Schubert calculus and scattering diagrams

Allen Knutson (Cornell)

CMS Summer Meeting 2018, Fredericton

Abstract

"How many lines in space pass through four, given, generic lines?" (answer: two) is a counting problem admitting many generalizations, to chains of subspaces, isotropic or Lagrangian subspaces; or, beyond counting to answers living in exotic cohomology theories (K, equivariant, quantum). In all these cases we have alternating-sum formulae for the manifestly nonnegative answers, admitting much computer experimentation in the search for manifesly nonnegative formulae.

I'll review the history, and talk about recent work (joint with Paul Zinn-Justin) that uses input from quantum integrable systems to give positive answers to more of these questions. Quiver varieties, cluster varieties, E_8, and triality will all make appearances.

These transparencies are available at `http://math.cornell.edu/~allenk/`

Conditions on subspaces.

Let $\mathrm{Gr}(k,n) := \{V \leq \mathbb{C}^n \ : \ \dim V = k\}$ be the **Grassmannian**. To specify a point $V \in \mathrm{Gr}(k,n)$, we pick a basis of V (here, row vectors), assemble its vectors into a $k \times n$ matrix, and (for uniqueness) pass to reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \quad \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \quad \begin{matrix} \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \\ \\ \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \quad \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Index these $\binom{n}{k}$ cases by bit strings λ, with 0s in the k pivot columns, and denote the corresponding subsets X°_λ, the **Bruhat cells**. So $X^\circ_\lambda \cong \mathbb{C}^{\#\text{pivot left of non-pivot}}$, and this even-real-dimensional cell decomposition gives bases of $H_*(\mathrm{Gr}(k,n))$, $H^*(\mathrm{Gr}(k,n))$, and more exotic cohomology theories (equivariant, K-theory, quantum — and combinations thereof).

"Which cell X°_λ contains my V?" We can answer this question geometrically:

$$\forall i = 0,\ldots,n, \quad \dim\left(V \cap \{[0 \cdots 0 * \cdots *] \in \mathbb{C}^n,\ i \text{ stars}\}\right) = i - \sum (\text{last } i \text{ bits in } \lambda)$$

i.e. "our k-plane intersects the standard i-plane so-and-so much."

Turning this-AND-this-condition into this-OR-this-condition.

Let A, B, C be general points in projective 3-space $\mathbb{CP}^3$, and consider

$$X \quad := \quad \left\{ L \in Gr(1, \mathbb{CP}^3) \cong Gr(2,4) \ : \ L \text{ meets } \overline{AB} \textit{ and } L \text{ meets } \overline{AC} \right\} \quad \subseteq Gr(2,4)$$
$$= \quad \{L \ni A\} \cup \{L \subset \text{the plane } \overline{ABC}\} \qquad \text{glued along } \{L \ni A \textit{ and } L \subset \overline{ABC}\}$$

These give us equations in cohomology and K-theory:

$$[X_{0101}]^2 \quad = \quad [X_{1001}] + [X_{0110}] \qquad \text{as classes in } H^*(Gr(2,4))$$

$$[X_{0101}]^2 \quad = \quad [X_{1001}] + [X_{0110}] - [X_{1010}] \qquad \text{as classes in } K(Gr(2,4))$$

In general, a product expansion $[X_\lambda][X_\mu] = \sum_\nu c_{\lambda\mu}^\nu [X_\nu]$ in the **Schubert basis** is about replacing $(g \cdot X_\lambda) \cap (h \cdot X_\mu)$, homologically, with a union of various $k \cdot X_\nu$.

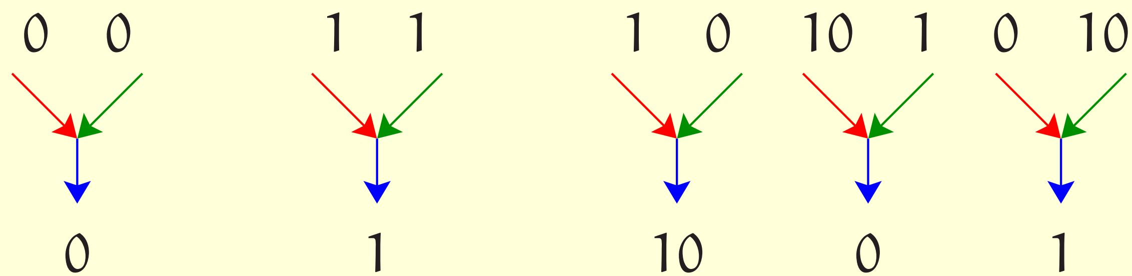
Theorem. [Lesieur '47, Kleiman '73]
In $H^*(Gr(k,n))$, and an analogous question for $H^*(G/P)$, each $c_{\lambda\mu}^\nu \geq 0$.

