

# THE LEFSCHETZ HYPERPLANE THEOREM FOR STACKS

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ABSTRACT. We use Morse theory to prove that the Lefschetz Hyperplane Theorem holds for compact smooth Deligne-Mumford stacks over the site of complex manifolds. For  $\mathfrak{Z} \subset \mathfrak{X}$  a hyperplane section,  $\mathfrak{X}$  can be obtained from  $\mathfrak{Z}$  by a sequence of deformation retracts and attachments of high-dimensional finite disc quotients. We use this to derive more familiar statements about the relative homotopy, homology, and cohomology groups of the pair  $(\mathfrak{X}, \mathfrak{Z})$ . We also prove some preliminary results suggesting that the Lefschetz Hyperplane Theorem holds for Artin stacks as well. One technical innovation is to reintroduce an inequality of Lojasiewicz which allows us to prove the theorem without any genericity or nondegeneracy hypotheses on  $\mathfrak{Z}$ .

## 1. INTRODUCTION

Consider an  $n$ -dimensional compact complex manifold  $X$  with a positive line bundle  $L$  and a global section  $s$  with zero locus  $Z \subset X$ . The Lefschetz hyperplane theorem (henceforth LHT) states that up to homotopy equivalence  $X$  is obtained from  $Z$  by attaching cells of dimension  $n$  or larger, or equivalently that  $\pi_i(X, Z) = 0$  for  $i < n$ . By the homological LHT we refer to the weaker statement that  $H_i(X, Z) = H^i(X, Z) = 0$  for  $i < n$ .

The LHT can fail for singular varieties, although it still holds for projective varieties as long as  $X - Z$  is a local complete intersection. See the book [GM88] for an exhaustive treatment of the classical LHT and its generalizations to singular varieties.

To motivate the introduction of stacks, we let  $X = \bar{\mathfrak{X}}$  be the coarse moduli space of a Deligne-Mumford (DM) stack  $\mathfrak{X}$ .  $X$  has local quotient singularities, so Poincaré-Lefschetz duality holds rationally for the pair  $(X, Z)$ . Then the fact that  $X - Z$  is affine and so  $H_i(X - Z) = 0$  for  $i > n$  implies that  $H_i(X, Z; \mathbb{Q}) = H^i(X, Z; \mathbb{Q}) = 0$  for  $i < n$ . It is natural to ask if this rational result is the shadow of an integral result for  $\mathfrak{X}$  itself. We will answer this question in the affirmative, but to prove it we will need to develop some basic tools of differential topology for stacks.

In Section 2 we show how the ideas of Bott's Morse theoretic proof of the LHT [Bot59] extend to the setting of DM stacks. We prove our main version of the LHT, which states that a stack can be constructed from a hyperplane section by a series of deformation retracts and attachments of high-dimensional disc quotients. Then we show how the usual statements

about relative homotopy groups and integral homology groups follow. The  
 25 homotopy groups of a stack are defined to be the homotopy groups of its  
 classifying space – this is discussed in more detail below.

In Section 3 we go back and carefully develop Morse theory of DM stacks.  
 The main lemmas of Morse theory have been proven for the *underlying space*  
 of a DM stack in [Hep09a], but we observe that the proofs work for the stack  
 30 itself. In this paper, a cell attachment refers to an honest 2-categorical  
 colimit. This is the strongest notion of gluing discussed in [Noo05] – it  
 doesn't exist unless one is gluing along embedded substacks, and even then  
 some care is required. Thus the main technical hurdle in Section 3 is showing  
 that a Morse function on  $\mathfrak{X}$  really does give a cell decomposition in the  
 35 strongest sense.

Finally, in Section 4 we discuss a purely algebraic proof of the homologi-  
 cal LHT for certain smooth stacks which are not DM. The argument applies  
 to real differentiable stacks admitting smooth proper maps from a compact  
 complex manifold, for instance global quotients of Kähler manifolds by com-  
 40 pact Lie groups. We also present some calculations which suggest that the  
 homological LHT holds for a much larger class of Artin stacks, although the  
 simple algebraic argument does not apply.

Another important point in this note is a classical result on the stability  
 of gradient descent flow. In Bott's original paper, he assumes that the zero  
 45 locus of  $s$  is “non-degenerate” in order to apply Morse theory. Here we show  
 that Bott's proof of the LHT works without any assumptions on the section.  
 The key observation is that the gradient flow of a function is stable as long  
 as it is close enough to a real analytic function near its critical loci. These  
 ideas were developed in the work of Lojasiewicz [Loj84], but to the author's  
 50 knowledge they are not widely applied in the Morse theory literature. We  
 discuss some of Lojasiewicz's results in Appendix A and refer to these ideas  
 often throughout the paper.

We refer to the references [BX03, BX11, Hei05, Noo05] for the foundations  
 of differentiable and topological stacks. The term Deligne-Mumford (DM)  
 55 stack will denote a stack over the site of complex manifolds admitting a  
 surjective representable étale map from a complex manifold  $X_0 \rightarrow \mathfrak{X}$  such  
 that  $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$  is finite. Such stacks are presentable by a proper étale  
 groupoid  $\mathfrak{X} \simeq [X_1 \rightrightarrows X_0]$ . Using a presentation, one can forget the complex  
 or differentiable structure of  $X_0, X_1$  and thus define forgetful functors from  
 60 stacks over complex manifolds to differentiable stacks to topological stacks.  
 We will use these forgetful functors as implicitly as they are used for complex  
 manifolds. We will denote the coarse moduli space of  $\mathfrak{X}$  by  $\tilde{\mathfrak{X}}$ . In Section  
 4, we will discuss differentiable Artin stacks, which we take to mean a stack  
 over the site of smooth manifolds admitting a smooth submersion from a  
 65 manifold.

The technical foundation for Morse theory of DM stacks has been laid by  
 Hepworth in [Hep09a]. In that paper he develops Morse functions, Riemann-  
 ian metrics, integration of vector fields, the strong topology on  $C^\infty(\mathfrak{X})$ , and

the main theorem of Morse theory for the underlying space  $\tilde{\mathfrak{X}}$ . We will build  
 70 on these foundations, showing that Morse theory gives cell decompositions  
 for the underlying stack and not just its coarse moduli space.

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 this project to me and for many useful conversations throughout.

## 2. HYPERPLANE THEOREM FOR DM STACKS

75 Let  $\mathfrak{X}$  be an  $n$ -dimensional Deligne-Mumford stack over the site of com-  
 plex manifolds and  $\mathfrak{L}$  a holomorphic line bundle on  $\mathfrak{X}$  with hermitian struc-  
 ture  $\mathfrak{h}$  of type  $k$  (defined below).  $s \in \Gamma(\mathfrak{X}, \mathfrak{L})$  will be a section of  $\mathfrak{L}$ , and  
 $\mathfrak{Z} \subset \mathfrak{X}$  will denote the zero locus of  $s$ . We assume furthermore that  $\tilde{\mathfrak{X}}$ , the  
 coarse moduli space of  $\mathfrak{X}$  treated as a topological stack, is compact.

Choose an étale atlas  $X_0 \rightarrow \mathfrak{X}$  such that  $\mathfrak{L}|_{X_0}$  admits a nowhere vanishing  
 holomorphic section  $\sigma$ . Then the  $(1, 1)$ -form

$$\Theta_0 = \bar{\partial}\partial \log \mathfrak{h}(\sigma, \sigma) \in \Omega^{1,1}(X_0)$$

80 descends to a hermitian global section of  $\Omega_{\tilde{\mathfrak{X}}}^{1,1}$ , the curvature form  $\Theta$ . In local  
 analytic coordinates we can write  $\Theta$  as  $H_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$ , and following [Bot59]  
 we call  $(\mathfrak{L}, \mathfrak{h})$  *type  $k$*  if the hermitian matrix  $H_{\alpha\beta}$  is positive on a space of  
 (complex) dimension  $k$  at every point. We call a hermitian line bundle of  
 type- $n$  *positive*. On compact complex manifolds, type  $k$  line bundles are  
 85  $(n - k)$ -ample in the sense of [Tot10], although the converse is not known.

