SCHUBERT CALCULUS AND PUZZLES
NOTES FOR THE OSAKA SUMMER SCHOOL 2012

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CONTENTS

1. INTERVAL POSITROID VARIETIES

In my first lecture I'll present a family of varieties interpolating between Schubert and Richardson, called "interval positroid varieties".

1.1. **Schubert varieties.** Definitions:

- $M_{k\times n} := k \times n$ matrices over $\mathbb{C}$.
- the **Stiefel manifold** $M^{\text{rank } k}_{k\times n}$ is the open subset in which the rows are linearly independent.
- the **Grassmannian** $Gr_k(\mathbb{C}^n) \cong GL(k)\backslash M^{\text{rank } k}_{k\times n}$ is the space of k-planes in $\mathbb{C}^n$.

Two matrices in $M^{\text{rank } k}_{k\times n}$ give the same k-plane if they're related by row operations. To kill that ambiguity, put things in **reduced row-echelon form**. From there we can associate

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a discrete invariant, a **bit string** like 0010101101 with 0s in the k pivot positions and 1s in the remaining $n-k$ (**careful:** not the reverse!). Let $\binom{n}{k}$ denote the *set* of such bit strings.

Examples:

(1) Most matrices in $M_{k\times n}^{\text{rank } k}$ give $000\ldots 11111$.
(2) If all the columns are zero except the last k, the bit string is $111\ldots 00000$.
(3) If the kth column is in the span of the first $k-1$, but there are no other dependencies, the bit string is $00\ldots 0101\ldots 1111$.

Proposition 1.1. (1) *There is a decomposition of the Grassmannian indexed by $\binom{\lambda\in n}{k}$, into complex cells. The codimension $\ell(\lambda)$ of the λ cell is the number of* **inversions** *in λ, where a 1 occurs somewhere left of a 0.*
(2) *These cells give bases for homology and cohomology.*
(3) *To figure out which cell a matrix is in, look at $\operatorname{rank}[1,i] :=$ the rank of the first i columns, for each $i \in [1,n]$.*
(4) *The closure of a cell (or, its preimage in the Stiefel manifold) satisfies a bunch of determinantal conditions.*
(5) *Fix λ. Let $P_\lambda \subset M_{k\times n}$ be the vector space of matrices where the ith row has 0s left of the ith 0 in λ. Then $P_\lambda \cap M_{k\times n}^{\text{rank } k} \to Gr_k(\mathbb{C}^n)$ has image X_λ.*

The closures of these **Bruhat cells** in the Grassmannian are called **Schubert varieties** and denoted $X_\lambda, \lambda \in \binom{n}{k}$. Hodge proved that the determinantal equations (vanishing of the **Plücker coordinates** on the affine cone over the Grassmannian) give prime ideals, i.e. define these as schemes.

Exercise 1.2. (1) *Fix $\lambda \in \binom{n}{k}$. Show that the rank conditions on those initial intervals $[1,i]$ for which 10 occur in positions $i, i+1$ of λ imply all of λ's other rank conditions.*
(2) *Let $B \leq GL(n)$ denote the upper triangular matrices. Show that the Bruhat cells are exactly the B-orbits (acting on $M_{k\times n}$ on the right).*
(3) *Show the same remains true if we replace B by $N = B'$, the upper triangular matrices with 1s on the diagonal.*

1.2. **Schubert calculus.** Let $[X_\lambda]$ always denote the element of *co*homology. Since we have a basis, we know $[X_\lambda][X_\mu] = \sum_\nu c_{\lambda\mu}^\nu [X_\nu]$. The study of these **Schubert structure constants** (and generalizations thereof) is **Schubert calculus**.

Theorem 1.3 (e.g. Kleiman 1973). *X_λ and $w_0 \cdot X_\mu$ intersect transversely, where $w_0 \in S_n \leq GL(n)$ reverses $M_{k\times n}$ left/right.*

Let $X^\mu := w_0 \cdot X_{w_0\cdot\mu}$ be the **opposite Schubert variety**, where $w_0\cdot\mu$ again means reverse left/right. Since w_0 lies in the connected group $GL(n)$, we have $[X_\mu] = [X^{w_0\cdot\mu}]$, so $[X_\lambda][X_\mu] = [X_\lambda][X^{w_0\cdot\mu}] = [X_\lambda \cap X^{w_0\cdot\mu}]$ (the last by Kleiman transversality).

First example: let $\lambda = \mu = 010$, each defining the space of matrices $\{[0 * *]\}$, so those points in the projective plane $Gr_1(\mathbb{C}^3)$ that lie on the yz line. Then $X_\lambda = X_\mu$ so they don't intersect transversely. But $X_\lambda \cap w_0 \cdot X_\mu = \{[0 * 0]\}$, the y point. Which is homologous to the Schubert point $\{[00*]\}$.

An intersection $X_\lambda^\mu := X_\lambda \cap X^\mu$ of a Schubert and opposite Schubert variety is called a **Richardson variety**[1]. Schubert calculus then becomes the question of computing the cohomology classes Poincaré dual to the X_λ^μ, in the Schubert basis. To determine the smallest Richardson variety containing a matrix, look at $\{\mathrm{rank}\ [1,i]\}$ and $\{\mathrm{rank}\ [j,n]\}$.

Proposition 1.4. *Define the* **Bruhat order** $\lambda \leq \mu$ *on* $\binom{n}{k}$ *by requring the* i*th* 1 *in* λ *to occur left of the* i*th* 1 *in* μ, $i = 1, \ldots, n-k$.

1. *If* $\lambda \leq \mu$, *then* $\dim X_\lambda^\mu$ *is nonempty of dimension* $\ell(\mu) - \ell(\lambda)$.
2. X_λ^λ *is a point.*
3. *If* $\lambda \not\leq \mu$, *then* $X_\lambda^\mu = \emptyset$.
4. *Let* $P_\lambda^\mu \subset M_{k\times n}$ *denote the vector space of matrices in which the* i*th row is supported between the* i*th* 0 *in* λ *and the* i*th* 0 *in* μ. *Then* $P_\lambda^\mu \cap M_{k\times n}^{\mathrm{rank}\ k} \to Gr_k(\mathbb{C}^n)$ *has image* X_λ^μ.

$$E.g.\ \lambda = 1100110111, \mu = 1101110101, P_\lambda^\mu = \begin{pmatrix} 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \end{pmatrix}$$

Corollary 1.5. $\int_{Gr_k(\mathbb{C}^n)} [X_\lambda][X^\mu] = \delta_{\lambda\mu}$. *In particular, if we define the more symmetric* **Schubert intersection numbers**

$$c_{\lambda\mu\nu} := \int_{Gr_k(\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu],$$

then $c_{\lambda\mu}^\nu = c_{\lambda\ \mu\ w_0\cdot\nu}$.

