Modern developments in Schubert calculus

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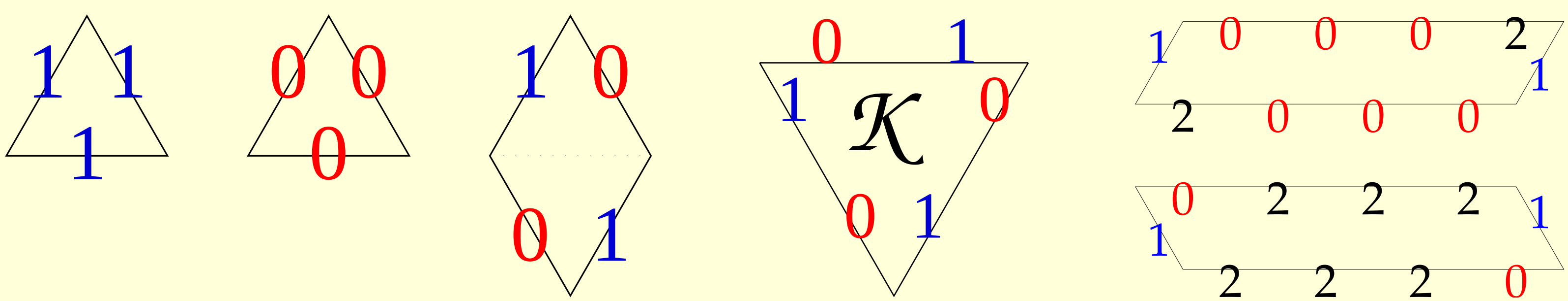
Abstract

Schubert calculus begins with the study of incidence conditions on k-planes. I'll recall its utility in eigenvalue inequalities, and describe many natural generalizations, most of which are unsolved.

I'll run down the progress that has been made in the 21st century, and then give combinatorial rules for many of these problems, in terms of counting "puzzles" made from various puzzle pieces.



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Morse theory on the space of projections.

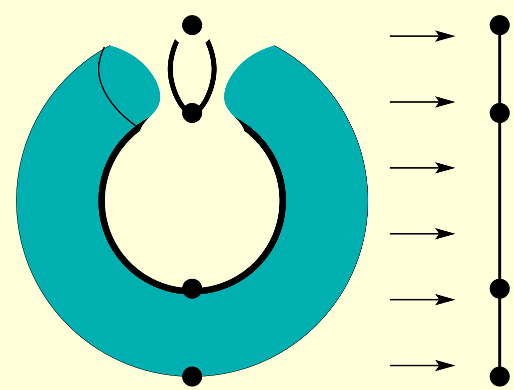
$$\text{Let } Gr_k(\mathbb{C}^n) = \{k\text{-planes in } \mathbb{C}^n\}$$

$$\cong \text{Hermitian matrices unitarily equivalent to } \begin{bmatrix} 1_{1_1} & & \\ & \ddots & \\ & & 0_{0_0} \end{bmatrix}$$

be the **Grassmannian**, bearing the functions $\rho_H : Gr_k(\mathbb{C}^n) \to \mathbb{R}$, $\pi \mapsto \mathrm{Tr}(\pi H)$.

Question 1. What does the Morse decomposition using ρ_H look like, for $H = \mathrm{diag}(n, n-1, n-2, \ldots, 1)$?

(Here's the standard picture of the Morse decomposition of a torus, where the function used is the height function. The four critical points from top to bottom have a point, $\mathbb{R}^1$, $\mathbb{R}^1$, and $\mathbb{R}^2$ gradient-flowing down into them, respectively.)



The Bruhat cells X_λ°, and the Schubert varieties X_λ.

Question 1. What does the Morse decomposition of $Gr_k(\mathbb{C}^n)$ using ρ_H look like, for $H = \mathrm{diag}(n, n-1, n-2, \ldots, 1)$?

