The totally nonnegative Grassmannian, juggling patterns, and the affine flag manifold

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Abstract

The common refinement of all the Bruhat decompositions of the Grassmannian is generally regarded as pathological; while the strata can be loosely indexed by matroids, they can have arbitrary singularity type (or be empty). Inspired by considerations of total nonnegativity, A. Postnikov discovered that the common refinement of *only the cyclic shifts* of the Bruhat decomposition is much better behaved, and gave many ways to index the strata.

I'll explain a new indexing of the strata, by "bounded juggling patterns". This suggests a connection to the affine Weyl group (of unbounded juggling patterns), and indeed, I'll show how to trace Postnikov's stratification to the (finite-codimensional) Bruhat decomposition on the affine flag manifold. Then I'll discuss the geometry of the closed strata.

The (awful) matroid decomposition of the complex Grassmannian.

Consider $k \times n$ complex matrices M, $k \leq n$, and for any k-tuple $\lambda \subseteq \{1, \ldots, n\}$ let $p_\lambda(M)$ denote the determinant of the maximal minor that uses the columns λ.

Let $\mathcal{C}(M) = \{\lambda : p_\lambda(M) \neq 0\}$. Then $\mathcal{C}(M)$ is automatically a **matroid**, meaning that any $\pi \cdot \mathcal{C}(M), \pi \in S_n$ has a unique Bruhat minimum (considering $C \subseteq S_n/(S_k \times S_{n-k})$). Let $\{\text{row-span}(M) \in Gr_k(\mathbb{C}^n) : \mathcal{C}(M) = C\}$ be the **matroid stratum** of the matroid C.

Q. What is the geometry of the matroid stratum for a fixed matroid C?
How about its closure?

A. *Any* singularity over $\mathbb{Z}$ can arise in the open stratum [N. Mnëv, 1988].

Q. For which matroids C is the matroid stratum even *nonempty?*

A. "The missing axiom of matroid theory is lost forever" [P. Vámos, 1978].

Equivalently, let $\mathcal{B}$ be the Bruhat decomposition, and consider $\bigcap_{\pi \in S_n}(\pi \cdot \mathcal{B})$, the **Gel′fand-Serganova decomposition**. What if we look at $\bigcap_{\pi \in P}(\pi \cdot \mathcal{B})$, for $P \subsetneq S_n$?

- $P = \{1\}$: the Bruhat cells are indexed by partitions, and their closures (Schubert varieties) are nice; e.g. irreducible, normal, Cohen-Macaulay (and nonempty!).

- $P = \{1, w_0\}$: the strata are indexed by pairs (λ, μ) of partitions, nonempty iff $\lambda \subseteq \mu$, and their closures, **Richardson varieties**, are similarly nice.

The (beautiful) matroid decomposition of the totally nonnegative real Grassmannian.

In [math.CO/0609764], A. Postnikov suggested restricting the question:
Consider only real matrices M, and assume $p_\lambda(M) \geq 0\ \forall\lambda$, a **total nonnegativity** condition. (If the left k columns of M are an identity matrix, this reduces to saying that *every* minor in the remaining $k \times (n-k)$ matrix is nonnegative.)
Then which C can arise?

Already in the $k = 2$ case one sees that the possibilities are much more restricted, because $p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}$.
So e.g. if $p_{13} = 0$, then $p_{12}p_{34} = p_{14}p_{23} = 0$, which isn't true for general M.

Define the **cyclic Bruhat decomposition** as $\bigcap_{\pi \in P}(\pi \cdot \mathcal{B})$, where $P = \{\text{powers of the cyclic rotation } \chi = (12\ldots n)\}$. It is a coarsening of the matroid decomposition, and (nonobviously) a refinement of the Richardson one.

Theorems (presented ahistorically).

1. [KLS, using work of Marsh-Rietsch] Each stratum in the cyclic Bruhat decomposition is irreducible, hence contains a unique dense matroid stratum.

2. [Postnikov] A matroid C arises as $\mathcal{C}(\cdot)$ of a totally nonnegative real matrix iff its matroid stratum is dense in a cyclic Bruhat stratum. In this case, the **totally nonnegative matroid stratum** is homeomorphic to an open ball.
Call such matroids **positroids**. Postnikov gives many ways to index them.

Bounded juggling patterns.