1.3. First positivity result.

Proposition 1.6. (1) *(Borel) Let* B *act on a nonempty projective scheme (or more generally, a complete one). Then there is a* B*-fixed point.*

2. *Any complete scheme over* $\mathbb{C}$ *has a natural fundamental class in homology: the* $\mathbb{N}$*-combination of the classes of its top-dimensional (geometric) components, where the coefficients are the lengths of the local rings at the generic points.*
3. *(Grothendieck, Mumford) Let* Y *be a projective scheme (in a moment, the Grassmannian) and* X *a subscheme. There is a functorially associated "Hilbert scheme" parametrizing a family of subschemes of* Y*, all of whom have the same homology class (and more specifically, the same "*K*-class", discussed later), and this moduli space is projective.*
4. *Hence every subscheme* X *of the Grassmannian is homologous to a schemy union* X' *of Schubert varieties.*
5. *Hence each* $c_{\lambda\mu}^\nu \geq 0$.

Proof. (1) Filter B by normal subgroups, so that the subquotients are 1-dimensional (most groups can't do this). Prove the theorem for those two groups, $(\mathbb{C}, +)$ and $(\mathbb{C}^\times, \cdot)$. Then use induction.

2. As stated, this is a definition, reducing to the case of X a variety (reduced and irreducible). Then we punt, appealing either to the statement that complex varieties are triangulable (1930s), or have resolutions of their singularities (1960s).
3. Beyond our scope.

[1]Warning: Richardson also studied nice nilpotent orbits in Lie algebras, and non-cognoscenti often guess incorrectly that "Richardson variety" refers to the closure of a Richardson orbit. It doesn't seem to have been used that way in the literature, or at least I prefer to believe that.

(4) By the functoriality, since B acts on $Gr_k(\mathbb{C}^n)$ it acts on X's Hilbert scheme. Let X' be a B-fixed point on there. Then X''s support must be a union of B-orbits, which are the Schubert varieties.
(5) Given a component of a scheme, we define its **length** to be the length of the local ring at the generic point. Then the homology class one associates is the homology class of the reduced scheme, times the length, a nonnegative integer.

□

Examples:

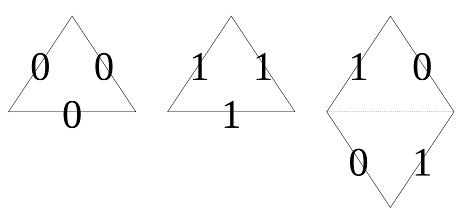
(1) Let X be a conic in the plane. Then the Hilbert scheme is the $\mathbb{P}^5$ of all conics. The B-invariant one is the double line $x^2 = 0$.
(2) Let X be a disjoint union of two lines in $\mathbb{P}^3$. First let them cross, giving a union of two lines with an extra point embedded at the intersection (reduced everywhere else). Then let those lines fall atop one another, giving a double (Schubert) line with an extra embedded (Schubert) point, which doesn't contribute to the homology class (but will contribute to the K-class).

    This hints at some other related subschemes in $\mathbb{P}^3$: plane conics union a disjoint point. Indeed, this Hilbert scheme has two components, one consisting of pairs of lines and the other of conics plus points. (It's connected, which is true for all Hilbert schemes of projective space, a theorem of Hartshorne.)
(3) A Schubert calculus example: $X^{1010}_{0101} \subset Gr_2(\mathbb{C}^4) \cong Gr_1(\mathbb{CP}^3)$, which interpreted projectively is the $\mathbb{P}^1 \times \mathbb{P}^1$ worth of lines that touch the xy line and zw line. Move the latter continuously to xz, and we get lines that touch xy and xz. That's the (reduced) scheme $X_{1001} \cup X_{0110}$, with class $1[X_{1001}] + 1[X_{0110}]$.

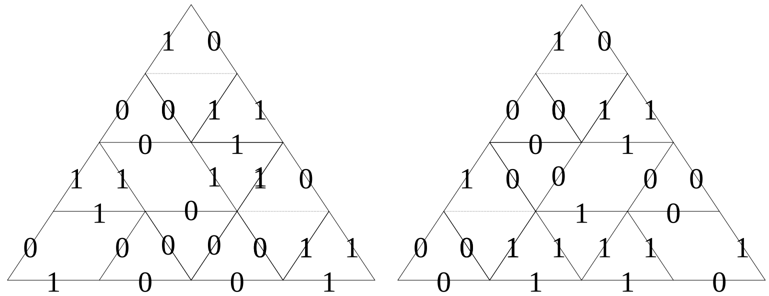
So any construction of subvarieties of a Grassmannian (or more generally, a space with an action of a unipotent group N with finitely many orbits) leads naturally to a "Schubert calculus" problem with nonnegative integer answers.

The real game, then: how to calculate these natural numbers in a *manifestly positive* way, i.e. by counting some combinatorial objects? There are many known ways (and sometimes-tricky bijections between them); the one we'll give is to use *puzzles* [KnTao03, KnTaoWood04].

The following three shapes, with edges labeled 0 or 1, are the **puzzle pieces**. They may be rotated but not reflected.



A **puzzle** is a size n triangle made of puzzle pieces, glued so as to have the edge labels match. The two possible puzzles with NW,NE sides labeled 0101.



Theorem 1.7. [KnTao03, KnTaoWood04] $c_{\lambda\mu}^{\nu} = \#\{$*puzzles with* λ, μ, ν *on their NW,NE,S sides respectively, all read left-to-right*$\}$. *Equivalently,* $c_{\lambda\mu\nu} = \#\{$*puzzles with* λ, μ, ν *on their NW,NE,S sides respectively, all read clockwise*$\}$.

So these two puzzles compute the example (3) above.

In the remainder we'll indicate a geometric proof of this theorem, following [K1], and discuss harder problems requiring more puzzle pieces.

Note that in addition to the obvious Z_3 rotational symmetry of puzzles, which matches half of the S_3 symmetry of Schubert intersection numbers, one can flip a puzzle over while exchanging 0s and 1s, which matches the symmetry coming from Grassmannian duality. However, it is hard to see directly that the puzzle product is commutative. (The nicest self-contained combinatorial proof of this is in [P08].)

1.4. **Interval rank varieties.** Before giving an answer to this, we generalize the problem just a little, beyond Richardson varieties to *interval rank varieties*.

Given a matrix $M \in M_{k\times n}$, let's compute all $\{\mathrm{rank}\ [i,j]\}_{i\leq j}$. These numbers are weakly increasing in j and $-i$, by 0 or 1 only, with the additional restriction that one doesn't see the pattern

$$\begin{matrix} r & r+1 \\ r & r \end{matrix}$$

anywhere in the matrix r.