[Buch '02, Brion '02]
In $K(Gr(k,n))$, and analogously for $K(G/P)$, each $(-1)^{\ell(\nu)-(\ell(\lambda)+\ell(\mu))} c_{\lambda\mu}^\nu \geq 0$.

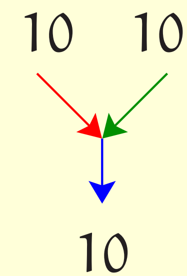
The (planar dual of the) puzzle formula for $c^{\gamma}_{\lambda\mu}$.

The first positive formula for $c^{\gamma}_{\lambda\mu}$ in H^* is the *Littlewood-Richardson rule*, stated in the '30s, proved by Schützenberger in the '70s. Many other rules have been given, but we'll focus on the *puzzle* rule, due to Terry Tao and me.

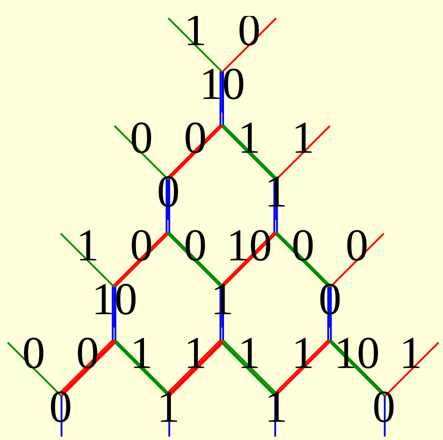
Consider honeycomb graphs with three **edge labels** 0, 1, 10, and the following allowed vertices (which may be rotated 180°, with orientations flipped):

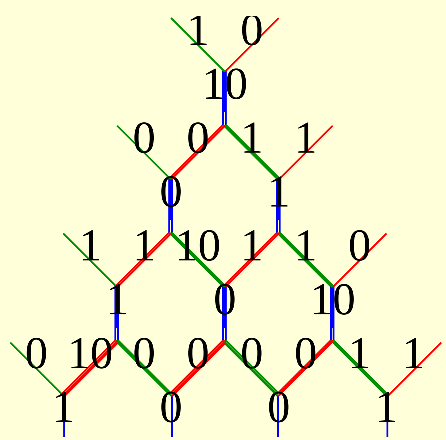


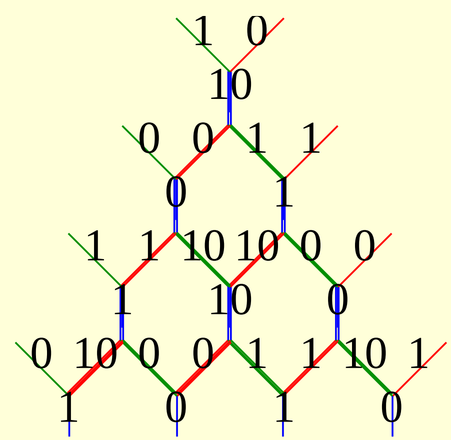
except don't rotate this "K-vertex" 180°:



Theorem. $\pm c^{\gamma}_{\lambda\mu}$ is the number of such graphs with λ, μ, ν as the boundary labels on NW,NE,S, all left-to-right.







Many flavors of Schubert calculus.

The basic, very solved, problem is to compute the product in $H^*(Gr(k,n))$. There are at least 5 avenues of generalization:

- K: to K-theory
- T: to equivariant cohomology (now $c_{\lambda\mu}^{\nu} \in H_T^*(pt) \cong \mathbb{Z}[y_1, \ldots, y_n]$)
- Q: to quantum cohomology
- G: to other "cominuscule flag manifolds" G/P (not defined here)
- F_d: to flag manifolds $Fl(k_1, k_2, \ldots, k_d;\ n) := \{V_1 \leq \ldots \leq V_d \leq \mathbb{C}^n\ :\ \dim V_i = k_i\}$

