The cyclic Bruhat decomposition of flag manifolds

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Abstract

If we intersect the n cyclic translates of the Bruhat decomposition of the Grassmannian $Gr(k,n)$, we get the celebrated **positroid stratification** studied by Lusztig, Postnikov, Williams, Rietsch, Knutson-Lam-Speyer... It is also the stratification by projected Richardson varieties [KLS], and its most natural flag manifold version is just the Richardson varieties $\{X^\pi_\sigma\}$.

Nonetheless, I'll look at the intersection of the n cyclic translates of the Bruhat decomposition, and index the strata with "cyclic flag pipe dreams". Alas: unlike on $Gr(k,n)$, these strata can be empty, or of bad dimension.

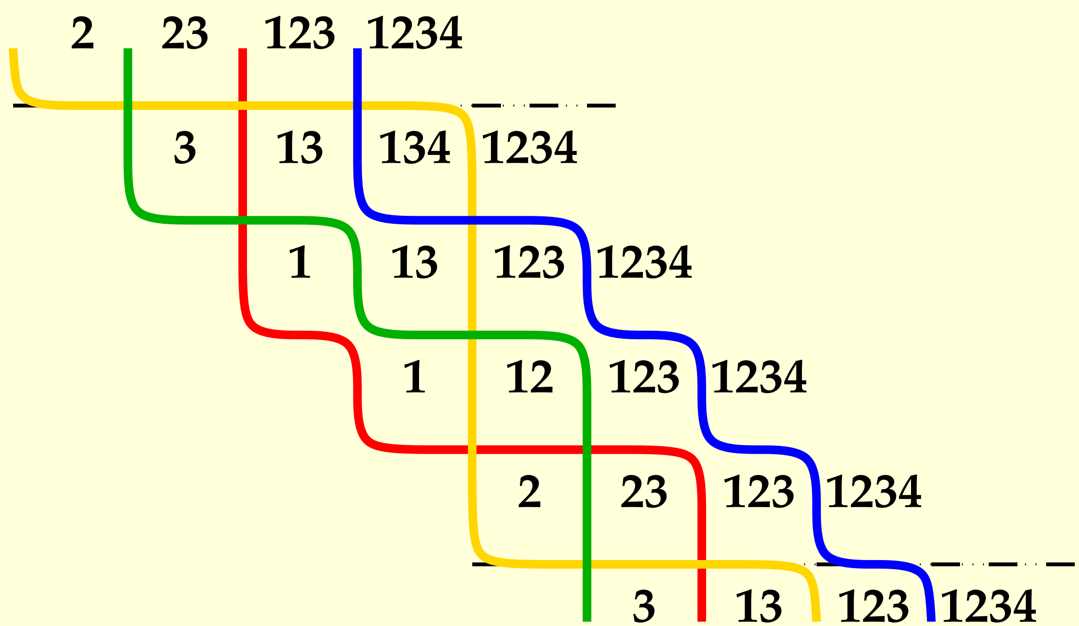
They are determined by $\left(\dim(F_k \cap \mathbb{C}^{[i,j]})\right)$ where $[i,j]$ varies over cyclic intervals; unfortunately their closures are *not* given by inequalities on those dimensions, and (relatedly) this decomposition is not a stratification. Taking $[i,j]$ only from (non-cyclic) intervals, as was useful for Schubert calculus [K], I *do* get $\neq\emptyset$ness, smoothness, irreducibility, and dimension.

Arrays of dimension jumps, with pipes.