Define a partial permutation matrix π to be one with *at most* one 1 in any row/column. To avoid confusion we refer to and draw the 1s as **dots** $\bullet$.

Proposition 1.8. *Such rank matrices* r *correspond 1:1 to upper triangular partial permutation matrices* π *under the correspondence*

$$r_{ij} = \big|[i,j]\big| - \#\big\{\textit{dots in } \pi \textit{ weakly SW of box } [i,j]\big\}, \qquad \forall\, 1 \leq i \leq j \leq n.$$

If M *has rank* k*, then* π *will have* $n-k$ *dots.*

As before, many of these rank conditions imply others. Define the **diagram**[2] of π by crossing out strictly South and West (but not Southwest) of each dot, and any rows or columns without dots, and taking the remaining boxes. (So each box with a dot is in the diagram.) Define the **essential set**[3] as the Northeast corners of the diagram boxes.

[2]This is closely related to the Rothe diagram used e.g. in Fulton's essential set definition [Fu92], but not the same, both for being flipped East/West and for the "strictly".

[3]It's actually possible to cut this down further, but we're using this name in analogy to [Fu92].

For example,

$$\pi = \begin{pmatrix} \cdot & \cdot & \cdot & \bullet & \cdot & \cdot \\ & \cdot & \bullet & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \bullet \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{pmatrix} \quad \text{has diagram} \quad \begin{pmatrix} + & + & - & \boxed{e} & | & \boxed{e} \\ & + & \boxed{e} & | & | & \boxed{} \\ & & + & + & + & - \\ & & & + & + & \boxed{e} \\ & & & & + & + \\ & & & & & + \end{pmatrix}$$

with $-, |, +$ to indicate the crossing-out, and es to indicate the essential boxes.

Proposition 1.9. *The rank conditions* $\operatorname{rank}[i, j] \leq r_{ij}$ *in the essential set imply all the other rank conditions.*

(More specifically, any other rank condition is implied by some single *essential condition; one doesn't have to cleverly combine them.)*

Exercise 1.10. (1) *Show that* π *has a unique essential rank condition, at* (i, j)*, iff its dots are the diagonal of some square with NE corner* (i, j)*.*

(2) *Find a* π *for* $n = 4$ *with three "essential" rank conditions, one of which is actually implied by the other two taken together.*

(3) *Show that the following are equivalent:*
- *All the essential rank conditions are in the first row.*
- *The dots are NW/SE, and in the first* $n - k$ *rows.*
- Π_r *is a Schubert variety.*

Theorem 1.11. *Let* π *be an upper triangular partial permutation with* $n - k$ *dots. Then the scheme* Π_π *defined by the rank conditions in proposition 1.8 is reduced and irreducible. Moreover, any intersection of such schemes is reduced.*

I hoped we would have time to prove this later, but we didn't. The quickest proof I know is in [Kn, §7.3], combined with [K1,].

If we simply *imposed* random rank conditions, with some matrix r, what different schemes could we get? Different rs can give the same scheme: anywhere $r_{ij} > r_{i,j+1}$, we can cut down r_{ij} to $r_{i,j+1}$ without changing the scheme, and anywhere $r_{ij} + 1 < r_{i,j+1}$, we can cut down $r_{i,j+1}$ to $r_{ij} + 1$ without changing the scheme, and similarly for j instead of i. So it's enough to consider matrices r that only increase by $0, 1$ as one goes North or East.

That leaves the forbidden pattern from before.

Proposition 1.12. *Let*

$$\begin{matrix} m & m+1 \\ m & m \end{matrix}$$

occur in the middle of a rank matrix, and Π_r *be the associated scheme. Then* $\Pi_r = \Pi_{r_1} \cup_{\Pi_{r_{12}}} \Pi_{r_2}$*, where*

$$r_1 = \begin{matrix} m & m \\ m & m \end{matrix}, \qquad r_2 = \begin{matrix} m & m+1 \\ m-1 & m \end{matrix}, \qquad r_{12} = \begin{matrix} m & m \\ m-1 & m \end{matrix}$$

in those spots.

Proof. Set-theoretically this is obvious. There is also a boring proof with determinants that lets one establish the scheme-theoretic statement. (The "quick proof" referred to four paragraphs above also gives a short proof of this.) ∎

2. VAKIL'S LITTLEWOOD-RICHARDSON RULE

2.1. **Combinatorial shifting.** Define the **shift** $\mathrm{sh}_{i\to j}$ in the following contexts:

- When applied to a number k, give k back unless $k = i$, in which case it becomes j.
- When applied to a set $S \subseteq [1, n]$, just apply $\mathrm{sh}_{i\to j}$ to every element $k \in S$, but don't shift i to j if j is "in the way", i.e. $j \in S$ already.
- When applied to a collection $\mathcal{P} \subseteq 2^{[1,n]}$, just apply $\mathrm{sh}_{i\to j}$ to every set $S \in \mathcal{P}$, but don't shift S to $\mathrm{sh}_{i\to j}S$ if $\mathrm{sh}_{i\to j}S$ is "in the way", i.e. $\mathrm{sh}_{i\to j}S \in \mathcal{P}$ already.

In particular the shift of a set or collection is always the same size as the original. Shifting was invented by Erdős-Ko-Rado [EKR61] to study extremal combinatorics of highly intersecting collections, and nowadays is also used to study simplicial complexes.

Exercise 2.1. (1) *Let $\mathcal{P} \subseteq \binom{n}{k}$ be a collection where every pair $S_1, S_2 \in \mathcal{P}$ intersects nontrivially. Show that $\mathrm{sh}_{i\to j}\mathcal{P}$ has the same property. Replace $\mathcal{P}$ by $\mathrm{sh}_{i\to n}\mathcal{P}$ for each i, to force $n \in S, \forall S \in \mathcal{P}$. Hence $|\mathcal{P}| \leq \binom{n-1}{k-1}$.*

(2) *Let $\mathcal{P}' = \binom{n}{k} \setminus \mathcal{P}$. Show $\mathrm{sh}_{i\to j}\mathcal{P}' = \binom{n}{k} \setminus \mathrm{sh}_{j\to i}\mathcal{P}$, i.e. backwards.*

2.2. **Geometric shifting.** Let $X \subseteq Gr_k(\mathbb{C}^n)$, and define

$$\mathrm{sh}_{i\to j}X := \lim_{t\to\infty} \begin{pmatrix} 1 & & & & & \\ & 1 & & & t & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \cdot X$$

where the t is in position (i, j). More specifically, define $F^\circ \subseteq \mathbb{P}^1 \times Gr_k(\mathbb{C}^n)$ as

$$F^\circ := \bigcup_{t\in\mathbb{A}^1} \{t\} \times \begin{pmatrix} 1 & & & & & \\ & 1 & & & t & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \cdot X,$$

then F as its closure (adding the $t = \infty$ fiber), and $\mathrm{sh}_{i\to j}X := F \cap (\{\infty\} \times Gr_k(\mathbb{C}^n))$.