Call a k-element subset $S \subseteq \mathbb{N}$ a **juggling state**, interpreted as the future times that the k balls being juggled are scheduled to next be caught. ($\mathbb{N} \ni 0$ = the present.)

If $S \ni 0$ but $S \not\ni m$, we **throw an** m by creating the new juggling state $S' = \{s - 1 : s \in S \setminus 0 \cup m\}$. Note that we can reconstruct m from (S, S').
If $S \not\ni 0$ (we are not catching a ball right now), then we allow the same recipe only for $m = 0$, called **throwing a** 0 or an **empty hand**.

Lemma. Let M be a $k \times n$ matrix of rank k, and greedily construct a basis of the column space starting from the left; call the columns used $S \subseteq \{0, \ldots, n-1\}$.
Rotate the leftmost column to the right end, and let the new greedy basis use columns S'. (Multiplying that column by $(-1)^{k-1}$ preserves total nonnegativity!)
Then there is a (unique) throw taking the juggling state S to the juggling state S'.

Define a **(bounded) juggling pattern** as a cyclic list $(\ldots, S_0, S_1, \ldots, S_{n-1}, S_n = S_0, \ldots)$ of juggling states such that each S_i can be reached from S_{i-1} by a throw (resp. a throw of height $\leq n$). Using the lemma, we can associate one to any matrix.
Theorems (juggling interpretation in KLS).
1. [Postnikov] For *any* matroid C, the periodic sequence (λ_i = Bruhat minimum of $\chi^{-i} \cdot C$) is a bounded juggling pattern. (The lemma shows this for $C = \mathcal{C}(M)$.)
2. [P] The map from positroids to their bounded juggling patterns is bijective.
3. [Oh] A positroid C can be recovered from its (λ_i) as $C = \bigcap_i \left(\chi^i \cdot \{\mu : \mu \geq \lambda_i\}\right)$.

Example: bounded juggling patterns for $\mathrm{Gr}_2(\mathbb{C}^4)$.

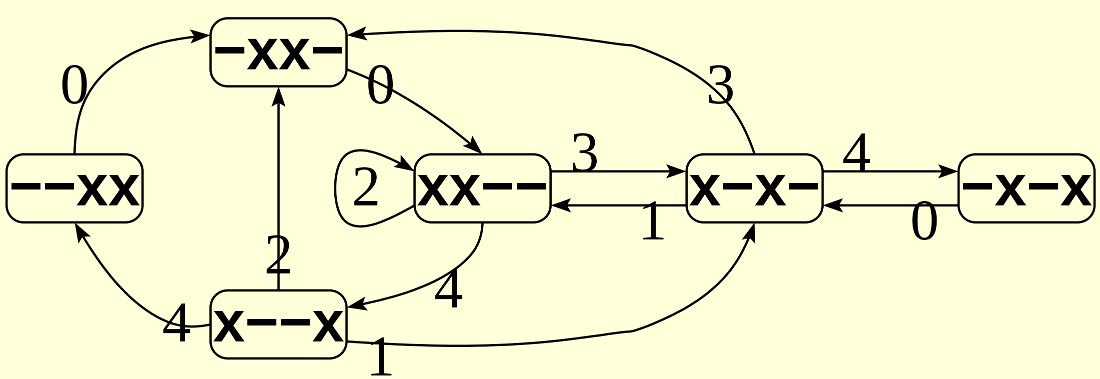
Mathematical juggling, in terms of cyclic lists of throws, dates from 1985. Jack Boyce and I independently invented juggling states in 1988, and denoted them as strings of **x** (catch) and **-** (empty hand), e.g. $\{0, 3, 4\} =$ **x--xx------...** or just **x--xx** .

Some juggling theorems.

1. The states in a pattern can be reconstructed from the (cyclic) list of throws.
2. The average of the throws is the number of balls, k.
3. [Buhler-Eisenbud-Graham-Wright 1994; much simpler proof by Thurston] The number of juggling patterns of length n with at most k balls is $(k+1)^n$.

It is much harder to count *bounded* juggling patterns, alas! [L. Williams 2005]

At left are all the states and throws for 2 balls, maximum throw 4. At right are the patterns (as lists of throws) indexing the positroid strata on $\mathrm{Gr}_2(\mathbb{C}^4)$, up to cyclic rotation.



dim	patterns
4	2222
3	3122
2	4112, 3302, 3131
1	4202, 4130, 4013
0	4400, 4040

Juggling patterns and affine permutations.

