A CATEGORIFICATION OF THE ATIYAH-BOTT LOCALIZATION FORMULA

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Contents

Let X be a smooth proper algebraic variety with a $\mathbb{C}^*$-action. The Atiyah-Bott localization theorem compares the topology of the fixed locus $X^{\mathbb{C}^*}$ with the topology of X. There are at least three versions of the localization theorem, which we state here in topological K-theory rather than in cohomology:

(1) The restriction map $K^i_{\mathbb{C}^*}(X) \to K^i_{\mathbb{C}^*}(X^{\mathbb{C}^*})$ is a map of finite $K_{\mathbb{C}^*}(\mathrm{pt})$-modules whose kernel and cokernel is torsion, i.e. it becomes an isomorphism after inverting finitely many elements of the ground ring,
(2) There is a decomposition of the identity $1_X = \sum_\alpha (\sigma_\alpha)_* \left(\frac{1_{Z_\alpha}}{e(N_{Z_\alpha} X)} \right)$ in $K_{\mathbb{C}^*}(X)$, where $\sigma_\alpha : Z_\alpha \hookrightarrow X$ are the connected components of the fixed locus and $e(-)$ denotes the Euler class, and
(3) The K-theoretic index localizes on the fixed loci Z_α, i.e. $\chi(X,E) = \sum_\alpha \chi(Z_\alpha, \frac{E_{Z_\alpha}}{e(N_{Z_\alpha} X)})$ for any equivariant class $E \in K_{\mathbb{C}^*}(X)$.

For any of these statements, one must invert some elemements of the base ring $K_{\mathbb{C}^*}(X)$ and work with localized K-theory.

There is, however, an isomorphism $K^i_{\mathbb{C}^*}(X) \simeq K^i_{\mathbb{C}^*}(X^{\mathbb{C}^*})$ as modules over $K_{\mathbb{C}^*}(X)$ which does not require localization. When the fixed loci Z_α consist of individual points, one constructs this isomorphism quite explicitly by proving that the closures of the Bialynicki-Birula strata of X form a basis for $K_{\mathbb{C}^*}(X)$ as a free $K_{\mathbb{C}^*}(\mathrm{pt})$-module. This version of the localization theorem can be elevated to a theorem on the derived category of equivariant coherent sheaves on X as an application of the main structure theorem of [?HL]. Using the Bialynicki-Birula stratification, one can construct "extension functors" from $\mathrm{D}^b(X^{\mathbb{C}^*}/\mathbb{C}^*)$ to $\mathrm{D}^b(X/\mathbb{C}^*)$ which induce an equivalence on algebraic (and also topologogical) K-theory.[1]

The difficulty in finding a categorification of (1-3) above rests mainly in the question of what procedure on the level of categories corresponds to "inverting elements of $K_{\mathbb{C}^*}(\mathrm{pt})$." In this note, we explain one approach, which is closer in spirit to completion than to localization. We construct a "completed" category $\mathrm{D}^b(X/\mathbb{C}^*)^\wedge$ containing $\mathrm{D}^b(X/\mathbb{C}^*)$ as a full subcategory. $\mathrm{D}^b(X/\mathbb{C}^*)^\wedge$ is a carefully chosen subcategory of the category of quasi-coherent complexes. $\mathrm{D}^b(X/\mathbb{C}^*)^\wedge$ is small

[1] In our notation if G is an algebraic group and X is a G-scheme, then the quotient X/G will always denote the quotient stack. In particular $\mathrm{D}^b(X/G)$ denotes the derived category of G-equivariant coherent sheaves on X.

enough that objects still have finite dimensional hypercohomology, but large enough that versions of (1),(2), and (3) can be formulated and proved in $K_0(\mathrm{D}^b(X/G)^\wedge)$.

0.1. **What's in this paper.** We actually work in a more general context. Instead of working with the Bialynicki-Birula stratification of a $\mathbb{C}^*$-action, we work with an arbitrary algebraic group G and a smooth scheme X with a stratification

$$X = X^{\mathrm{ss}} \cup \bigcup_\alpha S_\alpha$$

which is G-equivariant and induced a Θ-stratification of X/G (referred to as a KN-stratification in [?HL]). $X^{ss} \subset X$ is the open "semistable" locus. The strongest statements are for the situation when $X^{\mathrm{ss}} = \emptyset$. We formulate and prove a version of the "non-abelian" localization theorem of Witten,Kirwan, and Jeffrey, whose K-theoretic version in the guise of (3) was formulated by Teleman and Woodward.

Stratifications of this kind typically arise in geometric invariant theory. For a first read of this note, the reader can keep the following example in mind: $\lambda : \mathbb{C}^* \to G$ is a one parameter subgroup which is central in G, and X is a smooth variety such that the Bialynicki-Birula strata with respect to λ cover X. Then $X^{\mathrm{ss}} = \emptyset$ and $X = \bigcup_\alpha S_\alpha$ can be taken as the Bialynicki-Birula stratification, which will be G-equivariant in this case. The "centers" of the strata $Z_\alpha^{\mathrm{ss}} \subset S_\alpha$, discussed below, are just the connected components of the fixed loci $X^{\lambda(\mathbb{C}^*)}$.

In addition, we work over an arbitrary field.

1. Baric structures and completion

Recall [?achar] that a *baric structure* on a stable dg-category $\mathcal{D}$ is a collection of semiorthogonal decompositions $\mathcal{D} = \langle \mathcal{D}^{<w}, \mathcal{D}^{\geq w} \rangle$ such that $\mathcal{D}^{<w} \subset \mathcal{D}^{<w+1}$, or equivalently $\mathcal{D}^{\geq w} \subset \mathcal{D}^{\geq w-1}$. By definition this means that $\mathrm{RHom}(A, B) = 0$ for $A \in \mathcal{D}^{\geq w}$ and $B \in \mathcal{D}^{<w}$, and for every object $E \in \mathcal{D}$ we have an exact triangle

$$\beta^{\geq w}(E) \to E \to \beta^{<w}(E) \to,$$

with $\beta^{\geq w}(E) \in \mathcal{D}^{\geq w}$ and $\beta^{<w}(E) \in \mathcal{D}^{<w}$. The semiorthogonality implies that this exact triangle is unique and functorial, hence our introduction of the *baric truncation functors* $\beta^{\geq w}$ and $\beta^{<w}$.

Given a baric structure on an essentially small stable dg-category $\mathcal{D}$, one obtains a baric strcuture on the formal ind-completion $\mathrm{Ind}(\mathcal{D}) = \langle \mathrm{Ind}(\mathcal{D})^{<w}, \mathrm{Ind}(\mathcal{D})^{\geq w} \rangle$ defined uniquely in such a way that both factors are co-complete, and the baric truncations functors commute with filtered colimits.

Definition 1.1. Given an essentially small stable dg-category $\mathcal{D}$ with a baric structure, we define the *right baric completion* to be the full subcategory of $E \in \mathrm{Ind}(\mathcal{D})$ such that $\beta^{\geq w}(E) \in \mathcal{D}$ for all w.