Examples:

(1) Let X be a single point, the coordinate k-plane $\mathbb{C}^S$ that uses the k coordinates $S \subseteq \{1, \ldots, n\}$. Then $\mathrm{sh}_{i\to j}\{\mathbb{C}^S\} = \{\mathbb{C}^{\mathrm{sh}_{i\to j}S}\}$, a first link of the two notions.

(2) Let X be the divisor $p_S = 0$, given by the vanishing of the Plücker coordinate. Then $\mathrm{sh}_{i\to j}X = \{p_{\mathrm{sh}_{j\to i}S} = 0\}$, *backwards*. One should think of X as corresponding to $\binom{n}{k} \setminus \{S\}$, the set of coordinate subspaces lying in X.

(3) Generalizing both examples, let $C \subseteq 2^{\binom{n}{k}}$ be a collection of k-element subsets, and define $W_C \subseteq Gr_k(\mathbb{C}^n)$ as the vanishing set $\{p_S = 0, S \notin C\}$. Then $\mathrm{sh}_{i\to j}W_C = W_{\mathrm{sh}_{i\to j}C}$. ($W$ is for Neil White, who first considered these schemes.)

(4) Let $X = \mathbb{C}^{01} \coprod \mathbb{C}^{10} \subseteq Gr_1(\mathbb{C}^2)$, defined by the equation $p_{01}p_{10} = 0$. Then moving it by t as in the definition, it becomes $(p_{01} + tp_{10})p_{10} = 0$, so as $t \to \infty$ we get the double point $(p_{10})^2 = 0$.

By construction, F is a *flat family* over $\mathbb{P}^1$, with the effect that all of its fibers have the same Hilbert polynomial (as subvarieties of the Grassmannian, hence of projective space under the Plücker embedding). One way to determine F_∞ is to give an upper bound $(F_\infty)^+$ on it by determining (what are a priori) some of the equations that hold on it, and then to show that that upper bound has the same Hilbert polynomial as $F_0 = X$.

Proposition 2.2. *Let $X = Y \cap \{p_S = 0\}$, where Y is a $\mathrm{sh}_{i\to j}$-invariant variety, and $Y \supseteq \{\mathbb{C}^S, \mathbb{C}^{\mathrm{sh}_{j\to i} S}\}$ (remember, backwards). Then $\mathrm{sh}_{i\to j} X = Y \cap \{p_{\mathrm{sh}_{j\to i} S} = 0\}$.*

Proof. By the condition, both $p_S = 0$ and $p_{\mathrm{sh}_{j\to i} S} = 0$ define nonzero elements of Y's coordinate ring. Since Y is a variety, they both define non-zerodivisors. Modding out a non-zerodivisor of degree 1 replaces the Hilbert polynomial $h(d)$ by its difference $h(d) - h(d-1)$, so the two have the same Hilbert polynomial.

Following the equations gives the containment $\subseteq$, so by the equality of Hilbert polynomials they are equal. □

Corollary 2.3. *Let r be a rank matrix defining some Π_r, and let $a < b$. Assume that for*

- *one of the essential rank conditions is on $[a+1, b]$, and*
- *for the others $[i,j]$, $\mathrm{sh}_{b\to a}[i,j] = [i,j]$.*

Then $\mathrm{sh}_{a\to b}\Pi_r$ is a union of various $\Pi_{r'}$, which can be determined through repeated use of proposition 1.12.

We can always ensure these hypotheses hold, assuming Π_r is not a Schubert variety; let $[a+1, b]$ be the essential box with maximum a, then maximum b. (The not-Schubert condition says that $a + 1 \geq 2$.) Consequently, we have a combinatorial algorithm with which to degenerate any interval rank variety in the Grassmannian towards a union of Schubert varieties, and can thereby determines its homology class!

To spell this out further, we introduce also the combinatorial and geometric sweeping operations. The **combinatorial sweep** $\Psi_{i\to j} C$ of a collection C is just $C \cup \mathrm{sh}_{i\to j} C$. The **geometric sweep** $\Psi_{i\to j} X$ is the image of the projection to $Gr_k(\mathbb{C}^n)$ of the family F, so $\Psi_{i\to j} X \supseteq X \cup \mathrm{sh}_{i\to j} X$.

Proposition 2.4.
- *If X is irreducible, then so too are F and $\Psi_{i\to j} X$.*
- *If any two of $X, \mathrm{sh}_{i\to j} X, \Psi_{i\to j} X$ are equal, so is the third.*
- *If not, then $\dim \Psi_{i\to j} X = \dim X + 1$.*

Theorem 2.5. *Assume the setup of corollary 2.3. There is necessarily a dot in column b, which we'll call the wandering dot.*

Then $\Psi_{a\to b}\Pi_\pi = \Pi_\sigma$, where σ is constructed from π by moving the wandering dot up from whatever row k to row a, and the dot (if any) in row a down to that row k.

The π' are constructed from this σ. If there is no dot in column $b-1$, one π' comes from moving the wandering dot in σ left one square to column $b-1$.

For the other π', consider the dots in σ that are minimally NW of the wandering dot (nobody else is in between). Each π' comes from moving the wandering dot West, while one of them moves East, ending in the same two columns. (If this causes the wandering dot to end up in the lower triangle, discard this misbegotten π'.)

For example, let

$$\pi = \begin{pmatrix} \cdot & \cdot & \cdot & \bullet & \cdot & \cdot \\ & \cdot & \bullet & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \bullet \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{pmatrix} \quad \text{with diagram} \quad \begin{pmatrix} + & + & - & \boxed{e} & | & \boxed{e} \\ & + & \boxed{e} & | & | & \boxed{} \\ & & + & + & + & - \\ & & & + & + & \boxed{e} \\ & & & & + & + \\ & & & & & + \end{pmatrix}.$$

where the es are essential boxes. Let $(a, b) = (3, 6)$, above the SErnmost. Then

$$\sigma = \begin{pmatrix} \cdot & \cdot & \cdot & \bullet & \cdot & \cdot \\ & \cdot & \bullet & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & W \\ & & & \cdot & \cdot & \uparrow \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{pmatrix}, \quad \pi' = \begin{pmatrix} \cdot & \cdot & \cdot & \bullet & \cdot & \cdot \\ & \cdot & \bullet & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & W & \leftarrow \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \bullet & \cdot & \cdot \\ & \cdot & \rightarrow & \cdot & \cdot & \bullet \\ & & W & \cdot & \cdot & \leftarrow \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \rightarrow & \cdot & \bullet \\ & \cdot & \bullet & \cdot & \cdot & \cdot \\ & & \cdot & W & \cdot & \leftarrow \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{pmatrix}$$

where the W indicates the wandering dot, and the arrows are only to show where dots have most recently moved from.

