SCHUBERT POLYNOMIALS AND SYMMETRIC FUNCTIONS
NOTES FOR THE LISBON COMBINATORICS SUMMER SCHOOL 2012

ALLEN KNUTSON

CONTENTS

1. SYMMETRIC POLYNOMIALS

In this section $R = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ is the ring of polynomials in n variables, with integer coefficients.

A polynomial $p \in R$ is **symmetric** if it is unchanged under switching the variables around, i.e. if $p(x_1, \ldots, x_n) = p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for each permutation σ in the symmetric group S_n. It's enough to check that $p(\ldots, x_i, x_{i+1}, \ldots) = p(\ldots, x_{i+1}, x_i, \ldots)$ for each $i \in [1, n)$. Some examples:

- Any constant function (degree 0 polynomial) is symmetric.
- The sum $x_1 + \ldots + x_n$ of all the variables is symmetric.
- The sum $x_1^k + \ldots + x_n^k$ of all the kth powers is symmetric.
- The sum $\sum_{i \leq j} x_i x_j$ of all products, including the squares, is symmetric.
- The sum $\sum_{i<j} x_i x_j$ of all products, but *excluding* the squares, is also symmetric.

Let $R^{S_n} \leq R$ denote[1] the set of all symmetric polynomials. It is easy to see that it is closed under addition and multiplication, and in particular, is a subring.

> **Motivating question.** What polynomials "generate" R^{S_n}, analogous to the way that $x_1, \ldots, x_n$ generate R itself?

Polynomial rings such as R have a useful property: every polynomial p can be uniquely written as a sum $\sum_{\mathbb{N}} p_d$ of **homogeneous** polynomials p_d, in which every monomial in p_d has the same degree d. More generally, define a **graded** ring Q as one that contains a list $(Q_d)_{d \in \mathbb{N}}$ of subspaces, such that $Q_d Q_e \leq Q_{d+e}$ and every $q \in Q$ is uniquely the sum $q = \sum_{\mathbb{N}} q_d$, $q_d \in Q_d$.

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[1] More generally, one may write X^G to denote the fixed points of an action of a group G on a set X.

Exercise 1.1. R_k *has a* $\mathbb{Z}$*-basis of size* $\binom{n+k-1}{n}$. *(Hint: correspond a monomial like* $x_1^5x_2^3x_4^2x_5^2$*, for* $n=6$*, to a word like* $*****|***||**|**||$*.)*

Proposition 1.2. *The ring* R^{S_n} *of symmetric polynomials is a graded subring, i.e. the homogeneous pieces of a symmetric polynomial are themselves symmetric.*

Proof. If p is any polynomial of degree k, we can compute p's top homogeneous component p_k as $\lim_{t\to\infty} p(tx_1, tx_2, \ldots, tx_n)/t^k$. If p is symmetric, then the above ratio is symmetric for every t, so is symmetric in the limit. Then $p - p_k$ is again symmetric, and if not 0 is of strictly lower degree, so by induction on k its homogeneous pieces are symmetric. $\square$

Since the degree 1 part $(R^{S_n})_1$ of R^{S_n} is only 1-dimensional, namely multiples of $x_1 + x_2 + \ldots + x_n$, we will need our other generators to be higher degree. To narrow down the problem of giving generators, *let's insist that our generators be homogeneous.*

Being graded makes it easier for us to look at examples, like $n=2$, because we can look one homogeneous degree at a time. Here

$$(R^{S_2})_0 = \mathbb{Z}\cdot 1, \qquad (R^{S_2})_1 = \mathbb{Z}\cdot(x_1+x_2), \qquad (R^{S_2})_2 = \{ax_1^2 + ax_2^2 + bx_1x_2 : a, b \in \mathbb{Z}\}.$$

Exercise 1.3. *For each polynomial in* $(R^{S_2})_2$*, show that it can be written uniquely as a polynomial in* x_1+x_2 *and* $x_1^2+x_2^2$ *using rational coefficients. Find one that despite having integer coefficients itself, cannot be written as* $p(x_1+x_2, x_1^2+x_2^2)$ *where* $p(a,b) \in \mathbb{Z}[a,b]$ *has integer coefficients.*

So $\left\{x_1+x_2,\ x_1^2+x_2^2\right\}$ will not be a good generating set for us.

For each $k \leq n$, let e_k denote the polynomial

$$e_k = \sum_{1\leq i_1<i_2<\ldots<i_k\leq n} x_{i_1}\cdots x_{i_k},$$

called the k**th elementary symmetric polynomial**.

Theorem 1.4. *Let* p *be a symmetric polynomial with integer coefficients, i.e.* $p \in R^{S_n}$*. Then* p *is uniquely expressible as a polynomial in* $(e_1, \ldots, e_n)$ *with integer coefficients. Put another way, the homomorphism*

$$\begin{array}{rcl} \mathbb{Z}[e_1,\ldots,e_n] & \to & \mathbb{Z}[x_1,\ldots,x_n]^{S_n} \\ e_k & \mapsto & \displaystyle\sum_{1\leq i_1<i_2<\ldots<i_k\leq n} x_{i_1}\cdots x_{i_k} \end{array}$$

is an isomorphism of rings. Regarding e_k *as having degree* k*, it becomes an isomorphism of graded rings.*

To prove this, we define the *lexicographically first* monomial m in a nonzero polynomial p, and denote it $\operatorname{init} p$. It is the m with the highest power of x_1 available, then among ties it has the highest power of x_2 available, and so on. (Writing monomials $x_1^2x_2x_3^3$ like $x_1x_1x_2x_3x_3x_3$, this is almost dictionary order, except that in dictionaries x_1x_1 comes after x_1. It is indeed dictionary order when restricted to monomials of a fixed degree.)

Exercise 1.5. (1) $\operatorname{init} e_k = \prod_{i=1}^k x_i$.

(2) $\operatorname{init}(pq) = \operatorname{init} p \cdot \operatorname{init} q$.

(3) *If* p *is symmetric, and* $\operatorname{init} p = c\prod_i x_i^{m_i}$ $(c \in \mathbb{Z})$*, then* $m_1 \geq m_2 \geq \ldots \geq m_n$.

Proof. It is slightly simpler to break p into the sum of its homogeneous components, and treat each one separately. Which is to say, we reduce to the case that p is homogeneous of some degree k.

Let $\operatorname{init} p = c\prod_i x_i^{m_i}$, and $E = c\prod_{i=1}^n e_i^{m_i - m_{i+1}}$ (a polynomial, by exercise 1.5 (3)). Then $\operatorname{init} p = \operatorname{init} E$, so $\operatorname{init} p$ is lex-earlier than $\operatorname{init}(p - E)$.