The completion $\mathcal{D}^\wedge$ has the following equivalent characterizations:

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Lemma 1.2. *Assume that the baric structure on $\mathcal{D}$ is right bounded, meaning $\mathcal{D} = \bigcup_w \mathcal{D}^{\geq w}$. Then $\mathcal{D}^\wedge \subset \mathrm{Ind}(\mathcal{D})$ can be characterized alternatively as the category of objects which can be written as a filtered colimit $F = \mathrm{colim}_i P_i$ with $P_i \in \mathcal{D}$ satisfying either*

(1) $\forall w \in \mathbb{Z}$, $\beta^{\geq w}(P_i) \to \beta^{\geq w}(P_j)$ is an equivalence for i sufficiently large and all $i < j$, or
(2) $\forall w \in \mathbb{Z}$, $\mathrm{Cone}(P_i \to F) \in \mathrm{Ind}(\mathcal{D})^{<w}$ for i sufficiently large.

Proof. The fact that a filtered colimit of $P_i \in \mathcal{D}$ satisfying either of these conditions will have $\beta^{\geq w}(F) \in \mathcal{D}$ is an immediate consequence of the fact that $\beta^{\geq w}$ commutes with filtered colimits and $\beta^{\geq w}(P_i)$ stabilizes for $i \gg 0$.

Conversely, note that for any $F \in \mathrm{Ind}(\mathcal{D})$ we have a canonical diagram $\cdots \to \beta^{\geq w}(F) \to \beta^{\geq w-1}(F) \to \cdots$ coming from the canonical map $\beta^{\geq w}(\beta^{\geq w-1}(F)) \to \beta^{\geq w-1}(F)$ and the canonical isomorphism $\beta^{\geq w}(\beta^{\geq w-1}(F)) \simeq \beta^{\geq w}(F)$. For any $P \in \mathcal{D}$ the induced map

$$\mathrm{RHom}(P, \mathrm{colim}_w \beta^{\geq w}(F)) \to \mathrm{RHom}(P, F)$$

is an equivalence because P is a compact object of $\mathrm{Ind}(\mathcal{D})$ (so we may commute $\mathrm{RHom}(P, -)$ with filtered colimits), and $P \in \mathcal{D}^{\geq w}$ for sufficiently low w, which implies that $\mathrm{RHom}(P, \beta^{\geq w}(F)) \simeq \mathrm{RHom}(P, F)$ for all sufficiently low w. It follows, because $\mathrm{Ind}(\mathcal{D})$ is generated by $P \in \mathcal{D}$ that $\mathrm{colim}_w \beta^{\geq w}(F) \to F$ is an equivalence for any $F \in \mathrm{Ind}(\mathcal{D})$. Now if $F \in \mathcal{D}^{\wedge}$, then each $\beta^{\geq w}(F) \in \mathcal{D}$ by definition, so the presentation $F \simeq \mathrm{colim}_w \beta^{\geq w}(F)$ is an explicit presentation satisfying (1) and (2). □

2. Baric structures on equivariant derived categories

Let $X/G = X^{\mathrm{ss}} \cup \bigcup_\alpha S_\alpha/G$ be a Θ-stratification of a smooth quotient stack – we call $X^{\mathrm{us}} = \bigcup_\alpha S_\alpha$ the unstable locus. All we will need to know about these strata is that each contains a smooth locally closed "center" $Z_\alpha^{\mathrm{ss}} \subset S_\alpha$ which is fixed (pointwise) by a distinguished one paramater subgroup λ_α and equivariant with respect to the centralizer L_α of λ_α. We denote $\sigma_\alpha : Z_\alpha^{\mathrm{ss}}/L_\alpha \to X/G$ and $\iota_\alpha : S_\alpha \to X$.

We choose, once and for all, an integer $s_\alpha \in \mathbb{Z}$ and a positive integer $m_\alpha \in \mathbb{Z}$ for each index α in the stratification. Any G-equivariant complex restricted to Z_α^{ss} decomposes canonically into a direct sum of complexes whose homology sheaves are concentrated in a single λ_α-weight. We define

$$\mathrm{D}^b(X/G)^{\geq w} := \{F \in \mathrm{D}^b(X/G) | \forall \alpha, \mathcal{H}_*(F|_{Z_\alpha^{\mathrm{ss}}}) \text{ has weights } \geq m_\alpha w + s_\alpha\}$$

$$\mathrm{D}^b_{X^{\mathrm{us}}}(X/G)^{<w} := \left\{ F \in \mathrm{D}^b(X/G) \middle| \begin{array}{c} \mathrm{Supp}(F) \subset X^{\mathrm{us}} \text{ and} \\ \forall \alpha, \mathcal{H}_*(F|_{Z_\alpha^{\mathrm{ss}}}) \text{ has weights } < m_\alpha w + s_\alpha + \eta_\alpha \end{array} \right\}$$

where "weights" of a coherent sheaf on $Z_\alpha^{\mathrm{ss}}/L_\alpha$ always refers to λ_α-weights, and η_α is defined to be the weight of $\det(N^\vee_{S_\alpha} X)|_{Z_\alpha^{\mathrm{ss}}}$. Then categorical Kirwan surjectivity [?HL, ???] provides a baric structure

$$\mathrm{D}^b(X/G) = \left\langle \mathrm{D}^b_{X^{\mathrm{us}}}(X/G)^{<w}, \mathrm{D}^b(X/G)^{\geq w} \right\rangle. \quad (1) \quad \{\texttt{eqn:baric}\}$$

For any perfect complex, the weights of $F|_{Z_\alpha^{\mathrm{ss}}}$ are bounded above and below. It follows that the baric structure (1) is always right bounded, i.e. $\mathcal{D} = \bigcup \mathcal{D}^{\geq w}$, and is left bounded, i.e. $\mathcal{D} = \bigcup \mathcal{D}^{<w}$, if and only if $X^{\mathrm{ss}} = \emptyset$ and hence $X^{\mathrm{us}} = X$.

Example 2.1. A special case of this is when $X = Z_\alpha^{\mathrm{ss}}$ and λ_α is central in G. If we let $m_\alpha = 1$, then the baric structure (1) is just the direct sum decomposition of $\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)$ into subcategories of complexes whose homology has constant λ_α-weight.

Example 2.2. Another example is when $X = S$ consists of a single Θ-stratum, in which case $\mathrm{D}^b(S/G)$ receives a baric sructure. Among the properties established in [?HL] is that for a closed Θ-stratum $S \hookrightarrow X$, the functors $\iota_* : \mathrm{D}^b(S/G) \to \mathrm{D}^b(X/G)$ and $\sigma^* : \mathrm{D}^b(S/G) \to \mathrm{D}^b(Z^{\mathrm{ss}}/L)$ are both compatible with the baric structures which we've discussed.[2]

Because X/G is a quotient stack in characteristic 0, we have $\mathrm{QC}(X/G) = \mathrm{Ind}(\mathrm{D}^b(X/G))$, so it inherits a baric structure as well

$$\mathrm{QC}(X/G) = \left\langle \mathrm{QC}(X/G)^{<w}, \mathrm{QC}_{X^{\mathrm{us}}}(X/G)^{\geq w} \right\rangle$$

for all $w \in \mathbb{Z}$. The truncation functors $\beta^{\geq w} : \mathrm{QC}(X/G) \to \mathrm{QC}(X/G)^{\geq w}$ and $\beta^{<w} : \mathrm{QC}(X/G) \to \mathrm{QC}_{X^{\mathrm{us}}}(X/G)^{<w}$ commute with colimits by definition. The baric truncation functors can be computed by writing every $F \in \mathrm{QC}(X/G)$ as a filtered colimit $F = \mathrm{colim}_i P_i$ with P_i perfect. Then $\beta^{\geq w}(F) = \mathrm{colim}_i \beta^{\geq w}(P_i)$ and $\beta^{<w} = \mathrm{colim}_i \beta^{<w}(P_i)$.

