in S_n$ uses but **not** the "bump" , and connects $1 \ldots n$ on the East to $\pi(1) \ldots \pi(n)$ on the South, no two pipes crossing twice. Write $\delta \vDash \pi$.

Theorem [Klein-Weigandt]. To a bumpless $\delta \vDash \pi$, associate (noninjectively!) a coördinate space $\mathbb{C}^\delta \leq Mat_n$ with $m_{ij} = 0$ if $\delta(i,j) =$. There is an "iterated lex from Southeast" degeneration of $\overline{X}_\pi$ to a *schemy* union $\bigcup_{\delta \vdash \pi} \mathbb{C}^\delta$ whose multiplicities reflect the noninjectivity, and consequently, $\deg \overline{X}_\pi = \#\{\delta : \delta \vDash \pi\}$.

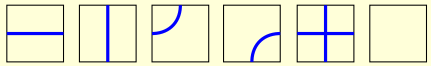
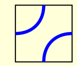
Example: computing the degree, and more, of $\overline{X}_{1432}$.



The degree d of an affine cone $X^k \subseteq V \cong \mathbb{C}^n$ can be interpreted cohomologically; if we write $H^*_{\mathbb{C}^\times}(V) \cong \mathbb{Z}[\hbar]$, then $[X] = d\hbar^{n-k}$. Why stop with the dilation action? Instead let's compute the **double Schubert polynomial**

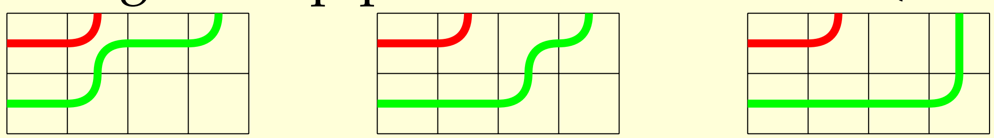
$$
\begin{aligned}
S_\pi \quad &:= \quad [\overline{X}_\pi] \in H^*_{T\times T}(Mat_n) \cong \mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n] \\
&= \quad \sum_{\delta \vdash \pi} \prod_{\boxplus \in \delta} (x_{row} - y_{col}) = \sum_{\delta \vDash \pi} \prod_{\square \in \delta} (x_{row} - y_{col})
\end{aligned}
$$

Generic pipe dreams, and a degree formula.

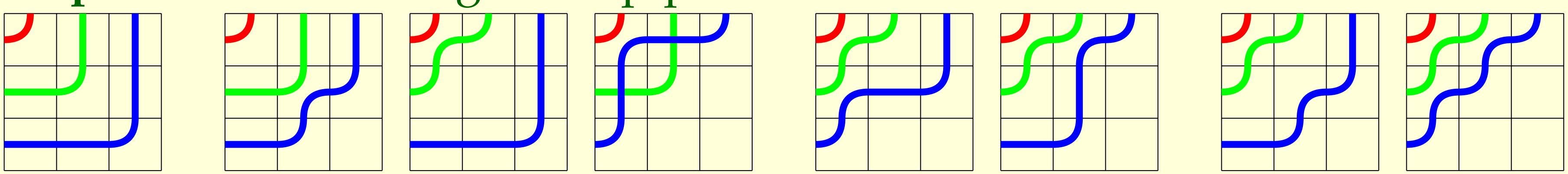
Consider the tiles  and yes *also* the bump .
Say a $k \times n$ rectangle filled with these is a **generic pipe dream** D for π, $D \vdash \pi$, if the connectivity from the West side to North is given by π, and the East and South labels are blank.

There is **no restriction** against pipes crossing twice, and if they do, we **simply follow them**, i.e. using ordinary *not* Demazure/greedy/nil Hecke product.

Example. These are the generic pipe dreams for $\pi = 24$ $(k = 2, n = 4)$.



Example. These are the generic pipe dreams for $\pi = 123$.



Define the **degree** of a generic pipe dream D to be $2^{\#\text{tiles with one or two elbows} - k}$, giving in this latter example a total degree of $1 + 2 + 2 + 2 + 4 + 4 + 8 + 8 = 31$.

Theorem. $\deg E_\pi = \sum_{D \vdash \pi} \deg(D)$. Using [K05], this gives a formula for $\deg(\text{the commuting scheme})$, OEIS sequence A029729 = 1, 3, **31**, 1145, 154881...