If $p - E = 0$, we are done. Otherwise it is again homogeneous of degree k. Since there are only finitely many monomials of degree k, and $\operatorname{init}(p - E)$ is lex-later than $\operatorname{init} p$, we can use induction to say that $p - E$ is a polynomial in the elementary symmetric polynomials. Hence p itself is also such a polynomial.

As for uniqueness, observe that each monomial $\prod_i e_i^{n_i}$ has a different initial term, $\prod_i x_i^{\sum_{j \le i} n_j}$. So we claim that we *must* use the monomial E in writing p, and then invoke induction to get the uniqueness of the expression of $p - E$. $\square$

Exercise 1.6. *Write $\sum_i x_i^n$ as a polynomial in the elementary symmetric polynomials, for $n \le 4$.*

It is important to note that it was by no means obvious that R^{S_n} would, itself, be isomorphic to a polynomial ring, as the following exercise shows.

Exercise 1.7. *Let $n = 2$, and $Z_2 = \{1, \tau\}$ act on R by $\tau \cdot x = -x$, $\tau \cdot y = -y$. Show that the τ-invariant subring R^{Z_2} is generated by x^2, xy, y^2, and is isomorphic to $\mathbb{Z}[a, b, c]/\langle b^2 - ac\rangle$. Show that this graded ring is not graded-isomorphic[2] to a polynomial ring.*

In fact there is a complete classification, due to C. Chevalley, of which finite groups G acting on R and preserving degree have R^G isomorphic to a polynomial ring.

2. Schubert polynomials

One moral of the story so far is that symmetric polynomials have many, many monomials in them, and that there are better ways to write down symmetric polynomials than the standard way (namely, by adding up monomials).

In this section we'll consider *partially* symmetric polynomials, that are symmetric under some exchanges of variables, but not all. Since we won't need full symmetry, we won't have to fix a finite number n of variables, and we let $R = \mathbb{Z}[x_1, x_2, \ldots]$ be the polynomial ring in an infinite set of variables.

Some references for this theory are [Man01, Mac91, BB93, BJS93, FS94].

2.1. Divided difference operators.

For $p \in R$, and $i \in \mathbb{N}$, define the action of the **simple reflection** operator r_i by

$$(r_i \cdot p)(x_1, \ldots, x_i, x_{i+1}, \ldots) := p(x_1, \ldots, x_{i+1}, x_i, \ldots)$$

so p is symmetric in x_i, x_{i+1} exactly if $r_i \cdot p = p$, or if $(1 - r_i) \cdot p = 0$.

It is interesting to note that if $q = (1 - r_i) \cdot p$, then $r_i \cdot q = -q$, and $q(x_1, \ldots, x_i, x_i, \ldots) = 0$. Using the long division algorithm for polynomials, we can write $q = (x_i - x_{i+1})q' + c$, where $\deg c < \deg(x_i - x_{i+1})$, i.e. $c \in \mathbb{Z}$. Setting $x_i = x_{i+1}$, we learn $0 = 0 + c$.

[2] In fact, there are no isomorphisms at all, even ones that ignore the grading. To show this, one can in some sense force a grading, as follows. Tensor both rings with $\mathbb{C}$, so we can use dimension arguments. Then prove that for any maximal ideal $\mathfrak{m}$ in a polynomial ring over $\mathbb{C}$, $\dim_{\mathbb{C}}(\mathfrak{m}^2/\mathfrak{m}^3) = \binom{\dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2)}{2}$, which is part of its being a "regular ring". But $\mathfrak{m} = \langle a, b, c\rangle \le R^{Z_2}$ doesn't satisfy this equality.

This allows us to define a **divided difference operator** ∂_i on $p \in R$ by

$$\partial_i p := \frac{p - r_i \cdot p}{x_i - x_{i+1}}$$

and we see that despite appearances, it is again a polynomial!

Exercise 2.1. (1) *If p is homogeneous of degree k, then $\partial_i p$ is homogeneous of degree $k-1$ (or zero).*

(2) *If $\partial_i p = 0$, then $\partial_i(pq) = p\partial_i q$.*

(3) *More generally (but less usefully) ∂_i satisfies the "twisted Leibniz rule":*

$$\partial_i(pq) = (\partial_i p)q + (r_i p)(\partial_i q).$$

In these ways, ∂_i behaves somewhat like a derivative.

(4) *Let p be a polynomial such that $\partial_i p = 0$ for all $i \neq n$. Show that p is a symmetric polynomial in $x_1, \ldots, x_n$.*

Let's start with $p = x_1^2 x_2$ and apply divided differences. This polynomial is symmetric in $x_3, x_4, \ldots$ (insofar as it doesn't involve any of them at all) so only ∂_1, ∂_2 can do anything interesting:

$$\begin{gathered}
\partial_1(x_1^2 x_2) = x_1 x_2 \partial_1 x_1 = x_1 x_2, \quad \partial_1 x_1 x_2 = 0 \\
\partial_2(x_1^2 x_2) = x_1^2 \partial_2 x_2 = x_1^2, \quad \partial_2 x_1^2 = 0 \\
\partial_2(x_1 x_2) = x_1 \partial_2 x_2 = x_1, \quad \partial_2 x_1 = 0 \\
\partial_1(x_1^2) = x_1 + x_2, \quad \partial_1(x_1 + x_2) = 0 \\
\partial_1 x_1 = \partial_2(x_1 + x_2) = 1
\end{gathered}$$

So from $x_1^2 x_2$, we can generate $x_1^2, x_1 x_2, x_1, x_1 + x_2, 1$ using divided difference operators. Here are some properties that speed up such calculations:

Exercise 2.2. (1) *Show $\partial_i^2 = 0$.*

(2) *If $|i - j| \neq 1$, show ∂_i and ∂_j commute.*

(3) *Show $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$.*

2.2. **Schubert polynomials: their definition and uniqueness.** We can embed S_n into S_{n+1} as those permutations that fix $n+1$. Taking the union over all n, we get a group S_∞ of those permutations π of $\mathbb{N}$ such that π moves only finitely many $i \in \mathbb{N}$, and conversely given a $\pi \in S_\infty$, for large enough n we have $\pi \in S_n$.