This notion is the right one from the perspective of descent.  $\Omega^{p,q}$  is a  
 sheaf on the Artin site of complex manifolds, so the construction of  $\Theta$  above  
 works when  $X_0 \rightarrow \mathfrak{X}$  is a surjective submersion and not necessarily étale. In  
 addition, the property of being positive definite on a  $k$ -dimensional subspace  
 90 is Artin-local, i.e. if  $\varphi : X \rightarrow Y$  is a surjective submersion, then a hermitian  
 form on  $Y$  is type  $k$  if and only if its pullback to  $X$  is type  $k$ . Thus we can  
 define a type  $k$  holomorphic line bundle on a smooth *Artin* stack.

**Lemma 1.** *Let  $\mathfrak{X}$  be a smooth DM stack of dimension  $n$  which can be pre-  
 sented as a global quotient and whose coarse moduli space is projective. Then  
 95 a line bundle  $\mathfrak{L}$  is positive if and only if  $\mathfrak{L}^k$  descends to an ample line bundle  
 on  $\tilde{\mathfrak{X}}$  for some  $k$ .*

*Proof.* The paper [Bai57] proves that a sufficiently high power of a positive  
 line bundle defines a projective embedding  $\tilde{\mathfrak{X}} \hookrightarrow \mathbb{C}\mathbb{P}^N$ .

Conversely, assume  $\mathfrak{X} = [X/G]$  for a smooth quasiprojective  $X$  and re-  
 100 ductive  $G$ .<sup>1</sup>  $\mathfrak{L}$  is positive if and only if  $\mathfrak{L}^k$  is positive for any  $k > 0$ , so we  
 may assume that  $\mathfrak{L}$  is pulled back from a very ample invertible sheaf on  
 $\tilde{\mathfrak{X}}$ . There is a smooth  $G$ -equivariant compactification  $X \subset \hat{X}$  such that  $\mathfrak{L}$   
 extends to a  $G$ -ample bundle on  $\hat{X}$  and  $X = \hat{X}^{ss}(\mathfrak{L})$  [Tel00]. Let  $K \subset G$  be

<sup>1</sup>This property is fairly common. For algebraic stacks it is equivalent to  $\mathfrak{X}$  having  
 enough locally free sheaves [?].

a maximal compact subgroup and  $\Phi : \hat{X} \rightarrow \mathfrak{k}^*$  the moment map corresponding to the linearization  $\mathcal{L}$ .  $\Phi^{-1}(0)$  is a manifold with locally free  $K$  action, and  $\mathfrak{X} \simeq [\Phi^{-1}(0)/K]$ , and a  $K$ -invariant positive Hermitian structure on  $\mathcal{L}$  descends to a positive Hermitian structure on  $\mathfrak{X}$  (see [GS82], Section 3).  $\square$

Furthermore, if  $q : \mathfrak{X} \rightarrow \bar{\mathfrak{X}}$  is the projection to the coarse moduli space, then  $q_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\bar{\mathfrak{X}}}$ , so an invertible sheaf on  $\bar{\mathfrak{X}}$  has the same global sections when pulled back to  $\mathfrak{X}$ .

**Corollary 2.1.** *Let  $\mathfrak{Z} \subset \mathfrak{X}$  be the zero locus of a section of a positive line bundle, then topologically  $\mathfrak{Z} = q^{-1}(H)$  for some hyperplane section  $H \subset \bar{\mathfrak{X}}$ .*

One can think of the generalization of the LHT to type  $k$  line bundles as a “large fiber” generalization as discussed in [GM88].

**Definition 2.2.** Let  $\phi : \mathfrak{X} \rightarrow \mathfrak{X}'$  is a map of stacks. Choose complex manifolds  $X_0$  and  $X'_0$  fitting into the commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0} & X'_0 \\ \text{surjective} \downarrow & & \downarrow \text{étale} \\ \mathfrak{X} & \xrightarrow{\phi} & \mathfrak{X}' \end{array}$$

We define the *minimal rank* of  $\phi$  to be  $\min_{x \in X_0} \text{rank}_{\mathbb{C}}(D_x\phi_0 : T_x X_0 \rightarrow T_{\phi_0(x)} X'_0)$ . It does not depend on the choice of  $X_0$  and  $X'_0$ .

A map having minimal rank  $r$  implies that all fibers have codimension at least  $r$ . Of course the converse does not hold: a finite map need not be étale, for instance.

**Example 2.3.** If  $X \rightarrow X'$  is a  $G$ -equivariant map of complex manifolds with  $G$  action, then the minimal rank of  $X/G \rightarrow X'/G$  is the same as  $X \rightarrow X'$ .

If  $\mathcal{L}$  is a positive invertible sheaf on  $\mathfrak{X}'$ , and  $\phi : \mathfrak{X} \rightarrow \mathfrak{X}'$  has minimal rank  $r$ , then  $\phi^*\mathcal{L}$  is type  $r$  on  $\mathfrak{X}$ . Hence if  $H \subset \bar{\mathfrak{X}}'$  is a hyperplane section, then the LHT implies that the relative homotopy groups  $\pi_i(\mathfrak{X}, \phi^{-1}(H))$ , interpreted as the relative homotopy groups of the classifying spaces, vanish for  $i < r$ .

Now we will prove the LHT by analyzing the “Morse” function  $f = \mathfrak{h}(s, s) \in C^\infty(\mathfrak{X})$ . We refer to [Hep09a] for a discussion of smooth functions on smooth DM stacks. In particular, he defines the strong topology on  $C^\infty(\mathfrak{X})$  and proves that Morse functions are dense. Define the closed topological substack  $\mathfrak{X}^{\epsilon] = f^{-1}[0, \epsilon]$  and the open differentiable substack  $\mathfrak{X}^{\epsilon) = f^{-1}[0, \epsilon)$ . Note that  $\mathfrak{X}^{0] = f^{-1}(\{0\}) = \mathfrak{Z}$  is precisely the vanishing locus of  $s$ .

Fixing a Riemannian metric on  $\mathfrak{X}$ , we first show that  $f$  satisfies a Łojasiewicz inequality near its global minimum. This property of  $f$  guarantees that the gradient descent flow of  $f$  retracts  $\mathfrak{X}^{\epsilon]$  onto  $\mathfrak{Z}$  for sufficiently small  $\epsilon$ , but we will not make this precise until Section 3.

**Claim 2.4.** *For a sufficiently small  $\epsilon > 0$ , there are constants  $C > 0$  and  $\rho \in (0, 1)$  such that*

$$|f|^\rho \leq C|\nabla f| \tag{L}$$

on the substack  $\mathfrak{X}^\epsilon$ .

*Proof.* Note that it suffices to verify this inequality on  $\bar{\mathfrak{X}}^\epsilon$ . By compactness of  $\bar{\mathfrak{Z}}$  it suffices to verify (L) around each  $p \in \bar{\mathfrak{Z}}$ . Choose an étale coordinate patch  $U \rightarrow \mathfrak{X}$  containing  $p \in \bar{\mathfrak{Z}}$  such that  $\mathfrak{L}|_U$  is trivial. Then  $f_U = \mathfrak{h}(s, s) = h \cdot |s|^2$  is the product of a nonvanishing smooth function  $h$  with the real analytic function  $|s|^2$ , and thus by Theorem A.5 below the inequality (L) holds in a neighborhood of  $p$ .  $\square$

Next we will perturb  $f$  to be Morse without affecting its values on  $\mathfrak{X}^{\epsilon/2}$ , so that our new function will continue to satisfy (L). Let  $g : \mathfrak{X} \rightarrow [0, 1]$  be a smooth function such that  $g|_{\mathfrak{X}^{\epsilon/2}} = 0$  and  $g|_{\mathfrak{X} - \mathfrak{X}^\epsilon} = 1$ .

**Claim 2.5.** *There is a small open neighborhood  $U \subset C^\infty(\mathfrak{X})$  containing 0 such that for any  $\eta \in U$ , the new function  $f' = f + g\eta$  has the properties*

- $f'$  has no critical points on  $\mathfrak{X}^\epsilon - \mathfrak{X}^{\epsilon/2}$
- $f'$  is positive on  $\mathfrak{X} - \mathfrak{X}^{\epsilon/2}$
- the hermitian form  $\bar{\partial}\partial \log f'$  is positive on a space of dimension  $k$  at every point in  $\mathfrak{X} - \mathfrak{X}^{\epsilon/2}$ .