In all cases, we have formulæ for the product and for most, geometric proofs of abstract positivity. But we have *combinatorial, manifestly positive* formulæ in only some cases:

- K [Buch '02]
- T [K-Tao '03]
- $Q \subset F_2$, $QT \subset TF_2$ [Buch-Kresch-Tamvakis '03]
- G [Thomas-Yong '09], some KG [Clifford-T-Y '12]
- F_2 [Buch-Kresch-Purbhoo-Tamvakis '16]
- TF_2 [Buch '15]
- KT [Pechenik-Yong '17, Wheeler–Zinn-Justin '17]
- KF_2, KTF_2 [K–Zinn-Justin, preprint]
- F_3, KF_3 [K–Zinn-Justin, preprint]

The rest of this talk is about the ingredients in my recent advances with PZ-J.

The "equivariant cohomology ring" $H_T^*(M)$ and its product.

Given $X, Y \subset M$ cycles in a manifold (all compact complex, say), we have a "cohomology ring" $H^*(M)$ containing "the classes $[X], [Y]$ of X, Y", and the product $[X] \cup [Y]$ measures our inability to make X, Y disjoint in M.

Another source of classes in $H^*(M)$ is from Euler classes: if $\mathcal{V} \to M$ is an oriented $\mathbb{R}^n$-bundle, then the **Euler class** $e(\mathcal{V}) \in H^n(\mathcal{V}) \cong H^n(M)$ measures our inability to make the zero section $M \hookrightarrow \mathcal{V}$ disjoint from itself.

Now let $T \cong (S^1)^n$ be the n**-dimensional torus** group, and assume $X, Y, M, \mathcal{V}$ all carry compatible T-actions. Then we want $[X], [Y], e(\mathcal{V}) \in H_T^*(M)$, some ring I won't fully define.

Main example: any complex representation V of T is a direct sum $\bigoplus_{i=1}^n \mathbb{C}_{\lambda_i}$, over some "weights" $\lambda_i \in T^* := \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^n$. Then $V \to pt$ is a T-equivariant vector bundle, with an **equivariant Euler class** $e(V \to pt) \in H_T^{2n}(pt)$ which should vanish iff some $\lambda_i = 0$.

No surprise, then, that $H_T^*(pt) = Sym(T^*)$ and $e(V \to pt) = \prod_{i=1}^n \lambda_i$.

For any T-space M, the stupid map $M \to pt$ is T-equivariant; ergo every $H_T^*(M)$ is an algebra over the pleasantly large base ring $H_T^*(pt) \cong \mathbb{Z}[y_1, \ldots, y_{\dim T}]$. Because the Bruhat cells are T-invariant, they are again a basis of $H_T^*(Gr(k,n))$, but with coefficients $H_T^*(pt)$.

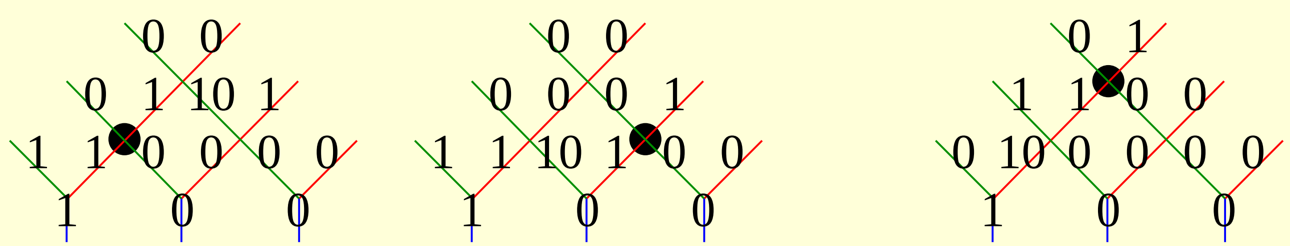
The (planar dual of the) puzzle formula for equivariant $c_{\lambda\mu}^{\nu}$.

Let $T \leq GL_n(\mathbb{C})$ hereafter be the diagonal matrices, with $H_T^*(pt) \cong \mathbb{Z}[y_1, \ldots, y_n]$. If we try to move the point X_{100} off of the line $X_{010} \subseteq Gr(1,3) \cong \mathbb{CP}^2$, we get

$$[X_{100}][X_{010}] = (y_1 - y_3)\ [X_{100}]$$

because the T-action on $\mathrm{Normal}_{X_{100}=\{[00*]\}}\left(X_{010} = \{[0 * *]\}\right)$ has weight $y_1 - y_3$.