[2] By this we mean a functor $\mathcal{C} \to \mathcal{D}$ which maps $\mathcal{C}^{\geq w}$ to $\mathcal{D}^{\geq w}$ and $\mathcal{C}^{<w}$ to $\mathcal{D}^{<w}$.

Definition 2.3. We define the $\mathrm{D}^b(X/G)^\wedge \subset \mathrm{QC}(X/G)$ to be the right baric completion of $\mathrm{D}^b(X/G)$ with respect to the baric structure (1). It consists of complexes such that $\beta^{\geq w}(F) \in \mathrm{D}^b(X/G)$ for all $w \in \mathbb{Z}$.

The general Lemma 1.2 implies that $\mathrm{D}^b(X/G)$ can be characterized alternatively as the category of complexes which can be written as a filtered colimit of perfect complexes $F = \mathrm{colim}_i P_i$ satisfying either

(1) $\forall w \in \mathbb{Z}$, $\beta^{\geq w}(P_i) \to \beta^{\geq w}(P_j)$ is an equivalence for i sufficiently large and all $i < j$, or
(2) $\forall w \in \mathbb{Z}$, $\mathrm{Cone}(P_i \to F) \in \mathrm{QC}_{X^{\mathrm{us}}}(X/G)^{<w}$ for i sufficiently large.

One consequence of this is that the subcategory $\mathrm{D}^b(X/G)^\wedge \subset \mathrm{QC}(X/G)$ does not depend on the initial choice of integers s_α or m_α used to define the baric structure on $\mathrm{D}^b(X/G)$.

Lemma 2.4. *$\mathrm{D}^b(X/G)^\wedge$ is a stable (i.e. pre-triangulated) dg-subcategory of $\mathrm{QC}(X/G)$. It contains $\mathrm{D}^b(X/G)$, it is a symmetric monoidal subcategory, and it is idempotent complete.*

Proof. Most of these properties are immediate from the definition and the fact that $\beta^{\geq w}$ is an exact functor of pre-triangulated dg-categories. Let us prove that $\mathrm{D}^b(X/G)^\wedge$ is symmetric monoidal: if F is perfect, then for any $E \in \mathrm{QC}(X/G)$

{eqn:trick} $$\beta^{\geq w}(F \otimes E) \simeq \beta^{\geq w}(F \otimes \beta^{\geq v}(E)) \tag{2}$$

for $v < n$ for some integer n which only depends on the highest weights of $F|_{Z_\alpha^{\mathrm{ss}}}$. This is because the weights of $F|_{Z_\alpha^{\mathrm{ss}}}$ are bounded above for each α, so we can choose a sufficiently large integer n such that $F \otimes \beta^{<v}(E) \in \mathrm{QC}_{X^{\mathrm{us}}}(X/G)^{<v+n}$ for all $E \in \mathrm{QC}(X/G)$. Thus (2) results from applying $\beta^{\geq w}$ to the exact triangle $F \otimes \beta^{\geq v}(E) \to F \otimes E \to F \otimes \beta^{<v}(E) \to$.

To deduce that $E \otimes F \in \mathrm{D}^b(X/G)^\wedge$ for $E, F \in \mathrm{D}^b(X/G)^\wedge$, we apply (2) twice. In particular we use that the highest weights of the perfect complexes $\beta^{\geq w}(F)|_{Z_\alpha^{\mathrm{ss}}}$ do not depend on w for w sufficiently large. We compute

$$\beta^{\geq w}(F \otimes E) \simeq \mathrm{colim}_v\, \beta^{\geq w}(\beta^{\geq v}(F) \otimes E) \simeq \mathrm{colim}_v\, \beta^{\geq w}(\beta^{\geq v}(F) \otimes \beta^{\geq u}(E))$$

where u is sufficiently low and does not depend on v. Now commuting the colimit and $\beta^{\geq w}$ once more, we can identify this with

$$\simeq \beta^{\geq w}(F \otimes \beta^{\geq u}(E)) \simeq \beta^{\geq w}(\beta^{\geq z}(F) \otimes \beta^{\geq u}(E))$$

where now z is sufficiently low. This will be perfect by hypothesis, hence $F \otimes E \in \mathrm{D}^b(X/G)^\wedge$. ☐

Recall that we say the Θ-strafitication of X/G is *complete* if X^{ss}/G and Z_α^{ss}/G admit projective good quotients for all α.

Lemma 2.5. *If the Θ-stratification of X/G is complete, then for any $E \in \mathrm{D}^b(X/G)$ and $F \in \mathrm{D}^b(X/G)^\wedge$, the complex $\mathrm{RHom}_{X/G}(E,F)$ has finite dimensional total cohomology.*

Proof. It suffices to prove this for $E = \mathcal{O}_X$. This is a consequence of the quantization commutes with reduction theorem [?HL, Theorem 3.29], which implies that $R\Gamma(F) = R\Gamma(\beta^{\geq w}(F))$ for w sufficiently low. ☐

3. The pushforward theorem

Note that if $j : U \subset X$ is an open union of strata, then $\mathrm{D}^b(U/G)$ also has a baric structure induced by the strata which lie in U. It follows from the exactness of the restriction functor $j^* : \mathrm{D}^b(X/G) \to \mathrm{D}^b(U/G)$ and the definitions of the categories $\mathrm{D}^b(U/G)^{\geq w}$ and $\mathrm{D}^b_{U^{\mathrm{us}}}(U/G)^{<w}$ that j^* is compatible with the baric structure. Our main result is the following:

{thm:main}

Theorem 3.1. *Let $j : U \subset X$ be an open complement of a union of strata. Then $j_* : \mathrm{QC}(U/G) \to \mathrm{QC}(X/G)$ maps $\mathrm{D}^b(U/G)^\wedge$ to $\mathrm{D}^b(X/G)^\wedge$, where the former is defined with respect to the strata which lie in U.*

Proof. It suffices by a simple inductive argument to assume that the complement of U consists of a single closed stratum $i : S \hookrightarrow X$. First of all, note that $R\underline{\Gamma}_S(\mathcal{O}_X) \in \mathrm{QC}(X/G)$ actually lies in $\mathrm{D}^b(X/G)^\wedge$ – this is [?HL, Lemma 3.37], and it is proved by considering $R\underline{\Gamma}_S(\mathcal{O}_X)$ as a colimit of Koszul complexes. It follows from the exact triangle $R\underline{\Gamma}_S\mathcal{O}_X \to \mathcal{O}_X \to j_*(\mathcal{O}_U) \to$ that $j_*(\mathcal{O}_U) \in \mathrm{D}^b(X/G)^\wedge$.