*Proof.* Multiplication by  $g$  induces a continuous map  $C^\infty \mathfrak{X} \rightarrow C^\infty \mathfrak{X}$ , so it suffices to prove that the set of  $\eta'$  for which  $f + \eta'$  satisfies the above properties is open in  $C^\infty \mathfrak{X}$  and contains 0.

Now choose a map  $V \rightarrow \mathfrak{X}$  and an open subset  $X_0 \subset V$  such that  $X_0 \rightarrow \mathfrak{X}$  is surjective étale and  $\text{cl}(X_0) \rightarrow \mathfrak{X}$  is proper. Then  $C^\infty(\mathfrak{X})$  is identified as a vector space with invariant functions in  $C^\infty(X_0)$ , and one can show that for such an atlas the strong topology on  $C^\infty(\mathfrak{X})$  is the subspace topology in  $C^\infty(X_0)$ . All three properties are open conditions on the derivatives of  $(f + \eta)|_{X_0}$  up to order 2, and they are satisfied by  $f$ , so the claim follows.  $\square$

Morse functions form a dense open subset of  $C^\infty(\mathfrak{X})$  [Hep09a], so there is some  $\eta \in U$ , where  $U \subset C^\infty \mathfrak{X}$  is defined in Claim 2.5, such that  $f + \eta$  is Morse. Consider the new function  $f' = f + g\eta$ .  $f'$  will satisfy all of the properties of Claim 2.5, and in addition it will be Morse on the substack  $\mathfrak{X} - \bar{\mathfrak{Z}}$ , because  $f'$  contains no critical points in  $\mathfrak{X}^\epsilon - \bar{\mathfrak{Z}}$  and  $f'$  agrees with  $f + \eta$  on  $\mathfrak{X} - \mathfrak{X}^\epsilon$ . From this point on we replace our function  $f$  with the new function  $f'$ .

Next we consider the index of the critical points of  $f$  away from  $\bar{\mathfrak{Z}}$ . For a critical point  $c$ , the tangent space  $T_c \mathfrak{X}$  and the Hessian  $H_c f$  are defined in [Hep09a] using the groupoid  $U \times_{\mathfrak{X}} U \rightrightarrows U$ , where  $U \rightarrow \mathfrak{X}$  is an étale coordinate patch containing  $c$ . The tangent space splits (non-canonically) as a representation of  $\text{Aut}_c$ ,  $T_c \mathfrak{X} \simeq T_c \mathfrak{X}_- \oplus T_c \mathfrak{X}_+$  on which  $H_c f$  is negative

175 definite and positive definite respectively. The index,  $\text{ind}_c$  is the isomorphism class of the representation  $T_c\mathfrak{X}_-$  of  $\text{Aut}_c$ .

Of course in a coordinate patch around  $c$ , the classical observation that the Hessian is negative definite on a space of dimension  $k$  still applies.

**Claim 2.6.** [Bot59] *At every critical point  $c \in \bar{X} - \bar{Z}$ , the index representation has  $\dim(\text{ind}_c) \geq k$ .*

Thus we have a function  $f : \mathfrak{X} \rightarrow \mathbb{R}$  that is Morse on the open substack  $\mathfrak{X} - \mathfrak{Z}$  and such that  $\mathfrak{Z} \subset \mathfrak{X}$  is a *stable* global minimum of  $f$ . Now from the results of Morse theory in Section 3, we have the main theorem:

**Theorem 2.7.** *As a topological stack,  $\mathfrak{X}$  can be obtained from  $\mathfrak{Z}$  by a finite sequence of deformation retracts and gluings of  $[D^r/G]$  along  $[S^{r-1}/G]$ , where  $r \geq k$ . Here we can assume for each cell attachment that  $G = \text{Aut}_c$  acting linearly on the unit disc in  $T_c\mathfrak{X}_-$  for some  $c \in \tilde{\mathfrak{X}} - \tilde{\mathfrak{Z}}$ .*

**Remark 2.8.** We only need  $\mathfrak{L}$  to be type  $k$  on the complement  $\mathfrak{X} - \mathfrak{Z}$ . Geometrically, if  $\mathfrak{Z}$  is the preimage of a hyperplane section  $H \subset \tilde{\mathfrak{X}}'$  under a map  $\phi : \mathfrak{X} \rightarrow \tilde{\mathfrak{X}}'$ , we only need  $k$  to be the minimal rank of the restriction  $\mathfrak{X} - \mathfrak{Z} \rightarrow \tilde{\mathfrak{X}}' - H$ . In particular  $\phi$  can have large fibers over  $H$ .

We will define more carefully what a deformation retract and a cell attachment are in Section 3 below, but suffice it so say that many corollaries about the various topological invariants follow.

195 Singular homology and cohomology are defined from the singular chain bi-complex of a simplicial manifold presenting  $\mathfrak{X}$ . The reader is referred to [Beh04] and [BX03] for an in-depth discussion and the equivalence between this and other notions of cohomology (i.e. de Rham or sheaf cohomology). One thing that's not described in [Beh04] is that  $H^i$  and  $H_i$  are homotopy 200 functors, but this follows from the usual argument, the key fact being the Poincaré lemma: by a local-to-global spectral sequence the projection  $p : \mathfrak{X} \times I \rightarrow \mathfrak{X}$  induces an isomorphism in homology and cohomology.

The homotopy groups are defined as  $\pi_i(\mathfrak{X}) = \pi_i(\mathbb{B}\mathfrak{X})$  for a classifying space  $\mathbb{B}\mathfrak{X}$ . We follow Noohi's treatment in [Noo08], in which the classifying 205 space is characterized up to weak equivalence by a universal property. He uses the "Haefliger-Milnor" construction to show the existence of classifying spaces in general, but in this note all stacks will admit presentations in which  $X_1$  and  $X_0$  are both metrizable – they are hoparacompact in the language of [Noo08] – in which case  $\mathbb{B}\mathfrak{X}$  is the geometric realization of the simplicial manifold  $X_p = X_0 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X_0$  (see [Seg68]) defined using an atlas  $X_0 \rightarrow \mathfrak{X}$ . 210

We recall the properties of  $\mathbb{B}\mathfrak{X}$  from [Noo08]. First there is an epimorphism  $\varphi : \mathbb{B}\mathfrak{X} \rightarrow \mathfrak{X}$  which is a universal weak equivalence, meaning that the base change along any map  $T \rightarrow \mathfrak{X}$  is a weak equivalence. This implies that

$\varphi$  induces isomorphisms on homology and cohomology, in other words the  
 215 homology of  $\mathbb{B}\mathfrak{X}$  agrees with the double complex homology defined above.<sup>2</sup>

Also the construction  $\mathbb{B}\mathfrak{X}$  is a *functor* from the category of (hoparacompact) stacks to the homotopy category of paracompact spaces. Finally, for  $\mathfrak{Z} \subset \mathfrak{X}$  an embedding, we can construct  $\mathbb{B}\mathfrak{Z}$  such that  $\mathbb{B}\mathfrak{Z} \subset \mathbb{B}\mathfrak{X}$  is a subspace. Thus we define the relative homotopy groups  $\pi_i(\mathfrak{X}, \mathfrak{Z}) := \pi_i(\mathbb{B}\mathfrak{X}, \mathbb{B}\mathfrak{Z})$ . We  
 220 omit base points from our notation, but the following theorem applies for any base point in  $\mathbb{B}\mathfrak{Z}$ .

**Corollary 2.9.** *We have  $\pi_i(\mathfrak{X}, \mathfrak{Z}) = 0$  for  $i < k$ . Hence from the long exact sequence of a pair  $\pi_i(\mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X})$  is an isomorphism for  $i < k - 1$  and surjective for  $i = k - 1$ .*

*Proof.* For a triple of topological stacks  $\mathfrak{Z} \subset \mathfrak{X} \subset \mathfrak{X}'$  we have a long exact sequence

$$\cdots \rightarrow \pi_i(\mathfrak{X}, \mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X}', \mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X}', \mathfrak{X}) \rightarrow \cdots$$

225 so if  $\pi_i(\mathfrak{X}', \mathfrak{X}) = \pi_i(\mathfrak{X}, \mathfrak{Z}) = 0$ , then  $\pi_i(\mathfrak{X}', \mathfrak{Z}) = 0$  as well. Thus the corollary follows by induction on the sequence of cell attachments and deformation retracts described in Theorem 2.7, once we verify that  $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$  for  $i < k$  when  $\mathfrak{X} \subset \mathfrak{X}'$  is a disc attachment or deformation retract.