One of our big goals is to prove the following:

Theorem 2.3. *There exists uniquely a way of assigning to each $\pi \in S_\infty$ a homogeneous* **Schubert polynomial** *$\mathcal{S}_\pi$ such that $\mathcal{S}_{\mathrm{id}} = 1$, and*

$$\partial_i \mathcal{S}_\pi = \begin{cases} \mathcal{S}_{\pi \circ (i \leftrightarrow i+1)} & \text{if } \pi(i) > \pi(i+1), \text{ a } \textbf{descent} \text{ of } \pi \\ 0 & \text{if } \pi(i) < \pi(i+1), \text{ an } \textbf{ascent} \text{ of } \pi. \end{cases}$$

Existence will take a bunch of doing, but uniqueness is pretty easy.

Let $\ell(\pi)$ denote the number of **inversions** of π, which are pairs $\{(i,j) : i < j, \pi(i) > \pi(j)\}$. (This will turn out to be the degree of the polynomial $\mathcal{S}_\pi$.) Notice that it is very dependent on the order $1 < 2 < 3 < \ldots < n$ – the number of inversions of the transposition $(i \leftrightarrow j)$ is $2|i-j|-1$. So it's very rare to see people in this line of work write down permutations as products of disjoint cycles, which is optimized for conjugation-invariant calculations;

rather we use **one-line** notation $\underline{\pi(1)\pi(2)\dots\pi(n)}$. Multiplying π by a simple reflection $(i \leftrightarrow i+1)$ on the right acts on *places*, switching the values at positions i and $i+1$. (Multiplying on the left acts on *values*, plucking the numbers i and $i+1$ out of their places and putting them back in, switched.)

Exercise 2.4. (1) *Show* $\ell(\pi \circ (i \leftrightarrow i+1)) = \ell(\pi) \pm 1$, *with the sign depending on whether* i *is an ascent or descent of* π.
(2) *Show* $\ell(\pi) = \ell(\pi^{-1})$.
(3) *What is the maximum value of* $\ell(\pi)$, $\pi \in S_n$*?*

Proof of theorem 2.3, uniqueness only. First we claim that if π is not the identity permutation, then $\mathcal{S}_\pi \neq 0$ and $\deg \pi > 0$. Pick a descent i of π. Then $\partial_i \mathcal{S}_\pi = \mathcal{S}_{\pi\circ(i\leftrightarrow i+1))}$, which by induction on $\ell(\pi)$ is nonzero of degree $\ell(\pi) - 1$. Hence $\mathcal{S}_\pi$ is nonzero of degree $\ell(\pi)$.

Now we claim that $\bigcap_i \ker \partial_i = \mathbb{Z}$. If $p \in \bigcap_i \ker \partial_i$, then p is symmetric in *all* our variables. But unless p is constant, being a finite sum of monomials it uses a variable x_m with m maximized, and thus isn't symmetric in x_m, x_{m+1}, contradiction.

Since we are given $\partial_i \mathcal{S}_\pi$ for all i, and $\mathcal{S}_\pi$ is homogeneous of degree $\neq 0$, it is uniquely determined. ☐

Exercise 2.5. (1) *Determine* $\mathcal{S}_{(i\leftrightarrow i+1)}$.
(2) *Determine* $\mathcal{S}_\pi$ *for* $\pi \in S_3$, *thought of as the evident subgroup of the group of finite permutations of* $\mathbb{N}$.

2.3. **Reduced words for permutations.** A **word** Q **for a permutation** $\pi \in S_n$ is a list $(q_1, q_2, \dots, q_\ell)$ of elements of $1, 2, \dots, n-1$, such that π is the ordered product of the transpositions $(q_i \leftrightarrow q_i + 1)$. To avoid (the exceedingly rare) confusion with one-line notation, we'll underline permutations in one-line notation, e.g. $Q = 232$ gives $\pi = \underline{1432}$. If we start with Q instead of π, we may write $\prod Q$ for π.

It is easy to see (and we will prove in a moment) that every permutation in S_n is a product of such "adjacent" transpositions. How many are needed?

Theorem 2.6. *The shortest possible words for* $\pi \in S_n$ *have exactly* $\ell(\pi)$ *letters.*

Proof. If π is the identity, then $\ell(\pi) = 0$, and indeed can be written as an empty product.

Otherwise π has some d with $\pi(d) > \pi(d+1)$, called a **descent** of π (a very special type of inversion). Then we can correspond the *other* inversions (i, j) of π with *all* the inversions of $\pi \circ (d \leftrightarrow d+1)$, taking $(i,j) \mapsto ((d \leftrightarrow d+1)\cdot i, (d \leftrightarrow d+1)\cdot j)$. Hence $\ell(\pi) = \ell(\pi \circ (d \leftrightarrow d+1)) + 1$.

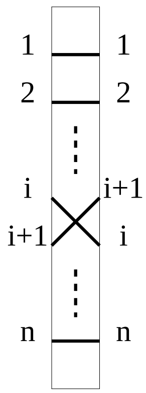
By induction, a shortest word for $\pi \circ (d \leftrightarrow d+1)$ has $\ell(\pi) - 1$ letters, so attach d at the end to get a word for π. This establishes the bound $|Q| \leq \ell(\pi)$ for the shortest Qs.

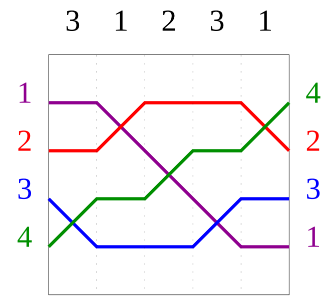
Now let Q be a shortest possible word for π, so each initial string of it is automatically shortest as well (for its respective product). Let Q' be Q with the last letter removed, and π' its product. By induction, $|Q'| = \ell(\pi')$, and the bijection on inversions again shows that $\ell(\pi) = 1 + \ell(\pi')$, which is $|Q|$. ☐

For this reason, $\ell(\pi)$ is called the **length** of π. A word Q is called **reduced** if $|Q| = \ell(\pi)$.

Exercise 2.7. *Show that* 12321, 13231, *and* 31231 *are reduced words for the same permutation in* S_4, *and find all the other reduced words for that permutation.*

2.3.1. *Wiring diagrams.* To each letter i (meaning $i \leftrightarrow i+1$) in a word Q, associate the following "card" pictured on the left. One very fruitful way to think about a word Q is in terms of the **wiring diagram** constructed by concatenating the cards of its letters, as pictured on the right. We can label the individual wires 1 through n on the left side, and carrying them through to the right side, we get $\prod Q$.





Proposition 2.8. *A word Q is reduced iff in its wiring diagram, no two wires cross twice. In this case, we can identify the crossings with the inversions of $\prod Q$, and call the wiring diagram* **reduced** *as well.*

Proof. If two wires cross twice, we can eliminate both crossings, to obtain a new diagram with the same connectivity. That one uses two fewer letters, so the original one was not reduced.