Take $F \in \mathrm{QC}(U)$, and write $F = \mathrm{colim}_w\, \beta_U^{\geq w}(F)$, so that we have $\beta_U^{\geq w}(F) \in \mathrm{D}^b(X/G)$ by hypothesis. Then by the categorical Kirwan surjectivity, we can fix any particular $s \in \mathbb{Z}$ and uniquely and functorially lift the complex $\beta_U^{\geq w}(F)$ to a perfect complex $F_w \in \mathrm{D}^b(X/G)$ such that the weights of $F_w|_{Z^{\mathrm{ss}}}$ lie in the window $[mw+s, mw+s+\eta)$ for each w. The quantization theorem [?HL, Theorem 3.29] implies that the restriction map

$$\mathrm{RHom}_{X/G}(F_w, F_{w-1}) \to \mathrm{RHom}_{U/G}(\beta_U^{\geq w}(F), \beta_U^{\geq w-1}(F)),$$

so we can lift the filtered system $\cdots \to \beta_U^{\geq w}(F) \to \beta_U^{\geq w}(F) \to \cdots$ uniquely to a filtered system $\cdots \to F_w \to F_{w-1} \to \cdots$ in $\mathrm{D}^b(X/G)$. Let us define $\tilde{F} := \mathrm{colim}_w\, F_w \in \mathrm{QC}(X/G)$.

Note that the cone $\mathrm{Cone}(F_w \to F_{w-1})$ is supported set theoretically on X^{us}. Furthermore the weights of $\mathrm{Cone}(F_w \to F_{w-1})|_{Z_\alpha^{\mathrm{ss}}}$ get lower and lower as $w \to -\infty$: for the Z_α^{ss} contained in U, this is because $\mathrm{Cone}(\beta_U^{\geq w}(F) \to \beta_U^{\geq w-1}(F)) \simeq \beta_U^{<w}(\beta_U^{\geq w-1}(F))$, and for the Z^{ss} in the stratum we are adding this follows from the weight bounds on F_w and F_{w-1} individually. Thus for any fixed v, $\beta^{\geq v}(F_w)$ stabilizes for w sufficiently low, and hence $\tilde{F} \in \mathrm{D}^b(X/G)^\wedge$. Finally, by construction we have a canonical equivalence $\tilde{F}|_U \simeq F$, so $\tilde{F} \otimes j_*(\mathcal{O}_U) \simeq j_*(F) \in \mathrm{D}^b(X/G)^\wedge$ because $j_*(\mathcal{O}_U) \in \mathrm{D}^b(X/G)^\wedge$ and the subcategory is closed under tensor products. □

Definition 3.2. Consider the closed subsets $X_{>\alpha} = \bigcup_{\beta>\alpha} S_\beta \subset X$. For any stratum $S_\alpha \subset X$, we define the object $R\underline{\Gamma}_{S_\alpha}(\mathcal{O}_X) \in \mathrm{QC}(X/G)$ to be the local cohomology complex for the close subset $S_\alpha \hookrightarrow X \setminus X_{>\alpha}$ pushed forward to X along the open immersion $X \setminus X_{>\alpha}$.

Corollary 3.3. *For all α, $R\underline{\Gamma}_{S_\alpha}\mathcal{O}_X \in \mathrm{D}^b(X/G)^\wedge$, as are $R\underline{\Gamma}_{X_{>\alpha}}\mathcal{O}_X$. All of these objects are idempotent for the symmetric monoidal structure. The the structure sheaf $\mathcal{O}_X$ thus has a filtration in $\mathrm{D}^b(X/G)^\wedge$* {cor:filtrat

$$R\underline{\Gamma}_{X_{>N}}\mathcal{O}_X \to R\underline{\Gamma}_{X_{>N-1}}\mathcal{O}_X \to \cdots \to R\underline{\Gamma}_{X_{>-1}}\mathcal{O}_X \to \mathcal{O}_X$$

whose associated graded is $\mathcal{O}_{X^{\mathrm{ss}}} \oplus \bigoplus_\alpha R\underline{\Gamma}_{S_\alpha}\mathcal{O}_X$.

Note that if $j : U \subset X$ is a G-equivariant open subset and $Y = X \setminus U$ its close complement, then one has a semiorthogonal decomposition

$$\mathrm{QC}(X/G) = \langle \mathrm{QC}(U/G), \mathrm{QC}_Y(X/G) \rangle, \qquad (3)$$ {eqn:tautolo

where the first factor is the essential image of the fully faithful functor j_*, and the second factor is the subcategory of complexes supported (set theoretically) on Y. For $F \in \mathrm{QC}(X/G)$ the exact triangle of this semiorthogonal decomposition is the local cohomology exact triangle $R\underline{\Gamma}_Y(F) \to F \to j_*(F|_U) \to$.

Corollary 3.4. *If $U \subset X$ is an open union of strata, then the semirthogonal decomposition (3) induces a semiorthogonal decomposition of $\mathrm{D}^b(X/G)^\wedge$ as well.*

Proof. This is immediate from Theorem 3.1, which implies that the local cohomology exact triangle $R\underline{\Gamma}_{X \setminus U}(F) \to F \to j_*(F|_U) \to$ lies in $\mathrm{D}^b(X/G)^\wedge$. □

4. Baric completion and K-theory

One can describe the effect of right baric completion on K-theory in general. For a stable dg-category with baric structure $\mathcal{D} = \langle \mathcal{D}^{<w}, \mathcal{D}^{\geq w} \rangle$ we introduce the notation $\mathcal{D}^{[w]} = \mathcal{D}^{\geq w} \cap \mathcal{D}^{<w+1}$, and let $\beta^{[w]}(F) = \beta^{\geq w} \beta^{<w+1}(F) \simeq \beta^{<w+1}\beta^{\geq w}$ denote the canonical projection onto this subcategory.

Lemma 4.1. *Let $\mathcal{D} = \langle \mathcal{D}^{<w}, \mathcal{D}^{\geq w} \rangle$ be a baric structure which is left bounded, i.e. such that $\mathcal{D} = \bigcup_w \mathcal{D}^{<w}$.*[3] *Then the functor*

$$\prod \beta^{[w]} : \mathcal{D}^\wedge \to \bigoplus_{w \geq 0} \mathcal{D}^{[w]} \oplus \prod_{w<0} \mathcal{D}^{[w]}$$

induces an isomorphism in K-theory

$$K_0(\mathcal{D}^\wedge) \simeq \bigoplus_{w \geq 0} K_0(\mathcal{D}^{[w]}) \oplus \prod_{w<0} K_0(\mathcal{D}^{[w]})$$

Proof. The boundedness hypothesis guarantees that the functor $\prod \beta^{[w]}$ actually has image in the full subcategory $\mathcal{C} := \bigoplus_{w \geq 0} \mathcal{D}^{[w]} \bigoplus \prod_{w<0} \mathcal{D}^{[w]}$ as claimed. K_0 commutes with arbitrary direct sums and products, so $K_0(\mathcal{C})$ agrees with the right hand side of the above equality.