The Weyl group of affine $GL(n)$ can be identified with periodic permutations of $\mathbb{Z}$:

$$\widehat{W} := \{f \in Sym(\mathbb{Z}) : \forall i \in \mathbb{Z}, f(i+n) = f(i) + n\}.$$

Call $f \in \widehat{W}$ a **juggling permutation** if $\forall i$, $f(i) \geq i$.

Theorem. If f is a juggling permutation, the periodic sequence $(f(i) - i)$ forms the throws of a juggling pattern. Every juggling pattern arises uniquely this way.

The condition $f(i) \geq i$ means balls land *after* they are thrown.
Feynman taught us that balls that travel backwards in time look like antimatter.

The Bruhat order on $\widehat{W}$ has components indexed by $\pi_1(GL(n)) \cong \mathbb{Z}$;
the kth one consists of $\{f \in \widehat{W} : avg(f(i) - i) = k\}$.

Theorem [KLS]. The inclusion of bounded juggling patterns of length n with k balls into the kth component of $\widehat{W}$ takes Postnikov's **cyclic Bruhat order** (the cyclic Bruhat strata ordered by closure) to an order ideal in the affine Bruhat order.

This simplifies descriptions of the cyclic Bruhat order given by Postnikov and by Williams.

Positroid varieties and their properties, I.

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a bounded juggling pattern of period n with k balls.
For each (i, j) with $i \leq j \leq i + n$, let $r_{ij} = \#\{k \in \mathbb{Z} \,:\, i \leq k \leq f(k) \leq j\}$.

Then the open cyclic Bruhat stratum is defined by

$$X_f^\circ := \left\{ V \in Gr_k(\mathbb{C}^n) : \forall i \leq j \leq i + n, \dim(V \cap \mathbb{C}^{[i,j]}) = r_{ij} \right\}$$

where $\mathbb{C}^{[i,j]}$ denotes the coordinate subspace using the coordinates in the cyclic interval $i, i+1, i+2, \ldots, j \bmod n$.

Define the **positroid variety** X_f by the same intersection, but with the closed conditions $\dim(V \cap \mathbb{C}^{[i,j]}) \geq r_{ij}$.

Theorem. [KLS]
1. The positroid variety X_f is the closure of $\overline{X_f^\circ}$, and is irreducible.
2. There is a Frobenius splitting of $Gr_k(\mathbb{C}^n)$ with respect to which all the $\{X_f\}$ are compatibly split.
3. Therefore, the equations above (vanishing of Plücker coordinates) define each X_f as a scheme. (Hodge-Pedoe proved this for Schubert varieties, hence for their cyclic shifts; hence for intersections thereof by the Frobenius splitting.)
4. Any irreducible variety of k-planes defined by intersection conditions with cyclic-interval coordinate subspaces is a positroid variety.

Examples of positroid varieties.

To repeat: any irreducible variety in $Gr_k(\mathbb{C}^n)$ defined by intersection conditions with cyclic-interval coordinate subspaces is a positroid variety.

Schubert varieties. Here the cyclic intervals are initial intervals $[0, j]$.

Richardson varieties. Here the cyclic intervals are either initial intervals $[0, j]$ or terminal intervals $[i, n-1]$.

Graph Schubert varieties. For $\pi \in S_n$, the **matrix Schubert variety** $\overline{X}_\pi$ is the closure in M_n of $B_-\pi B_+$, where $B_\pm$ are the usual Borel subgroups of $GL(n)$. The inclusion

$$\mathrm{graph} : M_n \hookrightarrow Gr_n(\mathbb{C}^{2n}), \quad M \mapsto \left\{ (\vec{v}, M\vec{v}) \in \mathbb{C}^{2n} : \vec{v} \in \mathbb{C}^n \right\}$$

of the big cell lets us define the **graph Schubert variety** $\overline{\overline{X}}_\pi := \overline{\mathrm{graph}(\overline{X}_\pi)}$.

Theorem [K] $\overline{\overline{X}}_\pi$ is a permuted Schubert variety iff π is a vexillary permutation.