Assume now that no two wires cross twice. Then by the Jordan curve theorem or just the intermediate value theorem, wire i and j cross at all iff (i, j) is an inversion of $\prod Q$. So the number of inversions is the number of crossings is the number of letters, hence $\ell(\pi) = |Q|$ and Q is therefore reduced. $\square$

Exercise 2.9. *Show the following conditions on a permutation π are equivalent:*

(1) $\pi(i) = i$ *for* $i \leq m$.
(2) *Some reduced word for π does not use the letters* $1, \ldots, m-1$.
(3) *No reduced word for π uses the letters* $1, \ldots, m-1$.

One can think of a wiring diagram as the superimposition of the graphs of n piecewise-linear functions $f_1, \ldots, f_n$ (with finitely many corners) on the interval $[0, |Q|]$, and such that

- $f_1(0) < f_2(0) < \ldots < f_n(0)$,
- $f_{\pi(1)}(|Q|) < f_{\pi(2)}(|Q|) < \ldots < f_{\pi(n)}(|Q|)$,
- at most two wires meet at a crossing, and such points are isolated, and
- no crossing is directly above another.

(More specifically, in the diagrams constructed above the values at the endpoints will be the numbers 1 through n.)

We can reverse this connection, associating a word Q to such a superimposition. Look at the (finitely many) crossings from left to right (at all different x). For each crossing $f_i(x) = f_j(x)$, if $f_i - f_j$ changes sign from $x - \epsilon$ to $x + \epsilon$, put in the letter k if $f_i(x)$ is the kth and $(k+1)$st largest numbers from $\{f_m(x)\}_{m=1}^n$.

In particular, if we change a wiring diagram by continuously moving around the wires, we can study how the associated word changes as we pass through diagrams that violate the latter two conditions listed.

Exercise 2.10. *If we replaced "f_i piecewise linear with finitely many corners" by "f_i differentiable", it becomes impossible to associate a word Q. Give a horrible counterexample to demonstrate this.*

2.3.2. *The moves.*

Theorem 2.11. *Any two reduced words for $\pi \in S_n$ can be connected by the "moves"*

- $i\,j \to j\,i$ *for* $|i-j| > 1$
- $i(i+1)i \to (i+1)i(i+1)$, *called the* **braid** *move.*

For example, from 1232 we can make 1323, 3123, and that's it; these are all the reduced words for $\underline{2431}$.

Proof. Let Q be a reduced word for π (not the identity), and m the least number appearing in Q. We'll use the moves to get Q to end with the sequence $m\ m+1\ \ldots\ \pi(m)-1$. Ripping that sequence off, we get a reduced word for a shorter element π' of S_n, which by induction can be turned into any other reduced word using the moves. (Moreover, m turns out to be $\min\{i : \pi(i) > i\}$, and m occurs only once in our modified Q.)

Use the commuting move to move the first occurrence of m forward past any $j > m+1$. We get stuck when m runs into either an m or $m+1$, or makes it to the end. If we get to $m\ m$, then the resulting word is not reduced, contradiction. If we push the m all the way to the end, then $\pi(i) = i$ for $i < m$, and $\pi(m) = m+1$, so we're done.

Assume then that we have a sequence $m\ m+1\ \ldots\ m+k$, and continue to bubble the numbers to its right back through this sequence. The only ways to get stuck are to push the sequence to the end, in which case we're done; for it to get longer by acquiring an $m+k+1$ in which case increment k and continue; or for some $j \in (m, m+k)$ to get stuck in the middle, in which case we have the sequence $m\ \ldots\ j-1\ j\ j+1\ j$. Use the braid move to trade that for $m\ \ldots\ j-1\ j+1\ j\ j+1$, then commute that to $j+1\ m\ \ldots\ j-1\ j\ j+1$. Resume pushing the sequence $m\ \ldots\ j-1\ j\ j+1$ to the right.

When we've done pushing forward this way, we find that the resulting word has only one m, and terminates with $m\ m+1\ \ldots\ m+k$, as claimed. (Hence $k = \pi(m)-1$.) ☐

Exercise 2.12. *Follow the algorithm just given, starting with $Q = 3142352$, and see how the wiring diagrams change.*

Corollary 2.13. *To any reduced word Q for a permutation π, we can define an operator ∂_π as the product $\partial_{q_1} \ldots \partial_{q_{\ell(\pi)}}$, and it is independent of Q (i.e. ∂_π is well-defined).*

Exercise 2.14. *Let $\pi \circ \rho = \sigma$ be a product of two permutations. Show that $\ell(\pi) + \ell(\rho) \geq \ell(\sigma)$, and*

$$\partial_\pi \partial_\rho = \begin{cases} \partial_\sigma & \text{if } \ell(\pi) + \ell(\rho) = \ell(\sigma) \\ 0 & \text{if } \ell(\pi) + \ell(\rho) > \ell(\sigma). \end{cases}$$

Exercise 2.15. *Show that* any *two words (not necessarily reduced) for π can be related by the commuting move, the braid move, and insertion/deletion of pairs $i\ i$.*

2.4. **Schubert polynomials: existence.**

Theorem 2.16. *Schubert polynomials exist, and can be computed as follows:*

(1) *If* $w_0^n := \underline{n\ n{-}1\ n{-}2 \ldots 3\ 2\ 1\ n{+}1\ n{+}2 \ldots}$*, define* $\mathcal{S}_{w_0^n} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1$.
(2) *If* $\pi \in S_n$*, define* $\mathcal{S}_\pi := \partial_{\pi^{-1} w_0^n} \mathcal{S}_{w_0^n}$.

In particular, the degree of $\mathcal{S}_\pi$ *is* $\ell(\pi)$.

Exercise 2.17. *If* $\pi \in S_n$*, show* $\ell\left(\pi^{-1} w_0^n\right) = \binom{n}{2} - \ell(\pi)$.

Proof. We first need to show that this definition is well-defined. At which point, the fact that it satisfies the recursion will be quite automatic.

The problem is that while there is a *least* n such that $\pi \in S_n$, and that's certainly the most efficient one to use to calculate $\mathcal{S}_\pi$, the definition doesn't insist that n be chosen that way. So we need to show that different choices n, n' give the same value for $\mathcal{S}_\pi$.