Theorem [KT '03, stated dually]. Introduce a tetravalent green-through-red vertex, either made of a Y atop λ, *hiding the vertical blue edge,* or with 1 crossing through 0. Now there are only n green and n red edges, which we each adorn with equivariant parameters $y_1, \ldots, y_n$. Then $c_{\lambda\mu}^{\nu}$ is a sum over scattering diagrams with boundary λ, μ, ν, each contributing $\prod_{\text{new crossings}}(y_i - y_j)$.



In the example above, $c_{100,010}^{100} = (y_1 - y_2) + (y_2 - y_3) = c_{010,100}^{100} = y_1 - y_3$.
In the pictures, there are $\bullet$s to point out the "equivariant" vertices.

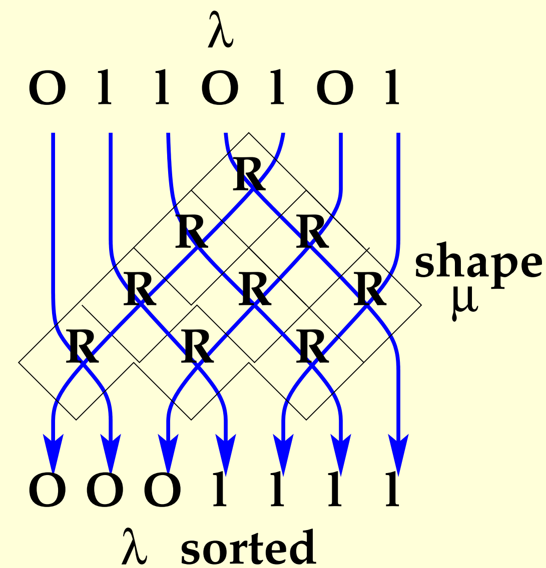
How equivariant cohomology is easier; or, exploiting symmetry.

Each coördinate k-plane $\mathbb{C}^\mu \in Gr(k,n)$ is T-fixed, so we can define

$$[X_\lambda]|_\mu := \int_{Gr(k,n)} [X_\lambda] \; \left[\{\mathbb{C}^\lambda\}\right] = \iota^* \left([X_\lambda]\right) \qquad \in H_T^*(pt) \cong \mathbb{Z}[y_1, \ldots, y_n]$$

where $\iota : \{\mathbb{C}^\mu\} \hookrightarrow Gr(k,n)$ is the (T-equivariant!) inclusion. There is a formula I won't give (due to Andersen-Jantzen-Soergel/Billey) for $[X_\lambda]|_\mu$ that can again be phrased as a sum over scattering diagrams, but now there are only 0, 1 labels, no 10s.

To help remember the difference between these AJS/Billey tetravalent vertices and the equivariant puzzle vertices from before, we color those red/green and these blue/blue (or later, red/red or green/green).



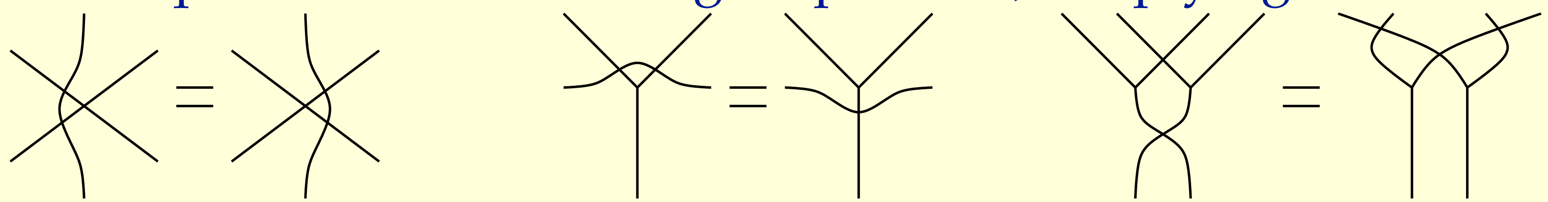
Equivariant localization to fixed points: To check a formula for equivariant $c_{\lambda\mu}^\nu$, *it's enough* to confirm $[X_\lambda]|_\sigma \, [X_\mu]|_\sigma = \sum_\nu c_{\lambda\mu}^\nu \, [X_\nu]|_\sigma$ for every σ!