We won't hack through the combinatorics to prove this from corollary 2.3. A different approach is taken in [K1], where this is related to Lascoux's "transition formula" for Schubert polynomials.

Exercise 2.6. (1) *Check the theorem in this example, using the algorithm from corollary 2.3.*
(2) *Do the same for all $n = 3$ examples.*
(3) *Do the same for all $n = 4, k = 2$ examples.*

2.3. **Vakil's degeneration order.** Geometric shifts were introduced[4] in [Va06], where he considers the following **degeneration order** of shifts:

$$\begin{gathered} n-1 \to n, \\ n-2 \to n, \quad n-2 \to n-1, \\ n-3 \to n, \quad n-3 \to n-1, \quad n-3 \to n-2, \\ \vdots \\ 1 \to n, \quad 1 \to n-1, \quad \ldots \quad 1 \to 4, \quad 1 \to 3, \quad 1 \to 2. \end{gathered}$$

Proposition 2.7 (essentially in Vakil)**.** *Let π be an upper triangular partial permutation matrix, and $a < b$. Assume*

- *the dots in rows $[1, a]$ are NW/SE,*
- *the dots in rows $[a + 1, n]$ are NW/SE, and*
- *the j dots in rows $[a + 1, n]$ are in the first such rows, $[a + 1, a + j]$.*

If $(a + 1, b)$ is not an essential box of π, then $\mathrm{sh}_{a\to b}\Pi_\pi = \Pi_\pi$, so the unique π' is just π.

Otherwise $\mathrm{sh}_{a\to b}\Pi_\pi \neq \Pi_\pi$, and we compute its components $\Pi_{\pi'}$ with theorem 2.5. There are at most two π'.

Let $(a, b)'$ be the next shift after (a, b), i.e. $(a, b)' = (a, b - 1)$ unless $a = b - 1$, in which case $(a, b)' = (a - 1, n)$. Then each π' satisfies the conditions of the first paragraph, for $(a, b)'$.

[4]Though not under that name – he didn't connect them with Erdős-Ko-Rado shifting theory.

Vakil only shows that the components of the shift have length 1, i.e. that the shift is generically reduced, whereas the above says that it's actually reduced. We'll need this actual reducedness later in order to compute in K-theory.

Example. Start with

$$\pi = \begin{pmatrix} \cdot & \bullet & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & & \cdot & \bullet \\ & & & \cdot \end{pmatrix}, \quad \text{diagram} \begin{pmatrix} - & e & | & e \\ & + & + & - \\ & & + & e \\ & & & + \end{pmatrix}, \quad (a,b) = (2,4), \quad \sigma = \begin{pmatrix} \cdot & \bullet & \cdot & \cdot \\ & \cdot & \cdot & W \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}.$$

Then the two π' are

$$\begin{pmatrix} \cdot & \bullet & \cdot & \cdot \\ & \cdot & W & \leftarrow \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix} \text{ which is Schubert,} \quad \text{and} \begin{pmatrix} \cdot & \rightarrow & \cdot & \bullet \\ & W & \cdot & \leftarrow \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix} \text{ with diagram} \begin{pmatrix} + & - & + & e \\ & e & + & + \\ & & + & + \\ & & & + \end{pmatrix}.$$

For the latter one take $(a,b) = (1,2)$, and get

$$\sigma = \begin{pmatrix} \cdot & W & \cdot & \cdot \\ & \uparrow & \cdot & \bullet \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}, \qquad \pi' = \begin{pmatrix} W & \leftarrow & \cdot & \cdot \\ & \cdot & \cdot & \bullet \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}.$$

The final result is thus

$$\left[\Pi_{\begin{pmatrix} \cdot & \bullet & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & & \cdot & \bullet \\ & & & \cdot \end{pmatrix}}\right] = \left[\Pi_{\begin{pmatrix} \cdot & \bullet & \cdot & \cdot \\ & \cdot & \bullet & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}}\right] + \left[\Pi_{\begin{pmatrix} \bullet & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \bullet \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}}\right]$$

or

$$[X^{1010}_{0101}] = [X_{1001}] + [X_{0110}].$$

2.4. **Partial puzzles.** Imagine a would-be puzzle with μ on NE and ν on South, both read left-right, but no other edge labels. Shear this to fit into the upper triangle where πs live, so the ν ends up on the diagonal and the μ on the upper right. For the rest of this section we'll draw puzzles in this way.

Now we can think about Vakil's order as specifying a sequence of squares in the matrix upper triangle, or rhombi in the puzzle triangle. Note that once one gets to the (a,a) triangles, they are uniquely fillable, so it's harmless to add them as (a,a) at the end of each line of the degeneration order.

Define an (a,b)**-partial puzzle** to be a filling of the puzzle triangle with puzzle pieces, but only through positions (a,b) as indexed in the above paragraph. Its **puzzle path** is the labels along its top, whose shape

- starts at $(0,0)$ in matrix coordinates (row,column),
- goes SE to (a,a),
- goes E to (a,b),

- has a **kink** up to $(a-1,b)$,
- goes E to $(a-1,n)$,
- finally up to $(0,n)$.

Note that in this correspondence, the edges of the matrix boxes may cut puzzle rhombi in half. As such, we have to ditch our rhombus puzzle piece in favor of a triangle with labels $1,0,R$ clockwise, and declare that a puzzle can't have any Rs on its boundary.[5] The main idea: we'll associate an interval rank variety to each puzzle path, such that the components of $\mathrm{sh}_{a\to b}$ correspond to the ways to fill in one more square in the matrix. The partial permutation matrix should have the properties in proposition 2.7, in particular being NW/SE in each half of the triangle (above/below the path).

The construction, in the upper half: first draw little **rays** inside the puzzle

- left from each 0
- up from each 0 and R
- left from the kink if it's R, in which case also make the next 1 to its left get an upward pointing ray.

We require there be the same number of up and left rays for the puzzle path to be **viable***.* If there are, put the dots on the intersections of these rays, in the only NW/SE way possible.

In the lower half, draw rays down from each 0. If the kink is $\vdash 1$, and there is a 0 to its right, the 0 gets a downward ray, and these two rays meet at a dot. Put dots on all remaining downward rays NW/SE, as N as possible (à la proposition 2.7).

Write this association puzzle path $\mapsto$ partial permutation matrix as $\gamma \mapsto \pi(\gamma)$.