For a deformation retract, the inclusion  $\mathbb{B}\mathfrak{X} \rightarrow \mathbb{B}\mathfrak{X}'$  is a homotopy equivalence, so  $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$  for all  $i$ .  
 230

Next let  $\mathfrak{X}' = \mathfrak{X} \cup_{[S/G]} [D/G]$  be a 2-categorical colimit, where  $D$  is a disc of dimension  $\geq k$ ,  $S$  its boundary sphere, and  $G$  a finite group acting linearly on the pair  $(D, S)$ . Let  $A \subset D$  be the complement of an open disc of half the radius of  $D$ , and consider the substack  $\mathfrak{X} \cup_{[S/G]} [A/G] \subset \mathfrak{X}'$ .

235 First observe that  $A \rightarrow [A/G]$  and  $D \rightarrow [D/G]$  are regular coverings, so  $\pi_i([A/G]) \rightarrow \pi_i([D/G])$  is an isomorphism for  $i < k - 1$  and surjective for  $i = k - 1$ ; hence  $([D/G], [A/G])$  is  $k - 1$ -connected. The interiors of  $\mathfrak{X} \cup_{[S/G]} [A/G]$  and  $[D/G]$  cover  $\mathfrak{X}'$ , so excision for homotopy groups implies that  $(\mathfrak{X}', \mathfrak{X} \cup_{[S/G]} [A/G])$  is  $k - 1$ -connected as well.

240 Finally, the equivariant map  $D \rightarrow D$  compressing  $A$  onto the boundary sphere  $S$  is equivariantly homotopic to the identity through a homotopy fixing  $S$ . Thus using the universal property of  $\mathfrak{X} \cup_{[S/G]} [D/G]$  we have a homotopy equivalence of pairs  $(\mathfrak{X}', \mathfrak{X} \cup_{[S/G]} [A/G]) \simeq (\mathfrak{X}', \mathfrak{X})$ , so we deduce  $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$  for  $i < k$ .  $\square$

245 Recall that  $H_*(\mathbb{B}\mathfrak{X}, \mathbb{B}\mathfrak{Z}) = H_*(\mathfrak{X}, \mathfrak{Z})$  and likewise for  $H^*$ , so

**Corollary 2.10.** *For the relative homology and cohomology, with integer coefficients, we have  $H^i(\mathfrak{X}, \mathfrak{Z}) = H_i(\mathfrak{X}, \mathfrak{Z}) = 0$  for  $i < k$ . As a consequence  $H^i(\mathfrak{X}) \rightarrow H^i(\mathfrak{Z})$  is an isomorphism for  $i < k - 1$  and injective for  $i = k - 1$ .*

<sup>2</sup>If  $X_p$  is a simplicial presentation for  $\mathfrak{X}$  and  $X'_p$  the pulled-back presentation of  $\mathbb{B}\mathfrak{X}$ , then  $X'_p \rightarrow X_p$  is a weak equivalence for all  $p$ , so the spectral sequence  $E_1^{p,q} = H^q(X_p) \Rightarrow H^{p+q}(\mathfrak{X})$  implies that  $H^*(\mathfrak{X}) \rightarrow H^*(\mathbb{B}\mathfrak{X})$  is an isomorphism.

250 **2.1. Topology of the underlying space.** For completeness we describe the consequences for the topology of the underlying spaces  $\bar{\mathfrak{Z}} \subset \bar{\mathfrak{X}}$ . In the introduction we mention that the homological LHT holds with rational coefficients for  $\bar{\mathfrak{Z}} \subset \bar{\mathfrak{X}}$ , but in fact we only need to invert the orders of the automorphism groups of points of  $\bar{\mathfrak{X}}$ .

**Proposition 2.11.** *The underlying space  $\bar{\mathfrak{X}}$  can be obtained from  $\bar{\mathfrak{Z}}$ , up to homotopy equivalence, by attaching finitely many disc quotients  $D^r/G$  along the boundary  $S^{r-1}/G$ . Here, as before,  $r \geq k$  and  $G = \text{Aut}_c$  acting linearly on the unit disc  $D^r \subset T_c\bar{\mathfrak{X}}_- \subset T_c\bar{\mathfrak{X}}$ .*

260 Using the Morse function  $f$  constructed above, we know that  $\bar{\mathfrak{X}}^{\epsilon}$  deformation retracts onto  $\bar{\mathfrak{Z}}$  for sufficiently small  $\epsilon$ . From this point the proposition is a direct application of the Morse lemmas 7.5-7.7 in [Hep09a], using the estimate on the index of the critical points of  $f$ .

**Corollary 2.12.**  *$H_i(\bar{\mathfrak{X}}, \bar{\mathfrak{Z}}; \mathbb{Z}[1/N]) = H^i(\bar{\mathfrak{X}}, \bar{\mathfrak{Z}}; \mathbb{Z}[1/N]) = 0$  for  $i < k$  as long as  $N$  is a common multiple of the orders of the isotropy groups of  $\bar{\mathfrak{X}}$  (of which there are finitely many because  $\bar{\mathfrak{X}}$  is compact).*

265 *Proof.* This follows from the fact that  $H_i(D/G, S/G; \mathbb{Z}[1/N]) = 0$  for  $i < k$ , which is really fact about group homology. One could also deduce the claim directly from Corollary 2.10 and the isomorphism  $H_*(\bar{\mathfrak{X}}; \mathbb{Z}[1/N]) \rightarrow H_*(\bar{\mathfrak{X}}; \mathbb{Z}[1/N])$ .  $\square$

### 3. MORSE THEORY OF DELIGNE-MUMFORD STACKS

270 Now we will prove the main theorem of the Morse theory for DM stacks used in the proof of Theorem 2.7.

**Theorem 3.1.** *Let  $f : \bar{\mathfrak{X}} \rightarrow \mathbb{R}$  be a smooth nonnegative function on a smooth Deligne-Mumford stack  $\bar{\mathfrak{X}}$  achieving its global minimum on the closed topological substack  $\bar{\mathfrak{Z}} = f^{-1}\{0\}$ . Assume that*

- 275
- $\bar{f} : \bar{\mathfrak{X}} \rightarrow \mathbb{R}$  is proper,
  - $f$  satisfies a Łojasiewicz inequality (L) at every  $p \in \bar{\mathfrak{Z}}$ , and
  - $f$  is Morse on the open substack  $\bar{\mathfrak{X}} - \bar{\mathfrak{Z}}$ , with critical points  $p_1, p_2, \dots, p_r$ .

280 *Then  $\bar{\mathfrak{X}}$  can be obtained from  $\bar{\mathfrak{Z}}$  by a finite sequence of deformation retracts and 2-categorical attachments of  $[D_i/\text{Aut}_{p_i}]$  along  $[S_i/\text{Aut}_{p_i}]$  where  $D_i \subset T_{p_i}\bar{\mathfrak{X}}_-$  is a small disc with boundary  $S_i$ .*

We fix a Riemannian metric  $g$  on  $\bar{\mathfrak{X}}$ , and we will study the flow along the vector field  $-\nabla f$ .<sup>3</sup>

**Fact 3.2.** [Hep09b] Given a vector field  $\xi$  on  $\bar{\mathfrak{X}}$  with compact support, there is a representable morphism  $\Phi : \bar{\mathfrak{X}} \times \mathbb{R} \rightarrow \bar{\mathfrak{X}}$  such that for any étale map

<sup>3</sup>On a DM stack a vector field corresponds to a consistent choice of vector field  $\xi_U$  on  $U$  for all étale  $U \rightarrow \bar{\mathfrak{X}}$ . The tangent bundle of an Artin stack is more complicated [Hep09b]

$M \rightarrow \mathfrak{X}$ , the base change

$$\begin{array}{ccc} M' & \longrightarrow & \mathfrak{X} \times \mathbb{R} \\ \Phi' \downarrow & & \downarrow \Phi \\ M & \longrightarrow & \mathfrak{X} \end{array}$$

satisfies  $D\Phi'(\frac{\partial}{\partial t})_{M'} = \xi_M$ . Furthermore there is an isomorphism  $e_\Phi : \Phi|_{\mathfrak{X} \times \{0\}} \Rightarrow \text{id}_{\mathfrak{X}}$ , and the flow morphism  $\Phi$  is uniquely determined by  $e_\Phi$ .