Identify $F\ell(n) := B_- \backslash GL(n)$, using $F_k :=$ span of top k rows of $M \in GL(n)$.
For each $i \leq j \leq i+n$, consider columns $[i,j] \bmod n$ of M, and record

$$J_{ij} := \{k \in [n] \ : \ \mathrm{rank}(\text{top } k \text{ rows in cols } [i,j]) > \mathrm{rank}(\text{top } k-1 \text{ rows in cols } [i,j])\}.$$

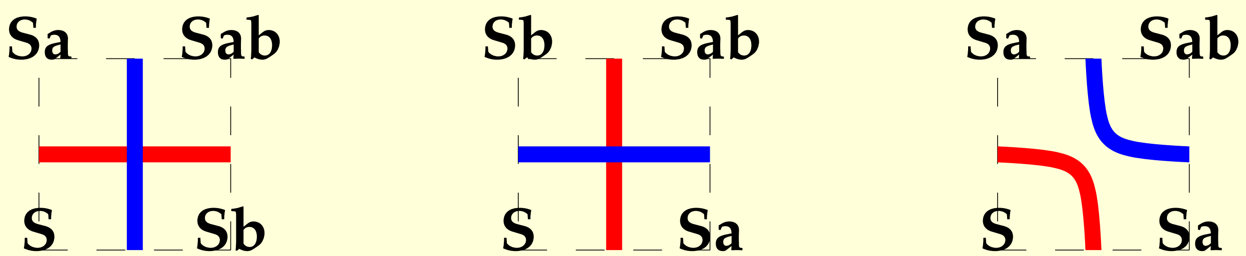
e.g. $$M = \begin{pmatrix} 0 & a & b & 0 \\ 0 & a & b & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \quad \mapsto$$



Theorem. J_{ij} increases by one element as you go North or East,
i.e. $\{(i,j) \ : J_{ij} \ni k\}$ is an order ideal above some k-**pipe**.

Tiles for CF pipe dreams (CF = "cyclic flag").

We can therefore think of these CF pipe dreams as being built out of tiles:



a<b

It's easy to show the elbows tile (the third type) doesn't occur with $a > b$.

The pipe labels on the vertical edges are weakly increasing in each row of $\mathcal{J}$:

$$a \leq a, \quad b \leq b, \quad a \leq b \quad \text{in the three tiles.}$$

For $\mathcal{J}$ an n-periodic assemblage of these tiles into a **CF pipe dream**, let $X(\mathcal{J})^\circ \subseteq F\ell(n)$ denote the corresponding locally closed subset of $F\ell(n)$.

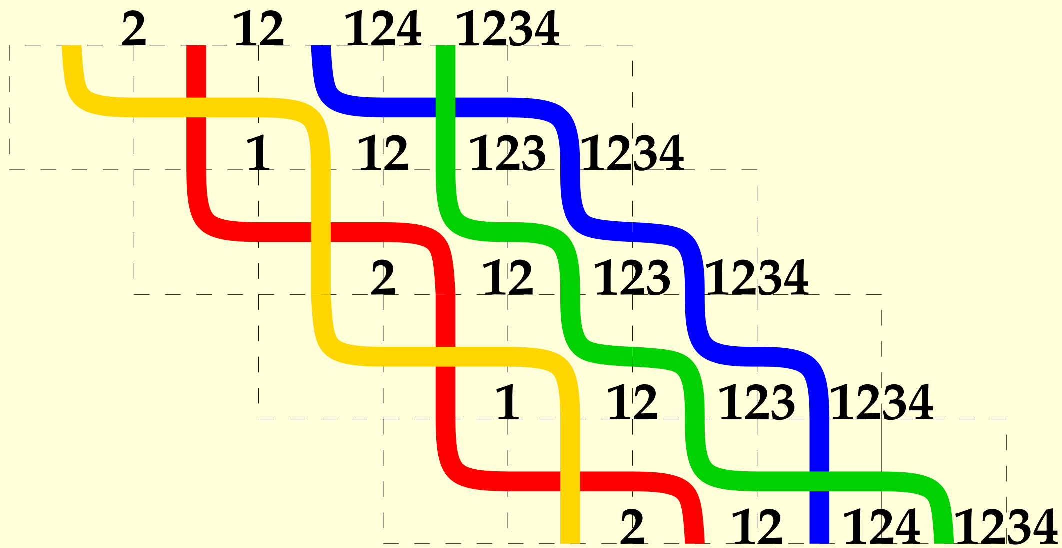
Based on the example of $Gr(k, n)$, I was moved to

> **Conjecture.** $X(\mathcal{J})^\circ$ is smooth and irreducible, with codimension given by the number of **horizontal tiles**, of the left type.
> (Which equals the number of vertical tiles, by the Jordan curve theorem.)