Theorem 2.8. *Let γ be a puzzle path.*

If the labels on the kink and the edge just leftward are not $_0 1$, *then*

1. *there is a unique way to put in two puzzle pieces,*
2. *the new path γ' has the same $\pi()$, and*
3. *$\Pi_{\pi(\gamma)}$ is $\mathrm{sh}_{a\to b}$-invariant.*

If those labels are indeed $_0 1$, then there are either one or two ways to fill two triangles and remain viable. The corresponding $\{\gamma'\}$ correspond to the components of $\mathrm{sh}_{a\to b}\Pi_{\pi(\gamma)} \neq \Pi_{\pi(\gamma)}$.

In this way, filling in the puzzle corresponds to keeping track of the components encountered during Vakil's degeneration.

The proof of this is a several-page case check [K1], quite straightforward yet somehow miraculous. The hard work was really in theorem 2.5.

3. Equivariant and K- extensions

3.1. K**-homology.** To begin with, let A be a commutative ring, and consider its finitely generated modules, under direct sum. The K**-homology group** $K_\bullet(A)$ is freely generated by the set of isomorphism classes $[M]$ of finitely generated A-modules M, "modulo exact sequences", meaning $[M_2] = [M_1] + [M_3]$ for any exact sequence $0 \to M_1 \to M_2 \to M_3 \to$

[5]Exercise: prove that if a "puzzle" without this new condition has the same number of 1s on the NW and NE sides, and no Rs there, then it has no Rs on the S side either.

0. In particular, $[M_1 \oplus M_3] = [M_1] + [M_3]$. If $A \to B$ is a ring homomorphism making B a finitely generated A-module, we get a map $K_\bullet(B) \to K_\bullet(A)$, contravariantly.

One can easily soup this up by assuming A is graded and only using graded modules, or a group G acts on A and its modules, etc., in which case we'd write $K_\bullet^G(A)$.

We can reinterpret this ring-theoretic construction in terms of affine schemes, where the statement becomes "Given an affine scheme X, define its K-homology using coherent sheaves, and this gives a *covariant* functor for finite maps (proper with finite fibers)." At that point one can leave out the word "affine" and obtain a K-homology theory for all schemes and finite maps. (Using higher sheaf cohomology, one can extend this to proper maps.)

The most important example to understand will be the following. Let $A = \mathbb{C}[x, y]$, considered as a bigraded ring. Then we have an exact sequence of bigraded modules and maps

$$0 \to A/\langle xy \rangle \to (A/\langle x \rangle) \oplus (A/\langle y \rangle) \to A/\langle x, y \rangle \to 0.$$

Hence $[A/\langle xy \rangle] = [A/\langle x \rangle] + [A/\langle y \rangle] - [A/\langle x, y \rangle]$.

3.1.1. *Comparison with ordinary homology.* Given a (closed) subscheme $X \subseteq Y$, we can associate a K-class $[X] \in K_\bullet(Y)$, really shorthand for $[\mathcal{O}_X]$ where $\mathcal{O}_X$ is the structure sheaf of X. In the example above, we have the equation

$$[\text{union of axes}] = [x\text{-axis}] + [y\text{-axis}] - [\text{origin}]$$

in $K_\bullet^{T^2}(\mathbb{C}^2)$. This differs from what we would expect in ordinary homology, where we wouldn't have the last term.

One can make this precise; filter $K_\bullet(Y)$ according to the dimension of the support of the sheaf. Then there is a map from the associated graded ring to $H_*(Y)$ (or really, to the Chow group).

In the example from before, we computed the homology class of X_{0101}^{1010} by degenerating it to $X_{1001} \cup X_{0110}$, glued along X_{1010}. Therefore we get the equation on K-classes,

$$[X_{0101}^{1010}] = [X_{1001}] + [X_{0110}] - [X_{1010}].$$

3.2. K**-cohomology.** Not only can we $\oplus$ sheaves, we can $\otimes$ them, suggesting we make $K_\bullet(Y)$ into a ring. But it turns out that $\otimes$ is not well-defined on K-equivalence classes, i.e. tensoring with M is not exact.

Of course, tensoring with a free module is just repeated direct sum, so that'd be okay. More generally tensoring with projective modules is fine; in scheme-theoretic language, use (finite-dimensional) vector bundles instead of all coherent sheaves. So define the K-**cohomology** $K^\bullet(Y)$ in the same way as $K_\bullet(Y)$, but only using exact sequences of vector bundles over Y.

Proposition 3.1. (1) $K^\bullet$ *is a contravariant functor from schemes to rings.*
(2) $K^\bullet(Y)$ *acts on* $K_\bullet(Y)$, *a "cap product" in* K*-theory.*
(3) $K_\bullet(Y)$ *comes with a "fundamental class"* $[Y]$.
(4) *There is a "Poincaré map"* $K^\bullet(Y) \to K_\bullet(Y)$, *taking a vector bundle* $[\mathcal{V}]$ *to its sheaf of sections, i.e. to* $[\mathcal{V}] \cap [Y]$.
(5) *If* Y *is smooth and proper, this is an isomorphism ("Poincaré duality").*

(6) *The Schubert varieties on the Grassmannian (or any other* G/P*) give bases for* K*-homology and* K*-cohomology.*

(The ontoness of the Poincaré map is not hard to see; it is based on Hilbert's theorem that modules over polynomial rings have free resolutions of finite length.)

So now we have K**-theoretic Schubert calculus** to compute: what is the cup product $[X_\lambda][X_\mu]$, where these denote the elements of K-*co*homology constructed using Poincaré duality?

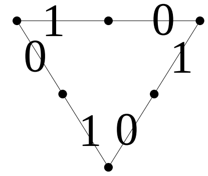
For the geometry, we need to soup up Vakil's proposition 2.7 using proposition 1.12: when $\mathrm{sh}_{i\to j}\Pi_\pi$ has two components, their intersection is another interval rank variety, whose class must therefore be subtracted, as in the example $[X^{1010}_{0101}] = [X_{1001}] + [X_{0110}] - [X_{1010}]$.

3.2.1. *Second positivity result.* The minuses may seem to kill the positivity statement $c^\nu_{\lambda\mu} \geq 0$ from before, but this can be fixed:

Theorem 3.2. [Buc02, Bri02] *In* $K^\bullet(Gr_k(\mathbb{C}^n))$, $(-1)^{|\nu|-|\lambda|-|\mu|}c^\nu_{\lambda\mu} \geq 0$.

The history is a little weird – Anders Buch gave an explicit combinatorial formula for $(-1)^{|\nu|-|\lambda|-|\mu|}c^\nu_{\lambda\mu}$ (in terms of tableaux, later bijected to K-puzzles, only much later explained geometrically), *after* which Michel Brion gave an abstract geometric proof that holds for general flag manifolds. Usually the geometry comes first (as, in this very story told, it has for all G/P other than Grassmannians).

The cleanest way to puzzlify this is with the K**-piece**, which is twice the size of other pieces (4x the area), and cannot be rotated. Try it out in the $[X_{0101}]^2$ case.