Obviously it is enough to take $n' = n + 1$, then use induction. The claim then becomes

$$\partial_{\pi^{-1} w_0^{n+1}} \mathcal{S}_{w_0^{n+1}} = \partial_{\pi^{-1} w_0^n} \mathcal{S}_{w_0^n}$$

Let us first check the case $\pi = w_0^n = \pi^{-1}$:

$$\begin{aligned}
\partial_{w_0^n w_0^{n+1}} \mathcal{S}_{w_0^{n+1}} &= \partial_n \partial_{n-1} \cdots \partial_1 \mathcal{S}_{w_0^{n+1}} \\
&= \partial_n \partial_{n-1} \cdots \partial_1 x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 \\
&= \partial_n \partial_{n-1} \cdots \partial_2 x_1^{n-2} x_2^{n-2} \cdots x_{n-1}^1 \\
&= \partial_n \partial_{n-1} \cdots \partial_i x_1^{n-2} x_2^{n-3} \cdots x_{i+1}^{n-i-2} x_{i+2}^{n-i-2} \cdots x_{n-1}^1 \\
&= x_1^{n-2} x_2^{n-3} \cdots x_{n-1}^0 \\
&= \mathcal{S}_{w_0^n}
\end{aligned}$$

Now observe that if Q is a reduced word for $\pi^{-1} w_0^n$, then $Q\ n\ n-1\ \ldots 1$ is a reduced word for $\pi^{-1} w_0^{n+1}$. (Hint: use the exercise.) Hence

$$\partial_{\pi^{-1} w_0^{n+1}} = \partial_{\pi^{-1} w_0^n} \partial_n \partial_{n-1} \cdots \partial_1$$

so applying $\partial_{\pi^{-1} w_0^n}$ to the equation just derived, we get that $\mathcal{S}_\pi$ is well-defined. □

The definition above is due to Lascoux and Schützenberger in 1973, though their approach was slightly different from that presented here (and their original motivation was very different).

Lemma 2.18. *If* $\ell(\pi) = \ell(\rho)$*, then* $\partial_\pi \mathcal{S}_\rho$ *is* 1 *if* $\pi = \rho$*,* 0 *if* $\pi \neq \rho$.

Proof. If $\ell(\pi) = \ell(\rho) = 0$, then $\pi = \rho =$ the identity permutation and we're done. Otherwise let i be a descent of π, so π has a reduced word of the form Qi. Then

$$\partial_\pi \mathcal{S}_\rho = \partial_{\pi(i \leftrightarrow i+1)} \partial_i \mathcal{S}_\rho$$

which is zero unless ρ also has a descent at i (as would be implied by $\pi = \rho$). When it is nonzero, we continue to $\partial_{\pi(i\leftrightarrow i+1)} \mathcal{S}_{\rho(i\leftrightarrow i+1)}$, and use induction. □

Recall that our stated goal in introducing these polynomials was to provide a more compact way to study partially-symmetric polynomials. The following theorem realizes half that dream:

Theorem 2.19. *Schubert polynomials are linearly independent.*

Proof. Let $\sum_\pi c_\pi \mathcal{S}_\pi = 0$, with only finitely many $c_\pi \neq 0$. If they are not all zero, let ρ be a term with $c_\rho \neq 0$ and $\ell(\rho)$ maximized. Then $0 = \partial_\rho \sum_\pi c_\pi \mathcal{S}_\pi = \sum_\pi c_\pi \partial_\rho \mathcal{S}_\pi$. By the recursion, $\partial_\rho \mathcal{S}_\pi$ is homogeneous of degree $\ell(\pi) - \ell(\rho) \leq 0$, so can only be nonzero if $\ell(\pi) = \ell(\rho)$ by the lemma. The equation is now just $0 = c_\rho$, contradiction. $\square$

We'll show later that they're a basis of $\mathbb{Z}[x_1, x_2, x_3, \ldots]$ (theorem 4.11).

3. Computing Schubert polynomials

Exercise 3.1. *Compute all the Schubert polynomials for S_4, starting with $\mathcal{S}_{\underline{4321}}$ and going down using divided difference operators.*

It was already somewhat odd that *divided* difference operators take polynomials to polynomials. It is perhaps even more surprising that the Schubert polynomials we define using these divided *difference* operators all have positive coefficients! While Schubert polynomials date to 1973, proof of the positivity only came in 1991, with two groups independently proving R. Stanley's combinatorial conjecture for the coefficients. (This is why his name is on two "independent" papers.)

Write $\pi' \gtrdot \pi$ if

(1) $\pi' = \pi(ab)$,
(2) $\pi(a) < \pi(b)$, and
(3) $\{c : a < c < b\} \cap \{c : \pi(a) < \pi(c) < \pi(b)\} = \emptyset$.

This is a **Bruhat covering relation**, in that one can take its transitive closure to get the **Bruhat partial order** on S_∞ (due again to Chevalley!). Note that $\ell(\pi') = \ell(\pi) + 1$.

Theorem 3.2 (Monk's rule). $\mathcal{S}_{(i\ i+1)} \mathcal{S}_\pi = \sum_{\pi' = \pi(ab), \pi' \gtrdot \pi, a \leq i < b} \mathcal{S}_{\pi'}$.

Proof. Both sides are homogeneous of degree $\ell(\pi) + 1 > 0$, so we just need to show they agree after application of any ∂_j.

We describe the case analysis, and leave the reader to check the many cases. If $j \neq i$, then $\partial_j \mathcal{S}_{(i\ i+1)} = 0$, so the left side becomes $\mathcal{S}_{(i\ i+1)} \partial_j \mathcal{S}_\pi$. Then we split into the cases that j is an ascent of π, or a descent. When it's a descent, the desired equation becomes a Monk's rule equation for $\pi(j\ j+1)$ and can be assumed by induction.

Those $\partial_{j \neq i}$ done, consider the application of ∂_i.

$$\partial_i \mathcal{S}_{(i\ i+1)} \mathcal{S}_\pi = \mathcal{S}_\pi + (r_i \cdot \mathcal{S}_{(i\ i+1)}) \partial_i \mathcal{S}_\pi = \mathcal{S}_\pi + (\mathcal{S}_{(i+1\ i+2)} - 2\mathcal{S}_{(i\ i+1)} + \mathcal{S}_{(i-1\ i)}) \partial_i \mathcal{S}_\pi$$

so again, induction lets us calculate each of the products on the right side. $\square$

Corollary 3.3. $x_i \mathcal{S}_\pi = \sum_{\pi' = \pi(ib), \pi' \gtrdot \pi, i < b} \mathcal{S}_{\pi'} - \sum_{\pi' = \pi(ai), \pi' \gtrdot \pi, a < i} \mathcal{S}_{\pi'}$.