But this turns out to be false, much like most conjectures about matroid strata!

Counterexamples: an empty stratum, and stratification failure.

$$\text{Which } \left\{ \begin{bmatrix} a & b & c & d \\ e & * & f & * \\ g & * & h & * \\ * & * & * & * \end{bmatrix} \right\} \to$$



?

The 2, 1, 2, 1 down the $i = j$ diagonal tell us $a = c = 0$, $b, d \neq 0$.

The 124 at $[1, 3]$ says that $\det \begin{bmatrix} a & b & c \\ e & * & f \\ g & * & h \end{bmatrix} = \det \begin{bmatrix} 0 & b & 0 \\ e & * & f \\ g & * & h \end{bmatrix} = b(eh - fg) = 0.$

But the 123 at $[3, 5]$ says that $\det \begin{bmatrix} c & d & a \\ f & * & e \\ h & * & g \end{bmatrix} = d(fg - eh) \neq 0$. So there are none.

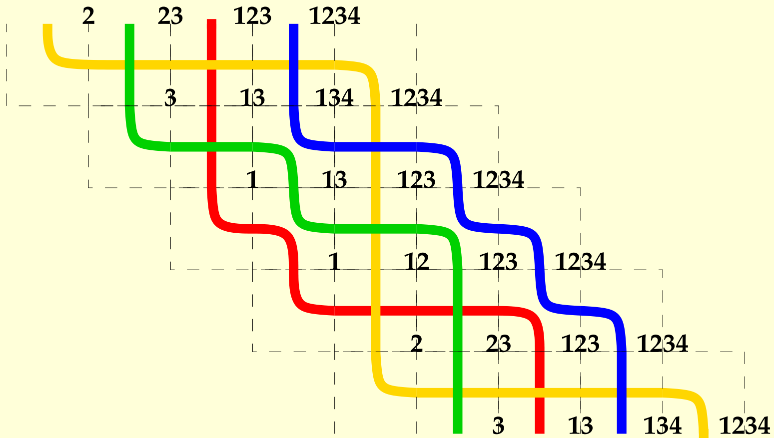
The same phenomenon, $b \neq 0 \implies eh - fg = 0$, leads to a stratum whose closure is not a union of strata.

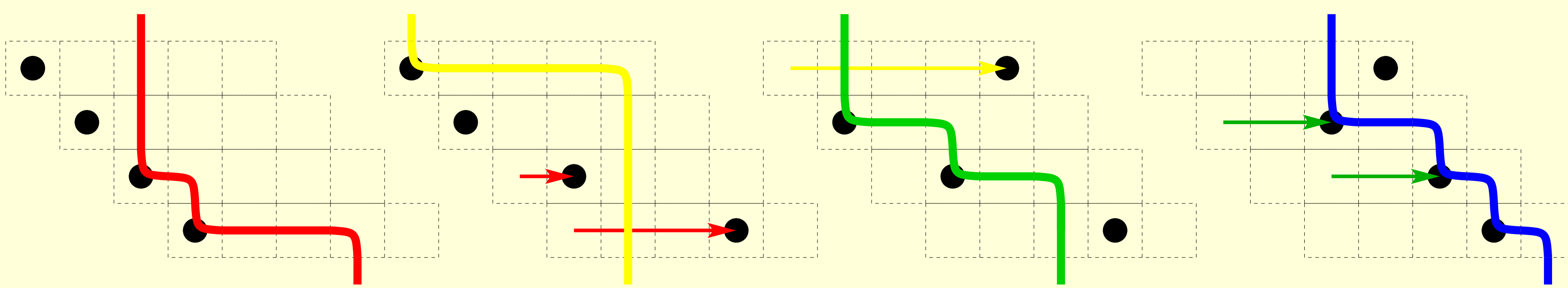
Moral: *When an inequality helps prove an equality, watch out!*

A flag of positroids.

The data defining $\mathcal{J}$ tells us which positroid stratum each k-plane F_k is in, i.e. we get an n-tuple of bounded affine permutations.