The variety associated to a generic pipe dream.

Let $D \vdash \pi$ be a generic pipe dream for $\pi : [k] \hookrightarrow [n]$. To each *edge* e in the $k \times n$ rectangle we associate the **flux**

$$\Phi_e = \sum_{\substack{\text{tiles } (i,j) \\ \text{right of } e \text{ if } e \text{ vertical} \\ \text{below } e \text{ if } e \text{ horizontal}}} X_{ij} Y_{ji} \qquad \text{so, a sub-sum of } (XY)_{ii} \text{ or } (YX)_{jj}$$

and define an analogue of the coördinate spaces associated to PDs and BPDs:

$$F_D := \left\{ \begin{matrix} (X,Y) \\ \pitchfork \\ M_{k\times n} \times M_{n \times k} \end{matrix} : \begin{matrix} X_{ij} = 0 \text{ if } D(i,j) \text{ one of } \boxminus, \boxplus, \boxed{|} \\ Y_{ji} = 0 \text{ if } D(i,j) \text{ is } \square \\ \Phi_e = \Phi_{e'} \text{ if } e, e' \text{ cross the same pipe} \\ \Phi_e = 0 \text{ if } e \text{ is a blank edge} \end{matrix} \right\} \begin{matrix} A + x_i - y_j \\ B - x_i + y_j \\ A + B \\ A + B \end{matrix}$$

At right are the $H^*_{T(k)\times T(n) \times T^2}$-weights, where T^2 acts on X, Y with weights A, B.

Theorem. F_D is a variety, and a c.i. of linear and quadratic equations, with $\deg(F_D) = \deg(D)$. One can reconstruct D from F_D's flux equations alone.

The main theorem: $E_\pi \rightsquigarrow \bigcup_D F_D$ (in top dimension at least).

Recall that **revlexing a variable** (thereby with it defining Gröbner degenerations of ideals) amounts to scaling that variable by t, whereas **lexing the variable** amounts to scaling that variable by t^{-1}, and in each case taking $t \to 0$.

We introduce T^***revlexing** in which we scale X_{ij} by t *while* scaling Y_{ji} by t^{-1}, leaving all other variables alone, then take $t \to 0$.

Theorem. T^*revlex the variety E_π once for each (i,j), starting from the Northwest X_{11}, Y_{11} and proceeding Southeast, rastering row by row. Then the limit scheme is the union $\bigcup_{D \vdash \pi} F_D$, plus possibly some lower-dimensional embedded junk.

We conjecture that that junk isn't there. (work in progress)

Corollary. As equivariant classes in $H^*_{B_-(k) \times B_+(n) \times T^2}(M_{k\times n} \times M_{n \times k})$, we have $[E_\pi] = \sum_{D \vdash \pi} [F_D]$, where

$$[F_D] = (A+B)^{-k} \prod_{(i,j)} \Big((A + x_i - y_j) \, (B - x_i + y_j) \, (A+B) \Big) .$$

This corollary is closely related to C. Su's "restriction formula" for $MO_w|_v$.

$\bigcup_{D \vdash \pi} F_D$ is not as pleasant as $\mathrm{init}(\overline{X}_\pi) = SR(\text{ordinary pipe dream complex})$: (1) it isn't "thin" as a cell complex, and (2) $F_D \cap F_{D'}$ may be reducible.

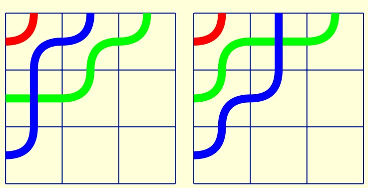
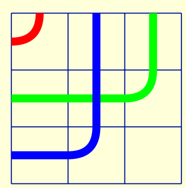
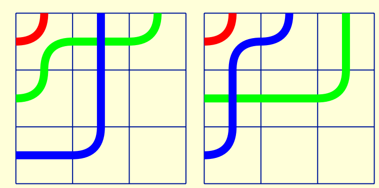
Recovering ordinary and bumpless pipe dreams.

Since the X_{ij} variables have T^1-weight A and the Y_{ji} have T^1-weight B,

$$[E_\pi] = B^{\max}\left[\text{the projection } \overline{X}_\pi \text{ of } E_\pi \text{ under } (X,Y) \mapsto X\right] + \text{l.o.t. in } B$$

That B-leading term (the double Schubert polynomial) arises for those D with the minimum number of ─, ┼, │. By Jordan curve we need $\ell(\pi)$ ┼, and then it's easy to see that ordinary PDs achieve the min, and that no other GPDs do.