Answer. There are $\binom{n}{k}$ critical points,
the diagonal matrices $\mathrm{diag}(\lambda)$ where λ has k 1s and $n-k$ 0s,
and for each λ the stratum $X_\lambda^\circ(H)$ flowing down into λ is a cell:

$$X_\lambda^\circ(H) \quad \cong \quad \mathbb{C}^{\#\{0\text{s before 1s in } \lambda\}} \quad \cong \quad \mathbb{C}^{k(n-k) \,-\, \ell(\lambda)}$$

Which stratum is a projection π in? If $V = \mathit{image}(\pi)$, take

$$V \quad \mapsto \quad (V \cap \mathbb{C}^0, V \cap \mathbb{C}^1, V \cap \mathbb{C}^2, \ldots, V \cap \mathbb{C}^n) \quad \stackrel{\text{jumps}}{\mapsto} \quad \binom{[n]}{k}.$$

So H doesn't matter, just its associated flag of partial sums of eigenspaces does.

Corollary [Hersch-Zwahlen, 1962]. If $\pi \in X_\lambda(H_e) = \overline{X_\lambda^\circ(H_e)}$,
where $e_1 \geq e_2 \geq \ldots \geq e_n$ are H_e's eigenvalues, then $\rho_H(\pi) \geq e_{\lambda_1} + \ldots + e_{\lambda_k}$,
with equality iff $\mathit{image}(\pi)$ is a sum of those eigenlines.

Inequalities on spectra of sums of Hermitian matrices.

Question ~~2~~ 3. For which $\lambda, \mu, \nu \in \binom{[n]}{k}$ is $X_\lambda(H_e) \cap X_\mu(H_f) \cap X_\nu(H_g) \neq \emptyset$ for all H_e, H_f, H_g?

If π is in that intersection, and $-H_g = H_e + H_f$, then

$$\begin{aligned} 0 &= \operatorname{Tr}(\pi(H_e + H_f + H_g)) = \operatorname{Tr}(\pi H_e) + \operatorname{Tr}(\pi H_f) + \operatorname{Tr}(\pi H_g) \\ &= \rho_{H_e}(\pi) + \rho_{H_f}(\pi) + \rho_{H_g}(\pi) \\ &\geq e_{\lambda_1} + \ldots + e_{\lambda_k} + f_{\mu_1} + \ldots + f_{\mu_k} + g_{\nu_1} + \ldots + g_{\nu_k} \end{aligned}$$

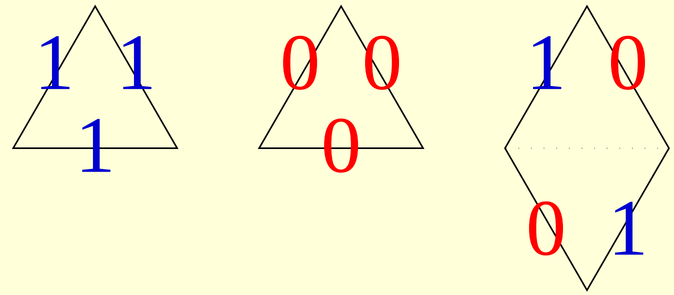
with equality iff H_e, H_f, H_g all commute with π, so, can be simultaneously block diagonalized [Totaro 1994, Helmke-Rosenthal 1995, Klyachko 1998].

Moreover, if the intersection is positive-dimensional one can tighten up λ, μ, ν (move 1s back, 0s forward) to get a stronger inequality. The inequalities constructed this way give the *only* conditions on $\vec{g}$ [Klyachko 1998]; one only needs the ones for which the intersection is a single point [Belkale 1999]; and one does indeed need all of those [Knutson-Tao-Woodward 2004].