The corollary is perhaps simpler-looking, but unlike Monk's rule it has the drawback of having minus signs. If we move them to the other side of the equation, and take a special case, we get

Theorem 3.4 (Lascoux's "transition formula"). *Let $\pi \neq \mathrm{Id}$, and i be the last descent of π. Let $j = \max\{i' : \pi(i') < \pi(i)\}$, so $j \geq i+1$. Let $\pi' = \pi(ij)$. Then*

$$\mathcal{S}_\pi = x_i \mathcal{S}_{\pi'} + \sum_{a<i, \pi'(ai) \gtrdot \pi'} \mathcal{S}_{\pi'(ai)}.$$

Proof. The condition on π' ensures that the only positive term in the corollary above will be $\mathcal{S}_\pi$. Then move the other terms to the other side of the equation. $\square$

There is a generalization of this to other Southeasternmost "essential boxes" (see exercise 4.8) that we might need later.

Corollary 3.5. *Schubert polynomials have positive coefficients.*

Exercise 3.6. *Compute all the Schubert polynomials for S_4, starting with $\mathcal{S}_{\mathrm{Id}} = 1$ and going* up *using the transition formula.*

In the next section we unwrap the recursion formula to a direct, *manifestly positive* formula for Schubert polynomials as a sum of monomials. That is, if some term in the formula contributes a certain monomial, we can say for sure that the monomial will actually appear, rather than being canceled by some other term in the sum.

4. A positive formula for Schubert polynomials

Define a **pipe dream** to be a filling of an $n \times n$ square with two types of tiles, **crosses** $+$ and **elbows** ╯╭, such that the Southeast triangle (including the antidiagonal) gets only elbows. In a moment we'll put an additional requirement on them.

To a pipe dream P, we can associate three objects:

- A permutation π_P in S_n, given by following the "pipes" from the left side to the top. (Label the top $123\ldots n$, carry the numbers SW to the left side, then read top to bottom.)
- A monomial $m_P := \prod x_i^{e_i}$, where e_i is the number of $+$s in the ith row.
- A word Q_P in S_n's generators, by "reading" P from the top row to the bottom row, right to left in each row, with a $+$ in location (i, j) giving a letter $i + j - 1$.

Exercise 4.1. *Use wiring diagrams to show that*

(1) Q_P *is a word for* π_P, *and*
(2) Q_P *is reduced iff no two pipes cross twice.*

We'll require this condition on pipe dreams, hereafter. (This is not always done in the literature, and when it is not, pipe dreams with this condition are called "reduced".)

Our goal is to prove the following:

Theorem 4.2. *Each Schubert polynomial is a sum of monomials, one for each pipe dream:*

$$\mathcal{S}_\pi = \sum_{P:\ \pi_P = \pi} m_P = \sum_{P:\ \pi_P = \pi} \prod_i x_i^{\#\{+s \text{ in row } i \text{ of } P\}}.$$

This is the interpretation from [BB93] of the less graphical formula from [BJS93, FS94]. Since we know reduced words have the same length, this formula gives homogeneous polynomials, as it should.

Exercise 4.3. *Check this theorem for each $\pi \in S_3$.*

First let's study the pipe dreams for a given π, following [BB93]. Make the pipe dreams into the vertices of a **pipe dream graph**, where P, Q are connected if P has only one $+$ that Q doesn't, and therefore vice versa (exercise: why "therefore"?). To find all the Q that P is connected to, look at each inversion $i < j$ of π. The pipes i, j must cross at some $+$ tile x in P, but may have a near-miss in some $\lrcorner\!\ulcorner$ tile m. If we move the crossing from x to m, we get a new pipe dream for π, and every Q arises from one of these **moves**.

Exercise 4.4. *Construct the pipe dream graph for $\pi = 1432$.*

This suggests that we could find all the pipe dreams for π, if we were able to find one, and if the pipe dream graph were connected. These are related problems; the easiest way to show a graph is connected is to show every vertex has a chain to some "home" vertex.

(In fact, there is more than a pipe dream *graph*, of vertices and edges – there is a natural way to put in triangles, tetrahedra, and higher-dimensional "simplices", in such a way that the resulting pipe dream *simplicial complex* is topologically a ball [KM04].)

4.1. **Pipe dream polynomials satisfy the transition formula.** The bijection in the following proof is due to Anders Buch (personal communication).

Proof sketch of theorem 4.2. Let $\mathcal{S}'_\pi$ be the polynomial given by the formula. First we show that the $\mathcal{S}'_\pi$ satisfy the transition formula (theorem 3.4), and then use induction to show that $\mathcal{S}_\pi = \mathcal{S}'_\pi$ for all $\pi \in S_\infty$.

Recall that in the transition formula we first let r be the last descent position where $\pi(r) > \pi(r+1)$, then choose s maximal such that $\pi(r) > \pi(s)$.

Given a pipe dream for π, find the unique crossing C where the string starting at column r crosses the string starting at column s. (Draw the pipe dream so that the permutation maps columns to rows.) Now do the following:

(1) Remove C. If there are no $\lrcorner\!\ulcorner$s above C (in the same column), then we are done.
(2) Otherwise replace the lowest $\lrcorner\!\ulcorner$ above C with a new crossing C'. If the diagram is now reduced, then we are done.
(3) If it is not, the strings passing through C' cross exactly one other place. Let C be this other crossing, and go back to step 1.

What must now be checked:
The resulting pipe dream has the same number of crossings in each column, except that it may be missing exactly one crossing in some column. When this happens, the crossing is always missing in column r, and we have a pipe dream representing ν. Otherwise the resulting pipe dream represents one of the permutations $\pi \circ (i \leftrightarrow r) t_{ir}$. □

4.2. **The bottom pipe dream of π (lets one prove that Schubert polynomials span).** Define the **Lehmer code** $c_\pi : \{1, \ldots, n\} \to \mathbb{N}$ of $\pi \in S_n$ to be the list of n numbers $c_\pi(i) := \#\{j > i : \pi(j) < \pi(i)\} \in \{0, \ldots, n-i\}$. For example, the Lehmer code of 426351 is 313110.