If we look at the A-leading term, then the minimization problem switches to the empty spaces $\square$. Any GPD with a bump can be drooped to lose the bump and invade/destroy one empty space. Also, double crossings can be replaced with double bumps and similarly drooped. So the GPDs with minimum $\#\{\square\}$ are the bumpless ones without double crossings (rotated 180°).

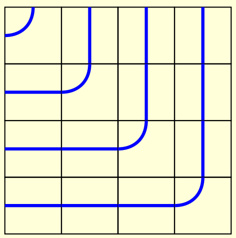
ordinary:  bumpless:  neither: 

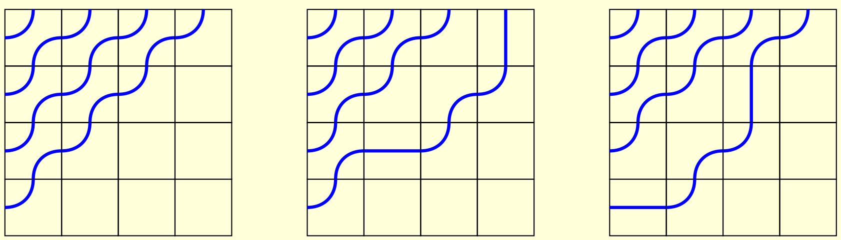
Of course, these bumpless GPDs are 180° off from the usual ones, which befits the projection $(X,Y) \mapsto Y$ of E_π being $w_0 \cdot \overline{X}_{w_0 \cdot \pi^{-1}}$.

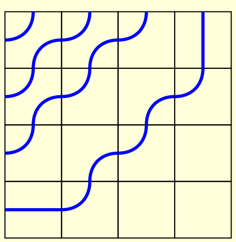
In particular, the corollary from the last slide implies both the PD and BPD formulæ for double Schubert polynomials.

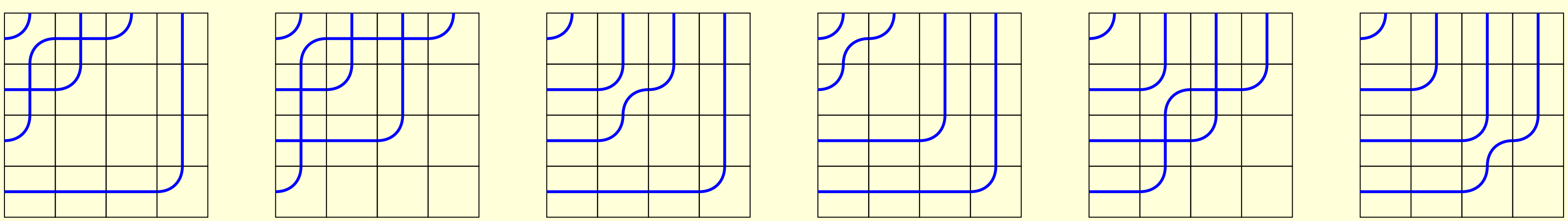
Showing off: the degree of the 4th commuting variety is 1145.

We compute it as $1 + (6)2 + (11)4 + (18)8 + (17)16 + (11)32 + (3)64 + 128$ (0.015 sec in Macaulay 2; $n = 5, 6$ take 0.21s, 9.74s to get 154881, 77899563).