Theorem 3.3. *In* $K^\bullet(Gr_k(\mathbb{C}^n))$, $(-1)^{|\nu|-|\lambda|-|\mu|}c^\nu_{\lambda\mu}$ *is the number of puzzles using the usual three pieces and now the* K*-piece, with* λ, μ, ν *on the NW, NE and S sides respectively, each left-to-right.*

K-puzzles have a very weird Z_3 rotational symmetry. Why weird? The obvious analogue of corollary 1.5 in K-theory is

$$\int\!\!\!\!\!\!K_{Gr_k(\mathbb{C}^n)} [X_\lambda][X^\mu] = \begin{cases} 1 & \text{if } \lambda \leq \mu \\ 0 & \text{if not} \end{cases}$$

where $\int\!\!\!\!\!\!K$ denotes the pushforward to a point in K-homology. But there is a less obvious analogue [Buc02,],

$$\int\!\!\!\!\!\!K_{Gr_k(\mathbb{C}^n)} [X_\lambda][X^\mu](1 - [X_{box}]) = \delta_{\lambda\mu}$$

with which one can interpret the number of K-puzzles with λ, μ, ν clockwise as the S_3-symmetric number

$$\int\!\!\!\!\!\!K_{Gr_k(\mathbb{C}^n)} [X_\lambda][X_\mu][X_\nu](1 - [X_{box}]).$$

3.3. **Equivariant K-theory.** As we said above, the definition of K-theory of a scheme X extends in a trivial way to G-equivariant K-theory (homology or cohomology). One extreme case is X a point, in which case a G-equivariant sheaf on X is just a G-representation, and $K^\bullet(X)$ is the representation ring $\mathrm{Rep}(G)$.

The natural group to use in Schubert calculus is B. However, since each the only irreps V of B are 1-dimensional (apply Borel's theorem to $\mathbb{P}V$), every rep is K-equivalent to a sum of 1-d reps, which are thus really reps of $B/[B,B] \cong T$. The effect is that $K^\bullet_B(pt) \cong K^\bullet_T(pt)$ canonically, and it's become traditional to use T-equivariant instead of B-equivariant K-theory.

Proposition 3.4. *The* K*-theoretic Schubert classes* $[\mathcal{O}_{X_\lambda}]$ *are a basis of* $K^\bullet_T(Gr_k(\mathbb{C}^n))$*, as a module over* $K^\bullet_T(pt) \cong \mathrm{Rep}(T)$*, a Laurent polynomial ring.*

I like to call the elements of $K^\bullet_T(pt)$ (and later, of $H^*_T(pt)$) "equivariant numbers".

To keep track of the relation of the additively written group $T^* := \mathrm{Hom}(T, \mathbb{C}^\times)$ and its multiplicative role inside $K^T_*(pt)$, it helps to denote the generators of $K^T_*(pt)$ by e^λ, not λ. In the application to Schubert calculus on Grassmannians, we'll write $y_i \in T^*$ for the representation taking $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i$.

Example: $Gr_1(\mathbb{C}^2) = \mathbb{P}^1$. The T-fixed points are $s = [*\ 0]$ and $n = [0\ *]$, but only the latter is Schubert. To compute the K_T-class $[s]$ as a combination of $[n]$ and $[\mathbb{P}^1]$, we use the following exact sequences of equivariant sheaves on $\mathbb{P}^1$:

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}(1) \otimes \mathbb{C}_{(-1,0)} \to \mathcal{O}_s \to 0$$

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}(1) \otimes \mathbb{C}_{(0,-1)} \to \mathcal{O}_n \to 0$$

In each sequence, the first map (being one from a trivial line bundle) corresponds to taking a section of the of the vector bundle in the middle, here $\mathcal{O}(1)$. But these sections are not T-invariant, so the map isn't equivariant unless we twist the target by the trivial-but-not-equivariantly-trivial line bundle of that weight, which is what the $\otimes\mathbb{C}_{(0,-1)}$ is denoting. Then the cokernel sheaf is the functions on the zero scheme of the section. Hence

$$[\mathcal{O}(1) \otimes \mathbb{C}_{(-1,0)}] = [\mathcal{O}_{\mathbb{P}^1}] + [\mathcal{O}_s]$$

or

$$[\mathcal{O}(1)]t_1^{-1} = [\mathbb{P}^1] + [s] \quad \text{and similarly} \quad [\mathcal{O}(1)]t_2^{-1} = [\mathbb{P}^1] + [n]$$

hence

$$[s] = t_1^{-1}t_2[n] + t_1^{-1}t_2[\mathbb{P}^1]$$

Fix this, with e^λs too

If $S \to T$ is a homomorphism of groups, there is an obvious map $K^\bullet_T(X) \to K^\bullet_S(Y)$. In the case that S is the trivial group, this factors through setting each $e^\lambda = 1$, or "$\lambda \mapsto 0$".

Theorem 3.5 (see [HL]). *If a torus* T *acts on a smooth projective variety* M*, then the map*

$$K^\bullet_T(M) \Big/ \left\langle \{e^\lambda - 1\}, \lambda \in T^* \right\rangle \to K^\bullet(M)$$

is an isomorphism.

While less relevant to us, it's also true that the natural **localization map**

$$K^\bullet_T(M) \to K^\bullet_T(M^T) \cong K^\bullet_T(M) \otimes K^\bullet_T(pt)$$

is an inclusion, as was used frequently in T. Ikeda's and T. Lam's lectures, where it was written $\alpha \mapsto (\alpha|_f)_{f \in M^T}$ *for* M^T *isolated, following* [KnTao03].

On the Grassmannian and other flag manifolds G/P, an abstract positivity result was proven in [AGriMil], and a little complicated to state. With more puzzle pieces, one can compute the K_T-class of X_λ^μ in their positive sense.

However, that is not Schubert calculus, because $[X^\mu] \neq [X_{w_0 \cdot \mu}]$ as *equivariant* classes. Before, we could turn $X_{w_0 \cdot \mu}$ into X^μ by multiplying it with w_0, which lives in the *connected* group $GL(n)$, so this preserved the homology and K-homology classes. (One can use a path in $GL(n)$ from 1 to w_0 to construct the homology.)

In any case the puzzle rule for the K_T-class of $[X_\lambda^\mu]$ becomes intricate enough that we won't bother detailing it here. One very unfortunate thing is that the matching rules are *no longer completely local;* one must sometimes look ahead along a row to see if a puzzle piece is allowed.

3.4. **Equivariant cohomology.** First we describe some properties, then some interpretation, and finally puzzles.