Exercise 4.5. (1) *Find the Lehmer codes for all of S_3.*
(2) *What is the sum of the Lehmer code of π?*
(3) *Show that π is uniquely determined by its Lehmer code.*

(4) *If one attaches a* 0 *to the end of the Lehmer code of* π*, the result is again a Lehmer code; of which new permutation?*

Theorem 4.6. *Let* $c : \mathbb{N} \to \mathbb{N}$ *be a function that is eventually* 0*. Fix* n *such that* $c(i) \neq 0 \implies i + c(i) \leq n$*. Then* c *is the Lehmer code of a unique permutation* $\pi \in S_n$*, and there is a* **bottom pipe dream** *for* π *with* $c(i)$ $+$*s in row* i*, all flush-left in each row, i.e. no* ╯╭$+$ *appears.*

Proof. ... □

The **(Rothe) diagram** of a permutation π is the set of matrix boxes left over after crossing out each box south or east (but not southeast) of each 1 in the permutation matrix, *including* crossing out the 1s themselves. ***** example, probably stolen from [KY] *****

Exercise 4.7.
(1) *Argue that each connected component of a diagram is shaped like a partition, so has one northwest, one northeast, and one southwest corner.*
(2) *Show that when we shove the diagram to the left, and put* $+$*s in its boxes, we get the bottom pipe dream of* π*.*
(3) *Show that the number of boxes is* $\ell(\pi)$*.*

Exercise 4.8. *The* **Fulton essential set** [Fu92] *of* π *is the set of SE corners in* π*'s diagram, which for this exercise we call the "weak SE diagram". Define the* **strict SW diagram** *by crossing out each box south or west (but not southwest) of each* 1 *in the permutation matrix, now* not *crossing out the* 1*s themselves, and define the* **new essential set** *as the NE corners in this diagram. What is the relation between these two essential sets? (Try a large random example, figure out the relation, then prove it.)*

Theorem 4.9.
(1) *The bottom pipe dream for* π *is the only pipe dream whose* $+$*s are flush-left.*
(2) *If a pipe dream is not the bottom pipe dream, one can move a* $+$ *in it down (and left). Consequently, the graph is connected.*
(3) *The bottom pipe dream contributes the unique lex-largest term in the formula in theorem 4.2.*

Proof.
(1) By counting the $+$s in a flush-left pipe dream, we compute π's (unique) Lehmer code.
(2) Look for the lowest, then leftmost, occurrence of ╯╭$+$. (If none occurs, we have the bottom pipe dream.) This pair stands atop a ladder of $+$$+$s, then atop a pair ╯╭╯╭. We can replace the ╯╭$+$ on top and ╯╭╯╭ on bottom with ╯╭╯╭ on top and $+$╯╭ on bottom, called a **ladder move** [BB93], obtaining another (reduced) pipe dream for π.
(3) Let p be a non-bottom pipe dream for π, and p' another constructed from p by a downwards ladder move. Then p is lex-smaller than p'.

□

Exercise 4.10. *Show that the following are equivalent:*

(1) $\mathcal{S}_\pi$ *is a monomial;*
(2) π *has all descents, then all ascents;*
(3) π*'s diagram has one component, a partition in the NW corner.*

When these hold, π *is called a* **dominant** *permutation. In* S_3*, which permutations are dominant?*

Theorem 4.11. *Schubert polynomials form a $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, x_2, x_3, \ldots]$.*

Proof. We know they are linearly independent (theorem 2.19), so we have only to show that they span.

Let p be a polynomial we wish to expand in Schubert polynomials. Let its lex-largest term be $b \prod_{i=1}^{n} x_i^{c_i}$, for some $b \in \mathbb{Z}$. For $n \in \mathbb{N}$ large enough, (c_i) the Lehmer code of some $\pi \in S_n$. Then $p - b\mathcal{S}_\pi$ has a smaller lex-largest term, and we can apply induction.

(Note that the permutation π in the algorithm above isn't well-defined, since n isn't. How does this fit with the Schubert polynomials being linearly independent?) $\square$

Since the Schubert polynomials form a $\mathbb{Z}$-basis, there are unique **structure constants** $c_{\pi\rho}^{\sigma} \in \mathbb{Z}$ such that

$$\mathcal{S}_\pi \mathcal{S}_\rho = \sum_\sigma c_{\pi\rho}^{\sigma} \mathcal{S}_\sigma.$$

It is an amazing theorem, proven using an algebro-geometric interpretation of these numbers, that the $(c_{\pi\rho}^{\sigma})$ are nonnegative. (Of course Monk's rule, theorem 3.2, is a case of this.) It is a long-standing problem to compute these structure constants in a combinatorial way, i.e. as counting special pipe dreams or somesuch.

Exercise 4.12. *Let $p \in \mathbb{Z}[x_1, x_2, \ldots]$ be symmetric in x_i, x_{i+1}. Show that the expansion of p in Schubert polynomials doesn't use any $\mathcal{S}_\pi$ with $\pi(i) > \pi(i+1)$.*

5. Schur polynomials and Schur functions

Let us return to our original motivation: symmetric polynomials. Say a permutation π is n**-Grassmannian** if $\pi(i) > \pi(i+1)$ implies $i = n$, i.e. π is either the identity or has only that one descent. There is a correspondence between n-Grassmannian permutations π and partitions $(\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ with at most n parts, taking $\lambda_{n-i} = \pi(i) - i$.

Exercise 5.1. *Prove that this is a bijection.*

Define a **Schur polynomial** $S_\lambda(x_1, \ldots, x_n)$ of a partition λ and a number of variables n (more than the number of parts of λ) to be the Schubert polynomial of the corresponding n-Grassmannian permutation. If n is less than the number of parts of λ, let $S_\lambda(x_1, \ldots, x_n) = 0$.

Proposition 5.2. *The Schur polynomial $S_\lambda(x_1, \ldots, x_n)$ is indeed a polynomial in $x_1, \ldots, x_n$, and symmetric.*

If we fix n, and let λ vary over the partitions with at most n parts, the resulting polynomials are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ of symmetric polynomials.

Of course, since this ring $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ is a polynomial ring in the elementary symmetric polynomials (e_i), the monomials in the (e_i) also form a basis, and there are other interesting bases besides these.

Proof. Since π has ascents at all $i \neq n$, this Schubert polynomial is symmetric in $x_1, \ldots, x_n$ and separately in $x_{n+1}, x_{n+2}, \ldots$ Hence it can't use the latter set of variables at all (being a polynomial, with finitely many terms).

By exercise 4.12, the expansion of any $f \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ into Schubert polynomials uses only these n-Grassmannian Schubert polynomials. $\square$

The interesting thing afforded by this point of view (partitions rather than permutations) is that we can fix λ and let n vary, even to ∞.

A **power series** in infinitely many variables $x_1, x_2, \ldots$ is an assignment of a scalar "coefficient" (in, say, $\mathbb{Z}$) to each monomial $\prod_i x_i^{d_i}$, where the "exponent vector" d has $d_i \geq 0$ and $\sum_i d_i < \infty$. If only finitely many coefficients are nonzero, this is the same data as a polynomial. The vector space of power series forms a ring under usual multiplication of monomials, which depends on the fact that there are only finitely many ways to decompose d as a sum $b + c$ of two other exponent vectors.