It thus suffices to show that $\prod \beta^{[w]}$ induces an isomorphism on K-theory. We can define a one-sided inverse functor $\phi : \mathcal{C} \to \mathcal{D}^\wedge$ mapping $\{A_w\} \mapsto \bigoplus_w A_w$. We have $(\prod_w \beta^{[w]}) \circ \phi \simeq \mathrm{id}_{\mathcal{C}}$, so the same holds after applying K_0. Conversely for any $F \in \mathcal{D}^\wedge$ and any w, we consider the exact triangle $\beta^{\geq w}(F) \to F \to \beta^{<w}(F) \to$ and the exact triangles $\beta^{[w-i]}(F) \to \beta^{<w-i+1}(F) \to \beta^{<w-i}(F)$ for $i \geq 1$. The direct sum of these exact triangles converges to an exact triangle in $\mathcal{D}^\wedge$, so we have

$$[F \oplus \bigoplus_{i \geq 1} \beta^{<w-i+1}(F)] = [\bigoplus_{i \geq 1} \beta^{<w-i+1}(F)] + [\beta^{\geq w}(F) \oplus \bigoplus_{i \geq 1} \beta^{[w-i]}(F)]$$

So choosing $w \gg 0$ large enough so that $\beta^{\geq w}(F) = 0$, we have $[F] = [\bigoplus_{i \geq 1} \beta^{[w-i]}(F)] \in K_0(\mathcal{D}^\wedge)$, and hence $\phi \circ (\prod \beta^{[w]}) \simeq \mathrm{id}_{K_0(\mathcal{D}^\wedge)}$. □

In our setting, the baric structure of $\mathrm{D}^b(X/G)$ will be left bounded if and only if $X^{\mathrm{ss}} = \emptyset$. Let us fix an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X/G)$ such that the weight of $\mathcal{L}|_{Z_\alpha^{\mathrm{ss}}}$ is < 0 for all α.

Example 4.2. If the stratification of X arises from geometric invariant theory, the G-ample bundle used to define the stratification will satsify this condition.

Example 4.3. If the stratification of X is the Bialynicki-Birula stratification associated to a central one parameter subgroup of G, then we can let $\mathcal{L} = \mathcal{O}_X \otimes \chi$ where χ is a character of G which pair negatively with this one-parameter-subgroup.

Given such an invertible sheaf, we regard both $K_0(X/G)$ and $K_0(Z_\alpha^{\mathrm{ss}}/L_\alpha)$ as $\mathbb{Z}[u^\pm]$-modules, where u acts by $\mathcal{L} \otimes (-)$. We also use $\mathcal{L}$ to fix our choice of parameters $m_\alpha = -\mathrm{wt}(L|_{Z_\alpha^{\mathrm{ss}}})$ in the definition of our baric structure of $\mathrm{D}^b(X/G)$ and $\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)$.

nm:K_theory}

Theorem 4.4. *Assume that $X^{\mathrm{ss}} = \emptyset$ and choose $\mathcal{L}$ and m_α as above. Then for any w we have a canonical equivalence*[4]

$$K_0(\mathrm{D}^b(X/G)^{[w]})(\!(u)\!) \to K_0(\mathrm{D}^b(X/G)^\wedge),$$

mapping $\sum_i [E_i] u^i \mapsto [\bigoplus_i L^{\otimes i} \otimes E_i]$. Futhermore if $K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^{[w]})$ is finitely generated for all α, then we have a canonical equivalence

$$K_0(\mathrm{D}^b(X/G)) \otimes_{\mathbb{Z}[u^\pm]} \mathbb{Z}(\!(u)\!) \to K_0(\mathrm{D}^b(X/G)^\wedge)$$

[3] This is equivalent to $\beta^{\geq w}(F) = 0$ for $w \gg 0$.

[4] The notaion $M(\!(u)\!)$ denotes the group $M[\![u]\!][u^{-1}]$, which differs from $M \otimes \mathbb{Z}(\!(u)\!)$ if M is not finitely generated.

given by the same formula.

Remark 4.5. It is possible to rephrase the condition on Z_α^{ss}/L_α in the theorem: Z_α^{ss}/L_α is a $\mathbb{G}_m$-gerbe over Z_α^{ss}/L'_α, where $L'_\alpha = L_\alpha/\lambda(\alpha)$. The Brauer group class of this gerbe is torsion. The condition in the statement of the theorem is equivalent to asking that the category of twisted perfect complexes on Z_α/L'_α, twisted by any power of this gerbe, has finitely generated K_0.

Example 4.6. Let $\lambda : \mathbb{G}_m \to G$ be a one parameter subgroup which is central in G, and let G act on a smooth variety X such that the Bialynicki-Birula stratification on X is exhaustive, and $K_0(X^{\lambda(\mathbb{G}_m)})$ is a finitely generated abelian group – for instance it could consist of isolated points, or it could admit a stratification by affine spaces. Then Theorem 4.4 implies that

$$K_0(\mathrm{D}^b(X/G)^\wedge) \simeq K_0(\mathrm{D}^b(X/G)) \otimes_{\mathbb{Z}[u^\pm]} \mathbb{Z}(\!(u)\!).$$

Before proving this theorem let us say a bit more about the structure of the category $\mathrm{D}^b(X/G)^{[w]}$.

{prop:fixed_

Proposition 4.7. *If $X^{ss} = \emptyset$, then $\mathrm{D}^b(X/G)^{[w]}$ has a finite semiorthogonal decomposition*

$$\mathrm{D}^b(X/G)^{[w]} = \langle \mathcal{A}_0^0, \ldots, \mathcal{A}_0^{m_0-1}, \mathcal{A}_1^0, \ldots, \mathcal{A}_1^{m_1-1}, \ldots, \mathcal{A}_N^0, \ldots, \mathcal{A}_N^{m_N-1} \rangle,$$

where the functor of restriction to Z_α^{ss}/L_α followed by projection onto the weight $m_\alpha w + i + s_\alpha$ summand defines an equivalence

$$\mathcal{A}_\alpha^i \simeq \{F \in \mathrm{D}^b(Z_\alpha^{ss}/L_\alpha) | \mathcal{H}_*(F) \textit{ is concentrated in weight } m_\alpha w + i + s_\alpha\}$$

for $i = 0, \ldots, m_\alpha - 1$. These equivalence combined with the inclusion into $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)$ defines a functor

$$\mathrm{D}^b(X/G)^{[w]} \xrightarrow{\mathrm{gr}} \bigoplus_{\alpha,i} \mathcal{A}_\alpha^i \to \bigoplus_\alpha \mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{[w]}$$

which induces an isomorphism in K-theory.

Proof. The semiorthogonal decomposition is a consequence of [?HL, ???]. We refer the reader to that paper for an explicit description of the categories $\mathcal{A}_\alpha^i$. Informally, the objects in $\mathcal{A}_\alpha^i$ arise from pulling back complexes concentrated in constant weight along a canonical map $\pi_\alpha : S_\alpha/G \to Z_\alpha^{ss}/L_\alpha$, then pushing forward to $X \setminus X_{>\alpha}$ and extending uniquely over the strata S_β using grade restriction rules. On the other hand, $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{[w]}$ consists by definition of complexes whose homology has weights concentrated in the interval $[m_\alpha w + s_\alpha, \ldots, m_\alpha w + m_\alpha - 1 + s_\alpha]$. Hence this category has a semiorthogonal decomposition (in fact a direct sum decomposition) whose summands are identified canonically with the $\mathcal{A}_\alpha^i$. The result follows from the fact that K-theory takes semiorthogonal decompositions to direct sums. □

Remark 4.8. One can define the inverse of the equivalence $K_0(\mathrm{D}^b(X/G)^{[w]}) \simeq \bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{[w]})$ a bit more explicitly by unravelling the main theorem of [?HL]. The image of the pullback functor $\pi_\alpha^* : \mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{[w]} \to \mathrm{D}^b(S_\alpha/G)^{[w]}$ generates and induces an equivalence on K-theory. We compose π_α^* with the pushforward functor $(\iota_\alpha)_* : \mathrm{D}^b(S_\alpha/G) \to \mathrm{D}^b((X \setminus X_{>\alpha})/G)^{[w]}$, followed by the functorial extension functor $\mathrm{D}^b((X \setminus X_{>\alpha})/G) \to \mathrm{D}^b(X/G)^{[w]}$ determined by a grade restriction rule to define a functor

$$\bigoplus_\alpha \mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{[w]} \to \mathrm{D}^b(X/G)^{[w]}.$$

This is an equivalence on K-theory, and in fact the image freely generates $K_0(\mathrm{D}^b(X/G))$ as a $\mathbb{Z}[u^\pm]$-module.