These are just $\binom{n}{k}$ equations in $H_T^*(pt) \cong \mathbb{Z}[y_1, \ldots, y_n]$.

Keys to the proof: The Yang-Baxter and bootstrap equations.

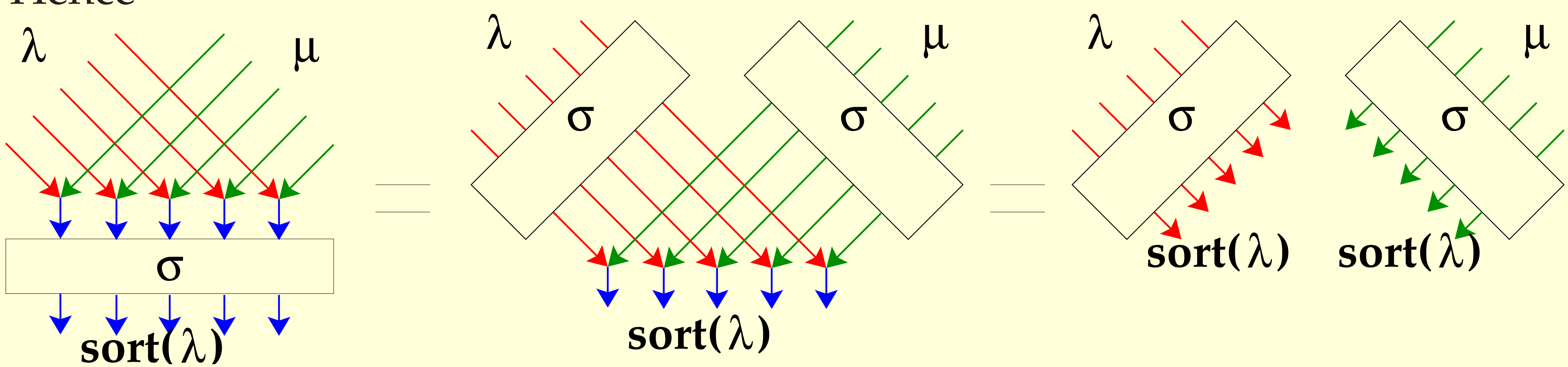
We can get the *entire* right side $\sum_\nu c_{\lambda\mu}^{\nu} [X_\nu]|_\sigma$ by gluing the puzzle picture atop the AJS/Billey picture! We need a proposition to perform moves on the result:

Proposition.
1. With any choice of orientations, colors, and boundary conditions, we have the first two equations on "scattering amplitudes," implying the third:



2. If a puzzle has the identity $0 \cdots 0\, 1 \cdots 1$ on the bottom, it must also have it on the NW and NE sides, and have scattering amplitude $= 1$.

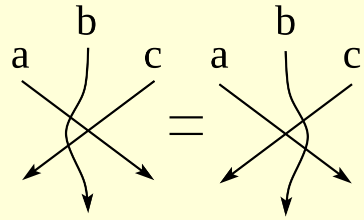
Hence



so there's our $[X_\lambda]|_\sigma\, [X_\mu]|_\sigma$. Of course proposition #1 above is a big case check.

Sources of "R-matrices", solutions of the Yang-Baxter equation.

Let V be a finite-dimensional vector space (with basis "0", "1", "10" in our main example), and $a, b, c \in \mathbb{C}$ parameters. Then the algebraic formulation of the **(rational) Yang-Baxter equation** on $R \in \mathrm{End}(V \otimes V)(u)$ is



$$R_{12}(a-b)R_{13}(a-c)R_{23}(b-c) = R_{23}(b-c)R_{13}(a-c)R_{12}(a-b)$$

This equation arose in statistical mechanics, as the key to giving exact formulæ for certain partition functions, establishing the field of "quantum integrable systems". In particular there's an immense physics literature on its solutions.

Jimbo and Drinfel′d exploit the "evaluation representations" (V, a) of the quantized current algebra $U_q(\mathfrak{g}[z])$ to construct solutions of YBE. For generic evaluation parameters a_1, a_2, the tensor product $(V, a_1) \otimes (V, a_2)$ of irreps is again irreducible (!), and isomorphic to the opposite product $(V, a_2) \otimes (V, a_1)$. The isomorphism, a rational function of $a_2 - a_1$, satisfies YBE.