285 The map  $\Phi$  is a group action of  $\mathbb{R}$  up to 2-isomorphism, and so it induces an  $\mathbb{R}$  action on the underlying space  $\mathfrak{X}$  and an  $\mathbb{R}$  action up to homotopy on the classifying space  $\mathbb{B}\mathfrak{X}$ .

We will use existence of flows to deconstruct the topology of  $\mathfrak{X}$  in three steps. First we show that  $\mathfrak{X}^{\epsilon^l}$  deformation retracts onto  $\mathfrak{Z}$  for sufficiently  
290 small  $\epsilon$ . Then we show that if the interval  $[a, b]$  contains no critical values of  $f$ , then  $\mathfrak{X}^{[b]}$  deformation retracts onto  $\mathfrak{X}^{[a]}$ , and finally we show how the topology changes as  $a$  crosses a critical value.

First of all what does a deformation retract mean for stacks? Say  $j : \mathfrak{Z} \rightarrow \mathfrak{X}$  is a closed immersion, then a *deformation retract* consists of a homotopy  
295  $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$  and a projection  $\pi : \mathfrak{X} \rightarrow \mathfrak{Z}$ , along with 2-morphisms

$$\begin{array}{ccc} \mathfrak{Z} \times I & \xrightarrow{j \times \text{id}} & \mathfrak{X} \times I \\ \text{pr}_1 \downarrow & \nearrow & \downarrow h \\ \mathfrak{Z} & \xrightarrow{j} & \mathfrak{X} \end{array} \qquad \begin{array}{ccc} \mathfrak{X} & \xrightarrow{h_0} & \mathfrak{X} \\ \downarrow & \text{id} & \downarrow \\ \mathfrak{X} & \xrightarrow{h_1} & \mathfrak{X} \\ \downarrow & j \circ \pi & \downarrow \end{array}$$

This definition is a categorification of the definition of deformation retract of spaces (or what's called a strong deformation retract by some authors), and implies that  $j$  is a homotopy equivalence. Both the geometric realization  
300 functor and the underlying space functor respect products, so if  $\mathfrak{X}$  deformation retract onto  $\mathfrak{Z}$  in the above sense, then  $\tilde{\mathfrak{X}}$  deformation retracts onto  $\tilde{\mathfrak{Z}}$  and  $\mathbb{B}\mathfrak{X}$  is homotopy equivalent to  $\mathbb{B}\mathfrak{Z}$ .

**Remark 3.3.** A strict deformation retract of topological groupoids induces a deformation retract of stacks in the above sense.

**Remark 3.4.** A morphism factoring through a substack does so uniquely  
305 up to unique isomorphism, so the morphism  $\pi$  could be omitted from the definition of a deformation retract, instead just requiring that  $h_1 : \mathfrak{X} \rightarrow \mathfrak{X}$  factor through  $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ .

Thus if  $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$  is a deformation retract of  $\mathfrak{X}$  onto  $\mathfrak{Z}$ , and  $\mathfrak{X}' \subset \mathfrak{X}$  is a substack (not necessarily closed) containing  $\mathfrak{Z}$  such that  $h|_{\mathfrak{X}' \times I} : \mathfrak{X}' \times I \rightarrow \mathfrak{X}$   
310 factors through  $\mathfrak{X}'$ , then  $\mathfrak{X}'$  deformation retracts onto  $\mathfrak{Z}$  as well via this restricted  $h$ .

**Remark 3.5.** For an embedding of topological spaces  $j : Z \rightarrow X$ , we can form the mapping cylinder  $M(j) := X \times \{0\} \cup Z \times [0, 1] \subset X \times [0, 1]$ . If

$j : \mathfrak{Z} \rightarrow \mathfrak{X}$  is an embedding of topological stacks, we choose a presentation  
 315 for  $\mathfrak{X}$  and form the mapping cylinder  $\mathfrak{M}(j) = [M(j_1) \rightrightarrows M(j_0)] \subset \mathfrak{X} \times [0, 1]$ .  
 Morita equivalent presentations give Morita equivalent mapping cylinders  
 because the mapping cylinder construction for spaces respects fiber products  
 and epimorphisms. Furthermore,  $\mathfrak{M}(j) \cong \mathfrak{X} \times \{0\} \cup_{\mathfrak{Z} \times \{0\}} \mathfrak{Z} \times [0, 1]$  is a pushout  
 320 in the 2-category of topological stacks. Thus we can simplify our definition  
 of a deformation retract further as *a map  $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$  such that  $h_1$  factors  
 through  $\mathfrak{Z}$ , along with an isomorphism  $h|_{\mathfrak{M}(j)} \cong \text{pr}_1|_{\mathfrak{M}(j)}$ .*

We are now ready to prove the

**Lemma 2.** *For  $\epsilon > 0$  sufficiently small,  $\mathfrak{X}^\epsilon$  contains no critical points of  
 $f$ , and the gradient descent flow of  $f$  provides a deformation retract of  $\mathfrak{X}^\epsilon$   
 325 onto  $\mathfrak{Z}$ .*

*Proof.* The key idea is to find, for  $\epsilon$  small enough, an atlas for  $\mathfrak{X}^\epsilon$  that  
 is *equivariant* for the action of the semigroup  $[0, \infty)$ . Start with an étale  
 atlas  $a : X_0 \rightarrow \mathfrak{X}$ .  $f|_{X_0}$  still satisfies the inequality (L) at every point in  
 $Z_0 = (f_{X_0})^{-1}\{0\}$ .

330 By Corollary A.4 there is an open subset  $U_0 \subset X_0$  containing  $Z_0$  such  
 that  $(-\nabla f)_{X_0}$  is integrable to a flow  $\Phi_0 : U_0 \times [0, \infty) \rightarrow U_0$  which extends  
 uniquely to a deformation retract  $h_0 : U_0 \times [0, \infty) \rightarrow U_0$  onto  $Z_0$ .

The étale map  $U_0 \rightarrow \mathfrak{X}$  is equivariant with respect to the action of the  
 semigroup  $[0, \infty)$ , so  $[0, \infty)$  acts on  $U_1 := U_0 \times_{\mathfrak{X}} U_0$  as well, and we would  
 like to lift the action to  $\infty$ , i.e. find a lift

$$\begin{array}{ccc}
 U_1 \times [0, \infty) & \xrightarrow{\Phi_1} & U_1 \\
 \downarrow & \dashrightarrow^{\exists! h_1} & \downarrow (s,t) \\
 U_1 \times [0, \infty) & \xrightarrow{(s,t)} & U_0 \times U_0 \times [0, \infty) \xrightarrow{h_0 \times h_0} U_0 \times U_0
 \end{array}$$

One shows the existence and uniqueness of  $h_1$  using the fact that, because  
 $\mathfrak{X}$  is DM,  $U_1 \rightarrow U_0 \times U_0$  is a proper immersion.

335 It suffices to show existence and uniqueness of  $h_1$  in a neighborhood of  
 each point  $(\infty, p) \in [0, \infty) \times U_1$ . If  $(x, y) \in U_0 \times U_0$  is the image of  $(\infty, p)$ , then  
 there is a compact neighborhood  $N$  of  $(x, y)$  such that  $(s, t)^{-1}(N)$  is a disjoint  
 union of connected compact subsets, each mapped homeomorphically onto  
 its image by  $(s, t)$ . Finally for a neighborhood of  $(\infty, p)$  whose image in  
 340  $U_0 \times U_0$  is contained in  $N$  there exists a unique lift.