Theorem [KLS] For $\pi \in Sym(\{0, \ldots, n-1\})$, the graph Schubert variety $\overline{\overline{X}}_\pi$ is the positroid variety for the n-ball juggling pattern of length $2n$ with throws $(n + \pi(0), n + \pi(1), \ldots, n + \pi(n-1), n, n, n, \ldots, n)$.

The relevant cyclic intervals $[i, j]$ are certain honest intervals with $i < n \leq j$.

More generally, Fomin-Zelevinsky's **double Bruhat cell** $B_-\pi B_+ \cap B_+\rho B_-$ is an open set in the positroid variety for $(n+\pi(0), \ldots, n+\pi(n-1), n+\rho(0), \ldots, n+\rho(n-1))$.

Juggling antimatter on the affine Grassmannian.

To see *why* the cyclic Bruhat order embeds in affine Bruhat order, we evidently need to allow for antimatter. This forces consideration of a Dirac sea of antiballs.

Let $\mathcal{P}ast \leq \mathbb{C}[[z^{-1}]][z]$ denote $z^{-1}\mathbb{C}[[z^{-1}]]$, and consider the space of "lattices" L

$$\mathcal{A}Gr_k := \left\{ L \leq \mathbb{C}[[z^{-1}]][z] \;\middle|\; L \geq z^{-n}L,\ \dim L/(L \cap \mathcal{P}ast) - \dim \mathcal{P}ast/(L \cap \mathcal{P}ast) = k \right\},$$

the kth component of the affine Grassmannian. Both $\dim$s should be finite.

The Bruhat decomposition of $\mathcal{A}Gr_k$ is naturally indexed by **virtual juggling states** $S \subseteq \mathbb{Z}$, where $|S \cap \mathbb{N}|$ (electrons) minus $|\mathbb{Z}_{<0} \setminus S|$ (positrons) equals k (both $|\cdot|$ finite). Any honest juggling state $S \subseteq \{0, \ldots, n-1\}$ with $|S| = k$ balls gives a virtual juggling state $S \cup \mathbb{Z}_{<0}$.

For $L \in \mathcal{A}Gr_k$, automatically $\dim(L \cap \mathbb{C}[z]) \geq k$. Define the **open Morse-Bott stratum**

$$\mathcal{A}Gr_k^\circ := \{L \in \mathcal{A}Gr_k : \dim(L \cap \mathbb{C}[z]) = k, \dim(L \cap z^n\mathbb{C}[z]) = 0\}$$

and its **collapsing** $\mathcal{A}Gr_k^\circ \twoheadrightarrow Gr_k(\mathbb{C}^n)$, $L \mapsto (L \cap \mathbb{C}[z])/(L \cap z^n\mathbb{C}[z]) \leq \mathbb{C}[z]/z^n\mathbb{C}[z] \cong \mathbb{C}^n$. This open set is the union of the finite-codimensional Bruhat cells corresponding to the $\binom{n}{k}$ juggling states $S \subseteq \{0, \ldots, n-1\}$, $|S| = k$.

The cyclic Bruhat decomposition of $\mathrm{Gr}_k(\mathbb{C}^n)$ from the affine Bruhat decomposition of AFlag_k°.

The periodic lists of lattices

$$\mathrm{AFlag}_k := \left\{ (\dots, L_0, L_1, \dots, L_{n-1}, L_n = L_0, \dots) : L_i \in \mathrm{AGr}_k, L_i \geq z^{-1} L_{i-1} \right\} \hookrightarrow \prod_{i=0}^{n-1} \mathrm{AGr}_k$$

form the kth component of the affine flag manifold for $GL(n)$.
Its finite-codimensional Bruhat cells are indexed by $\{f \in \widehat{W} : avg(f(i) - i) = k\}$.

Let $\mathrm{AFlag}_k^\circ := \{(L_i) \in \mathrm{AFlag}_k : \forall i, L_i \in \mathrm{AGr}_k^\circ\}$.
Easy theorem: $\mathrm{AFlag}_k^\circ = \bigcup$ Bruhat cells associated to bounded juggling patterns.