Remark 4.9. A word of caution: The restriction functor $\sigma^* : \mathrm{D}^b(X/G) \to \bigoplus_\alpha \mathrm{D}^b(\mathcal{Z}_\alpha^{ss}/L_\alpha)$ is not compatible with the baric structures on the respective categories. For a complex $F \in \mathrm{D}^b(X/G)^{<w}$ the weights of $F|_{Z_\alpha^{ss}/L_\alpha}$ by definition are $< m_\alpha w + s_\alpha + \eta_\alpha$, whereas $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{<w}$ consists of

complexes whose weights are $< m_\alpha w + s_\alpha$. As a consequence σ_α^* does not map $\mathrm{D}^b(X/G)^{[w]}$ to $\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^{[w]}$, and this functor is not suitable for comparing the K-theory of these two categories. We will, however, study the restriction map in **??** below.

Proof of Theorem 4.4. The choice of $m_\alpha = -\operatorname{wt}(\mathcal{L}|_{Z_\alpha^{\mathrm{ss}}})$ implies that $L \otimes (-)$ is an equivalence $\mathrm{D}^b(X/G)^{\geq w} \to \mathrm{D}^b(X/G)^{\geq w-1}$ and likewise for $\mathrm{D}^b(X/G)^{<w}$ and $\mathrm{D}^b(X/G)^{[w]}$. The first claim is thus an immediate consequence of the lemma above.

For the second claim, Proposition 4.7 implies that $K_0(\mathrm{D}^b(X/G)^{[w]})$ is finitely generated if $K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^{[w]})$ is finitely generated for all α. One can use the fact that the baric structure on $\mathrm{D}^b(X/G)$ is bounded along with the observation above to show that $K_0(\mathrm{D}^b(X/G)) \simeq K_0(\mathrm{D}^b(X/G)^{[w]}) \otimes \mathbb{Z}[u^\pm]$, where the equivalence maps $\sum_i [E_i] u^i \mapsto [\bigoplus L^{\otimes i} \otimes E_i]$. If $K_0(\mathrm{D}^b(X/G)^{[w]})$ is a finitely generated abelian group, then the canonical map

$$K_0(\mathrm{D}^b(X/G)^{[w]}) \otimes \mathbb{Z}[u^\pm] \otimes_{\mathbb{Z}[u^\pm]} \mathbb{Z}(\!(u)\!) \simeq K_0(\mathrm{D}^b(X/G)^{[w]}) \otimes \mathbb{Z}(\!(u)\!) \to K_0(\mathrm{D}^b(X/G)^{[w]})(\!(u)\!)$$

is an equivalence, hence the claim. □

5. Expressions involving the centers of the strata

We work in the same context as the previous section, so $X^{\mathrm{ss}} = \emptyset$, and $\mathcal{L} \in \operatorname{Pic}(X/G)$ is an appropriately chosen invertible sheaf. The usual formulation of Atiyah-Bott localization involves statements involving the centers Z_α^{ss}/L_α of the strata. The key observation is the following

{lem:units} **Lemma 5.1.** *Let E be a locally free sheaf on $Z_\alpha^{\mathrm{ss}}/L_\alpha$ whose weight 0 piece is trivial. Then $e(E) := \sum_i (-1)^i [\bigwedge^i E^*] \in K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^\wedge)$ is a unit.*

Proof. First decompose $E = E^+ \oplus E^-$ into summands of positive and negative weight respectively. Because $e(E) = e(E^+) \cdot e(E^-)$, it suffices to prove the lemma for each individually. $(E^+)^*$ has strictly negative weights, and hence the object $\operatorname{Sym}((E^+)^*) := \bigoplus_{n \geq 0} \operatorname{Sym}^n((E^+)^*)$ lies in $\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^\wedge$. The usual formal computation showing that $\operatorname{Sym}((E^+)^*) \otimes \bigwedge((E^+)^*) \sim \mathcal{O}_{Z_\alpha^{\mathrm{ss}}} \in K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^\wedge)$ is actually rigorous because these complexes are well-defined in the completed category. On the other hand $e(E^-) = (-1)^{\operatorname{rank}(E^-)} \det(E^-)^\vee \otimes e((E^-)^*)$. The invertible sheaf is a unit, and now the previous argument shows that $e(E^-)$ is a unit as well with

$$e(E^-)^{-1} = (-1)^{\operatorname{rank}(E^-)} \det(E^-) \otimes \operatorname{Sym}(E^-)$$

□

Remark 5.2. It follows from this that we can define $e(E)$ for any complex $E \in \mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)$ whose homology vanishes in weight 0. To do this, we choose a presentation as a finite complex of locally free sheaves $\to \cdots E_1 \to E_0 \to \cdots$. Because the homology vanishes in weight 0 we may discard the weight zero piece of each locally free sheaf E_i in this presentation, so we may assume that $E_i^0 = 0$. Then we define $e(E) = \prod_i e(E_i)^{(-1)^i}$. This is the unique extension of e to a group homomorphism $K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^{\neq 0}) \to K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^\wedge)^\times$.

Proposition 5.3. *The restriction functor σ^* induces an equivalence*

$$K_0(\mathrm{D}^b(X/G)^\wedge) \simeq \bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^\wedge) \simeq K_0(\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^{[w]})(\!(u)\!).$$

Proof. Note that even though the restriction functor $\sigma_\alpha^* : \mathrm{D}^b(X/G) \to \mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)$ is not compatible with the baric structures, it is compatible with the baric structures up to a finite shift in weights, and hence it maps $\mathrm{D}^b(X/G)^\wedge$ to $\mathrm{D}^b(Z_\alpha^{\mathrm{ss}}/L_\alpha)^\wedge$.

We apply Theorem 4.4 directly to the stack $\bigsqcup_\alpha Z_\alpha^{ss}/L_\alpha$ itself to obtain an isomoprhism

$$\bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge) \simeq \bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{[w]})(\!(u)\!).$$

Then we compose this with the isomoprhism of Proposition 4.7 and Theorem 4.4

$$\bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{[w]})(\!(u)\!) \simeq K_0(\mathrm{D}^b(X/G)^{[w]})(\!(u)\!) \simeq K_0(\mathrm{D}^b(X/G)^\wedge).$$

Finally we compose this with the restriction functor to $\bigoplus_\alpha \mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge$. If one traces through these maps, one finds that the composition $\bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge) \to \bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge)$ is multiplication by $\bigoplus e(N_{S_\alpha}X)$. Hence by Lemma 5.1 the restriction functor $K_0(\mathrm{D}^b(X/G)^\wedge) \to \bigoplus_\alpha K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge)$ differs from a known equivalence by multiplication by a unit, and it is therefore also an equivalence. □

We now reformulate our version of the localization theorem in the more familiar terms of [?AB].