In this interpretation, the k-pipe says which dots move when going from the k-ball affine permutation to the $(k+1)$-ball affine permutation.





Question. For which $\mathcal{J}$ is $\overline{X(\mathcal{J})}$ determined as a set by intersecting the flag manifold with the preimages of those positroid varieties?

Partial flag manifolds, and the loop amplituhedron.

The same recipes work on $F\ell(n_1 < n_2 < \ldots < n_m = n)$, except now we have $n_i - n_{i-1}$ many n_i-pipes, and they mustn't cross each other.

In the case $Gr(k, n)$, in row $i \in \mathbb{Z}$ the pipe labels on the n vertical edges go $k, k, \ldots, k, n, n, \ldots, n$. If we let $\pi(i) := i + \#k$s in that row, we get the corresponding bounded affine permutation defining the positroid stratum.

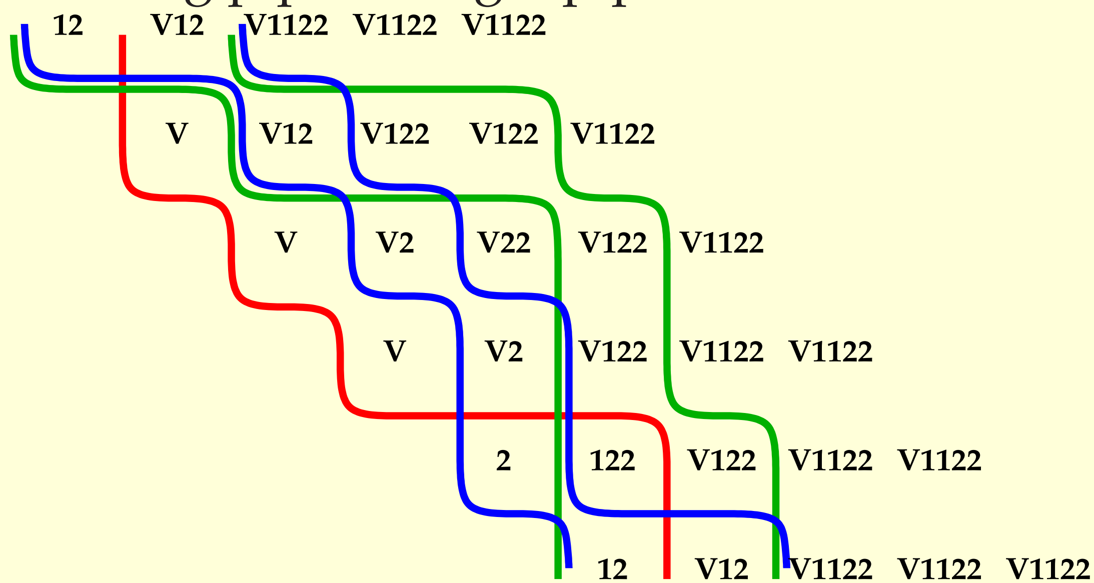
The **ℓ-loop amplituhedron** $\mathcal{A}_{k,\ell}$ is the $Gr(2,4)^\ell$-bundle over $Gr(k, k+4)$.

So $\mathcal{A}_{k,1} \cong F\ell(k, k+2, k+4)$, and $\mathcal{A}_{k,\ell} \hookrightarrow F\ell(k, k+2, k+4)^\ell$, from which it inherits a cyclic Bruhat decomposition. Now there are k pipes labeled V, and 2 pipes labeled each of $1, \ldots, \ell$, *that can lie along one another* (but not along V-pipes). If we don't bother drawing the remaining pipes, we get pipe dreams like this:

$$M_V = \begin{bmatrix} 0 & * & 1 & 2 & 0 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} * & * & 1 & 2 & 0 \\ * & * & 1 & 2 & 0 \end{bmatrix} \qquad \mapsto$$

$$L_2 = \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & * \end{bmatrix}$$



Interval rank flag strata and IF pipe dreams.