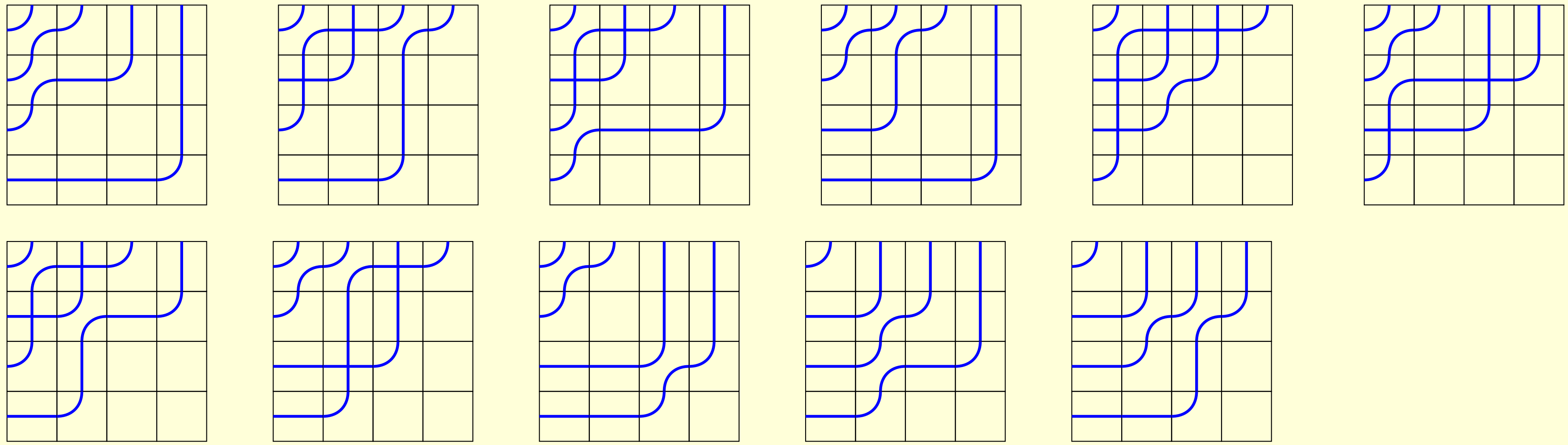
1:  This is the only bumpless pipe dream.

64:  The left one is the only ordinary pipe dream.

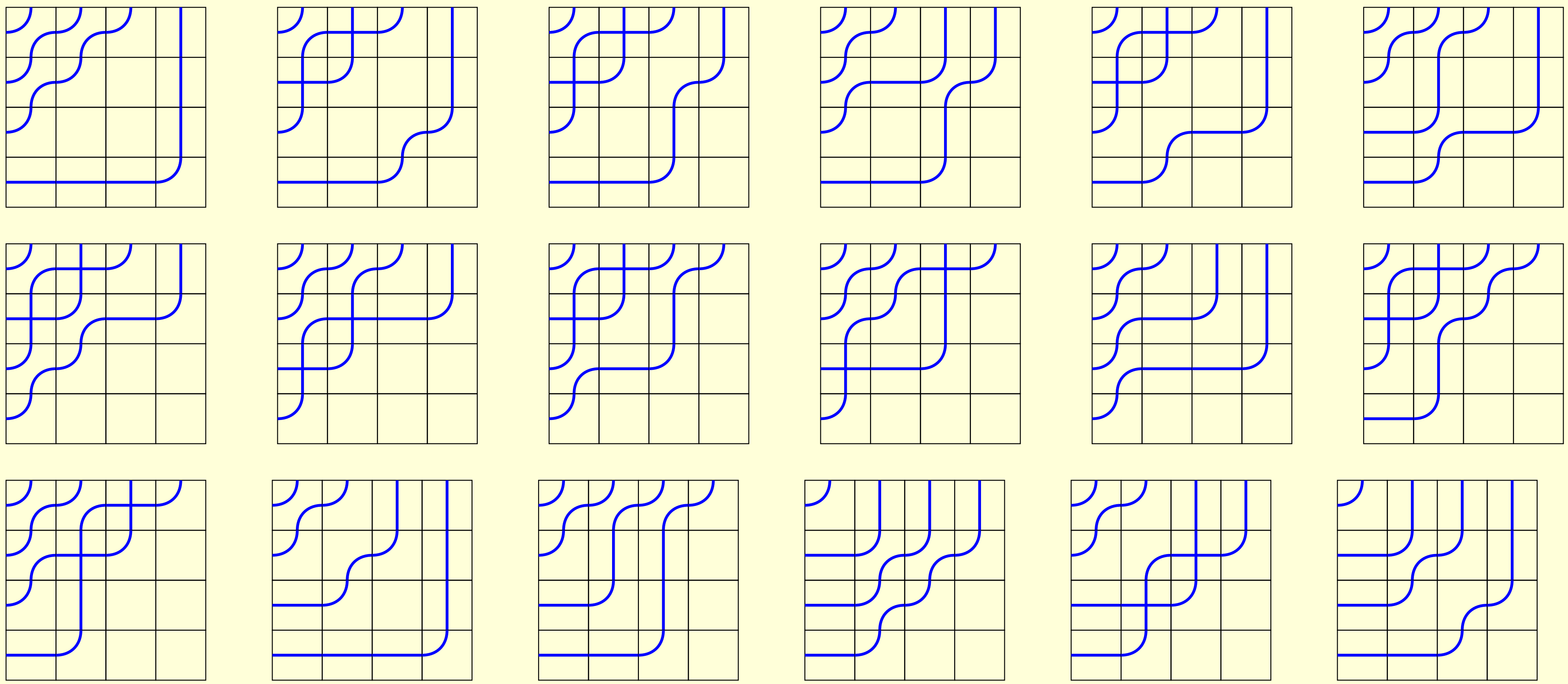
128:  The highest-degree pipe dream is not ordinary.