1. (1) Equivariant cohomology H_G^* is a functor from {G-spaces and G-equivariant maps} to (supercommutative) rings, with a natural transformation to ordinary H^*.
2. (2) Since every G-space has a canonical G-equivariant map to the (G-invariant) point, and triangles connecting these commute, we could instead say H_G^* takes values in $H_G^*(pt)$-algebras.
3. (3) If M is compact with a G-invariant even-dimensional cell decomposition, then $H_G^*(M)$ is a free module over $H_G^*(pt)$ with a basis indexed by the cells.
4. (4) If G is a torus T, then $H_T^*(pt)$ is the symmetric algebra in the weight lattice T^* of T (where weights are given degree 2, making it commutative).
5. (5) If M is a complex projective manifold, then over $\mathbb{Q}$ the natural map
$$H_T^*(M)/\langle \lambda \in T^* \rangle \to H^*(M)$$
is an isomorphism, and the localization map $H_T^*(M) \to H_T^*(M^T) \cong H^*(M^T) \otimes H_T^*(pt)$ is injective.

We now interpret the ring structure. What is the ordinary cohomology ring structure? The product $[X] \cup [Y]$ for $X, Y \subseteq Z$ measures the difficulty in disentangling X from Y (say, for X, Y, Z compact oriented manifolds). A topologist would say we should perturb Y to miss X as much as possible, i.e. become transverse, and then $[X] \cup [Y] = [X \cap Y]$.

Now assume a group acts on Z, and X, Y are invariant. Then we may not be able to perturb Y *while keeping it invariant.* In keeping with not actually defining equivariant cohomology, we won't derive the following result from first principles. (But just as homology is related to the associated graded of K-theory, one should see this as the leading-order terms as $t \to 1$ of the formula after proposition 3.4.)

Proposition 3.6. *Let* T *act on* 1*-dimensional spaces* $\mathbb{C}, V$ *with weights* $0, \lambda \in T^*$*, so and* n, s *denote the* T*-fixed points* $[* \ 0]$, $[0 \ *]$. *Then as classes in* $H_T^2(\mathbb{P}(\mathbb{C} \oplus V))$,
$$[n] = [s] + \lambda \, [\mathbb{P}(\mathbb{C} \oplus V)].$$

Here $[n]$ and $[s]$ are degree 2 because the points are real codimension 2 inside the line $\mathbb{P}(\mathbb{C} \oplus V))$, and the third term is degree 2 because we put T^* into $H_T^2(pt)$. This formula accords with the fact that one can T-equivariantly deform n to s iff λ happens to be 0. Or, that passing from equivariant to ordinary corresponds to imposing $\lambda = 0$.

Corollary 3.7. *If we pick the shift* $\mathrm{sh}_{a \to b}$ *of an interval rank variety* Π_π *as in corollary 2.3, then*

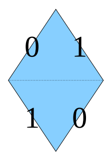
$$[\Pi_\pi] = [\mathrm{sh}_{a\to b}\Pi_\pi] + (y_a - y_b)[\Psi_{a\to b}\Pi_\pi].$$

Proof. Recall the total space F of the family degenerating Π_π to its shift. This has a projection to $\mathbb{P}^1$, and when we pull back the H^*_T equation from 3.6, we get

$$[\{0\} \times \Pi_\pi] = [\{\infty\} \times \mathrm{sh}_{a\to b}\Pi_\pi] + (y_a - y_b)[F].$$

One must then check that the projection $F \twoheadrightarrow \Psi_{a\to b}\Pi_\pi$ has degree 1, which is in [K1] and I will skip here. Pushing the equation above forward along that projection, we get the desired equation. ☐

The effect is, we should get an extra correction term with a factor of the equivariant number $y_a - y_b$ each time $\mathrm{sh}_{a\to b}\Pi_\pi \neq \Pi_\pi$. We know when the latter happens, from theorem 2.8: the labels on the kink and efge immediately left must be $_0 1$. The **equivariant puzzle piece** we create to fit in there looks like



and may not be rotated. To determine the factor $y_a - y_b$ it contributes, we drop lines SW and SE from it, coming out of the puzzle at the South edges a, b. Then the $H^*_T(pt)$ structure constant is

$$c^\nu_{\lambda\mu} = \sum_P \prod_{\text{eqvt rhombi } \rho \text{ in } P} (y_{a(\rho)} - y_{b(\rho)}).$$

The proof of this formula in [KnTao03] worked backwards from the "most equivariant" case $c^\lambda_{\lambda\lambda}$, and gaily divided by factors $y_a - y_b$ throughout in its derivation. Such a derivation becomes impossible if one specializes to ordinary cohomology in advance, taking each $y_i \mapsto 0$.

4. Other partial flag manifolds

There is a truncation of the cohomology ring of G/P in [BeKu06] that is more useful for their applications to inequalities in linear algebra problems. (Each nonvanishing structure constant in ordinary Schubert calculus implies a certain inequality [BS], but this list of inequalities is very redundant. To have a less redundant list, it's actually nice to replace most of the structure constants by zero.)

Unlike ordinary cohomology, this Belkale-Kumar product is not functorial under projections $G/P \twoheadrightarrow G/Q$, so each flag manifold must be handled separately. Let us consider only GL_n/P, where the reductive part of P is $\prod_{i=1}^m GL(n_i)$. Then the Schubert classes can be indexed not by bit strings λ, μ, ν as on the Grassmannian, but by words π, ρ, σ in $1 \ldots m$ where the letter i is used n_i times.

Theorem 4.1. (1) [BeKu06] *If the B-K structure constant* $d^\sigma_{\pi\rho}$ *is nonzero, then it matches the actual structure constant* $c^\sigma_{\pi\rho}$.

(2) [KP] $d^{\sigma}_{\pi\rho}$ *is the number of puzzles with boundary labels* π, ρ, σ, *made of* (i,i,i)*-triangles and* (i,j,i,j)*-rhombi where* $i > j$.
(3) [KP] *It factors as* $\prod_{i<j\leq m} c^{\sigma_{ij}}_{\pi_{ij}\rho_{ij}}$, *where* π_{ij} *is* π *with all other letters removed (likewise* ρ, σ*).*

Around 2000 (so, long before [BeKu06]) I circulated among a small number of people, a conjectural puzzle rule for actual Schubert calculus on GL_n/P. I'm still very pleased with this beautiful conjecture and quite annoyed that it's wrong – already for 3-step flag manifolds in 5-space, it defines a noncommutative ring. Ç'est la vie. However, for quantum Schubert calculus purposes Anders Buch was interested in 2-step flag manifolds [BuKrTam03], where this conjecture seems to be correct! One can think of it with all triangular pieces

$$(0,0,0),(1,1,1),(2,2,2),(1,0,10),(2,0,20),(2,1,21),(2,(10),2(10)),((21),0,(21)0)$$

with edge-labels $0,1,2,10,20,21,2(10),(21)0$. As before only single numbers can appear on the boundary of a puzzle. Examples appear in [BuKrTam03].

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