Thus we get a map of groupoids  $h : [0, \infty) \times (U_1 \rightrightarrows U_0) \rightarrow (U_1 \rightrightarrows U_0)$   
 providing a deformation retract of  $U_1 \rightrightarrows U_0$  onto the subgroupoid  $Z_1 \rightrightarrows$   
 $Z_0$ . By Remark 3.3  $h$ , gives a deformation retract of the open substack  
 $\mathfrak{U} = [U_1 \rightrightarrows U_0]$  onto  $\mathfrak{Z}$ . Because  $\mathfrak{Z}$  is compact,  $\mathfrak{X}^\epsilon$  is contained in  $\mathfrak{U}$  for  
 345 sufficiently small  $\epsilon$ , so by Remark 3.4 the result follows.  $\square$

Next we study what happens when there are no critical points between  $a$   
 and  $b$ .

**Proposition 3.6.** *If  $f$  has no critical points in  $\mathfrak{X}^{[a,b]}$ , then there is a deformation retract of  $\mathfrak{X}^{[a,b]}$  onto  $\mathfrak{X}^a$ .*

*Proof.* Let  $\xi$  be a vector field with compact support such that  $\xi \cdot f \leq 0$  and  $\xi \cdot f = -1$  on  $\mathfrak{X}^{[a,b]}$ . Let  $\Phi : \mathfrak{X} \times \mathbb{R} \rightarrow \mathfrak{X}$  be the flow of  $\xi$  and form the composition

$$h : \mathfrak{X}^{[a,b]} \times [0, 1] \xrightarrow{(\text{id}_{\mathfrak{X}^{[a,b]}}, \max(0, t(f-a)))} \mathfrak{X}^{[a,b]} \times [0, \infty) \xrightarrow{\Phi} \mathfrak{X}$$

350  $\bar{f}$  decreases along the flow lines of  $\bar{\Phi}$  and decreases with constant rate 1 in  $\mathfrak{X}^{[a,b]}$ , so it follows that  $h : \mathfrak{X}^{[a,b]} \times [0, 1] \rightarrow \mathfrak{X}$  factors through  $\mathfrak{X}^{[a,b]}$  and that  $h_1$  factors through  $\mathfrak{X}^a$ .<sup>4</sup> The isomorphism  $e_\Phi : \Phi|_{\mathfrak{X} \times \{0\}} \Rightarrow \text{id}_{\mathfrak{X}}$  induces an isomorphism  $h|_{\mathfrak{M}(j)} \simeq \text{pr}_1|_{\mathfrak{M}(j)}$  because  $\max(0, t(f-a)) = 0$  on the substack  $\mathfrak{M}(j) \subset \mathfrak{X}^{[a,b]} \times I$ .  $\square$

355 **Remark 3.7.** An argument identical to that in theorem 3.1 of [Mil63] or theorem 7.5 of [Hep09a] can also be used to show that  $\mathfrak{X}^a \simeq \mathfrak{X}^b$  as stacks.

**Proposition 3.8.** *Let  $c \in \mathbb{R}$  be a critical value of  $f$ , and assume that  $f$  is Morse near  $f^{-1}\{c\}$ . Denote the critical points in  $f^{-1}\{c\}$  by  $p_1, \dots, p_r$ . Then for  $\epsilon$  sufficiently small there is a closed topological substack*

$$\mathfrak{X}^{c-\epsilon] \subset \mathfrak{X}' \subset \mathfrak{X}^{c+\epsilon]}$$

such that  $\mathfrak{X}^{c+\epsilon]}$  deformation retracts onto  $\mathfrak{X}'$ , and

$$\mathfrak{X}' \simeq \mathfrak{X}^{c-\epsilon] \cup \coprod_i [S_i / \text{Aut}_{p_i}] \coprod \coprod_i [D_i / \text{Aut}_{p_i}],$$

where  $D_i$  and  $S_i$  are the disc and sphere of radius  $\epsilon$  in the index representation of  $\text{Aut}_{p_i}$ .

360 *Proof.* Again, the idea of the proof is identical to the discussion in [Mil63] or [Hep09a], one just has to check that the argument gives a cell attachment of stacks and not just underlying spaces.

For each  $p_i$ , identify an open substack containing  $p_i$  with  $[U_{p_i} / \text{Aut}_{p_i}]$ , where  $U_{p_i}$  is a ball around the origin in  $T_{p_i}\mathfrak{X}$ , and coordinates have been chosen in  $T_{p_i}\mathfrak{X} = T_{p_i}\mathfrak{X}_+ \oplus T_{p_i}\mathfrak{X}_-$  such that the function  $f|_{U_i} = c + |u_+|^2 - |u_-|^2$ . Choose an  $\epsilon$  smaller than the radii of all of these Morse coordinate patches, and define the closed substack of  $\mathfrak{X}^{c+\epsilon]}$

$$\mathfrak{X}' = \{f \leq c - \epsilon\} \cup \bigcup_i \{u_+ = 0 \text{ and } |u_-|^2 \leq \epsilon\}$$

Where we have used the slightly informal notation  $\{u_+ = 0 \text{ and } |u_-|^2 \leq \epsilon\}$  for the closed substack  $[D_i / \text{Aut}_{p_i}]$ .

365 Theorem 16.9 of [Noo05] gives a criterion for checking that  $\mathfrak{X}'$  is in fact a 2-categorical union. First of all,  $\mathfrak{X}'$  is manifestly a gluing in the language of [ibid.], in that  $\mathfrak{X}' \setminus \coprod_i [S_i / \text{Aut}_{p_i}] \simeq \mathfrak{X}^{c-\epsilon] \sqcup \coprod_i [(D_i \setminus S_i) / \text{Aut}_{p_i}]$ . The

<sup>4</sup>A map  $\psi : \mathfrak{Y} \rightarrow \mathfrak{X}$  factors through  $\mathfrak{X}^{[a,b]}$  if and only if  $f \circ \psi : \mathfrak{Y} \rightarrow \mathbb{R}$  factors through  $(-\infty, b]$ , so it suffices to check on the level of underlying spaces.

second criterion – the existence of an atlas  $a : X'_0 \rightarrow \mathfrak{X}'$  such that the pair of invariant subspaces  $a^{-1}([S_i/\text{Aut}_{p_i}]) \subset a^{-1}([D_i/\text{Aut}_{p_i}])$  satisfies the local left lifting property (LLLP) with respect to the class of covering maps – is tautologically satisfied because every pair satisfies the LLLP with respect to local homeomorphisms.<sup>5</sup> So we have really verified that a union of two substacks of a DM stack is always a 2-categorical union.

The rest of the argument for proceeds exactly as in [Mil63] and [Hep09a]. One introduces an auxiliary function  $F$  such that:  $F$  agrees with  $f$  outside of the Morse coordinate patches (on which  $F \leq f$ ),  $\{F \leq c + \epsilon\} = \mathfrak{X}^{c+\epsilon}$ , and  $F(p_i) < c - \epsilon$ . Then one uses Proposition 3.6 to retract  $\mathfrak{X}^{c+\epsilon}$  onto  $\{F \leq c - \epsilon\}$ , a substack which differs from  $\mathfrak{X}^{c-\epsilon}$  only within the Morse coordinate patches. Finally one constructs a manifestly  $\text{Aut}_{p_i}$  equivariant deformation retract of  $\{F \leq c - \epsilon\}$  onto  $\mathfrak{X}'$  in each  $[U_i/\text{Aut}_{p_i}]$  which restricts to the identity on  $\mathfrak{X}^{c-\epsilon} \cap [U_i/\text{Aut}_{p_i}]$ . These can then be glued to give the global deformation retract of  $\{F \leq c - \epsilon\}$  onto  $\mathfrak{X}'$ .  $\square$

#### 4. HOMOLOGICAL LEFSCHETZ THEOREM FOR ARTIN STACKS

The LHT for Artin<sup>6</sup> stacks is beyond the current reach of Morse theoretic techniques. The theory of integration of vector fields on Artin stacks has been worked out in [Hep09b], but Morse theory has not been developed yet.

In this section we describe a local-to-global proof of the homological LHT that applies to differentiable Artin stacks admitting a presentation by compact manifolds.

In the complex category, proper Artin stacks are close to being gerbes, so in order to have a version of the LHT with broader applications, we will leave the category of complex stacks. For instance, the following theorem applies to a global quotient of a compact complex manifold by a compact Lie group.