Proposition 5.3. *For any partition λ and $n \in \mathbb{N}$,*

$$S_\lambda(x_1, \ldots, x_{n+1})|_{x_{n+1}=0} = S_\lambda(x_1, \ldots, x_n).$$

Consequently, $S_\lambda(x_1, x_2, \ldots)$ *is a well-defined power series, called a* **Schur function**, *symmetric in all variables.*

This will be a trivial consequence of the following well-known formulæ for Schur polynomials and Schur functions. Define a **semistandard Young tableau on** λ as a filling of the squares $\{(i,j) : j \leq \lambda(i)\}$ with natural numbers, weakly increasing in columns j and strictly increasing in rows i. For example, here are the five SSYT on the shape $\lambda = (4,3,2)$ with entries ≤ 3.

$$\begin{array}{llll} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \\ 3 & 3 & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & \\ 3 & 3 & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \\ 3 & 3 & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & \\ 3 & 3 & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & \\ 3 & 3 & & \end{array}$$

Theorem 5.4.

$$S_\lambda(x_1, \ldots, x_n) = \sum_\tau \prod_{i=1}^n x_i^{\#\text{ of is in } \tau}$$

where τ runs over the semistandard Young tableaux τ of shape λ, with entries $\leq n$.

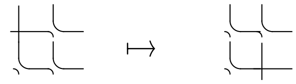
Hence

$$S_\lambda(x_1, \ldots, \ldots) = \sum_\tau \prod_{i=1}^n x_i^{\#\text{ of is in } \tau}$$

where τ runs over all *semistandard Young tableaux τ of shape λ.*

Proof. This is very similar to the pipe dream formula, suggesting that there is a correspondence between these SSYT and pipe dreams for π, the n-Grassmannian permutation corresponding to λ.

To associate a pipe dream to an SSYT on λ, start with $+$s in the boxes of λ, and *upside-down* ⌟⌜s in the rest of the first n rows. Then move each $+$ due Southeast until its new row matches the SSYT entry. The SSYT conditions ensure that boxes do go South not North, and no box outraces the boxes east or south of it. In particular, we can move the $+$s one by one, ordered so as to only move a $+$ after the boxes to its East and South move first, using only the move



which preserves the connectivity. Now flip this upside-down pipe dream right side up.

Because of the flip, this actually gives the formula $\sum_\tau \prod_{i=1}^n x_{n+1-i}^{\#\text{ of is in }\tau}$. But since π is n-Grassmannian this is symmetric in $x_1, \ldots, x_n$, so we can switch $x_i \leftrightarrow x_{n+1-i}$. $\square$

In particular, S_λ is homogeneous of degree $|\lambda|$, the number of boxes in the partition.

5.1. **The ring of symmetric functions.** A power series is a **symmetric function** if the coefficient on $\prod_i x_i^{d_i}$ is invariant under permutation of the variables. (So a symmetric function can only be a polynomial if it is constant.)

To what extent is such a thing a "function" of its variables? If we restrict to power series of bounded degree (i.e. $\sum d_i$ is bounded, for those monomials $\prod_i x_i^{d_i}$ with nonzero coefficient), we can specialize their variables x_i to values, so long as all but finitely many are set to 0. The ring of symmetric functions $\mathbf{Symm}$ is defined as this set of bounded-degree symmetric power series.

Exercise 5.5. *Define a* **monomial symmetric function** m_λ *for each partition* $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ *(with any finite number of parts), with coefficient* 1 *on* $\prod_i x_i^{d_i}$ *iff* d *is a permutation of* λ*, and* 0 *otherwise. In particular* m_λ *is homogeneous of degree* $|\lambda|$*. Show that the monomial symmetric functions are a* $\mathbb{Z}$*-basis for* $\mathbf{Symm}$*.*

Given a $\mathbb{Z}$-basis $\mathcal{B}$ for a ring (or module), one can speak of elements being $\mathcal{B}$**-positive** if their expansion in $\mathcal{B}$ has coefficients in $\mathbb{N}$. For example, theorem 5.4 shows that Schur functions are "monomial-positive".

Exercise 5.6. *Let* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *be three* $\mathbb{Z}$*-bases for* R*, such that* $\mathcal{A}$ *is* $\mathcal{B}$*-positive and* $\mathcal{B}$ *is* $\mathcal{C}$*-positive. Then* $\mathcal{A}$ *is* $\mathcal{C}$*-positive.*

Theorem 5.7. *The Schur functions are a* $\mathbb{Z}$*-basis for* $\mathbf{Symm}$*.*

Proof. First, we show they span. Let f be in $\mathbf{Symm}$, with a bound N on the total degree $\sum e_i$ of its summands. Let f' be the specialization $f|_{x_{N+1}=x_{N+2}=\ldots=0}$, which is now a symmetric polynomial in $x_1, \ldots, x_N$, and hence can be written uniquely as a $\mathbb{Z}$-combination of Schur polynomials with at most N rows:

$$f' = \sum_\lambda c_\lambda \mathbf{S}_\lambda(x_1, \ldots, x_N)$$

Even better, by degree considerations, each $|\lambda| \leq N$.

We want to show that $g := f - \sum_\lambda c_\lambda \mathbf{S}_\lambda$ is the symmetric function 0; we know that $g|_{x_{N+1}=x_{N+2}=\ldots} = 0$. Writing g as a combination $\sum d_\lambda m_\lambda$ of monomial symmetric functions, degree considerations again allow us to assume $|\lambda| \leq N$, so $d_\lambda \neq 0$ implies λ has at most N rows. Hence $m_\lambda|_{x_{N+1}=x_{N+2}=\ldots} \neq 0$. But then $g|_{x_{N+1}=x_{N+2}=\ldots} = \sum_\lambda d_\lambda m_\lambda|_{x_{N+1}=x_{N+2}=\ldots} \neq 0$, contradiction.

Linear independence is similar; any nontrivial linear relation $\sum_\lambda d_\lambda \mathbf{S}_\lambda = 0$ only involves λ with some bounded number of rows, and we can specialize to that many variables to reach a contradiction. $\square$

Exercise 5.8. *Find a total order on partitions, such that the change-of-basis matrix expressing a Schur function as a sum of monomial symmetric functions is upper triangular with* 1*s on the diagonal. Use this to give a different proof that the Schur functions are a* $\mathbb{Z}$*-basis of* $\mathbf{Symm}$*.*

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