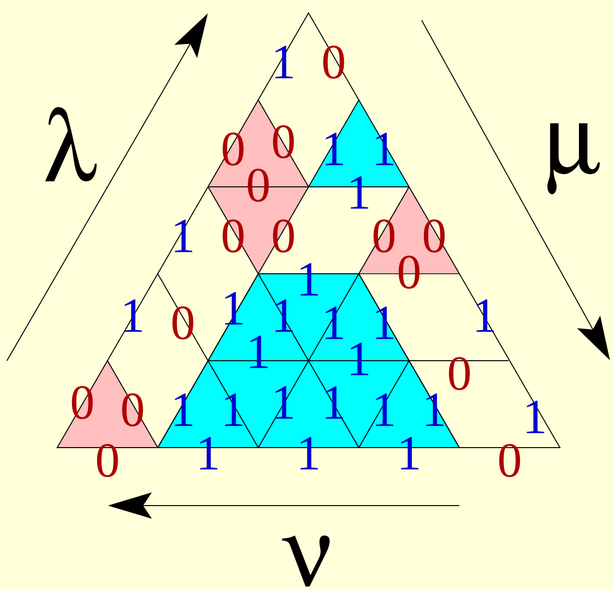
So when *is* the intersection a nonempty set of points (where each point corresponds to a k-plane)? We expect points when $\ell(\lambda) + \ell(\mu) + \ell(\nu) = k(n-k)$.

Puzzles.

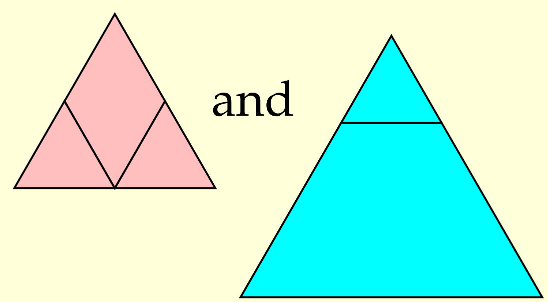
Consider the following three **puzzle pieces**, with edges labeled by 0s and 1s. They may be rotated but not reflected (unless 0 and 1 are exchanged).



Theorem [K-Tao-Woodward 2004]. If $\ell(\lambda) + \ell(\mu) + \ell(\lambda) = k(n-k)$, and $X_\lambda(H_e) \cap X_\mu(H_f) \cap X_\nu(H_g)$ has dimension 0 (as expected), then its cardinality (counted with multiplicities) is the number of puzzles with λ, μ, ν clockwise around the outside. Otherwise there are no such puzzles.



Fun fact:



appear inside!

The cohomology ring of the Grassmannian.

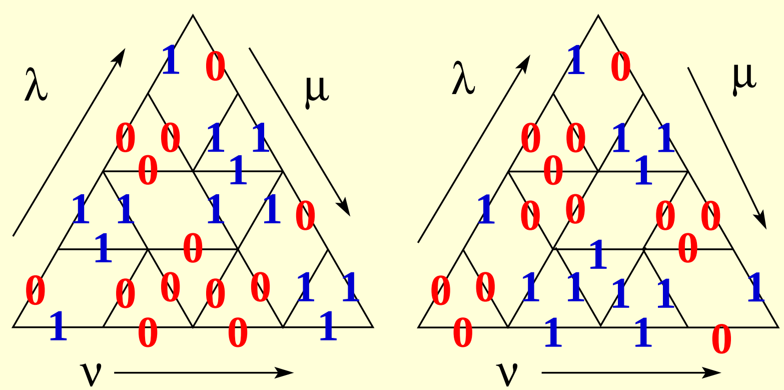
<u>Question ~~3~~ 2.</u> What does $X_\lambda(H) \cap X_\mu(H')$ look like?

Example. Let $\lambda = \mu = 0101$, so $X_\lambda(H)$ is the space of $\mathbb{CP}^1$s touching the projective line L made from the top two eigenlines of H. Likewise define L' from H'.

- If $L = L'$, then $X_\lambda(H) = X_\mu(H')$ is 3-dimensional.
- If $L \cap L' = \emptyset$, then $X_\lambda(H) \cap X_\mu(H') \cong L \times L'$, so 2-dimensional.
- If $L \cap L' = p$, then

$$X_{0101}(H) \cap X_{0101}(H') = X_{1001}(H'') \cup X_{0110}(H'''), \quad \text{union along } X_{1010}(H'''').$$

Theorem. If the codimensions add, the homology class of the result is well-defined. The Poincaré dual classes $[X_\lambda]$ give a basis of cohomology, and puzzles compute the structure constants in that basis: $[X_\lambda][X_\mu] = \sum(\#\text{puzzles with } \nu \text{ on bottom})[X_\nu]$.