{prop:pushfo

Proposition 5.4. *The pushforward functor* $(\sigma_\alpha)_* : \mathrm{QC}(Z_\alpha^{ss}/L_\alpha) \to \mathrm{QC}(X/G)$ *maps* $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge$ *to* $\mathrm{D}^b(X/G)^\wedge$.

Proof. Because the functor $(\iota_\alpha)_* : \mathrm{D}^b(S_\alpha/G) \to \mathrm{D}^b(X/G)$ is compatible with the baric structure, if suffices to show that the pushforward $(\sigma_\alpha)_* : \mathrm{QC}(Z_\alpha^{ss}/L_\alpha) \to \mathrm{QC}(S_\alpha/G)$ maps $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge$ to $\mathrm{D}^b(S_\alpha/G)^\wedge$. It suffices to show that $(\sigma_\alpha)_*$ maps $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{<w}$ to $\mathrm{D}^b(S_\alpha/G)^{<w+n}$ for some fixed integer n, independent of w.

In order to study this, we use a different presentation of the stack $S_\alpha/G \simeq Y_\alpha^{ss}/P_\alpha$, where $Y_\alpha^{ss} \to Z_\alpha^{ss}$ is the Bialynicki-Birula stratum associated to the distinguished one parameter subgroup λ_α, and $P_\alpha \subset G$ is the parabolic subgroup associated to λ_α. Then the section $\sigma_\alpha : Z_\alpha^{ss}/L_\alpha \to Y_\alpha^{ss}/P_\alpha$ factors as closed immersion $Z_\alpha^{ss}/L_\alpha \hookrightarrow Y_\alpha^{ss}/L_\alpha$ followed by the projection $Y_\alpha^{ss}/L_\alpha \to Y_\alpha^{ss}/P_\alpha$.

The stack Y_α^{ss}/L_α is also a Θ-stratum with center Z_α^{ss}/L_α, so $\mathrm{D}^b(Y_\alpha^{ss}/L_\alpha)$ has a baric structure according to the weights of $F|_{Z_\alpha^{ss}}$. The conormal bundle of Z_α^{ss} in Y_α^{ss} has negative weights, so the pushforward functor $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha) \to \mathrm{D}^b(Y_\alpha^{ss}/L_\alpha)$ maps $\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^{<w} \to \mathrm{D}^b(Y_\alpha^{ss}/L_\alpha)^{<w}$.

The map $Y_\alpha^{ss}/L_\alpha \to Y_\alpha^{ss}/P_\alpha$ is representable and affine, admitting a presentation by the map $(P_\alpha/L_\alpha) \times Y_\alpha^{ss}/P_\alpha \to Y_\alpha^{ss}/P_\alpha$. The scheme P_α/L_α is isomorphic to an copy of affine space which is attracted to a single fixed point under the action of $\lambda_\alpha(t)$ as $t \to 0$. It follows that under the grading induced by λ_α we have $\mathcal{O}_{P_\alpha/L_\alpha} = k \oplus \bigoplus_{w<0} A_w$. Using this one can show that the pushforward $\mathrm{QC}(Y_\alpha^{ss}/L_\alpha) \to \mathrm{QC}(Y_\alpha^{ss}/P_\alpha)$ maps $\mathrm{QC}(Y_\alpha^{ss}/L_\alpha)^{<w}$ to $\mathrm{QC}(Y_\alpha^{ss}/P_\alpha)^{<w}$, and it also maps $\mathrm{D}^b(Y_\alpha^{ss}/L_\alpha)$ to $\mathrm{D}^b(Y_\alpha^{ss}/P_\alpha)^\wedge$. This implies (using the criteria of Lemma 1.2) that the pushforward functor maps $\mathrm{D}^b(Y_\alpha^{ss}/L_\alpha)^\wedge$ to $\mathrm{D}^b(Y_\alpha^{ss}/P_\alpha)^\wedge$. □

Proposition 5.5. *The complex* $e(N_{Z_\alpha^{ss}}X)$ *is a unit in* $K_0(\mathrm{D}^b(Z_\alpha^{ss}/L_\alpha)^\wedge)$*, and in* $K_0(\mathrm{D}^b(X/G)^\wedge)$ *we have,*

$$[R\underline{\Gamma}_{S_\alpha}\mathcal{O}_X] = (\sigma_\alpha)_*\left(\frac{e(\mathcal{O}_{Z_\alpha^{ss}} \otimes \mathfrak{g}^{\lambda_\alpha \neq 0})}{e(N_{Z_\alpha^{ss}}X)}\right).$$

where $\mathfrak{g}^{\lambda_\alpha\neq 0}$ *denotes the direct summand of* $\mathfrak{g}$ *on which* λ_α *acts with non-zero weight.*

Remark 5.6. Note that the tangent complex of the stack X/G is a two term complex $\mathcal{O}_X \otimes \mathfrak{g} \to TX$, and the tangent complex of Z_α^{ss}/L_α is a two term complex $\mathcal{O}_{Z_\alpha^{ss}} \otimes \mathfrak{g}^{\lambda_\alpha=0} \to TZ_\alpha^{ss}$. Therefore $e(\mathcal{O}_{Z_\alpha^{ss}} \otimes \mathfrak{g}^{\lambda_\alpha\neq 0})/e(N_{Z_\alpha^{ss}}X) = e(T_{\sigma_\alpha}[-1])^{-1}$, where T_{σ_α} is the relative tangent complex of the map $\sigma_\alpha : Z_\alpha/L_\alpha \to X/G$, and hence $T_{\sigma_\alpha}[-1]$ is the "virtual normal bundle" of the map σ_α. When λ_α is central, and in particular when G is abelian, $\mathfrak{g} = \mathfrak{g}^{\lambda_\alpha=0}$, so this formula simplifies to $(\sigma_\alpha)_*(e(N_{Z_\alpha^{ss}}X)^{-1})$, which is closer to the usual form of the Atiyah-Bott localization formula.

Proof. By Theorem 3.1 it suffices to prove the claim for a single closed Θ-stratum $\iota : S \hookrightarrow X$. Using the description of the local cohomology complex as a colimit $R\underline{\Gamma}_S(\mathcal{O}_X) = \operatorname{colim}_n \underline{\mathrm{RHom}}_X(\mathcal{O}_X/I_S^n, \mathcal{O}_X)$, one can deduce that it has a bounded below filtration whose associated graded is

$$\begin{aligned} \underline{\mathrm{RHom}}_X(\iota_*(\mathrm{Sym}(N_S^\vee X)), \mathcal{O}_X) &\simeq \iota_*(\underline{\mathrm{RHom}}_S(\mathrm{Sym}(N_S^\vee X), \iota^!(\mathcal{O}_X)) \\ &\simeq \iota_*\left(\det(N_S X) \otimes \mathrm{Sym}(N_S X)[-c]\right) \end{aligned}$$

where $c = \operatorname{codim}(S, X)$. Thus factoring the map from the center of the strata as $\sigma : Z^{\mathrm{ss}}/L \xrightarrow{\sigma} Y^{\mathrm{ss}}/P \simeq S/G \xrightarrow{\iota} X/G$, is suffices to show that