**Theorem 4.1.** *Let  $\mathfrak{L}$  be a hermitian line bundle on a differentiable Artin stack  $\mathfrak{X}$ , and let  $s$  be a section. Assume that there exists a proper submersion  $X_0 \rightarrow \mathfrak{X}$  from a compact complex manifold  $X_0$  such that the pullback  $L_0 = \mathfrak{L}|_{X_0}$  is holomorphic of type  $k$  and the section  $s|_{X_0}$  is holomorphic.*

*As before let  $\mathfrak{Z} \subset \mathfrak{X}$  be the topological substack on which  $s$  vanishes. Then  $H^i(\mathfrak{X}, \mathfrak{Z}) = H_i(\mathfrak{X}, \mathfrak{Z}) = 0$  for  $i < k$ .*

*Proof.* From the proper submersion  $X_0 \rightarrow \mathfrak{X}$  we get a closed, full, saturated sub-groupoid  $Z_\bullet \subset X_\bullet$  presenting  $\mathfrak{Z} \subset \mathfrak{X}$ . Let  $\{X_p\}$  and  $\{Z_p\}$  denote the respective simplicial nerves.

We can form the smooth function  $f = |s|^2$  on  $\mathfrak{X}$ , and as before  $\mathfrak{Z}$  is precisely the zero locus of  $f$ . In particular  $Z_p$  is the zero locus of  $f|_{X_p} =$

<sup>5</sup>Although it is not necessary here, one could even find an atlas  $a : X'_0 \rightarrow \mathfrak{X}'$  for which  $a^{-1}([D_i/\text{Aut}_{p_i}]) = D_i$  and likewise for  $S_i$

<sup>6</sup>An Artin stack is just a differentiable stack admitting a surjective submersion from a manifold

405  $(f|_{X_0})|_{X_p}$ . Additionally  $Z_p$  is the preimage of  $Z_0$  under any of the  $p + 1$  simplicial maps  $X_p \rightarrow X_0$ .

Now the function  $f|_{X_0}$  is the norm squared of a section of a hermitian line bundle of type  $k$ . Thus as in Section 2 above we can perturb  $f|_{X_0}$  to a function which we call  $\phi_0$  which

- 410
- satisfies a Lojasiewicz inequality near its global minimum  $Z_0$ ,
  - is Morse(-Bott) away from  $Z_0$  with critical points of index  $\geq k$ .

Define the functions  $\phi_p = \phi_0|_{X_p}$  using the  $0^{\text{th}}$  simplicial face map  $X_p \rightarrow X_0$  to restrict to  $X_p$ . Note that  $\phi_0$  is no longer invariant (does not necessarily descend to a smooth function on  $\mathfrak{X}$ ), so we must be explicit about which  
 415 simplicial maps we use. Both bulleted properties are preserved when one restricts along a surjective submersion, so the  $\phi_p$  are functions with Lojasiewicz global minima  $Z_p = \phi_p^{-1}\{0\}$  that are Morse-Bott away from  $Z_p$  with critical manifolds of index  $\geq k$ . It follows from non-degenerate Morse theory that  $H_i(X_p, Z_p) = H^i(X_p, Z_p) = 0$  for all  $i < k$ .

420 From the bicomplex computing  $H_*(\mathfrak{X}, \mathfrak{Z})$  we have a homological spectral sequence

$$E_{p,q}^1 = H_q(X_p, Z_p) \Rightarrow H_{p+q}(\mathfrak{X}, \mathfrak{Z}) \quad (1)$$

so the vanishing of  $H_i(X_p, Z_p)$  implies  $H_i(\mathfrak{X}, \mathfrak{Z}) = 0$  for  $i < k$ . A similar spectral sequence implies the result for cohomology.  $\square$

**Example 4.2.** This theorem applies to global quotients even when the  
 425 group does not act holomorphically. For instance let  $G = \mathbb{Z}/2\mathbb{Z}$  acting on  $\mathbb{C}\mathbb{P}^l$  by complex conjugation  $[z_0 : \cdots : z_l] \mapsto [\bar{z}_0 : \cdots : \bar{z}_l]$ . By identifying  $\mathbb{C}^{l+1} = \mathbb{C} \otimes \mathbb{R}^{l+1}$ , real subspaces of  $\mathbb{R}^{l+1}$  induce complex subspaces of  $\mathbb{C}^{l+1}$ , and thus we have an embedding  $\mathbb{R}\mathbb{P}^l \subset \mathbb{C}\mathbb{P}^l$  as the fixed locus of  $G$ .

Complex conjugation acts equivariantly on  $\mathcal{O}_{\mathbb{C}\mathbb{P}^l}(1)$ , and an invariant holomorphic section corresponds to a form with real coefficients  $s = r_0 z_0 +$   
 430  $\cdots r_l z_l$ . The zero locus of  $s$  is a hypersurface  $H \subset \mathbb{C}\mathbb{P}^l$  induced by a real hypersurface in  $\mathbb{R}^{l+1}$ , and by a real coordinate change we may assume that  $H = \{[0 : * : \cdots : *]\}$ .

Let  $p = [1 : 0 : \cdots : 0] \in \mathbb{C}\mathbb{P}^l$ . Then  $\mathbb{C}\mathbb{P}^l - \{p\}$  deformation retracts equivariantly onto  $H$ , so we have

$$\begin{aligned} H_G^i(\mathbb{C}\mathbb{P}^l, H) &= H_G^i(\mathbb{C}\mathbb{P}^l, \mathbb{C}\mathbb{P}^l - \{p\}) = H_G^i(\mathbb{C}^l, \mathbb{C}^l - \{0\}) \\ &\simeq H^{i-l}(\mathbb{B}[*/\mathbb{Z}/2]; \Omega) \end{aligned}$$

In this last expression, we have used the Thom isomorphism for the bundle  
 435  $[\mathbb{C}^l/G] \rightarrow [*/G]$ , and  $\Omega$  is a local system on  $\mathbb{B}[*G]$  which depends on the orientability of this bundle. Thus we have  $H_G^i(\mathbb{C}\mathbb{P}^l, H) = 0$  for  $i < l$ .

**Example 4.3.** Non-holomorphic group actions also arise naturally as subgroups of the group of unit quaternions acting by left multiplication on  $\mathbb{P}(\mathbb{H}^l)$ , where choose either  $i, j$ , or  $k$  as the complex structure on  $\mathbb{H}^l$ .

440 **Example 4.4.** Let  $X$  be a projective variety of dimension  $n$  and  $\mathfrak{X} \rightarrow X$  a gerbe for a compact group  $G$ , and let  $Z \subset X$  be a hyperplane section. The

theorem implies that the cohomology of  $\mathfrak{X}$  agrees with the cohomology of the restricted gerbe  $\mathfrak{Z} \rightarrow Z$  in degree  $< n - 1$ .

In this case we can also see this by comparing the Leray-Serre spectral sequence  $E_2^{p,q} = H^p(X; H^q(BG)) \Rightarrow H^{p+q}(\mathfrak{X})$  and the corresponding sequence for  $\mathfrak{Z} \rightarrow Z$ . The result follows because, by the classical LHT, the restriction map on the  $E^2$  page  $H^p(X; H^q(BG)) \rightarrow H^p(Z; H^q(BG))$  is an isomorphism for  $p < n - 1$ .

Unlike Theorem 4.1, the Leray-Serre argument in the example above does not require  $G$  to be compact, which suggests that Theorem 4.1 might hold for a much larger class of Artin stacks. We also observe that the homological LHT also applies to global quotients by reductive groups, giving further evidence for a more general statement than Theorem 4.1.