If we only study intervals $[i,j] \subseteq [n]$ of columns, rather than *cyclic* intervals, we get a coarser decomposition into **IF strata**, indexed by triangular (rather than periodic) **IF pipe dreams**. It still is finer than the Richardson stratification.

Associated to an IF pipe dream $\mathcal{J}$ are two permutations π and σ, from the lists of pipes crossed across the North side and then down the East side.

Theorems. Let $\mathcal{J}$ be an IF pipe dream, and π and σ as above.

- $X(\mathcal{J})_\circ$ is nonempty, smooth, and irreducible.
- $X(\mathcal{J})_\circ \subseteq X_{\pi^{-1}}^{\sigma^{-1}}$
- $\operatorname{codim}\left(X(\mathcal{J})_\circ \subseteq X^{\sigma^{-1}}\right) = $ #vertical tiles.

  $\operatorname{codim}\left(X(\mathcal{J})_\circ \subseteq X_{\pi^{-1}}\right) = $ #horizontal tiles (equivalent to the previous).

Not everything is great: the same counterexample still works to show that this coarser decomposition is not a stratification.

David Speyer and I are trying to relate this decomposition to Deodhar's.

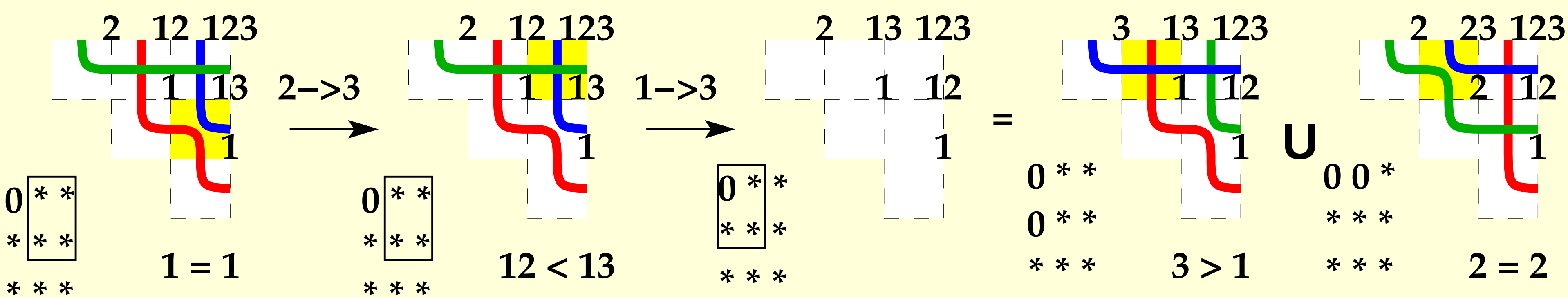
On the Grassmannian, this was the stratification I used in `arXiv:1408.1261` to extend Vakil's "geometric Littlewood-Richardson rule" to equivariant K-theory. On there, though, it was *coarser* than the projected Richardson stratification.

Towards a geometric L-R rule for IF pipe dreams.

I defined the **geometric shift** Ш_α of $X \subseteq P\backslash G$ as $\lim_{t\to\infty} \exp(te_\alpha) \cdot X$, connecting a construction from Vakil with Erdős-Ko-Rado combinatorial shifting.
Here, $\exp(te_\alpha)\cdot$ means adding t times column i to column j, and taking the limit; the rank conditions on column j thereby move backwards to column i.

Vakil gave a list of shifts to apply to (initially Richardson, eventually Schubert) varieties in $Gr(k, n)$. His list rasters the rows of the pipe dream bottom to top, and right to left within rows; we indicate his $\{(i, j)\}$ below at yellow tiles.

If the corners of a yellow tile have NW $<$ SE, the shift switches those sets (and otherwise does nothing). The resulting array of subsets may be combinatorially illegal, reflecting the geometry that $\text{Ш}_{i\to j}\, \overline{X(\mathcal{J})}$ has become reducible.



This is the calculation $[X_{213}][X^{312}] = [X_{231}] + [X_{312}]$ (don't forget the inverting!).
The main holdup: **what does $\text{Ш}_{i\to j}$ do to the equations-from-inequations?**