{eqn:simp_1}
$$\sigma_*\left(\frac{e(\mathcal{O}_{Z^{\mathrm{ss}}} \otimes \mathfrak{g}^{\lambda \neq 0})}{e(N_{Z^{\mathrm{ss}}} X)}\right) = \det(N_S X) \otimes \mathrm{Sym}(N_S X)[-c] \in K_0(\mathrm{D}^b(S/G)^\wedge) \tag{4}$$

The computation at the end of the proof of Lemma 5.1 shows identifies the restriction $\det(N_S X|_{Z^{\mathrm{ss}}}) \otimes \mathrm{Sym}(N_S X|_{Z^{\mathrm{ss}}})[-c]$ with $e(N_S X|_{Z^{\mathrm{ss}}})^{-1}$, because $N_S X|_{Z^{\mathrm{ss}}}$ is a locally free sheaf concentrated in negative weights by construction. On the other hand, we have a short exact sequence $0 \to \mathfrak{g}^{<0} \to (N_{Z^{\mathrm{ss}}} X)^{<0} \to N_S X|_{Z^{\mathrm{ss}}} \to 0$, so

$$e(N_{Z^{\mathrm{ss}}} X)^{-1} = e((N_{Z^{\mathrm{ss}}} X)^{>0})^{-1} e(\mathcal{O}_{Z^{\mathrm{ss}}} \otimes \mathfrak{g}^{<0})^{-1} e(N_S X|_{Z^{\mathrm{ss}}})^{-1}.$$

By the projection formula, in order to verify (4) it suffices to show that

$$\sigma_*\left(e((N_{Z^{\mathrm{ss}}} X)^{>0})^{-1} e(\mathcal{O}_{Z^{\mathrm{ss}}} \otimes \mathfrak{g}^{>0})\right) = \mathcal{O}_S \in K_0(\mathrm{D}^b(S/G)^\wedge),$$

which we now verify.

By the projection formula it suffices to show: 1) that the pushforward $\mathrm{D}^b(Z^{\mathrm{ss}}/L)^\wedge \to \mathrm{D}^b(Y^{\mathrm{ss}}/L)^\wedge$ maps $e((N_{Z^{\mathrm{ss}}} X)^{>0})^{-1}$ to $\mathcal{O}_{Y^{\mathrm{ss}}} \in K_0(\mathrm{D}^b(Y^{\mathrm{ss}}/L)^\wedge)$, then 2) that the pushforward $\mathrm{D}^b(Y^{\mathrm{ss}}/L)^\wedge \to \mathrm{D}^b(Y^{\mathrm{ss}}/P)^\wedge$ maps $e(\mathcal{O}_{Y^{\mathrm{ss}}} \otimes \mathfrak{g}^{>0})$ to $\mathcal{O}_{Y^{\mathrm{ss}}} \in K_0(\mathrm{D}^b(Y^{\mathrm{ss}}/P)^\wedge)$:

Step 1: The map $\pi : Y^{\mathrm{ss}} \to Z^{\mathrm{ss}}$ is a locally trivial fibration of affine spaces with the section given by $\sigma : Z^{\mathrm{ss}} \to Y^{\mathrm{ss}}$. Under scaling action of the distinguished one parameter subgroup λ, $\pi_* \mathcal{O}_{Y^{\mathrm{ss}}}$ is negatively graded with weight 0 piece isomorphic to $\mathcal{O}_{Z^{\mathrm{ss}}}$. Using the equivalence between the category of equivariant quasi-coherent sheaves on Y^{ss} and quasi-coherent equivariant sheaves of $\pi_*\mathcal{O}_{Y^{\mathrm{ss}}}$-modules on Z^{ss}, we see that the filtration of $\mathcal{O}_{Y^{\mathrm{ss}}}$ by λ-weights has as its associated graded $\sigma_*(\mathrm{Sym}(N_{Z^{\mathrm{ss}}}^\vee Y^{\mathrm{ss}}))$, so these classes are equal in $K_0(\mathrm{D}^b(Y^{\mathrm{ss}}/L)^\wedge)$.[5] On the other hand $N_{Z^{\mathrm{ss}}}^\vee Y^{\mathrm{ss}} \simeq ((N_{Z^{\mathrm{ss}}} X)^{>0})^\vee$, so $\mathrm{Sym}(N_{Z^{\mathrm{ss}}}^\vee Y^{\mathrm{ss}}) = e((N_{Z^{\mathrm{ss}}} X)^{>0})^{-1}$ by Lemma 5.1.

Step 2: As discussed in the proof of Proposition 5.4, the map $Y^{\mathrm{ss}}/L \to Y^{\mathrm{ss}}/P$ is affine – it is the relative Spec of the sheaf of algebras $\mathcal{O}_{Y^{\mathrm{ss}}} \otimes_k \mathcal{O}_{P/L} \in \mathrm{QC}(Y^{\mathrm{ss}}/P)$. The object $\mathcal{O}_{Y^{\mathrm{ss}}} \otimes \mathfrak{g}^{>0} \in \mathrm{D}^b(Y^{\mathrm{ss}}/L)$ is the pullback of the complex of the same name in $\mathrm{D}^b(Y^{\mathrm{ss}}/P)$, so by the projection formula it suffices to show that

$$[\mathcal{O}_{Y^{\mathrm{ss}}} \otimes_k \mathcal{O}_{P/L}] \otimes e(\mathcal{O}_{Y^{\mathrm{ss}}} \otimes \mathfrak{g}^{>0}) = [\mathcal{O}_{Y^{\mathrm{ss}}}] \in K_0(\mathrm{D}^b(Y^{\mathrm{ss}}/P)^\wedge).$$

Evidently, all of these classes are pulled back from $\mathrm{D}^b(\mathrm{pt}/P)^\wedge$, so it suffices to verify the identity $[\mathcal{O}_{P/L}] e(\mathfrak{g}^{>0}) = [k] \in K_0(\mathrm{D}^b(\mathrm{pt}/P)^\wedge)$. So $\mathcal{O}_{P/L}$ has a filtration whose associated graded is $\mathrm{Sym}((\mathfrak{g}^{>0})^*)$, which implies $[\mathcal{O}_{P/L}] = e(\mathfrak{g}^{>0})^{-1}$ and thus our identity.

□

Our final statement of Atiyah-Bott localization is thus

[5] The latter sum converges because the weights of $\mathrm{Sym}^n(N_{Z^{\mathrm{ss}}}^\vee Y^{\mathrm{ss}})$ approach $-\infty$ as $n \to \infty$.

Corollary 5.7. *We have a decomposition of the unit* $[\mathcal{O}_X] \in K_0(\mathrm{D}^b(X/G)^\wedge)$ *as a finite sum of idempotents*

$$[\mathcal{O}_X] = \sum_\alpha [R\underline{\Gamma}_{S_\alpha}(\mathcal{O}_X)] = \sum_\alpha (\sigma_\alpha)_* \left(\frac{e(\mathcal{O}_{Z_\alpha^{\mathrm{ss}}} \otimes \mathfrak{g}^{\lambda_\alpha \neq 0})}{e(N_{Z_\alpha^{\mathrm{ss}}} X)} \right),$$

where $\sigma_\alpha : Z_\alpha^{\mathrm{ss}}/L_\alpha \to X/G$ *are the centers of the strata.*