Define the locally closed subset of $\prod_{i=0}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n)$

$$\mathrm{Jugg} := \{(V_0, V_1, \dots, V_n = V_0) : \forall i, V_i \in \mathrm{Gr}_k(\mathbb{C}^n); V_i \geq \mathrm{shift}(V_{i-1})\}$$

where shift is the principal nilpotent operator taking $\vec{e}_n \mapsto \vec{e}_{n-1} \mapsto \dots \mapsto \vec{e}_1 \mapsto \vec{0}$.

Theorem [KLS]. The Bruhat decompositions relate via Jugg:
1. The composite map $\mathrm{AFlag}_k^\circ \hookrightarrow \prod_{i=0}^{n-1} \mathrm{AGr}_k^\circ \twoheadrightarrow \prod_{i=0}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n)$ has image Jugg, so induces a stratification of it.
2. The map $\mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \prod_{i=0}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n)$, $V \mapsto (V, \chi \cdot V, \dots, \chi^{n-1} \cdot V)$ lands inside Jugg.
3. The cyclic Bruhat stratification on $\mathrm{Gr}_k(\mathbb{C}^n)$ is the pullback of the one on Jugg.

Positroid varieties and their properties, II.

Hodge-Pedoe constructed a degeneration of $Gr_k(\mathbb{C}^n)$ to a union of projective spaces, one for each maximal chain in the lattice of partitions $\{\lambda \subseteq k \times (n-k)\}$.

Given any subvariety $X \subseteq Gr_k(\mathbb{C}^n)$, we can follow it under this degeneration and ask whether it, too, becomes a union of $\mathbb{P}^m$s glued together along subspaces, like simplices in some simplicial complex. If so, call that the **Hodge complex** $\Delta(X)$ of X.

Theorems. (#1-2 historic, #3-4 technical lemmas, #5-6 payoff)
1. [Hodge-Pedoe] The Schubert variety X_λ has a Hodge complex, whose faces correspond to chains of partitions containing λ.
2. [Björner-Wachs] These Hodge complexes are homeomorphic to balls. Hence the degenerations are Cohen-Macaulay, hence each X_λ is too by semicontinuity.
(#1 and #2 hold for Richardson varieties too, with facets indexed by skew-tableaux.)
3. [Marsh-Rietsch]
Each X_f° is the isomorphic image of an open Richardson stratum in GL_n/B.
4. [KLS] Hence by irreducibility, each X_f is the birational image of a Richardson variety in GL_n/B. Using [Brion-Lakshmibai, prop. 1] we show this map is crepant.
5. [KLS] Each positroid variety has a Hodge complex, whose facets correspond to saturated chains in Bergeron-Sottile's k-Bruhat order. These complexes are homeomorphic to balls, so positroid varieties are Cohen-Macaulay.
6. [KLS] If $X_f \subsetneq X_g$ is a containment of positroid varieties, then $\Delta(X_f) \subseteq \partial\Delta(X_g)$.
As a corollary, each positroid variety X_g is normal.

Positroid varieties and their properties, III.

Given any subvariety $X \subseteq Gr_k(\mathbb{C}^n)$, we can look at the coefficients of $[X] = \sum_\lambda c_\lambda [X_\lambda]$ from expansion of its Chow (or homology) class into Schubert classes. They are automatically in $\mathbb{N}$. Under the natural map $\mathcal{Symm} \twoheadrightarrow A_*(Gr_k(\mathbb{C}^n))$ taking Schur functions to Schubert classes, we can ask what symmetric functions map to $[X]$.

Theorem [KLS]. The Chow class of a positroid variety X_f is represented by the affine Stanley symmetric function (introduced in [Lam 2006]) corresponding to the affine permutation f.

Corollary [conjectured by K around 2003]. The Chow class of a graph Schubert variety $\overline{\overline{X}}_\pi$, $\pi \in S_n$, is represented by the ordinary Stanley symmetric function of π.

Additional results:

1. We have a combinatorial description of $\Delta(X_f)$ in terms of "increasing skew-tableaux", generalizing the case that X_f is a Richardson variety in $Gr_k(\mathbb{C}^n)$.

2. We have a formula for the multiplicity of each T-fixed point on a positroid variety, as a sum of reciprocals of integers. (Multiplicities are 1 exactly at smooth points.) This extends to a formula for the Hilbert series of the tangent cone.

3. Many results extend to arbitrary projections $G/B \twoheadrightarrow G/P$ of Richardson varieties, using Chirivì's extension of the Hodge-Pedoe degeneration.