ν left to right this time!

Even before actually computing these structure constants, one can prove using algebraic geometry that they must be positive. (This uses the homogeneity of the Grassmannian – it's not true for the blowup of $\mathbb{CP}^2$, for example.)

Schubert calculus and 2^5 generalizations.

We've turned the question into computing the cohomology ring of the Grassmannian, in the Schubert basis. The Schubert varieties $\{X_\lambda\}$ give bases of other cohomology theories, and for other homogeneous spaces, suggesting many generalizations:

- **K**: K-theory, where $[X_{0101}]^2 = [X_{1001}] + [X_{0110}] - [X_{1010}]$.
- **T**: Torus-equivariant cohomology, where $[X_{10}]^2 = (y_2 - y_1)[X_{10}]$.
  This has coefficients in $\mathbb{Z}[y_1, \ldots, y_n]$.
- **Q**: Quantum cohomology, where $[X_{10}]^2 = q[X_{01}]$, with coefficients in $\mathbb{Z}[q]$.
- **F**: Larger flag manifolds $GL_n(\mathbb{C})/P$, isomorphic to spaces of Hermitian matrices with more different eigenvalues.
- **G**: (Co)minuscule flag manifolds for other groups, like the Lagrangian Grassmannian.

For each and every combination, we can ask to prove **A**bstract positivity or a **C**omputational rule. (As the K-example shows, even defining what positivity to expect can be subtle.)

Note that *non-manifestly-positive* computational rules are known for every one of these problems, so checking conjectures in small examples is easy.

Recent progress on Abstract and Computational positivity in Schubert calculus.

$H^*(Gr_k(\mathbb{C}^n))$ C [Littlewood-Richardson 1934, Thomas 1974, Schützenberger 1977]

K-theory
C [Buch 2002]

T-equivariant
A see TFG
C [K-Tao 2003]

Quantum
A [Mihalcea 2006]
C reduces† to **F**C [Buch-Kresch-Tamvakis 2003]

Flag
A see FG
C [Coşkun ?]

Groups
A see FG
C [Thomas-Yong 2009] (extending [Pragacz 1991, Worley 1984]

QG
C reduces† to **FG**C [Chaput-Manivel-Perrin 2008]

FG
A [Kleiman 1973]

KFG
A [Brion 2002]

TFG
A [Graham 2001]

KTQG
C reduces to **KTFG**C [Buch-Mihalcea 2011]

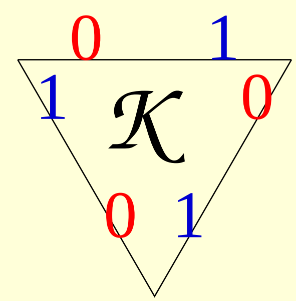
KTFG
A [Anderson-Griffeth-Miller 2011]

† The [BKT] result says that quantum cohomology of Grassmannians can be positively computed inside ordinary cohomology of 2-step flag manifolds.

Work by [Belkale-Kumar 2006] and [Ressayre 2010] shows that for eigenvalue inequalities, what is really relevant is a subproblem of **FG** called the **Belkale-Kumar product**, in which many coefficients of the actual product are set to 0.

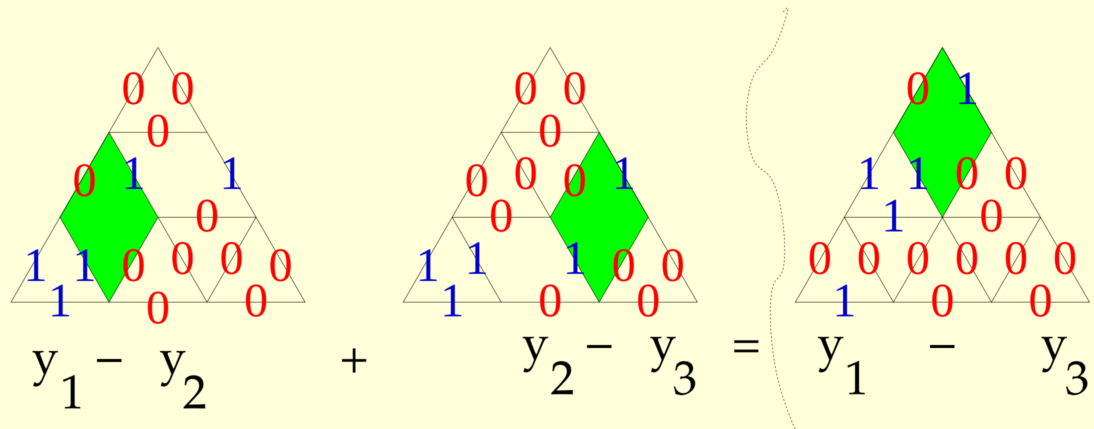
Puzzle rules for many of these.

We already gave the three puzzle pieces used to compute $H^*(Gr_k(\mathbb{C}^n))$.



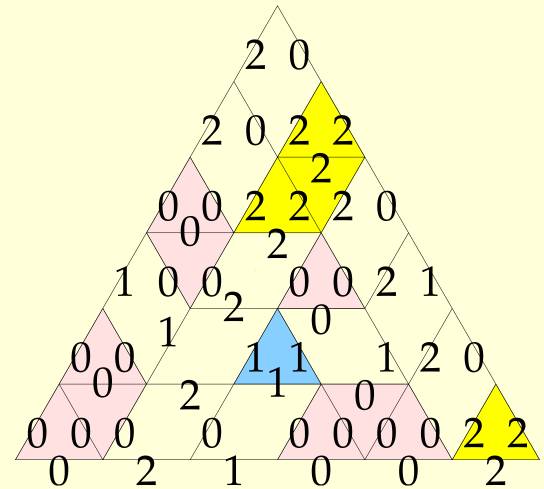
For K-theory, one new piece is needed. It increases $\ell(\nu) - (\ell(\lambda) + \ell(\mu))$ by 1, and contributes a factor of -1. It may not be rotated.

For H_T^*, a backwards vertical rhombus called the "equivariant piece" is needed. It *decreases* $\ell(\nu) - (\ell(\lambda) + \ell(\mu))$ by 1, and contributes a factor of $y_i - y_j$ that depends on its location.

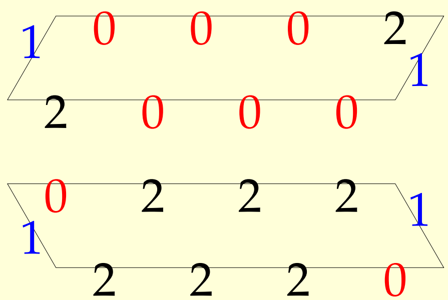


Equivariant K-theory is unsolved, but I'll be talking about a closely related puzzle-solvable problem tomorrow.

Puzzle rules for many of these,][.



The Belkale-Kumar product on a d-step flag manifold needs puzzle pieces with edge labels $0, \ldots, d$, that otherwise look like the three used in $H^*(Gr_k(\mathbb{C}^n))$. Interestingly, one can correspond such a puzzle to a tuple of $\binom{d+1}{2}$ ordinary puzzles. [K-Purbhoo]



There is no (living) conjecture for puzzles to compute H^* of flags in general, but there *is* one for 2-step flag manifolds, which I gave in 1999 and expect to be proven soon. In addition to the $3 + \binom{3}{2}$ pieces above, there are two extensible kinds shown at right.

This rule was checked by Buch, Kresch, and Tamvakis up to $n = 16$, which is especially impressive in that the nonpositive rules are only calculable by computer up to about $n = 9$ in practice.

Recall that 2-step flag manifold Schubert calculus includes quantum Schubert calculus of Grassmannians, itself equivalent to tensor products of $U_q(\mathfrak{gl}_n)$-representations at q a root of unity, or fusion products of affine GL_n reps.