**Corollary 4.5.** *Let  $G$  be a compact Lie group and let the complexification  $G_{\mathbb{C}}$  act on a compact complex manifold  $X$ , and let  $(\mathfrak{L}, \mathfrak{h})$  be a  $G_{\mathbb{C}}$ -equivariant hermitian line bundle of type  $k$  with invariant global section  $s$  vanishing on  $Z \subset X$ , then  $H_i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}]) = H^i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}]) = 0$  for  $i < k$ .*

*Proof.*  $G_{\mathbb{C}}$  deformation retracts onto  $G$ , and so the pair  $(X \times (G_{\mathbb{C}})^{\times p}, Z \times (G_{\mathbb{C}})^{\times p})$  deformation retracts onto  $(X \times (G)^{\times p}, Z \times (G)^{\times p})$  for any number of factors  $p$ . Thus by the spectral sequence (1),  $H_i([X/G], [Z/G]) \rightarrow H_i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}])$  is an isomorphism, and  $H_i([X/G], [Z/G]) = 0$  for  $i < k$  by Theorem 4.1.  $\square$

## APPENDIX A. STABILITY OF GRADIENT DESCENT FLOW

We have used Morse theory to prove the LHT without requiring the section of the line bundle to be non-degenerate as in [Bot59]. In Lemma 2, we reduce the problem of showing that  $(\mathfrak{X}, \mathfrak{Z})$  is a neighborhood deformation retract to the purely local statement that the gradient descent flow of a function near its minimum is *stable*. The key technique is due to Lojasiewicz [Loj63, Loj65, Loj84], and we review it here. He proved that real analytic functions satisfy the inequality (L), and applied it to show that analytic subvarieties of  $\mathbb{R}^n$  are neighborhood deformation-retracts. Many authors have extended his work, for instance in [KMP00].

**Definition A.1.** Let  $X$  be a Riemannian manifold and  $f$  a smooth function on  $X$ . We say that  $f$  satisfies a *Lojasiewicz inequality* at a point  $p$  if there are constants  $\rho \in (0, 1)$  and  $C > 0$  such that

$$|f - f(p)|^{\rho} \leq C|\nabla f| \tag{L}$$

in some neighborhood of  $p$

As a consequence of (L),  $f$  takes the value  $f(p)$  at any critical point near  $p$ . In particular if  $Z$  is the global minimal set of  $f$  and (L) holds at every point of  $Z$ , then  $Z$  is isolated from the rest of the critical locus of  $f$ .

**Proposition A.2.** *If  $f$  achieves its global minimum along  $Z \subset X$  and (L) holds at  $p \in Z$ , then there is an open set  $U \subset X$  containing  $p$  such that*

- 480 (1) the flow  $\phi_t(x)$  of  $-\nabla f$  is defined for all  $x \in U$  and all  $t \geq 0$ , and  $\phi_t(x) \in U$ .
- (2) The map  $U \times [0, \infty) \rightarrow U$  given by  $(x, t) \mapsto \phi_t(x)$  extends uniquely to a continuous map  $U \times [0, \infty] \rightarrow U$ . In particular the flow of  $-\nabla f$  deformation retracts  $U$  onto  $U \cap Z$ .

**Remark A.3.** The proposition is equally valid for a local minimum. Also, 485 the argument below shows that any gradient like vector field for  $f$  will have the same stability property.

*Proof.* For simplicity we will assume that  $f \geq 0$  and  $f(p) = 0$ . Let  $\phi_t(x_0)$  be an integral curve starting at  $x_0 \in X$ . As long as  $\phi_t(x_0)$  stays in a region in which (L) holds, an arc length integral gives a uniform bound

$$\text{dist}(x_0, \phi_t(x_0)) \leq \frac{C}{1-\rho} f(x_0)^{1-\rho} \quad \text{for } t \geq 0 \quad (2)$$

490 Start with a relatively compact open ball  $B_{r'}$  of radius  $r'$  around  $p$  on which the inequality (L) holds. Now we can find a smaller ball  $B_r \subset B_{r'}$  such that the difference in radii  $r' - r$  is larger than  $Cf(x_0)^{1-\rho}/(1-\rho)$  for any point  $x_0 \in B_r$ . It follows from (2) that any flow line starting in  $B_r$  stays within  $B_{r'}$  under the (positive time) flow of  $-\nabla f$ , and thus by 495 the escape lemma [Lee03] the flow is defined on  $B_r$  for all  $t \geq 0$ . We let  $U = \bigcup_{t \geq 0} \phi_t(B_r)$ .

Inequality (L) implies that  $f(\phi_t(x_0)) \rightarrow 0$  as  $t \rightarrow \infty$ . Combining this with (2) shows that  $\phi_t(x_0)$  is Cauchy as  $t \rightarrow \infty$  and remains in a compact region, so  $\phi_\infty(x) = \lim_{t \rightarrow \infty} \phi_t(x)$  is a well defined function on  $U$ . In fact 500 (2) shows that  $(x, t) \mapsto \phi_t(x)$  is uniformly continuous as  $t \rightarrow \infty$ , and so the gradient descent flow extends uniquely to a continuous map  $U \times [0, \infty] \rightarrow U$ . Finally,  $\phi_s(\phi_\infty(x)) = \lim_{t \rightarrow \infty} \phi_{s+t}(x) = \phi_\infty(x)$ , so  $\phi_\infty(U) \subset Z$  and the map constructed above is a deformation retract of  $U$  onto  $U \cap Z$ .

□

505 An immediate corollary of Proposition A.2 is the global statement that

**Corollary A.4.** *Let  $f$  be a smooth nonnegative function on  $X$  with global minimum  $Z = f^{-1}\{0\} \subset X$ . If the inequality (L) holds at each  $p \in Z$ , then there is an open neighborhood  $U$  of  $Z$  on which the negative gradient flow extends uniquely to a map  $U \times [0, \infty] \rightarrow U$  which is a deformation retract 510 of  $U$  onto  $Z$ .*

In addition if  $Z$  is compact then  $X^{\text{cl}}$  deformation retracts onto  $Z$  for sufficiently small  $\epsilon$ .

We have shown above that the Lojasiewicz inequality implies a stable gradient flow, but we are left with the question of which functions satisfy 515 (L). First of all, functions satisfying (L) have the following properties.

- (1) if  $h$  is a smooth function with  $h(p) \neq 0$  and  $f$  satisfies (L) with  $f(p) = 0$ , then so does  $hf$

- 520 (2) Let  $f$  and  $g$  satisfy (L) with  $f(p) = g(p) = 0$ . If the angle between  $\nabla f$  and  $\nabla g$  is bounded away from  $\pi$  near  $p$  then  $f + g$  satisfies (L),<sup>7</sup> and if the angle between  $\nabla f^2$  and  $\nabla g^2$  is bounded away from  $\pi$  then  $fg$  satisfies (L).
- 525 (3) if  $\pi : X \rightarrow Y$  is an open mapping near  $p \in X$  and  $f \circ \pi$  satisfies (L) at  $p$  then  $f$  satisfies (L) at  $\pi(p) \in Y$ . If  $\pi$  is submersive at  $p$ , then the converse is true as well. Letting  $\pi$  be the identity map shows that (L) is independent of the metric.
- 530 (4) if  $\pi : X \rightarrow Y$  is proper and surjective onto a neighborhood of  $q \in Y$  and  $f \circ \pi$  satisfies (L) at every point in the fiber  $\pi^{-1}\{q\}$ , then  $f$  satisfies (L) at  $q$ .

One might wonder in light of property (2) above if arbitrary algebraic combinations of functions will still satisfy (L), but this turns out to be false. For instance if  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  vanishes to all orders at 0, then  $f(x) = (1 + \rho)x - x = \rho x$  does not satisfy (L) at 0 even though  $(1 + \rho)x$  and  $x$  do. Thus while inequality (L) has convenient dynamical properties, it lacks algebraic properties. It is thus surprising that many functions do satisfy the Lojasiewicz inequality:

540 **Theorem A.5.** *Let  $f$  be smooth function on  $X$  which in some choice of local coordinates is a non-vanishing smooth function times a real analytic function which vanishes at  $p$ , then  $f$  satisfies property (L) at  $p$ .*

*Proof.* Using real analytic resolution of singularities there is a proper surjective analytic map  $\pi : Y \rightarrow X$  such that  $f \circ \pi = h(y)y_1^{\alpha_1} \cdots y_n^{\alpha_n}$  where  $y_1, \dots, y_n$  are local analytic coordinates on  $Y$  and  $h$  is a smooth function which does not vanish near  $\pi^{-1}\{p\}$ . Applying property (2) above, or by a direct calculation, one shows that the monomial  $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$  satisfies (L) along its zero locus, and thus by property (1) so does  $f \circ \pi$ . Finally by property (4)  $f$  satisfies (L) at  $p$ .  $\square$

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<sup>7</sup>More precisely the condition is  $(\nabla f, \nabla g) \geq (-1 + \epsilon)|\nabla f||\nabla g|$  in a neighborhood of  $p$  for some small  $\epsilon$

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