"Geometric representation theory" gives a construction of these representations, on the homology of Nakajima quiver varieties [Ginzburg-Vasserot, Nakajima, Varagnolo].

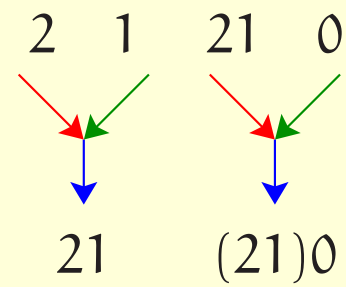
More recently, [Maulik-Okounkov] have constructed the R-matrices directly from the quiver varieties (including, cotangent bundles $T^*Gr(k,n)$...)

Settling puzzle conjectures, and discovering new puzzle pieces.

Tao and I happened on puzzles in '97, while studying Horn's conjecture on sums of Hermitian matrices. Klyachko used Grassmannian Schubert calculus instead. Were puzzles and Schubert calculus directly related? They were! So I looked into whether puzzles could compute products on d-step flag manifolds.

I came up with (what I felt was) a very beautiful puzzle rule for all d, circulated it to a few people... and then found counterexamples at $d = 3$.

To specify a Schubert class in $H^*(Fl(n_1, n_2; \mathbb{C}^n))$ requires words in $0, 1, 2$. That suggests there should be labels $10, 20, 21$ internal to the puzzle. But a little experimentation shows that one also needs $2(10)$ and $(21)0$: so eight total.



Buch conjectured, beyond my 23 suggested 3-step labels (YX) with $\forall Y > \forall X$, there should be four labels like $3(((32)1)0)$ "protected by three parentheses."

Theorem [K-ZJ]. 1. $\lim_{q \to 0}$(the $U_q(\mathfrak{sl}_3[z^\pm]) \circlearrowright \mathbb{C}^3$ R-matrix) gives $d = 1$ puzzles.
2. A certain $q \to 0$ limit of the $U_q(\mathfrak{so}_8[z^\pm]) \circlearrowright \mathbb{C}^8$ R-matrix recovers $d = 2$ puzzles, giving the K(2-step) rule (for which there'd been no conjecture), and K_T(2-step).
3. A certain $q \to 0$ limit of the $U_q(\mathfrak{e}_6[z^\pm]) \circlearrowright \mathbb{C}^{27}$ R-matrix confirms Buch's fix in H^*(3-step), also giving the K(3-step) rule (for which there'd been no conjecture).
4. We can't take $q \to 0$ of the $U_q(\mathfrak{e}_8[z^\pm]) \circlearrowright \mathfrak{e}_8 \oplus \mathbb{C}$ R-matrix. End of the line?

Epilogue: what about $d \geq 4$ and/or q *not* to 0?

To each directed graph, up to "mutation class", is associated a "cluster variety". Each *finite* mutation class contains exactly one Dynkin diagram (all types, not just ADE as more common). While (the cone over) every Grassmannian $Gr(k, n)$ is a cluster variety, the only ones of finite mutation type are $Gr(2, d+4)$ of type A_d, and $Gr(3, 5 \leq n \leq 8)$ of types A_2, D_4, E_6, E_8.

The $Gr(3, *)$ family is telling us the internal labels of puzzles, whereas the $Gr(2, *)$ seems to correspond to the boundary labels ($0, 1, \ldots, d$ indexing a basis of $\mathbb{C}^{d+1} \circlearrowleft A_d$).

What's the connection? Is there a story for $Gr(\geq 4, *)$?

Following Maulik-Okounkov's geometric construction R-matrices using quiver varieties, Zinn-Justin and I looked into their "stable basis" $\{[MO_\lambda]\}$, an analogue of Schubert classes living on the *cotangent bundle* $T^*Fl(n_1, \ldots, n_d;\ n)$.

Theorem [K-ZJ]. The right puzzles can compute the product in the basis $\{[MO_\lambda]/[\text{zero section}]\}$, for $d \leq 4$, where the q appears as the equivariant parameter for dilation of the cotangent fibers.

We can take $q \to 0$ in the puzzle-provided *answers* to $d = 4$, but not the in the *rule*. But something even worse is happening – the coefficients in the rule are not all (suitably) positive. Positivity was a happy accident for $d \leq 3$.