2: 

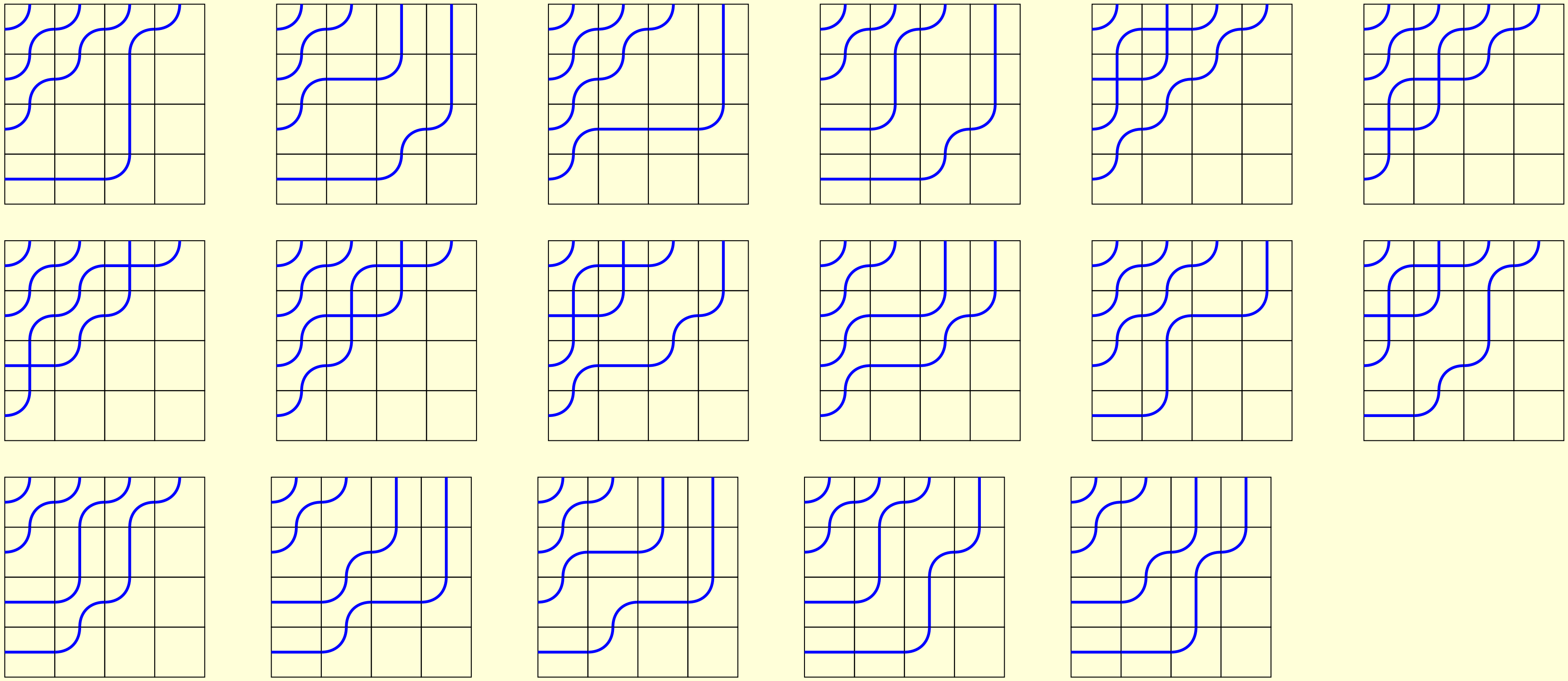
4:



8:



16:



32: