# Monoidal Functors, Species and Hopf Algebras

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# Foreword by Kenneth Brown and Stephen Chase

Ever since they were graduate students at Cornell University, Marcelo Aguiar and Swapneel Mahajan have shown a remarkable ability to grasp and develop highly complex and abstract theories without losing sight of their more down-to-earth aspects. They are able to find unexpected connections between apparently different subjects which almost always unify them and illuminate their mysteries. And they are also wonderful communicators, with an ability to find just the right example, or just the right picture, to explain a complicated idea.

It was, therefore, very gratifying to us, their former thesis advisors, that they began to combine their talents several years ago. Their first joint publication [12] uses geometric methods, inspired by Jacques Tits's projection operators in the theory of buildings, to study a variety of Hopf algebras arising in combinatorics. That monograph introduces fundamentally new ideas and obtains many new results, in addition to unifying and simplifying the earlier theories.

Another giant step towards a deeper understanding of these "combinatorial Hopf algebras" is taken in the present work. Its heart is Part II, in which the crucial notion of a Hopf monoid in the monoidal category of linear species is introduced (a linear species being a functor from finite sets to vector spaces over a field). The authors give due credit for this concept to others, but by means of their detailed theory of these objects, and the wealth of examples they construct from geometric and combinatorial data, they make it their own. Hopf monoids in species appear to retain some of the information that is lost when one passes directly from these data to the relevant Hopf algebras; their connection to these Hopf algebras is thus perhaps reminiscent of that between derived categories and the classical invariants of homological algebra.

But how does one obtain the combinatorial Hopf algebras from Hopf monoids in species? It is here that the highly categorical Part I of the monograph plays its starring role. In it the authors introduce and study in great detail the notion of a bilax functor on a braided monoidal category; namely, a functor which is both lax and colax and satisfies appropriate compatibility conditions linking these two structures. The point is that, just as a (co)lax functor preserves (co)monoids, a bilax functor preserves bimonoids. In Part III of the monograph the relevant bilax functors on species—labeled "Fock functors"—are introduced, studied, and applied to obtain bialgebras (arising as bimonoids in the monoidal category of graded vector spaces), which are then shown to be the desired Hopf algebras.

The monograph, especially Parts I and III, introduces a wealth of further ideas, concepts, and results that will undoubtedly exert a strong influence on future research in algebraic combinatorics. In particular, the authors present the notion of a 2-monoidal category (a category with two monoidal structures linked by an interchange law), and show that it provides a natural setting for the theory of bilax functors; they also prove, and apply, the fact that linear species comprise, in two different ways, such a 2-monoidal category. Moreover, their theorem in Part I that the familiar construction of a chain complex from a simplicial module constitutes, by means of the Alexander–Whitney and Eilenberg–Zilber maps, a bilax functor; their construction in Part III of important quantum groups from Hopf monoids in colored species; and many other issues discussed throughout the monograph indicate that it has a reach extending far beyond the combinatorial questions it was originally intended to address.

It is our great pleasure to introduce you to the exciting world of Monoidal functors, species, and Hopf algebras.

Ithaca, NY — *Kenneth S. Brown*
March 2009 — *Stephen U. Chase*

# Foreword by André Joyal

The theory described in this book is at the crossroads between category theory, algebra, and combinatorics. Its main goal is to unify and clarify a large number of constructions of Hopf algebras found in the literature and to reveal many new connections between them. The book contains a systematic description of the relevant aspects of category theory together with many examples of Hopf algebras. It includes the tensor algebra, the shuffle algebra, the symmetric algebra, the exterior algebra, the divided power algebra, the algebra of symmetric functions, the algebra of permutations introduced by Malvenuto and Reutenauer, the algebra of quasi-symmetric functions introduced by Gessel and the algebra of noncommutative symmetric functions introduced by Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon. It also includes $q$-deformations of these Hopf algebras as well as multi-parameters deformations and decorated versions. The book contains other interesting material which is not discussed in this foreword.

The notions of algebra, coalgebra, bialgebra and Hopf algebras are special cases of the general notions of monoid, comonoid, bimonoid and Hopf monoid in a monoidal category. The notion of bilax (monoidal) functor plays a central role in the theory. Its importance arises from the fact that the image of a bimonoid by a bilax functor is again a bimonoid. A strong braided monoidal functor is bilax, but a bilax functor may not be strong. For example, in algebraic topology, the chain complex functor

$$\mathcal{C}\colon \mathsf{sMod} \to \mathsf{dgMod}$$

from the category of simplicial modules to the category of differential graded modules is bilax but not strong. Its lax structure is given by the Eilenberg–Zilber map

$$\varphi\colon \mathcal{C}(X) \otimes \mathcal{C}(Y) \to \mathcal{C}(X \otimes Y)$$

and its colax structure by the Alexander–Whitney map

$$\psi\colon \mathcal{C}(X \otimes Y) \to \mathcal{C}(X) \otimes \mathcal{C}(Y).$$

Most bilax functors considered in the book preserve Hopf monoids in addition to preserving bimonoids. This means that a bilax functor can be used for constructing new Hopf monoids from old ones. A stronger notion is that of Hopf lax (monoidal) functor. The image of a Hopf monoid under a Hopf lax functor is always a Hopf monoid.

The Hopf algebras considered in the book are all constructed from a small number of *primeval* Hopf monoids living in the category of vector species $\mathsf{Sp}$. This generalizes the constructions by Stover of Hopf algebras from *twisted Hopf algebras* in the sense of Barratt. If $\Bbbk$ is a fixed field, then a *vector species* $\mathbf{p}$ is a functor

$$\mathbf{p}[-]\colon \mathsf{Set}^{\times} \to \mathsf{Vec}$$

where $\mathsf{Set}^\times$ is the category of finite sets and bijections and $\mathsf{Vec}$ is the category of $\Bbbk$-vector spaces and linear maps. The *Cauchy product* of two vector species $\mathbf{p}$ and $\mathbf{q}$ is defined by the formula

$$(\mathbf{p} \cdot \mathbf{q})[I] = \bigoplus_{S \sqcup T = I} \mathbf{p}[S] \otimes \mathbf{q}[T]$$

where the sum is indexed over all decompositions $S \cup T = I$, $S \cap T = \emptyset$. The category $(\mathsf{Sp}, \cdot)$ of vector species equipped with the Cauchy product is monoidal and a twisted $\Bbbk$-algebra is a monoid in this monoidal category (the authors prefer not to use this terminology). The category is symmetric monoidal with symmetry $\beta$ which interchanges the factors. Most primeval Hopf monoids in the symmetric monoidal category $(\mathsf{Sp}, \cdot, \beta)$ can be constructed combinatorially. For example, if $\mathrm{L}[I]$ denotes the set of linear orders on a finite set $I$, then the functor $\mathrm{L}\colon (\mathsf{Set}^\times, \sqcup) \to (\mathsf{Set}, \times)$ is bilax. Its lax structure

$$\mathrm{L}[S] \times \mathrm{L}[T] \to \mathrm{L}[S \sqcup T]$$

is defined by concatenating $(l_1, l_2) \mapsto l_1 \cdot l_2$ and its colax structure

$$\mathrm{L}[S \sqcup T] \to \mathrm{L}[S] \times \mathrm{L}[T]$$

by restricting $l \mapsto (l|_S, l|_T)$. It then follows by a general result of Chase that the vector species $\mathbf{L} = \Bbbk\mathrm{L}$ has the structure of a bimonoid in $(\mathsf{Sp}, \cdot, \beta)$ (it is actually a Hopf monoid). There is a similar Hopf monoid structure on the exponential species $\mathbf{E}$, the species of partitions $\mathbf{\Pi}$ and the species of compositions (ordered partitions) $\mathbf{\Sigma}$. Other examples can be obtained by dualizing, where the dual $\mathbf{p}^*$ of a vector species $\mathbf{p}$ is defined by setting $\mathbf{p}^*[I] = \mathbf{p}[I]^*$ for every finite set $I$. The Hopf monoids $\mathbf{E}$ and $\mathbf{\Pi}$ are self-dual.

For each vector space $V$, the *evaluation functor*

$$e_V \colon \mathsf{Sp} \to \mathsf{gVec}$$

defined by setting

$$e_V(\mathbf{p}) = \mathbf{p}(V) = \bigoplus_n \mathbf{p}[n] \otimes_{\mathrm{S}_n} V^{\otimes n},$$

is strong monoidal and braided, where $\mathsf{gVec}$ is the category of graded vector spaces. It thus takes a Hopf monoid $\mathbf{h}$ in $\mathsf{Sp}$ to a graded Hopf algebra $\mathbf{h}(V)$ which is commutative (resp. cocommutative) if $\mathbf{h}$ is commutative (resp. cocommutative). For example, the Hopf algebra $\mathbf{L}(V)$ is the tensor algebra $\mathcal{T}(V)$, the Hopf algebra $\mathbf{L}^*(V)$ is the shuffle algebra $\mathcal{T}^\vee(V)$, and the Hopf algebra $\mathbf{E}(V)$ is the symmetric algebra $\mathcal{S}(V)$.

The functor $e_V$ is called the *decorated bosonic Fock functor* and is denoted $\overline{\mathcal{K}}_V$. If $V = \Bbbk$, it is called the *bosonic Fock functor* and denoted $\overline{\mathcal{K}}$. By definition, we have

$$\overline{\mathcal{K}}(\mathbf{p}) = \bigoplus_n \mathbf{p}[n]_{\mathrm{S}_n},$$

where $\mathbf{p}[n]_{\mathrm{S}_n}$ denotes the space of $\mathrm{S}_n$-coinvariants of $\mathbf{p}[n]$.

The Hopf algebra $\overline{\mathcal{K}}(\mathbf{\Pi})$ is the algebra of symmetric functions $\Lambda$ (when $\Bbbk$ is of characteristic 0), and it is self dual, since $\mathbf{\Pi}$ is self-dual. The Hopf algebra $\overline{\mathcal{K}}(\mathbf{\Sigma})$ is the algebra of noncommutative symmetric functions $\mathrm{N}\Lambda$ and the Hopf algebra $\overline{\mathcal{K}}(\mathbf{\Sigma}^*)$ is the algebra of quasi-symmetric functions $\mathrm{Q}\Lambda$.

The *full Fock functor* $\mathcal{K}\colon \mathsf{Sp} \to \mathsf{gVec}$ and its decorated version $\mathcal{K}_V$ are respectively defined by setting

$$\mathcal{K}(\mathbf{p}) = \bigoplus_n \mathbf{p}[n] \quad \text{and} \quad \mathcal{K}_V(\mathbf{p}) = \bigoplus_n \mathbf{p}[n] \otimes V^{\otimes n}.$$

These functors are bilax but not strong monoidal. The Hopf algebra $\mathcal{K}(\mathbf{L}^*)$ is the algebra of permutations $\mathrm{S}\Lambda$.

Other examples of bilax functors can be obtained by dualizing. Every bilax functor $\mathcal{F}\colon \mathsf{Sp} \to \mathsf{gVec}$ has a dual $\mathcal{F}^\vee\colon \mathsf{Sp} \to \mathsf{gVec}$ obtained by composing with the functors

$$(-)^*\colon \mathsf{Sp} \to \mathsf{Sp} \quad \text{and} \quad (-)^*\colon \mathsf{gVec} \to \mathsf{gVec}.$$

Assume (for simplicity) that each space $\mathbf{p}[n]$ is finite-dimensional. By dualizing the Fock functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ we obtain two new bilax functors

$$\mathcal{K}^\vee(\mathbf{p}) = \bigoplus_n \mathbf{p}[n] \quad \text{and} \quad \overline{\mathcal{K}}^\vee(\mathbf{p}) = \bigoplus_n \mathbf{p}[n]^{\mathrm{S}_n},$$

where $\mathbf{p}[n]^{\mathrm{S}_n}$ denotes the space of $\mathrm{S}_n$-invariants in $\mathbf{p}[n]$. Each of these functors has a decorated version,

$$\mathcal{K}_V^\vee(\mathbf{p}) = \bigoplus_n \mathbf{p}[n] \otimes V^{\otimes n} \quad \text{and} \quad \overline{\mathcal{K}}_V^\vee(\mathbf{p}) = \bigoplus_n (\mathbf{p}[n] \otimes V^{\otimes n})^{\mathrm{S}_n}.$$

The functors $\mathcal{K}_V$ and $\mathcal{K}_V^\vee$ are isomorphic but they differ by their bilax structure. For example, $\mathcal{K}_V(\mathbf{E})$ is the tensor algebra $\mathcal{T}(V)$ while $\mathcal{K}_V^\vee(\mathbf{E})$ is the shuffle algebra $\mathcal{T}^\vee(V)$. The functor $\overline{\mathcal{K}}_V^\vee$ is strong monoidal and braided.

A lax or a colax functor can be braided. For example, the chain complex functor is braided lax, since the Eilenberg–Zilber map preserves the natural symmetry $X \otimes Y \cong Y \otimes X$, but it is not braided colax, since the Alexander–Whitney map does not preserve this symmetry. Dually, a bilax functor can be braided colax without being braided lax. For example, the functor $\mathcal{K}$ is braided colax but it is not braided lax. This is why the Hopf algebra $\mathcal{K}(\mathbf{\Pi})$ is cocommutative but not commutative, although the Hopf monoid $\mathbf{\Pi}$ is bicommutative. Similarly, the Hopf algebra $\mathcal{K}(\mathbf{L}^*)$ is not commutative although the Hopf monoid $\mathbf{L}^*$ is commutative.

The category of vector species $\mathsf{Sp}$ admits another monoidal structure in addition to the Cauchy product: the *Hadamard product* $\mathbf{p} \times \mathbf{q}$ of two vector species $\mathbf{p}$ and $\mathbf{q}$ is defined by setting

$$(\mathbf{p} \times \mathbf{q})[I] = \mathbf{p}[I] \otimes \mathbf{q}[I]$$

for every finite set $I$. It turns out that the Hadamard product functor

$$\times\colon \mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$$

is bilax. It follows that the Hadamard product $\mathbf{h}_1 \times \mathbf{h}_2$ of two Hopf monoids $\mathbf{h}_1$ and $\mathbf{h}_2$ is again a Hopf monoid. It follows also that the functor $\mathbf{h} \times -\colon \mathsf{Sp} \to \mathsf{Sp}$ is bilax for any bimonoid $\mathbf{h}$. In particular, the functor $\mathbf{L} \times -\colon \mathsf{Sp} \to \mathsf{Sp}$ is bilax; hence so is its composite with the functor $\overline{\mathcal{K}}\colon \mathsf{Sp} \to \mathsf{gVec}$. The obvious natural isomorphism

$$\mathcal{K}(\mathbf{p}) \cong \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{p})$$

preserves the bilax structure on these functors. It follows that the Hopf algebra of permutations

$$\mathrm{S}\Lambda = \mathcal{K}(\mathbf{L}^*) \cong \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{L}^*)$$

is self-dual, since the Hopf monoid $\mathbf{L} \times \mathbf{L}^*$ is self-dual.

The notion of *fermionic Fock functor* can be obtained by replacing the symmetry isomorphism $U \otimes V \xrightarrow{\cong} V \otimes U$ between graded vector spaces with a graded symmetry

$$\beta_{-1} \colon U \otimes V \xrightarrow{\cong} V \otimes U$$

defined by setting

$$\beta_{-1}(x \otimes y) = (-1)^{mn}\, y \otimes x$$

for $x \in U_m$ and $y \in V_n$. This defines a symmetric monoidal category $(\mathsf{gVec}, \beta_{-1})$. A commutative monoid in $(\mathsf{gVec}, \beta_{-1})$ is a graded commutative algebra.

If $V$ is a vector space, let $V\langle n\rangle$ denote the graded vector space with $V$ concentrated in degree $n$. We then have an $\mathrm{S}_n$-equivariant isomorphism

$$V\langle 1\rangle^{\otimes n} = \mathbf{E}^-[n] \otimes V^{\otimes n}\langle n\rangle$$

in the category $(\mathsf{gVec}, \beta_{-1})$, where $\mathbf{E}^-[n]$ denotes the sign representation of $\mathrm{S}_n$. The functor

$$e_V^- \colon (\mathsf{Sp}, \beta) \to (\mathsf{gVec}, \beta_{-1})$$

defined by setting

$$e_V^-(\mathbf{p}) = \mathbf{p}(V\langle 1\rangle) = \bigoplus_n \mathbf{p}[n] \otimes_{\mathrm{S}_n} V\langle 1\rangle^{\otimes n} = \bigoplus_n (\mathbf{p}[n] \otimes \mathbf{E}^-[n] \otimes V^{\otimes n})_{\mathrm{S}_n}$$

is strong monoidal and braided. It thus takes a Hopf monoid $\mathbf{h}$ in $\mathsf{Sp}$ to a graded Hopf algebra $\mathbf{h}(V\langle 1\rangle)$ which is graded commutative (resp. cocommutative) if $\mathbf{h}$ is commutative (resp. cocommutative). For example, the Hopf algebra $e_V^-(\mathbf{E})$ is the exterior algebra on $V$.

If $V = \Bbbk$, the functor $e_V^-(\mathbf{E})$ is called the *fermionic Fock functor* and is denoted $\overline{\mathcal{K}}_{-1}$. By definition, we have

$$\overline{\mathcal{K}}_{-1}(\mathbf{p}) = \bigoplus_n (\mathbf{p}[n] \otimes \mathbf{E}^-[n])_{\mathrm{S}_n}.$$

The dual fermionic functor $\overline{\mathcal{K}}_{-1}^{\vee} \colon (\mathsf{Sp}, \cdot) \to (\mathsf{gVec}, \beta_{-1})$ defined by setting

$$\overline{\mathcal{K}}_{-1}^{\vee}(\mathbf{p}) = \bigoplus_n (\mathbf{p}[n] \otimes \mathbf{E}^-[n])^{\mathrm{S}_n}$$

is also strong monoidal and braided.

Deformation of algebras, coalgebras, bialgebras and Hopf algebras is an important topic discussed in the book. If $q \in \Bbbk$ is nonzero and if $U$ and $V$ are graded $\Bbbk$-vector spaces, then the map

$$\beta_q \colon U \otimes V \xrightarrow{\cong} V \otimes U,$$

defined by setting

$$\beta_q(x \otimes y) = q^{mn}\, y \otimes x$$

for $x \in U_m$ and $y \in V_n$, is a braiding. This defines a braided monoidal category $(\mathsf{gVec}, \beta_q)$. The braiding $\beta_q$ is a symmetry if $q = \pm 1$. If $\mathbf{p}$ and $\mathbf{q}$ are vector species, then the map

$$\beta_q \colon \mathbf{p} \cdot \mathbf{q} \xrightarrow{\cong} \mathbf{q} \cdot \mathbf{p},$$

defined by setting

$$\beta_q(x \otimes y) = q^{mn}\, y \otimes x$$

for $x \in \mathbf{p}[m]$ and $y \in \mathbf{q}[n]$, is a braiding. This defines a braided monoidal category $(\mathsf{Sp}, \beta_q)$. A Hopf monoid in $(\mathsf{Sp}, \beta_q)$ is called a *q-Hopf monoid*. For example, the Hopf monoid $\mathbf{L}$ admits a $q$-deformation $\mathbf{L}_q$ in which the coproduct $\delta_q \colon \mathbf{L} \to \mathbf{L} \cdot \mathbf{L}$ is defined by setting

$$\delta_q(l) = q^{\mathrm{sch}(S,T)}(l|_S, l|_T)$$

for every linear order $l$ on $S \sqcup T$. Here $\mathrm{sch}(S,T)$ is the distance between the orders $(l|_S) \cdot (l|_T)$ and $l$. The Hopf monoid $\mathbf{\Sigma}$ has a similar deformation $\mathbf{\Sigma}_q$. Other $q$-Hopf monoids can be obtained by dualizing.

The Fock functors

$$\overline{\mathcal{K}}_V, \mathcal{K}_V \colon (\mathsf{Sp}, \beta_q) \to (\mathsf{gVec}, \beta_q)$$

are respectively braided strong and bilax. They thus take a $q$-Hopf monoid to a $q$-Hopf algebra. For example, $\overline{\mathcal{K}}_V(\mathbf{L}_q^*)$ is the $q$-shuffle algebra $\mathcal{T}_q^\vee(V)$, a deformation of the shuffle algebra $\mathcal{T}^\vee(V)$. The algebra $\mathrm{S}\Lambda_q = \mathcal{K}(\mathbf{L}_q^*)$ is a deformation of the Hopf algebra of permutations $\mathrm{S}\Lambda$. More generally, there are *deformed Fock functors*

$$\mathcal{K}_q, \mathcal{K}_q^\vee, \colon (\mathsf{Sp}, \beta_p) \to (\mathsf{gVec}, \beta_{pq})$$

obtained by respectively deforming the colax structure of the functor $\mathcal{K}$ and the lax structure of the functor $\mathcal{K}^\vee$. There are also canonical isomorphisms of bilax functors

$$\mathcal{K}_q \cong \overline{\mathcal{K}}(\mathbf{L}_q \times -) \quad \text{and} \quad \mathcal{K}_q^\vee \cong \overline{\mathcal{K}}^\vee(\mathbf{L}_q^* \times -)$$

and it may be deduced that the $q$-Hopf algebra

$$\mathrm{S}\Lambda_q = \overline{\mathcal{K}}(\mathbf{L}_q \times \mathbf{L}^*)$$

is self-dual.

The book contains many other interesting subjects not discussed in this foreword. For example, it describes the relations between the various Fock functors using the properties of the norm transformation. It develops a theory of up and down operators on species and a theory of creation and annihilation operators on Fock spaces. It introduces the notion of 2-monoidal category and more generally of $n$-monoidal category. These are related to a notion introduced by Balteanu, Fiedorowicz, Schwänzl and Vogt. An example of a 2-monoidal category is provided by $(\mathsf{Sp}, \cdot, \times)$, but many more examples are given in the book.

The book of Aguiar and Mahajan is a quantum leap toward the mathematics of the future. I strongly recommend it to all researchers in algebra, topology and combinatorics.

Montréal *André Joyal*
August 2009

# Introduction

This research monograph is divided into three parts. Broadly speaking, Part I belongs to the realm of category theory, while Parts II and III pertain to algebraic combinatorics, although the language of the former is present and apparent throughout. Four appendices supplement the main text.

In this introduction, we explain informally the main ideas in this monograph and provide pointers to important results in the text. We also indicate the chapter most relevant to a particular discussion. More details are given in the introduction of individual chapters.

## Contents of Part I

Part I is of a general nature; it contains the material on monoidal categories on which the constructions of the later parts are laid out. Except for a few references to later parts for the purpose of examples, Part I is independent and self-contained. Our main goal here is to develop the basic theory of *bilax monoidal functors.* These are functors between braided monoidal categories which allow for transferring bimonoids in one to bimonoids in the other. These functors possess a rich theory, more so than one may perhaps anticipate. Further, a systematic study of bilax monoidal functors naturally leads us to the exciting world of higher monoidal categories. The results here constitute the beginnings of a theory which should find applications in a variety of settings and which should therefore be of wide interest.

**Monoidal categories** (Chapter 1)**.** Our starting point is a review of basic notions pertaining to monoidal category theory. We do not assume any previous knowledge of the subject. As our goal in Part III is the construction of certain Hopf algebras, in Chapter 1 there is a special emphasis on the notion of *Hopf monoid* (and the related notions of monoid, comonoid, and bimonoid). This notion can be defined in a braided monoidal category. When the latter is the category of (graded) vector spaces, one obtains the notion of (graded) Hopf algebra. In Part II we deal with the category of species, and Hopf monoids therein are our main concern. Chapter 1 thus lays out notions that will occupy us throughout the monograph and sets up the corresponding notation.

**Graded vector spaces** (Chapter 2)**.** Our next step is a review of basic notions pertaining to graded vector spaces. We discuss three monoidal structures on this category, namely, the Cauchy, Hadamard and substitution products. Our emphasis is on the Cauchy product since it is Hopf monoids with respect to this product which yield the notion of graded Hopf algebras. We discuss $q$-Hopf algebras, a notion obtained by deforming the braiding by a parameter $q$, and also $Q$-Hopf algebras, which are higher dimensional generalizations obtained by considering braidings parametrized by a matrix $Q$. Examples include the Eulerian $q$-Hopf algebra of Joni and Rota, and Manin's quantum linear space.

We discuss basic Hopf algebras such as the tensor algebra, the shuffle algebra, the symmetric algebra and other relatives, and explain how they relate via symmetrization and abelianization. These relations can be understood via freeness and cofreeness properties of these Hopf algebras. We also discuss comparatively less familiar objects such as the tensor algebra on a coalgebra and the quasi-shuffle bialgebra.

We also consider graded vector spaces with the added structure of boundary maps. These include chain complexes (which are used later to discuss an important example of a bilax monoidal functor) and graded vector spaces with creation-annihilation operators.

In Part II we are mainly concerned with analogues of these constructions for species. The link to graded vector spaces is made in Part III, where we study how monoidal properties of species translate to monoidal properties of graded vector spaces. Many results of Chapter 2 can then be seen as particular instances of results on species. Thus, Chapter 2 provides us in a nutshell some of the main ideas that will occupy us in later parts of this monograph.

**Lax and colax monoidal functors** (Chapter 3)**.** There are two types of functors between monoidal categories: lax and colax. A lax monoidal functor $\mathcal{F}$ is equipped with a transformation

$$\varphi_{A,B} \colon \mathcal{F}(A) \bullet \mathcal{F}(B) \to \mathcal{F}(A \bullet B)$$

satisfying certain associativity and unital conditions. The transformation $\varphi$ neefd not be an isomorphism. Colax monoidal functors $(\mathcal{F}, \psi)$ are the dual notion. These notions go back to the dawn of monoidal category theory with Bénabou [36]. Observe that if one ignores the objects $A$ and $B$ in the above definition, then a lax monoidal functor specializes to a monoid. This can be stated more precisely: A monoid is equivalent to a lax monoidal functor from the one-arrow category. In contrast to monoids, lax monoidal functors can be composed. The composite of lax monoidal functors is again lax (Theorem 3.21). This implies that a lax monoidal functor preserves monoids. More precisely, if $(A, \mu)$ is a monoid and $(\mathcal{F}, \varphi)$ is a lax monoidal functor, then $\mathcal{F}(A)$ is a monoid whose product is given by the composite

$$\mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{A,A}} \mathcal{F}(A \bullet A) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(A).$$

Colax monoidal functors correspond to comonoids in the same manner as lax monoidal functors correspond to monoids. A strong monoidal functor is a lax monoidal functor $(\mathcal{F}, \varphi)$ for which the transformation $\varphi$ is an isomorphism. This is the notion of monoidal functor most frequently used in the literature. In this situation, the distinction between lax and colax monoidal functors disappears.

We also consider adjunctions between monoidal functors. One of the main results here states that the right adjoint of a colax monoidal functor carries a canonical lax structure, and the left adjoint of a lax monoidal functor carries a canonical colax structure (Proposition 3.84). This, as well as some related results, can be obtained as special cases of general results of Kelly on adjunctions between categories with structure [195], but we provide direct proofs.

In category theory, along with categories and functors, there are natural transformations. This gives rise to the 2-category $\mathsf{Cat}$. In the context of monoidal categories and (co)lax functors, there is a corresponding notion for natural transformations. We simply call them morphisms between (co)lax functors. This gives rise to the 2-categories $\mathsf{lCat}$ and $\mathsf{cCat}$ corresponding to the lax and colax cases respectively. The fact that 1-cells in a 2-category can be composed corresponds to the fact that the composite of (co)lax monoidal functors is again (co)lax.

**Bilax monoidal functors** (Chapter 3)**.** If the monoidal category is braided, then the notions of lax and colax monoidal functor can be combined into that of a bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$, much in the same manner as the notion of bimonoid combines the notions of monoid and comonoid. Recall that bimonoids are monoids in the category of comonoids (or comonoids in the category of monoids). We provide a similar characterization of bilax monoidal functors (Proposition 3.77). Many results for bimonoids carry over to bilax monoidal functors in this manner. If the transformations $\varphi$ and $\psi$ are isomorphisms, then we say that the functor is bistrong. Just as for lax and colax monoidal functors, the composite of bilax monoidal functors is again bilax (Theorem 3.22). Further, just as for monoids and comonoids, a bimonoid is equivalent to a bilax functor from the one-arrow category. It then follows that a bilax functor preserves bimonoids.

There are two other types of functors between braided monoidal categories: braided lax and braided colax. Unlike the bilax case, these have appeared frequently in the literature. They preserve commutative monoids and cocommutative comonoids respectively. If the underlying lax structure of a braided lax functor is strong, then we say that the functor is braided strong. It is important to point out that in the strong situation, the distinction between bilax, braided lax and braided colax disappears. Thus, a bistrong monoidal functor is the same thing as a braided strong monoidal functor (Proposition 3.46). This nontrivial result may explain the lack of references in the literature to the notion of bilax monoidal functors: to encounter this notion one must look beyond the familiar setting of strong (and braided strong) monoidal functors.

A number of examples of bilax monoidal functors and morphisms between them are given in the monograph; a summary is provided in Tables 3.1, 3.2 and 3.3.

**The op and cop constructions** (Chapter 3)**.** Recall that to any monoid $A$ in a braided monoidal category, one can associate the opposite monoids $A^{\mathrm{op}}$ and ${}^{\mathrm{op}}A$ by precomposing the product with the braiding or its inverse. Similarly, to any comonoid $C$, one associates the opposite comonoids ${}^{\mathrm{cop}}C$ and $C^{\mathrm{cop}}$ by postcomposing the coproduct with the braiding or its inverse. We refer to these as the op and cop constructions.

The same construction can be carried out for (co)lax monoidal functors. To $(\mathcal{F}, \varphi)$, we associate $(\mathcal{F}, \varphi^b)$ and $(\mathcal{F}, {}^b\varphi)$, and similarly to $(\mathcal{F}, \psi)$, we associate $(\mathcal{F}, \psi^b)$ and $(\mathcal{F}, {}^b\psi)$. These are obtained by conjugating the (co)lax structures with the braiding or its inverse. For example,

$$\varphi^b \colon \mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\beta} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{B,A}} \mathcal{F}(B \bullet A) \xrightarrow{\mathcal{F}(\beta^{-1})} \mathcal{F}(A \bullet B).$$

These constructions can be combined to obtain the following important result. If $(\mathcal{F}, \varphi, \psi)$ is a bilax monoidal functor, then so are $(\mathcal{F}, \varphi^b, \psi^b)$ and $(\mathcal{F}, {}^b\varphi, {}^b\psi)$ (Proposition 3.16). The images of a bimonoid $H$ under these functors yield bimonoids which are related to $\mathcal{F}(H)$ via the op and cop constructions. The precise result is given in Proposition 3.34.

If $\mathcal{F}$ is braided lax, then $\varphi = \varphi^b = {}^b\varphi$. A similar statement holds for braided colax functors. If a bilax monoidal functor is both braided lax and braided colax, then we say that it is braided bilax. In this case, conjugation does not yield anything new.

**Normal bilax monoidal functors** (Chapter 3)**.** A bimonoid for which the unit and counit maps are inverses is necessarily trivial (isomorphic to the unit object). In contrast, there are many interesting bilax functors for which such a property holds. We refer to them as *normal bilax monoidal functors*. It is a weakening of the notion of bistrong functors (Proposition 3.45). The Fock functors which form the focus of our attention in Part III are normal. We provide other examples as well.

Our terminology is motivated by the normalized chain complex functor discussed in Chapter 5. Needless to say, the normalized chain complex functor is an example of a normal bilax functor. The class of normal bilax functors satisfies some interesting properties (Proposition 3.41). These are related to the notion of a Frobenius monoidal functor, which has been considered in the literature. Some of these properties, for the example of the normalized chain complex functor, have also appeared in the literature.

**Hopf lax monoidal functors** (Chapter 3)**.** At this point a natural question presents itself. If lax, colax and bilax monoidal functors correspond to monoids, comonoids and bimonoids, then what class of functors corresponds to Hopf monoids? A starting point is provided by the following result: The image of a Hopf monoid under a bistrong monoidal functor is again a Hopf monoid, in such a way that the antipode of the latter is the image of the antipode of the former (Proposition 3.50).

The answer we offer is the following. Between bilax monoidal functors and bistrong monoidal functors, there is an intermediate class of functors that preserves Hopf monoids but modifies antipodes in a predictable manner, much as the rest of the structure is modified by a bilax monoidal functor. We call them *Hopf lax monoidal functors*. We show that a normal bilax functor is Hopf lax if and only if it is bistrong (Proposition 3.60). It is worth pointing out that the analogy of Hopf lax monoidal functors with Hopf monoids is less straightforward than that of bilax monoidal functors with bimonoids, and the result on preservation of Hopf monoids (Theorem 3.70) requires a considerable amount of work. Familiar results for Hopf monoids admit generalizations to Hopf lax monoidal functors: the antipode of a Hopf lax monoidal functor is unique (Proposition 3.56), a morphism of bilax monoidal functors preserves antipodes when they exist (Proposition 3.59).

The antipode of a Hopf lax functor is related to the identity natural transformation through certain convolution formulas (Propositions 3.62–3.66). They are further developed from a more abstract point of view in Section D.4 (reviewed under *Monoids and the simplicial category*).

The results on Hopf lax functors, though interesting, are less relevant to the applications in later parts of the monograph. Indeed, the bilax full Fock functors are not Hopf lax, though they preserve Hopf monoids for other reasons. It is possible that a yet more general class of functors can be identified, so that it includes our Hopf lax functors as well as the Fock functors, and so that functors in this class preserve Hopf monoids.

**The image functor** (Chapter 3)**.** Recall that the image of a morphism of bialgebras is a bialgebra. More generally, the image of a morphism of bimonoids in an abelian monoidal category is itself a bimonoid. We extend this result by showing that the image of a morphism of bilax monoidal functors from a monoidal category to an abelian monoidal category is itself a bilax monoidal functor (Theorem 3.116). The diagram below shows the image $\Im_\theta$ of a transformation $\theta$, and the factorization of the latter.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, Rightarrow, "\theta"] \arrow[dr, Rightarrow] & & \mathcal{G} \\ & \Im_\theta . \arrow[ur, Rightarrow] & \end{tikzcd}$$

This result finds applications throughout the text. For example, the bosonic and fermionic Fock functors of Part III arise in this manner.

We obtain a concise proof of the above fact by viewing a morphism between two bilax monoidal functors from $\mathsf{C}$ to $\mathsf{D}$ as a bilax monoidal functor from $\mathsf{C}$ to the category of arrows in $\mathsf{D}$ (Proposition 3.111). There is another functor called the image functor which goes from the category of arrows in $\mathsf{D}$ to $\mathsf{D}$. It is constructed by choosing monic-epi factorizations in $\mathsf{D}$. Further, it is bistrong. Composing these two functors yields the required bilax functor.

**Bilax monoidal functors in homological algebra** (Chapter 5)**.** The notion of bilax monoidal functor between braided monoidal categories is of central importance to this work. Chapter 5 discusses what may be the most classical example of a bilax monoidal functor in mathematical nature. The familiar construction of a chain complex out of a simplicial module defines a functor between symmetric monoidal categories. The classical maps of Eilenberg–Zilber and Alexander–Whitney provide the lax and colax structures that turn it into a bilax monoidal functor (Theorem 5.6). While not formulated in these exact terms in the literature, this result pertains to the folklore of simplicial algebra. It was brought to our attention by Clemens Berger.

It is important to remark that we work with ordinary morphisms of chain complexes, not chain homotopy classes. If we pass to the homotopy category of chain complexes, then the chain complex functors become bistrong. In this situation the bilax axiom simplifies, and one does not need to confront it explicitly. In addition, this suffices for the applications to the construction of products in (co)homology. This may perhaps explain the lack of treatment in the literature of general bilax monoidal functors.

We state a number of well-known results which may be seen as consequences of Theorem 5.6, mainly regarding the existence of products in (co)homology. We also discuss the possibility of obtaining a one-parameter deformation of the chain complex functor. This can be done successfully provided that the boundary maps are set aside (Theorem 5.17).

**2-monoidal categories** (Chapter 6)**.** A careful analysis of bilax functors shows that they really belong to the world of higher monoidal categories. More precisely, they should be viewed as functors not between braided monoidal categories but rather between 2-monoidal categories, which are more general. The latter are categories with two compatible tensor products. The braiding is now replaced by an *interchange law*, which roughly speaking, specifies a way to interchange the order of the two tensor products. The familiar braiding axioms are replaced by a different set of 12 axioms.

There are many interesting examples of 2-monoidal categories and related objects ranging from directed graphs to bimodules over commutative algebras to lattices. Numerous other examples, primarily constructed out of species and graded vector spaces, along with applications are discussed in later parts of the monograph.

The definition of a 2-monoidal category raises a natural question: Despite the theorems that one may prove with such a definition, how does one know that there ought to be exactly 12 axioms? We provide an answer to this question by showing that a 2-monoidal category is an instance of a general notion in higher category theory, namely, it is a pseudomonoid in a certain monoidal 2-category (Proposition 6.73). This interpretation explains the origin of each axiom, so to speak. Further support is lent to this idea when one discovers that bilax functors are then nothing but appropriate morphisms between pseudomonoids (Proposition 6.75).

We mention that there are two other types of functors between 2-monoidal categories, namely double lax functors and double colax functors. These are generalizations of the notions of braided lax and braided colax functors. Just as bilax functors correspond to bimonoids, double (co)lax functors correspond to what we call double monoids and double comonoids. The latter, as expected, are generalizations of the familiar concepts of commutative monoids and cocommutative comonoids. Thus there are three different types of functors between 2-monoidal categories.

**The Eckmann–Hilton argument** (Chapter 6)**.** The classical Eckmann–Hilton argument says that if a set has two "mutually compatible" binary operations, then the two operations coincide and are commutative. This type of argument appears a number of times in the text; a summary of the results is provided below.

| | | | |
|---|---|---|---|
| strong 2-monoidal category | $\longleftrightarrow$ | braided monoidal category | (Proposition 6.11) |
| double monoid | $\longleftrightarrow$ | commutative monoid | (Proposition 6.29) |
| double lax monoidal functor | $\longleftrightarrow$ | braided lax monoidal functor | (Proposition 6.59) |

A 2-monoidal category is said to be strong if the structure morphisms defining it are all isomorphisms. The first result above is due to Joyal and Street and says that a strong 2-monoidal category is equivalent to a braided monoidal category. This may be regarded as a categorical version of the Eckmann–Hilton argument: a 2-monoidal category is a category with two "mutually compatible" products and a braided monoidal category is a category with a "commutative" product. Working under this equivalence, double monoids are equivalent to commutative monoids, and double lax functors are equivalent to braided lax functors. The first of these results applied to the category of sets is essentially the classical Eckmann–Hilton argument.

**Higher monoidal categories** (Chapter 7)**.** The pseudomonoid interpretation for a 2-monoidal category not only sheds light on the notion of bilax monoidal functors, but also allows us to dive deeper into the world of monoidal categories. There are two simple constructions related to pseudomonoids in monoidal 2-categories; namely, the lax and colax constructions. They are discussed separately in Appendix C (reviewed under *Pseudomonoids and the looping principle*). These constructions allow us to systematically climb up the ladder of higher monoidal categories.

After 2-monoidal categories, we define 3-monoidal categories, which are monoidal categories with three compatible tensor products: A combination of any two of these products yields a 2-monoidal category, and further there is a set of 8 axioms that must be satisfied. Among these axioms there is one that stands out; we call it the *interchange axiom*. It is reminiscent of the relations in the standard presentation of the braid group.

Just as there are two types of monoidal functors between monoidal categories, and three types of functors between 2-monoidal categories, there are four types of monoidal functors between 3-monoidal categories. These are straightforward to define, using the previous definitions.

Higher monoidal categories, contrary to what one may expect, are built out of 1-, 2- and 3-monoidal categories in a rather straightforward manner, meaning that, there are no further axioms to worry about. The same is true of the monoidal functors that relate them. In general, there are $n+1$ different types of functors between two $n$-monoidal categories. At the level of objects, there are $n+1$ different types of monoids in a $n$-monoidal category, one for each type of functor.

**The contragredient construction and self-duality** (Chapters 3, 6 and 7)**.** We provide a framework to deal with the notion of duality for monoidal categories, monoidal functors and transformations between them. The basic idea is explained below.

Let $\mathsf{C}$ be a category equipped with a contravariant functor $(-)^*\colon \mathsf{C} \to \mathsf{C}$ which induces an adjoint equivalence of categories. A specific example to keep in mind is the duality functor on finite-dimensional vector spaces, which sends a space to its dual. We say that an object $V$ in $\mathsf{C}$ is self-dual if $V \cong V^*$. Now let $\mathsf{C}$ and $\mathsf{D}$ be categories each equipped with such a functor, and let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor. Define the contragredient $\mathcal{F}^\vee$ to be the composite

$$\mathsf{C} \xrightarrow{*} \mathsf{C} \xrightarrow{\mathcal{F}} \mathsf{D} \xrightarrow{*} \mathsf{D}.$$

Further, we say that $\mathcal{F}$ is self-dual if $\mathcal{F}^\vee \cong \mathcal{F}$. Continuing with the above setup, let $\mathcal{F}$ and $\mathcal{G}$ be functors from $\mathsf{C}$ to $\mathsf{D}$, and let $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ be a natural transformation. Define the contragredient $\theta^\vee\colon \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee$ by

$$\mathcal{G}^\vee(A) = \mathcal{G}(A^*)^* \xrightarrow{(\theta_{A^*})^*} \mathcal{F}(A^*)^* = \mathcal{F}^\vee(A).$$

We say that $\theta\colon \mathcal{F} \Rightarrow \mathcal{F}^\vee$ is self-dual if $\theta^\vee \cong \theta$.

These notions can be extended to monoidal categories, braided monoidal categories and more generally to higher monoidal categories. This, in particular, allows us to define a self-dual monoidal category and a self-dual braided monoidal category. The contragredient of a lax functor is a colax functor and viceversa, while the contragredient of a bilax functor is a bilax functor (Proposition 3.102). This is the categorical version of the familiar statement that the dual of an algebra is a coalgebra and viceversa, and the dual of a bialgebra is a bialgebra. This setup allows us to define a self-dual bilax functor, a self-dual transformation between bilax functors, and so on. Further, one can establish results along the lines of: A self-dual bilax functor preserves self-dual bimonoids (Proposition 3.107).

We encounter plenty of examples of the contragredient construction in later parts of the monograph. The main examples of self-dual functors are summarized in Table 3.4.

**Types of monoid and monoidal functors** (Chapter 4)**.** The analogies between the notion of associative monoid and that of lax monoidal functor, and between the notion of commutative monoid and that of braided lax monoidal functor, which form the crux of Chapter 3, can be expanded to other types. Just as there are other types of monoid besides associative and commutative, there are other types of monoidal functors. This is the subject of Chapter 4. We first provide motivation to these ideas by explicitly introducing a number of types of monoids in monoidal categories and monoidal functors between monoidal categories. We then treat these notions in full generality by making use of the notion of operad. The necessary background on operads is given in Appendix B. For each operad $\mathbf{p}$, there is a notion of $\mathbf{p}$-monoid (in a monoidal category) and a notion of $\mathbf{p}$-lax monoidal functor (between monoidal categories). For general operads $\mathbf{p}$, the monoidal categories are required to be symmetric and linear.

The question arises as to how monoids (of a given type) transform under monoidal functors (of another type). The answer is contained in Theorem 4.28. It describes, more generally, the structure on a composite of two monoidal functors of such general types: if $\mathcal{F}$ is $\mathbf{p}$-lax and $\mathcal{G}$ is $\mathbf{q}$-lax, then the composite $\mathcal{G}\mathcal{F}$ is $(\mathbf{q} \times \mathbf{p})$-lax (under minor linearity hypotheses on the functors and categories). Here $\mathbf{q} \times \mathbf{p}$ stands for the Hadamard product of operads. Transformation of monoids is then a special case, obtained by viewing monoids as functors from the one-arrow category.

## Contents of Part II

The main actors in Part II are Joyal's *species* [181]. This part is devoted to a careful study of the monoidal category of species, Hopf monoids therein, and the discussion of several examples.

Recall that a Hopf monoid in the category of graded vector spaces is the same as a graded Hopf algebra. Our ultimate goal is to provide a solid conceptual framework for the study of a large number of Hopf algebras of a combinatorial nature, which include the Hopf algebra of symmetric functions, quasi-symmetric functions, noncommutative symmetric functions, and others of prominence in the recent literature, as well as a host of new ones.

The following principle is central to our approach: A proper understanding of these objects and their interrelationships requires the consideration of a more general setting; namely, that of Hopf monoids in the monoidal category of species. This category is richer than that of graded vector spaces. The precise link between the two categories is made in Part III.

**Species** (Chapter 8)**.** Informally, a species is a family $\mathbf{p}$ of vector spaces, one space $\mathbf{p}[I]$ for each finite set $I$, which is natural in $I$ with respect to bijections. By contrast, a graded vector space is simply a sequence of vector spaces, one space for each nonnegative integer. There is an alternative definition of a species as follows. A species is a graded vector space whose degree $n$ component is equipped with an action of the symmetric group $\mathrm{S}_n$ for each $n$.

We are interested in studying algebraic notions in the category of species, such as monoids, Hopf monoids, and other related notions. These are the analogues of graded algebras, graded Hopf algebras, and other familiar objects pertaining to the category of graded vector spaces. Connected and positive species, which play a useful role in the theory, are the analogues of connected and positively graded vector spaces.

There are various monoidal structures on species. We are mainly interested in the *Cauchy* product (the monoids and Hopf monoids alluded above are with respect to this product), but the *Hadamard* and the *substitution* product play an important role as well. They are written

$$\mathbf{p} \cdot \mathbf{q}, \quad \mathbf{p} \times \mathbf{q}, \quad \text{and} \quad \mathbf{p} \circ \mathbf{q}$$

respectively. These are analogues of monoidal structures on graded vector spaces of the same name that are discussed in Chapter 2. The notions of (co)monoids, bimonoids and Hopf monoids in the monoidal category of species (with respect to the Cauchy product) can be made explicit. Roughly, a monoid is a species $\mathbf{p}$ equipped with maps

$$\mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I],$$

one such map for each decomposition $I = S \sqcup T$ of a finite set $I$ into disjoint subsets $S$ and $T$, and dually a comonoid is a species $\mathbf{p}$ equipped with maps

$$\mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T].$$

Similar descriptions hold for bimonoids and Hopf monoids.

Familiar properties of graded bialgebras and Hopf algebras hold also for bimonoids and Hopf monoids. For example, one can define the dual $\mathbf{p}^*$ of a species $\mathbf{p}$ by $\mathbf{p}^*[I] = \mathbf{p}[I]^*$. This association is natural in $\mathbf{p}$, so it gives rise to a (contravariant) functor on species. Further, this duality functor is bistrong, so the dual of a Hopf monoid is again a Hopf monoid.

However, bimonoids possess certain unique features not to be seen for bialgebras. The first instance is the compatibility axiom itself (Remark 8.8). Another instance is the interplay between the Cauchy and the Hadamard products on species. We show that the Hadamard product is a bilax monoidal functor with respect to the Cauchy product (Proposition 8.58). As a consequence, the Hadamard product of two Hopf monoids is again a Hopf monoid. In contrast, the Hadamard product of two graded Hopf algebras fails to be a graded Hopf algebra, in general. This is significant; we come back to this point at various places in this introduction, in particular under *Hopf algebras from geometry*.

**Deformations of Hopf monoids** (Chapter 9)**.** The standard braiding $\beta$ on the monoidal category of species is defined by interchanging the tensor factors. Analogous to the situation for graded vector spaces, one can deform this braiding as follows.

$$\mathbf{p}[S] \otimes \mathbf{q}[T] \to \mathbf{q}[T] \otimes \mathbf{p}[S] \qquad\qquad x \otimes y \mapsto q^{|S||T|} y \otimes x.$$

This defines the braiding $\beta_q$ where $q$ is any scalar. Letting $q = 1$ recovers the previous case. Note that this braiding is a symmetry if and only if $q = \pm 1$.

A useful way to think about the coefficient in the braiding is as follows. We give the idea in rough terms. Let $S|T$ be a "state" in which every element of $S$ precedes every element of $T$. Let $q$ be the cost of changing the relative order between two elements. Then the coefficient is the cost of going from state $S|T$ to state $T|S$.

A Hopf monoid with respect to the braiding $\beta_q$ is called a $q$-Hopf monoid. The preceding theory of Hopf monoids in species generalizes in a natural manner to the deformed setting. For example, duality continues to a bistrong functor. It turns out that the Hadamard product of a $p$-Hopf monoid and a $q$-Hopf monoid is a $pq$-Hopf monoid.

In addition to the above, one can also define a *signature functor* on species. It twists the action of the symmetric group by tensoring with its one-dimensional sign representation. This functor is bistrong provided the braidings are chosen carefully: if $\beta_q$ is used in the source category, then use $\beta_{-q}$ in the target category (Proposition 9.9). As a consequence, the signature functor takes a $q$-Hopf monoid to a $(-q)$-Hopf monoid. In particular, it switches Hopf monoids and $(-1)$-Hopf monoids.

**The exponential species and the species of linear orders** (Chapters 8, 9 and 11)**.** The exponential species $\mathbf{E}$ and the species of linear orders $\mathbf{L}$ are the simplest interesting and nontrivial examples of Hopf monoids in species. As a species, $\mathbf{E}$ is the graded vector space whose graded components consist of the trivial representations of the symmetric groups, while the graded components of $\mathbf{L}$ consist of the regular representations of the symmetric groups. These Hopf monoids accompany us through all our constructions. They are basic examples of universal objects: $\mathbf{L}$ is the free monoid on one generator, and $\mathbf{E}$ is the free commutative monoid on one generator. Their products and coproducts are simple to describe. We also describe their antipodes explicitly, and explain the different ways in which they can be derived.

Since we have given the rough idea of a monoid and comonoid in species, we indicate the product and coproduct of the species $\mathbf{L}$. The product is as follows: given linear orders on $S$ and $T$, their product is their common extension to $S \sqcup T$ in which the elements of $S$ precede the elements of $T$. The coproduct is as follows: given a linear order on $I$, its coproduct is its restriction to $S$ tensored with its restriction to $T$.

The species $\mathbf{L}$ and $\mathbf{E}$ can be combined in various ways to obtain new examples of Hopf monoids. Many of these have a rich geometric flavor as we will see later. As an example, combining the Hadamard product construction with duality yields the Hopf monoid $\mathbf{L}^* \times \mathbf{L}$. This Hopf monoid is self-dual (isomorphic to its dual). This follows from the compatibility between duality and the Hadamard product.

The exponential species admits a signed version; we call it the signed exponential species and denote it by $\mathbf{E}^-$. As a species, its graded components consist of the sign representations of the symmetric groups. As a $(-1)$-Hopf monoid, it is the free commutative monoid on one generator, where commutative is now to be interpreted in the graded sense (with respect to the braiding $\beta_{-1}$). This object is intimately tied to the signature functor. The signature functor sends a species $\mathbf{p}$ to $\mathbf{p} \times \mathbf{E}^-$, that is, to its Hadamard product with $\mathbf{E}^-$. The bistrong structure of this functor arises from the bimonoid structure of $\mathbf{E}^-$ and the bilax structure of the Hadamard product. Since $\mathbf{E}$ is the unit object for the Hadamard product, it follows from this construction that the signature functor sends $\mathbf{E}$ to $\mathbf{E}^-$.

The situation for $\mathbf{L}$ is even more interesting. It admits a one-parameter deformation to a $q$-Hopf monoid which we denote by $\mathbf{L}_q$. Letting $q = 1$ recovers $\mathbf{L}$, while $\mathbf{L}_{-1}$ is the signature functor applied to $\mathbf{L}$. Note that $\mathbf{L}$ and $\mathbf{L}_{-1}$ are identical as species since tensoring the regular representation with the sign representation again yields the regular representation. However, their Hopf structures are different and in a sense cannot be compared since one is a Hopf monoid while the other is a $(-1)$-Hopf monoid.

**Universal constructions** (Chapter 11)**.** The free monoid on a species $\mathbf{q}$ is given by

$$\mathcal{T}(\mathbf{q}) = \mathbf{L} \circ \mathbf{q}.$$

This shows that the Cauchy and substitution products are intimately related. (This relation is encountered again in Section B.4, where the substitution product is studied in detail.) We refer to $\mathcal{T}$ as the free monoid functor; it is the species analogue of the tensor algebra for graded vector spaces. The space $\mathcal{T}(\mathbf{q})[I]$ is graded over compositions of the set $I$, much as the degree $n$ component of the tensor algebra of a graded vector space is graded over compositions of $n$. Let $\mathbf{X}$ be the species whose value is the base field on singletons and 0 on all other finite sets. It is the unit object for the substitution product. The species $\mathbf{L}$ can be recovered as $\mathbf{L} = \mathcal{T}(\mathbf{X})$; a little more loosely, one may say that $\mathbf{L}$ is the free monoid on one generator, as already mentioned.

If $\mathbf{q}$ is a positive comonoid, then $\mathcal{T}(\mathbf{q})$ can be turned into a Hopf monoid, and its structure can be explicitly described. In addition, $\mathcal{T}(\mathbf{q})$ is the free Hopf monoid on the positive comonoid $\mathbf{q}$ (Theorem 11.9). This result is supplemented with a discussion of the monoidal properties of the corresponding adjunction.

There are commutative versions of these constructions. The free commutative monoid (Hopf monoid) on a positive species (comonoid) $\mathbf{q}$ is

$$\mathcal{S}(\mathbf{q}) = \mathbf{E} \circ \mathbf{q}.$$

The species $\mathbf{E}$ can be recovered as $\mathbf{E} = \mathcal{S}(\mathbf{X})$.

Since examples of both free or cofree Hopf monoids arise naturally, it is worth considering the dual constructions to those mentioned above. Thus we also discuss the cofree comonoid (Hopf monoid) on a positive species (monoid), and their co-commutative versions. The corresponding functors are denoted $\mathcal{T}^\vee$ and $\mathcal{S}^\vee$. The former is the species analogue of the tensor coalgebra functor for graded vector spaces.

Interestingly, the functor $\mathcal{T}$ admits a one-parameter deformation, which we denote by $\mathcal{T}_q$. It takes values in the category of $q$-Hopf monoids. Further, if $\mathbf{q}$ is a positive comonoid, then $\mathcal{T}_q(\mathbf{q})$ is the free $q$-Hopf monoid on $\mathbf{q}$. The deformation $\mathbf{L}_q$ can be recovered as $\mathbf{L}_q = \mathcal{T}_q(\mathbf{X})$. The commutative version of this construction for $q = 1$ is the functor $\mathcal{S}$. Similarly, there is an interesting commutative version of this construction for $q = -1$. We call the corresponding functor $\Lambda$. It takes values in the category of $(-1)$-Hopf monoids. Further, if $\mathbf{q}$ is a positive comonoid, then $\Lambda(\mathbf{q})$ is the free commutative $(-1)$-Hopf monoid on $\mathbf{q}$. The signed exponential species $\mathbf{E}^-$ can be recovered as $\mathbf{E}^- = \Lambda(\mathbf{X})$. We also discuss the dual functors $\mathcal{T}_q^\vee$ and $\Lambda^\vee$.

We briefly touch upon related functors such as the free Lie algebra functor, the universal enveloping algebra functor and the primitive element functor for species, along with the Poincaré-Birkhoff-Witt and Cartier-Milnor-Moore theorems for species which appear in the works of Joyal and Stover. We describe the coradical filtrations and primitive elements of the Hopf monoids which arise as values of the functors $\mathcal{T}^\vee$, $\mathcal{S}^\vee$ and $\Lambda^\vee$.

**The Coxeter complex. The break, join, and projection maps** (Chapter 10)**.** Symmetries of a set (bijections from the set to itself), or equivalently, the symmetric groups play a pivotal role in the theory of species. Recall that symmetric groups are Coxeter groups of type $A$. It turns out that key features of the theory of Coxeter groups when specialized to the example of type $A$ can be formulated in the language of species.

To any Coxeter group is associated a simplicial complex which is called the Coxeter complex. We highlight three important properties of these objects.

- The join of Coxeter complexes is a Coxeter complex.
- The star of any face in a Coxeter complex is a Coxeter complex.
- The set of faces of a Coxeter complex is a monoid, whose product is given by the projection maps of Tits.

A set species is a family $\mathrm{P}$ of sets, one set $\mathrm{P}[I]$ for each finite set $I$, which is natural in $I$ with respect to bijections. The family of Coxeter complexes of type $A$ can be assimilated into one object which we denote by $\Sigma$. It is a set species with added structure. The component $\Sigma[I]$ is the set of faces of the Coxeter complex associated to the symmetric group on $|I|$ letters. An important observation specific to type $A$ is as follows. The star of a vertex in a Coxeter complex of type $A$ is isomorphic to the join of two smaller Coxeter complexes of type $A$. More precisely, for a decomposition $I = S \sqcup T$, there is a canonical identification

$$\mathrm{Star}(S|T) \cong \Sigma[S] \times \Sigma[T]$$

between the star of the vertex $S|T$ in $\Sigma[I]$ and the join of the complexes $\Sigma[S]$ and $\Sigma[T]$. We use

$$\begin{tikzcd} \mathrm{Star}(S|T) \arrow[r, shift left, "b_{S|T}"] & \Sigma[S] \times \Sigma[T] \arrow[l, shift left, "j_{S|T}"] \end{tikzcd}$$

to denote the inverse isomorphisms of simplicial complexes. We refer to $b_{S|T}$ and $j_{S|T}$ as the *break* and *join* maps, respectively. Further, for any vertex $S|T$ of $\Sigma[I]$, there is a map

$$p_{S|T} \colon \Sigma[I] \to \mathrm{Star}(S|T)$$

which sends a face to its projection on the vertex $S|T$. It is called the *Tits projection*. These maps may be used to turn $\Sigma[I]$ into a monoid.

The break, join and projection maps are at the basis of the understanding of a number of combinatorial Hopf algebras. This geometric point of view was advocated in our previous work [12]. The compatibilities between these maps were listed as a set of coalgebra and algebra axioms [12, Sections 6.3.1 and 6.6.1]. These maps as well as their compatibilities find their most natural expression in this monograph, in the context of species.

**Hopf monoids from geometry** (Chapter 12)**.** In this chapter, we construct many examples of Hopf monoids in species and analyze them in considerable detail. Their nature is primarily geometric: they are associated to the Coxeter complex of type $A$ in various ways. For this reason they admit explicit descriptions in terms of familiar combinatorial objects. The geometry of the complex, through the break, join and projection maps, is at the basis of the understanding of the algebraic structure of these objects, as evidenced by the results we present. These Hopf monoids arise by combining the species $\mathbf{X}$, $\mathbf{E}$, and $\mathbf{L}$ in various ways. They can be related by morphisms of Hopf monoids as indicated in diagram (0.1). The duality functor acts on this diagram by reflection across the diagonal.

$$
\begin{tikzcd}
\mathcal{T}(\mathbf{X}^*) \arrow[r] \arrow[dd] & \mathcal{T}(\mathbf{E}^*_+) \arrow[r] \arrow[dd] \arrow[dddrr] & \mathcal{T}(\mathbf{L}^*_+) \arrow[rr] \arrow[dd] & & \mathbf{L}^* \times \mathbf{L} \arrow[dr, "\cong"] & \\
& & & & & \mathbf{L} \times \mathbf{L}^* \arrow[dd] \\
\mathcal{S}(\mathbf{X}^*) \arrow[r] \arrow[dddrrr, "\cong"'] & \mathcal{S}(\mathbf{E}^*_+) \arrow[r] \arrow[ddrr, "\cong"'] & \mathcal{S}(\mathbf{L}^*_+) \arrow[ddrrr] & & & \\
& & & \mathcal{S}^\vee(\mathbf{L}_+) \arrow[rr] \arrow[d] & & \mathcal{T}^\vee(\mathbf{L}_+) \arrow[d] \\
& & & \mathcal{S}^\vee(\mathbf{E}_+) \arrow[rr] \arrow[d] & & \mathcal{T}^\vee(\mathbf{E}_+) \arrow[d] \\
& & & \mathcal{S}^\vee(\mathbf{X}) \arrow[rr] & & \mathcal{T}^\vee(\mathbf{X})
\end{tikzcd}
\tag{0.1}
$$

Each of these Hopf monoids admits an explicit description in terms of familiar combinatorial objects. For instance, according to the definition of substitution, $\mathcal{T}(\mathbf{E}^*_+)$ is the species of set compositions. The product of this Hopf monoid is concatenation and the coproduct is dual to the quasi-shuffle of set compositions. Similarly, $\mathcal{S}(\mathbf{L}^*_+)$ is the species of *linear set partitions*. The other species and their products and coproducts can be described in similar combinatorial terms.

Some of the morphisms in (0.1) arise simply from functoriality; this is the case of the map $\mathcal{T}(\mathbf{E}^*_+) \to \mathcal{T}(\mathbf{L}^*_+)$, for instance. Some, like the map $\mathbf{L} \times \mathbf{L}^* \to \mathcal{T}^\vee(\mathbf{L}_+)$ (which relates pairs of linear orders to linear set compositions) arise from universal constructions (cofreeness of $\mathcal{T}^\vee(\mathbf{L}_+)$ in this case). Others, like the map $\mathcal{T}(\mathbf{E}^*_+) \to \mathcal{S}^\vee(\mathbf{L}_+)$ (which relates set compositions to linear set partitions), are specific to the species under consideration. In these cases, a combinatorial description can be of limited use.

Our main point in this chapter is that, in spite of their combinatorial appearance, proper understanding of these Hopf monoids and the morphisms among them demands the consideration of their geometric nature. Each of these species arises from the Coxeter complex of type $A$, and the products and coproducts that turn them into Hopf monoids can always be expressed in terms of the break, join and projection maps. For instance, set compositions correspond to *faces* of the complex, and linear set partitions to *directed flats*. Table 12.1 summarizes the combinatorial and geometric description of these species.

**A bilinear form on chambers. Varchenko's result** (Chapter 10)**.** Consider a hyperplane arrangement in which each hyperplane is assigned a weight. For any pair of chambers (top-dimensional faces) $C$ and $D$ in the arrangement, define $\langle C, D\rangle$ to be the product of the weights of the hyperplanes which separate $C$ and $D$. This defines a bilinear form on the space spanned by chambers. Varchenko obtained a factorization of the determinant of this bilinear form, see equation (10.129). The special case when the hyperplane arrangement is the braid arrangement and all weights are equal was treated earlier by Zagier.

It follows from Varchenko's result that for generic weights on the hyperplanes the bilinear form on chambers is nondegenerate. There is a useful generalization of this result in which one puts weights on half-spaces instead of hyperplanes (Lemma 10.27). We use this result to deduce rigidity results of a very general kind in two different contexts. The first context is that of the norm transformation between $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ and its higher dimensional generalization. The second context is that of the norm transformation between the deformed full Fock functors of Part III. This is explained in more detail below, under *Relations among the universal objects. The norm transformation* and *Relations among the Fock functors. The norm transformation.*

**Hopf monoids from combinatorics** (Chapter 13)**.** In this chapter, we discuss Hopf monoids that are based on combinatorial structures such as relations, preposets, posets, graphs, rooted forests, planar rooted forests, set-graded posets, closure operators, matroids and topologies. The origin of many of these ideas can be found in the paper of Joni and Rota [179]. The emphasis of this chapter is on the construction of interesting morphisms from these objects to objects such as $\mathcal{T}^\vee(\mathbf{E}_+)$ and $\mathcal{T}^\vee(\mathbf{X})$ which occur to the bottom right of diagram (0.1). By the universal constructions of Chapter 11, the latter are cofree objects, hence morphisms of the above kind can be constructed by minimal principles.

Many of these combinatorial objects can be interpreted geometrically. For example, posets can be viewed as appropriate unions of chambers (top-dimensional cones, to be precise) in the Coxeter complex of type $A$. This interpretation extends to preposets. This forges a link with the ideas of Chapters 10 and 12. We then observe that the morphisms relating the Hopf monoid of posets to the Hopf monoids based on linear orders, linear set partitions and set compositions, initially constructed using purely combinatorial or algebraic motivations, have simple geometric descriptions. The distinction between combinatorics and geometry (reflected in our chapter titles) is mainly for organizational purposes. The above examples reinforce the fact that either viewpoint may be used profitably according to the situation.

**Relations among the universal objects. The norm transformation** (Chapter 11)**.** To understand the relations between various universal objects (for example, to relate $\mathbf{L}$ and $\mathbf{E}$), one needs to properly understand how the functors $\mathcal{T}_q$, $\mathcal{T}_q^\vee$, $\mathcal{S}$ and $\mathcal{S}^\vee$, $\Lambda$ and $\Lambda^\vee$ relate to one another. We now explain this.

First of all, the functors $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$, as well as $\mathcal{S}$ and $\mathcal{S}^\vee$, and $\Lambda$ and $\Lambda^\vee$ are related through duality. Second, there are natural transformations $\pi\colon \mathcal{T} \Rightarrow \mathcal{S}$ and $\pi^\vee\colon \mathcal{S}^\vee \Rightarrow \mathcal{T}^\vee$ (projection onto coinvariants and inclusion of invariants). The former is called the *abelianization* and the latter is its dual. Similarly, there are natural transformations $\pi_{-1}\colon \mathcal{T}_{-1} \Rightarrow \Lambda$ and $\pi_{-1}^\vee\colon \Lambda^\vee \Rightarrow \mathcal{T}_{-1}^\vee$ called the *signed abelianization* and its dual.

It is not possible to relate the functors $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ in general since they take values on different type of objects. The former is defined on positive comonoids while the latter is defined on positive monoids. However, one can restrict both functors to the category of positive species, and in that case, there is a natural transformation

$$\kappa_q\colon \mathcal{T}_q \Rightarrow \mathcal{T}_q^\vee.$$

This is called the *q-norm transformation* (for its relation to the norm map in group theory). As mentioned under *The image functor*, transformations can be factored. The factorization of the $q$-norm can be explicitly understood for $q = \pm 1$. It is as follows.

$$\begin{tikzcd} \mathcal{T} \arrow[r, Rightarrow, "\kappa"] \arrow[d, Rightarrow, "\pi"'] & \mathcal{T}^\vee \\ \mathcal{S} \arrow[r, Rightarrow, "\mathrm{id}"'] & \mathcal{S}^\vee \arrow[u, Rightarrow, "\pi^\vee"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_{-1} \arrow[r, Rightarrow, "\kappa_{-1}"] \arrow[d, Rightarrow, "\pi_{-1}"'] & \mathcal{T}^\vee_{-1} \\ \Lambda \arrow[r, Rightarrow, "\mathrm{id}"'] & \Lambda^\vee \arrow[u, Rightarrow, "\pi^\vee_{-1}"'] \end{tikzcd}$$

Thus, in the restricted setting of positive species, the functors $\mathcal{S}$ and $\mathcal{S}^\vee$ are identical and equal to the image of the norm. Similarly, the functors $\Lambda$ and $\Lambda^\vee$ are identical and equal to the image of the signed norm.

Applying the diagram on the left to the species $\mathbf{X}$ yields a commutative diagram of Hopf monoids relating $\mathbf{L}$, its dual $\mathbf{L}^*$, and $\mathbf{E}$. In particular, the dual of $\mathbf{E}$ is itself. This can also be checked directly. Similarly, applying the diagram on the right to the species $\mathbf{X}$ yields a commutative diagram of $(-1)$-Hopf monoids relating $\mathbf{L}_{-1}$, its dual $\mathbf{L}^*_{-1}$, and $\mathbf{E}^-$ (which is self-dual).

The generic case of the $q$-norm is quite different. In fact, we deduce from Varchenko's result that $\kappa_q$ is an isomorphism if $q$ is not a root of unity (Theorem 11.35). Under this hypothesis, for any positive species $\mathbf{q}$, the $q$-Hopf monoids $\mathcal{T}_q(\mathbf{p})$ and $\mathcal{T}_q^\vee(\mathbf{p})$ are isomorphic. It follows by letting $\mathbf{p} = \mathbf{X}$ that the $q$-Hopf monoid $\mathbf{L}_q$ is self-dual (Proposition 12.6).

An interesting situation is $q = 0$. In this case, $\kappa_0$ is the identity and $\mathcal{T}_0 = \mathcal{T}_0^\vee$. It follows that the 0-Hopf monoid $\mathbf{L}_0$ is self-dual in a canonical manner.

**The Schubert cocycle** (Chapters 9 and 11)**.** Let $l$ be a linear order on $I$, and let $I = S \sqcup T$ be a decomposition of $I$ into disjoint subsets $S$ and $T$. Let

$$\mathrm{sch}_{S,T}(l) := |\{(i,j) \in S \times T \mid \; i > j \text{ according to } l\}|.$$

We call this the *Schubert cocycle*. If we view $l$ as a list, then $\mathrm{sch}_{S,T}(l)$ counts the number of minimum adjacent transpositions required to bring the elements of $S$ to the beginning of the list.

The Schubert cocycle is equivalent to a standard combinatorial statistic, which we call the Schubert statistic. Our interest in this notion stems from its relevance to deformation theory. In this regard, the following two properties of the Schubert cocycle are significant.

For any decomposition $I = R \sqcup S \sqcup T$, and for any linear order $l$ on $I$,

$$\mathrm{sch}_{R,S\sqcup T}(l) + \mathrm{sch}_{S,T}(l|_{S\sqcup T}) = \mathrm{sch}_{R\sqcup S,T}(l) + \mathrm{sch}_{R,S}(l|_{R\sqcup S}),$$

where the vertical bar denotes restriction of the linear order. This is the cocycle condition. It can be understood as follows. Both sides count the number of minimum adjacent transpositions required to rearrange the list $l$ so that elements of $R$ are at the beginning, followed by elements of $S$, followed by elements of $T$.

Now consider a pair of decompositions $I = S \sqcup T = S' \sqcup T'$ into disjoint subsets and let $A$, $B$, $C$, and $D$ be the resulting intersections:

$$A = S \cap S', \quad B = S \cap T', \quad C = T \cap S', \quad D = T \cap T'.$$

Then for any linear order $l$ on $S$, and linear order $m$ on $T$,

$$\mathrm{sch}_{S',T'}(l \cdot m) = \mathrm{sch}_{A,B}(l) + \mathrm{sch}_{C,D}(m) + |B||C|,$$

where $l \cdot m$ denotes the common extension of $l$ and $m$ to $I$ in which the elements of $l$ precede the elements of $m$. This is the multiplicative property of the cocycle. Note that the last term on the right is precisely the exponent of the coefficient of $\beta_q$ (under *Deformations of Hopf monoids*).

The $q$-Hopf monoid $\mathbf{L}_q$ can be constructed by deforming $\mathbf{L}$ via the Schubert cocycle: Keep the product the same as before, but modify the coproduct by multiplying it by the coefficient

$$q^{\mathrm{sch}_{S,T}(l)}.$$

The fact that $\mathbf{L}_q$ is coassociative is equivalent to the cocycle condition. The product-coproduct compatibility is equivalent to the multiplicative property of the cocycle.

This idea is the driving force behind a very general construction, namely, the construction of the functor $\mathcal{T}_q$ by deforming $\mathcal{T}$. For this construction, one extends the Schubert cocycle to set compositions (which are more general combinatorial objects than linear orders), and then proceeds in the same way as above. The situation for $\mathbf{L}_q$ can be seen as a special case by specializing the general construction to the species $\mathbf{X}$.

**Cohomology of comonoids in species** (Chapter 9)**.** An important cohomology theory for associative algebras is Hochschild cohomology. The coefficients for this theory can be chosen to be in any bimodule over that algebra. Dually, there is a Hochschild cohomology for coalgebras with coefficients in a bicomodule.

Cohomology of a comonoid in species with coefficients in a bicomodule can be defined in the same manner. We are interested in the special case when the comonoid is the exponential species $\mathbf{E}$ and the bicomodule is any linearized comonoid (this roughly means that the coproduct is well-behaved on a basis). We develop this theory in explicit terms with emphasis on low-dimensional cocycles. If, in addition, we have a linearized bimonoid (as opposed to just a comonoid), then we can also define the notion of multiplicative cocycles. We explain how the given linearized comonoid can be deformed using a 2-cocycle on it. If we are in the setup of linearized bimonoids, then the same deformation can be carried out provided the 2-cocycle is multiplicative.

The bimonoid of linear orders $\mathbf{L}$ is linearized. As indicated by the terminology, the Schubert cocycle is a 2-cocycle on $\mathbf{L}$. Further, it is multiplicative. These facts follow from the properties of the Schubert cocycle mentioned earlier. In fact, the Schubert cocycle is the unique multiplicative cocycle (of twist 1) on $\mathbf{L}$ (Theorem 9.27). We mentioned earlier how $\mathbf{L}_q$ can be constructed from $\mathbf{L}$ using the Schubert cocycle. This can now be seen as an instance of a general construction.

**Antipode formulas** (Chapter 11)**.** Recall that a Hopf monoid is a bimonoid with an antipode. As for graded vector spaces, a connected bimonoid in species is automatically a Hopf monoid. The antipode can be expressed as an alternating sum, where each summand is a composite of an iterated coproduct with an iterated product. This is the species analogue of Takeuchi's antipode formula for connected Hopf algebras. Since the sum is alternating, cancellations may take place (and in concrete examples, many often do). By contrast, we would like an explicit formula for the structure constants of the antipode on a given basis. Obtaining such a formula requires understanding of these cancellations; this is often a challenging combinatorial problem.

We solve this problem for any Hopf monoid which is a universal object, that is, for any Hopf monoid which is the image under either of the functors $\mathcal{T}_q$ or $\mathcal{S}$ or $\Lambda$ or their duals (Theorems 11.38–11.43). It may be surprising to note that the cancellations hinge on a simple result related to Tits projection maps (Lemma 11.37). Though our treatment of the functors $\mathcal{T}$, $\mathcal{S}$ and $\Lambda$ and their duals is largely combinatorial, it is clear that projection maps do play a central role in their construction.

Instances of the antipode formulas for universal objects include antipode formulas for all objects in (0.1) except for $\mathbf{L} \times \mathbf{L}^*$ and its dual. For example, consider the six objects in the top left of this diagram. The first two are the exponential and linear order species; thus the general result recovers the antipodes of these basic objects. The antipode formulas for the remaining four objects are explicitly written down in Theorems 12.21, 12.34, 12.44 and 12.51.

We also provide cancellation-free antipode formulas for some other Hopf monoids which do not fit the above framework; see Theorems 12.17 and 12.18 for the Hopf monoid $\mathbf{L} \times \mathbf{L}^*$, and Theorems 13.4 and 13.5 for the Hopf monoids of planar rooted forests and rooted forets. The proof is very similar to the one employed for universal objects suggesting that there may be a more general framework for writing down antipode formulas which includes these examples as well.

**Colored species and $Q$-Hopf monoids** (Chapter 14)**.** *Colored species* are higher dimensional analogues of species. They are also called *multisort species*. Roughly, an $r$-colored species associates a vector space to each ordered decomposition $I = S^1 \sqcup \cdots \sqcup S^r$ of a finite set $I$. We view the category of colored species as an analogue of the category of multigraded vector spaces.

There is a higher dimensional generalization of the preceding theory in which species are replaced by colored species. This context provides more flexibility for the definition of braidings and bilax structures. For each square matrix $Q$ of size $r$, we define a braiding $\beta_Q$ on $r$-colored species. A Hopf monoid in this braided monoidal category is called a $Q$-Hopf monoid.

A matrix $Q$ is called *log-antisymmetric* if

$$q_{ij} q_{ji} = 1 \quad \text{for } 1 \leq i, j \leq r,$$

where $q_{ij}$ denotes the $ij$-th entry of $Q$. The significance of these matrices is as follows. The braiding $\beta_Q$ is a symmetry if and only if $Q$ is log-antisymmetric. Note that the only log-antisymmetric matrices of size one are $[1]$ and $[-1]$.

One can construct a colored version of the functor $\mathcal{T}$; we denote it by $\mathcal{T}_Q$. It takes values in the category of $Q$-Hopf monoids. Further, if $\mathbf{q}$ is a positive colored comonoid, then $\mathcal{T}_Q(\mathbf{q})$ is the free $Q$-Hopf monoid on $\mathbf{q}$. Setting $Q = [q]$ recovers the functor $\mathcal{T}_q$. Similarly, one can construct a functor $\mathcal{S}_Q$ whenever $Q$ is log-antisymmetric. This recovers $\mathcal{S}$ when $Q = [1]$ and $\Lambda$ when $Q = [-1]$. If $\mathbf{q}$ is a positive colored comonoid, then $\mathcal{S}_Q(\mathbf{q})$ is the free commutative $Q$-Hopf monoid on $\mathbf{q}$. We provide antipode formulas for these $Q$-Hopf monoids in Theorems 14.18 and 14.20. The functors $\mathcal{T}_Q$, $\mathcal{S}_Q$ and their duals fit into the following commutative diagram (if one restricts them to positive colored species).

$$\begin{tikzcd} \mathcal{T}_Q \arrow[r, Rightarrow, "\kappa_Q"] \arrow[d, Rightarrow, "\pi_Q"'] & \mathcal{T}_Q^\vee \\ \mathcal{S}_Q \arrow[r, Rightarrow, "\mathrm{id}"'] & \mathcal{S}_Q^\vee \arrow[u, Rightarrow, "\pi_Q^\vee"'] \end{tikzcd}$$

The top horizontal transformation $\kappa_Q$ is the colored norm. The vertical transformations are the colored abelianization and its dual. Thus, $\mathcal{S}_Q$ is the image of the colored norm $\kappa_Q$.

Let $\mathbf{X}_{(r)}$ be the colored species which is nonzero (and equal to the base field) only if the ordered decomposition is into singletons. This is a colored version of $\mathbf{X}$. Then

$$\mathbf{L}_Q = \mathcal{T}_Q(\mathbf{X}_{(r)}) \quad \text{and} \quad \mathbf{E}_Q = \mathcal{S}_Q(\mathbf{X}_{(r)})$$

yields colored analogues of the linear order species and the exponential species (the latter for $Q$ log-antisymmetric). These may be regarded as the simplest interesting Hopf monoids in the category of $r$-colored species. We also consider colored analogues of some of the other Hopf monoids occurring in diagram (0.1).

There is a colored analogue of the signature functor for any log-antisymmetric matrix $Q$. It sends a colored species $\mathbf{p}$ to $\mathbf{p} \times \mathbf{E}_Q$, its Hadamard product with the colored exponential species. This functor continues to be bistrong.

The generic case of the $Q$-norm is quite different from the log-antisymmetric case. We deduce from Varchenko's result that $\kappa_Q$ is an isomorphism if no monomial in the $q_{ij}$'s equals one (Theorem 14.17).

## Contents of Part III

In Part II, we systematically studied the monoidal category of species and Hopf monoids therein along with plenty of interesting examples constructed from combinatorial or geometric data. The goal of this part is to link the setting of Hopf monoids in species with that of graded Hopf algebras. This connection is made by means of certain bilax monoidal functors which we term *Fock functors*. The theory of bilax monoidal functors presented in Part I is extensively applied to understand how concepts involving species and graded vector spaces relate to one another via the Fock functors. This means that instead of looking at properties specific to Hopf monoids and Hopf algebras we study the Fock functors themselves. This is another principle which is central to our approach.

**Sets versus numbers.** A graded vector space is a family of vector spaces indexed by nonnegative integers. Recall that a species is a family of vector spaces indexed by finite sets (with further compatibilities). Thus species correspond to sets in the same way as graded vector spaces correspond to numbers. The passage from species to graded vector spaces via the Fock functors in the most naive sense amounts to replacing a set by its cardinality.

The Fock functors retain a lot of information, which is why their study is important, but at the same time, they forget or lose some information, which is why species are nicer to work with than graded vector spaces. There are a number of operations on sets for which there may or may not be any analogue for numbers. For example, disjoint union of sets corresponds to addition of numbers. This is significant because the former corresponds to the Cauchy product on species while the latter corresponds to the Cauchy product on graded vector spaces. However, note for example that intersection of sets has no analogy for numbers. Some interesting examples of well-known objects associated to numbers and the corresponding objects associated to sets can be found in Table 13.4.

**Fock functors** (Chapter 15)**.** The parallel between the categories of species and of graded vector spaces is reinforced by the existence of several functors between the two:

$$\mathcal{K} \quad \text{and} \quad \mathcal{K}^\vee, \qquad \text{and} \qquad \overline{\mathcal{K}} \quad \text{and} \quad \overline{\mathcal{K}}^\vee.$$

We refer to them collectively by the term *Fock functors*. For further distinction, we refer to $\mathcal{K}$ and $\mathcal{K}^\vee$ as *full Fock functors* and to $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ as *bosonic Fock functors*. The construction of these functors and the study of their properties and the relations among them constitute the main results of this chapter. The motivation for the terminology comes from Fock spaces which are studied in physics. A connection with these spaces is made later in Chapter 19 when we consider decorated versions of the Fock functors.

We show that the full Fock functors are bilax, while the bosonic Fock functors are bistrong (Theorems 15.3 and 15.6). By applying the Fock functors, one obtains four graded Hopf algebras out of each Hopf monoid in species (Theorem 15.12). These constructions of Hopf algebras were introduced by Stover [346] and further studied by Patras et al, without reference to monoidal functors. The categorical formulation is more general (for it allows the construction of other type of algebras from the corresponding type of monoids in species) and allows for a better understanding of the transfer of properties from one context to the other. Consider for instance the fact that if $\mathbf{h}$ is a commutative Hopf monoid, then of the graded Hopf algebras $\mathcal{K}(\mathbf{h})$ and $\mathcal{K}^\vee(\mathbf{h})$ only the latter is necessarily commutative. This is understood as follows: the lax monoidal functor $\mathcal{K}^\vee$ is braided, but the functor $\mathcal{K}$ is not (Propositions 15.26 and 15.28). The statement for functors is more general: it implies not only that $\mathcal{K}^\vee$ preserves commutative monoids, but also Lie monoids (and in fact any type of monoid defined from a symmetric operad, according to the results of Section 4.4). There are two other reasons why the categorical formulation is important. First, it allows us to formalize the relations between the various constructions in terms of morphisms of monoidal functors. Second, it is ripe for far-reaching generalizations, as witnessed by the results of Chapters 19 and 20.

The functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ both associate a graded vector space to a species in a very simple manner: the degree $n$ component of $\mathcal{K}(\mathbf{p})$ is $\mathbf{p}[n]$, where $[n]$ denotes the set $\{1, \ldots, n\}$, and that of $\overline{\mathcal{K}}(\mathbf{p})$ is $\mathbf{p}[n]_{\mathrm{S}_n}$, the space of coinvariants under the action of the symmetric group $\mathrm{S}_n$. Thus, roughly, the first functor forgets the action of the symmetric group while the second mods it out. The interest is in the bilax structure of $\mathcal{K}$ and $\overline{\mathcal{K}}$. This is constructed out of the functoriality of species with respect to bijections together with two basic ingredients: the unique order-preserving bijections

$$\{1, \ldots, t\} \xrightarrow{\cong} \{s+1, \ldots, s+t\} \qquad \text{and} \qquad S \xrightarrow{\cong} \{1, \ldots, s\},$$

where $S$ is a set of integers of cardinality $s$. These combinatorial procedures are called shifting and standardization; they interact precisely as prescribed by the axioms of bilax monoidal functors. This provides a conceptual explanation for the repeated occurrence of shifting and standardization in the construction of Hopf algebras in combinatorics. Shifting gives rise to the lax structure of $\mathcal{K}$ and $\overline{\mathcal{K}}$ and standardization to their colax structure. Their roles can be switched; this gives rise to the other bilax functors $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$. The degree $n$ component of $\mathcal{K}^\vee(\mathbf{p})$ is $\mathbf{p}[n]$ and that of $\overline{\mathcal{K}}^\vee(\mathbf{p})$ is $\mathbf{p}[n]^{\mathrm{S}_n}$, the space of invariants.

We study abstract properties of the Fock functors, with emphasis on the implications to the Hopf algebra constructions. We show that the full Fock functors do not preserve duality whereas the bosonic Fock functors do (at least in characteristic 0) (Corollary 15.25). We study how (co)commutativity transfers from the context of species to that of graded vector spaces. We show that $\mathcal{K}$ is braided as a colax monoidal functor, but not as a lax monoidal functor (Proposition 15.26). As a consequence, the Hopf algebra $\mathcal{K}(\mathbf{h})$ associated to a Hopf monoid $\mathbf{h}$ will be cocommutative if so is $\mathbf{h}$, but may not be commutative even when $\mathbf{h}$ is. An interesting example is that of the commutative Hopf monoid $\mathbf{L}^*$: the Hopf algebra of permutations, which is the corresponding object under the functor $\mathcal{K}$, is far from commutative (Example 15.17). Commutativity is preserved by the functor $\mathcal{K}$ in a weaker sense, however. There is an isomorphism of bilax monoidal functors

$$(\mathcal{K}, \varphi^b, \psi^b) \Rightarrow (\mathcal{K}, \varphi, \psi)$$

given by the half-twist transformation (Proposition 15.30). The lax and colax structures of the functor on the left are obtained from those of $\mathcal{K}$ by conjugation with the braidings of species and graded vector spaces (the general construction is discussed in Chapter 3). As a consequence, while the Hopf algebras $\mathcal{K}(\mathbf{h}^{\mathrm{op}})$ and $\mathcal{K}(\mathbf{h})^{\mathrm{op}}$ may not be equal, they are always canonically isomorphic. In particular, if $\mathbf{h}$ is commutative, the Hopf algebra $\mathcal{K}(\mathbf{h})$ is endowed with a canonical anti-involution. This feature has been observed for several combinatorial Hopf algebras on a case by case basis. It finds now a unified explanation.

We also study how the Fock functors interact with the primitive element functors $\mathcal{P}$ on both contexts (from Hopf monoids to Lie monoids and from graded Hopf algebras to graded Lie algebras). It is convenient to work with the functor $\mathcal{K}^\vee$ since being braided lax it preserve Lie monoids. We show that

$$\mathcal{K}^\vee\big(\mathcal{P}(\mathbf{h})\big) \subseteq \mathcal{P}\big(\mathcal{K}^\vee(\mathbf{h})\big) \qquad \text{and} \qquad \overline{\mathcal{K}}^\vee\big(\mathcal{P}(\mathbf{h})\big) = \mathcal{P}\big(\overline{\mathcal{K}}^\vee(\mathbf{h})\big)$$

as graded Lie algebras (Proposition 15.35).

The Fock functor $\mathcal{K}^\vee$ is Zinbiel-lax monoidal (Proposition 15.40). This provides a concrete example for the operad-lax monoidal functors of Chapter 4. This property is responsible for the existence of Zinbiel and dendriform structures on algebras constructed from associative or commutative monoids in species via $\mathcal{K}^\vee$. This includes some important examples of such algebras in the literature (Proposition 15.41, Examples 15.42 and 15.43).

**Deformations of the Fock functors** (Chapter 16)**.** The theory of Chapter 15 can be greatly generalized. The first step involves the introduction of a parameter $q$ and the construction of $q$-deformations of all the objects; we explain this presently. The next step involves a further generalization to colored species and is the topic of Chapter 20.

The key idea here is to directly deform the Fock functors, rather than each object individually. We regard this as one of the main strengths of our functorial approach. This is achieved with the aid of the Schubert statistic. It is used to twist the lax or colax structure of the functors with appropriate powers of $q$ in much the same way as the Hopf monoid of linear orders was deformed using the Schubert cocycle. The resulting bilax monoidal functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$ map from the braided monoidal category of species to the braided monoidal category of graded vector spaces (Theorems 16.1 and 16.2). The braidings are deformed by powers of $p$ and $pq$ respectively. One may choose not to deform the braiding on the category of species at all; this amounts to putting $p$ to be 1. For simplicity of exposition, we work in this setup for the present discussion.

The deformed Fock functors applied to a Hopf monoid produce $q$-Hopf algebras, which for $q = 1$ recover the Hopf algebras produced by the undeformed Fock functors. Thus functoriality of the construction guarantees that every Hopf algebra arising from a Hopf monoid in species can be coherently deformed. We offer this as an answer to a question raised by Thibon (in personal conversations and talks), that all "combinatorial Hopf algebras" should be the limiting case of a "quantum group".

Recall that the functor $\mathcal{K}$ is braided colax. However, its deformation $\mathcal{K}_q$ is not braided colax in general. We show that conjugating the colax structure of $\mathcal{K}_q$ with the braidings yields the functor $\mathcal{K}_{q^{-1}}$ (Proposition 16.26). This is again a result of a very general nature, and explains the nature of cocommutativity present in the $q$-Hopf algebras associated to $\mathcal{K}_q$.

The bosonic Fock functors admit signed analogues which we refer to as *fermionic Fock functors*. They are denoted by

$$\overline{\mathcal{K}}_{-1} \quad \text{and} \quad \overline{\mathcal{K}}^\vee_{-1}.$$

We describe them briefly. The degree $n$ component of $\overline{\mathcal{K}}_{-1}(\mathbf{p})$ is $\mathbf{p}[n]_{\mathrm{S}_n}$, where coinvariants are now taken under a twisted action of the symmetric group $\mathrm{S}_n$: tensor the usual action with the one-dimensional sign representation. The other functor $\overline{\mathcal{K}}^\vee_{-1}$ is defined similarly by using invariants instead of coinvariants. The bilax structures of the fermionic Fock functors are related to those of $\mathcal{K}_{-1}$ and $\mathcal{K}^\vee_{-1}$ and involve the Schubert statistic.

The bosonic and fermionic Fock functors are related to each other by the signature functor on species. This provides an important link between the bosonic and fermionic worlds, and results in one can be transferred to the other via properties of the signature functor. For example, since the signature functor and the bosonic Fock functor are both bistrong, it follows that so is the fermionic Fock functor.

**Relations among the Fock functors. The norm transformation** (Chapters 15 and 16)**.** The Fock functors in general are distinct. There are however various relations among these functors that allow us to relate the corresponding Hopf algebras in a natural manner. First of all, the functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$, as well as $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$, and $\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^\vee_{-1}$ are related through the contragredient construction (Propositions 15.8 and 16.3). Thus,

$$\mathcal{K}_q^\vee(-) = \mathcal{K}_q\big((-)^*\big)^*,$$

and similarly for the other pairs of functors. Second, there are natural transformations $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee \Rightarrow \mathcal{K}^\vee$ (projection onto coinvariants and inclusion of invariants). These are morphisms of bilax monoidal functors (Theorems 15.3 and 15.6). Similarly, there are morphisms of bilax monoidal functors $\mathcal{K}_{-1} \Rightarrow \overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^\vee_{-1} \Rightarrow \mathcal{K}^\vee_{-1}$. Third, there are isomorphisms of bilax monoidal functors as follows

$$\mathcal{K}_q \cong \overline{\mathcal{K}}\big(\mathbf{L}_q \times (-)\big) \qquad \text{and} \qquad \mathcal{K}_q^\vee \cong \overline{\mathcal{K}}^\vee\big(\mathbf{L}_q^* \times (-)\big).$$

This result is given in Proposition 16.6; its signed analogue involving the fermionic Fock functor is given in Proposition 16.22. The special case $q = 1$ is discussed earlier in Propositions 15.9 and 15.10. There is thus no reason to view either one of $\mathcal{K}$, $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}_{-1}$ (or one of $\mathcal{K}^\vee$, $\overline{\mathcal{K}}^\vee$ and $\overline{\mathcal{K}}_{-1}^\vee$) as more fundamental than the others.

There is yet another relation among the functors that is particularly important in regard to later generalizations. Namely, there is a morphism of bilax monoidal functors

$$\kappa_q \colon \mathcal{K}_q \Rightarrow \mathcal{K}_q^\vee$$

(Proposition 16.15). This is called the *$q$-norm transformation*. For a species $\mathbf{p}$, the map $(\kappa_q)_{\mathbf{p}} \colon \mathcal{K}_q(\mathbf{p}) \to \mathcal{K}_q^\vee(\mathbf{p})$ is the action of the elements

$$\sum_{\sigma \in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)} \sigma$$

on each space $\mathbf{p}[n]$, where $\mathrm{inv}(\sigma)$ denotes the number of inversions of the permutation $\sigma$. For the species of linear orders, this map has been studied by several authors in a variety of contexts (see the references in Example 16.17). The result of Zagier or of Varchenko mentioned earlier implies that it is generically invertible, and from here we deduce that over a field of characteristic 0 and if $q$ is not a root of unity, the $q$-norm transformation is an isomorphism of bilax monoidal functors (Theorem 16.18). This is a rigidity result of a very general nature. The isomorphism between the tensor algebra of a vector space and the $q$-shuffle algebra of Duchamp, Klyachko, Krob, and Thibon, is one very special case (Example 16.31). Theorem 16.18 gives such a result for *every* Hopf algebra arising from a Hopf monoid in species. When the above hypotheses fail, the image of the transformation $\kappa_q$ is truly a new bilax monoidal functor $\mathfrak{I}_q$, which we call the *anyonic Fock functor*. Its study appears very intriguing. A first connection with *Nichols algebras* is encountered at this point.

The parameter values $\pm 1$ are quite interesting. The $q$-norm in this case is given by symmetrization or antisymmetrization. So it is indeed an instance of the norm map in group theory. Hence, the image in this situation can be understood in terms of invariants and coinvariants. Recall that the bosonic and fermionic functors are defined precisely in this manner. This leads to the following diagrams.

$$\begin{tikzcd} \mathcal{K} \arrow[r, Rightarrow, "\kappa"] \arrow[d, Rightarrow] & \mathcal{K}^\vee \\ \overline{\mathcal{K}} \arrow[r, Rightarrow, "\bar\kappa"'] & \overline{\mathcal{K}}^\vee \arrow[u, Rightarrow] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{K}_{-1} \arrow[r, Rightarrow, "\kappa_{-1}"] \arrow[d, Rightarrow] & \mathcal{K}_{-1}^\vee \\ \overline{\mathcal{K}}_{-1} \arrow[r, Rightarrow, "\bar\kappa_{-1}"'] & \overline{\mathcal{K}}_{-1}^\vee \arrow[u, Rightarrow] \end{tikzcd}.$$

In characteristic 0, the induced transformations $\bar\kappa$ and $\bar\kappa_{-1}$ are isomorphisms. Thus, in this case, one may view the isomorphic functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ as naturally associated to $\kappa$: they are the coimage and image of this morphism, respectively. The same statement holds for $\kappa_{-1}$. To summarize, in characteristic 0, for the values $\pm 1$, the anyonic Fock functor $\mathfrak{I}_q$ specializes to the bosonic and fermionic Fock functors.

**The free Fock functor and 0-bialgebras** (Chapter 16)**.** As already mentioned, the Fock functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$ are duals (contragredients) of each other. An interesting phenomenon occurs at $q = 0$, namely, these two functors coincide. In other words, $\mathcal{K}_0 = \mathcal{K}_0^\vee$; thus this resulting functor, which we call the free Fock functor, is self-dual. Further, its lax structure coincides with that of $\mathcal{K}$ and its colax structure coincides with that of $\mathcal{K}^\vee$. These results are summarized in Proposition 16.4.

By the preceding theory, the free Fock functor sends $p$-Hopf monoids to 0-Hopf algebras. A rigidity result of Loday and Ronco says that any connected 0-bialgebra is free as a graded algebra and cofree as a graded coalgebra. The free Fock functor applied to any connected $p$-bimonoid $\mathbf{h}$ yields a connected 0-bialgebra $\mathcal{K}_0(\mathbf{h})$. As lax functors, $\mathcal{K}_0 = \mathcal{K}$, and as colax functors $\mathcal{K}_0^\vee = \mathcal{K}^\vee$. It follows that $\mathcal{K}(\mathbf{h})$ is a free graded algebra and $\mathcal{K}^\vee(\mathbf{h})$ is a cofree graded coalgebra, for any connected $p$-bimonoid $\mathbf{h}$ (Proposition 16.11). Results of this kind have been recently obtained by Livernet [233] and our functorial approach serves to further clarify them.

**Hopf algebras from geometry** (Chapter 17)**.** Many of the Hopf algebras associated to the Hopf monoids in (0.1) are familiar and have received a great deal of attention in the recent literature. For simplicity, let us work over a field of characteristic 0 and concentrate on the following small portion of diagram (0.1), which gives rise to the most familiar Hopf algebras.

$$\begin{tikzcd}
\mathcal{T}(\mathbf{E}^*_+) \arrow[r] \arrow[d] & \mathbf{L} \times \mathbf{L}^* \arrow[d] \\
\mathcal{S}(\mathbf{E}^*_+) \arrow[r] & \mathcal{T}^\vee(\mathbf{E}_+)
\end{tikzcd}$$

Applying each of the functors $\mathcal{K}$, $\overline{\mathcal{K}}$, $\mathcal{K}^\vee$, and $\overline{\mathcal{K}}^\vee$ one obtains a commutative square of graded Hopf algebras. In view of the natural transformations $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee \Rightarrow \mathcal{K}^\vee$, these four squares can be assembled into two commutative cubes.

$$\begin{tikzcd}
 & \mathrm{N}\Pi \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{S}\Pi \arrow[dl] \arrow[dd] \\
\mathrm{N}\Lambda \arrow[rr] \arrow[dd] & & \mathrm{S}\Lambda \arrow[dd] & \\
 & \Pi \arrow[rr] \arrow[dl] & & \mathrm{Q}\Pi \arrow[dl] \\
\Lambda \arrow[rr] & & \mathrm{Q}\Lambda &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Q}\Pi^* \arrow[rr] \arrow[dd] & & \mathrm{S}\Pi^* \arrow[dd] \\
\mathrm{N}\Lambda \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{S}\Lambda \arrow[ur] \arrow[dd] & \\
 & \Pi^* \arrow[rr] & & \mathrm{N}\Pi^* \\
\Lambda \arrow[ur] \arrow[rr] & & \mathrm{Q}\Lambda \arrow[ur] &
\end{tikzcd}$$

The above mentioned natural transformations are responsible for the morphisms between the back and front faces. As already mentioned, in characteristic 0, the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ are naturally isomorphic. This explains the common face of the cubes. The duality between the back faces is due to the corresponding relation between $\mathcal{K}$ and $\mathcal{K}^\vee$.

We have used the notation of [12], where all these objects are studied. Thus $\Lambda$, $\mathrm{N}\Lambda$, $\mathrm{Q}\Lambda$, and $\mathrm{S}\Lambda$ are the Hopf algebras of symmetric functions, noncommutative symmetric functions, quasi-symmetric functions, and permutations, respectively, while $\Pi$ is the Hopf algebra of symmetric functions in noncommuting variables. They provide examples of how graded Hopf algebras can be obtained through a combination of the functors $\mathcal{T}$ or $\mathcal{S}$, the Hadamard product, and duality, with the Fock functors.

Since the Hadamard product on graded vector spaces is not bilax, the Hopf algebras associated to the Hopf monoid $\mathbf{L} \times \mathbf{L}^*$ via the Fock functors cannot be described in terms of the Hopf algebras associated to $\mathbf{L}^*$ and $\mathbf{L}$. This provides evidence for our claim that proper understanding of these Hopf algebras requires the consideration of Hopf monoids in species.

The claim is further substantiated by the fact that many of the Hopf monoids we consider are universal, that is, they are special values of the functors $\mathcal{T}$, $\mathcal{S}$, or their duals, on the category of species. Depending on which Fock functors we use, these universal properties may be lost in the passage from Hopf monoids to Hopf algebras and therefore can never be fully appreciated if one is confined to the world of graded vector spaces and graded Hopf algebras. For instance, the species $\mathbf{L}$ is the free monoid on one generator, and can therefore be regarded as an analogue of the polynomial algebra in one variable. By contrast, the Hopf algebra of permutations which is its image under one of the full Fock functors has no analogous property.

Duality is another property that may be lost in the passage from Hopf monoids to Hopf algebras. For example, $\mathcal{T}(\mathbf{E}_+^*)$ are $\mathcal{T}^\vee(\mathbf{E}_+)$ are dual as Hopf monoids, but $\mathrm{N}\Pi$ and $\mathrm{Q}\Pi$ (which are their images under $\mathcal{K}$) are not dual as graded Hopf algebras.

Two questions regarding graded Hopf algebras are often interesting and difficult: the determination of a linear basis of the space of primitive elements and the determination of the structure constants of the antipode. For most of the Hopf algebras associated to the Hopf monoids in (0.1), the first question was answered in [12]. In Part II, we consider the same questions directly for the Hopf monoids, which are more fundamental objects. In view of the results of Chapter 15, the answers to these questions for a Hopf algebra of the form $\overline{\mathcal{K}}(\mathbf{h})$ can be told from the answers for the Hopf monoid $\mathbf{h}$.

**Hopf algebras from combinatorics** (Chapter 17)**.** The Hopf monoids of Chapter 13 give rise to another long list of Hopf algebras. Several of these Hopf algebras have received much attention in the recent literature. We mention in particular work of Gessel and Malvenuto in connection to the Hopf algebras of posets; of Ehrenborg in connection to the Hopf algebras of set-graded posets; of Sagan, Schmitt, and Stanley in connection to the Hopf algebras of graphs; of Connes–Kreimer and Grossman–Larson in connection to the Hopf algebras of forests; of Crapo and Schmitt in connection to the Hopf algebras of matroids. Similarly, the morphisms arising from universal constructions yield well-known generating functions for the corresponding combinatorial objects. They include the enumerator of poset partitions, the enumerator of descents, the chromatic function for graphs and its variant for labeled graphs, the quasi-symmetric flag function, a generating function for matroids, and so on.

**Adjoints of the Fock functors** (Chapter 18)**.** This chapter is devoted to the construction of the various adjoints of the Fock functors. We view each of $\mathcal{K}$, $\overline{\mathcal{K}}$, $\mathcal{K}^\vee$, and $\overline{\mathcal{K}}^\vee$ as a functor at three levels: one from species to graded vector spaces, another from monoids to graded algebras, and another from comonoids to graded coalgebras. A complete analysis of the left and right adjoints of each of these functors (which may exist or not) is presented. This includes important notions such as the free monoid on a graded algebra (relative to each of the four functors). These notions are related through composition of adjunctions. A summary of the results is given in Tables 18.1 and 18.2. The free monoid on a graded algebra is not to be confused with the free monoid on a species discussed earlier. This chapter also serves to formalize the analogy between the tensor and symmetric algebra functors on graded vector spaces and the corresponding functors $\mathcal{T}$ and $\mathcal{S}$ on species.

**Decorated Fock functors** (Chapter 19)**.** The Fock functors as discussed thus far admit a decorated version: one can define a Fock functor for each vector space $V$, with $V = \Bbbk$ recovering the undecorated case (Theorems 19.2 and 19.33). We denote them by adding a subscript $V$ to the previous notation. In a sense, one may view the result of applying a decorated Fock functor, say $\mathcal{K}_V$, to a species $\mathbf{p}$ as a version of the graded vector space $\mathcal{K}(\mathbf{p})$ in which the given combinatorial structure determined by the species $\mathbf{p}$ has been decorated with elements of the vector space $V$.

The exponential species $\mathbf{E}$ admits a decorated version which we denote by $\mathbf{E}_V$. It recovers $\mathbf{E}$ for $V = \Bbbk$ and retains all its important features. For example, it is the free commutative monoid as well as the cofree cocommutative comonoid on $\mathbf{X}_V$. The latter is the species which is $V$ on singletons and 0 otherwise. Further, the dual of $\mathbf{E}_V$ is $\mathbf{E}_{V^*}$. In particular, if $V$ is finite-dimensional, then $\mathbf{E}_V$ is self-dual (the self-duality depending on a choice of an isomorphism $V \cong V^*$). The significance of the decorated exponential species in the present context is brought about by the relation

$$\mathcal{K}_V(-) = \mathcal{K}\big((-) \times \mathbf{E}_V\big)$$

This allows us to quickly generalize the theory of undecorated Fock functors to the decorated setting. For instance, the bilax structure of $\mathcal{K}_V$ arises from that of the Hadamard product plus the bimonoid structure of $\mathbf{E}_V$. A similar relation holds for the other decorated functors as well.

The decorated Fock functors give a systematic procedure of decorating any graded Hopf algebra that arises from a Hopf monoid in species.

**Fock spaces. Up-down and creation-annihilation operators** (Chapter 19)**.** Fix a vector space $V$. Classical full Fock space is the underlying space of the tensor algebra on $V$. Similarly, bosonic Fock space is the underlying space of the symmetric algebra on $V$, while fermionic Fock space is the underlying space of the exterior algebra on $V$. In physical terms, $V$ stands for the quantum states of a single particle, while the Fock spaces describe quantum states with a variable number of particles. The terms bosonic and fermionic are used depending on whether the particles are bosons or fermions. These spaces carry certain operators called *creation* and *annihilation*. The former increases the number of particles by 1, while the latter decreases the number of particles by 1. Further, these operators satisfy canonical commutation relations, see equations (19.4) and (19.6).

The first observation is that bosonic and fermionic Fock spaces are the values of the decorated bosonic and decorated fermionic Fock functors respectively on the exponential species. The second observation is that the exponential species is naturally equipped with what we call *up-down operators*. The third observation is that Fock functors convert up-down operators on species to creation-annihilation operators on graded vector spaces. This explains the existence of such operators on Fock spaces.

Following Guţă and Maassen [158] and Bożejko and Guţă [64], we define generalized Fock spaces to be the values of the decorated Fock functors on any species with up-down operators. In particular, we add to their constructions by paying attention to the monoidal properties of the functors. Propositions 19.16, 19.21 and 19.38 serve to illustrate this point.

We then introduce the notion of a species with balanced operators. It is a species with up-down operators in which the operators are required to satisfy further compatibilities. The motivating example is that of the exponential species. The point is that the decorated bosonic and fermionic Fock functors convert a species with balanced operators to a graded vector spaces with creation-annihilation operators which satisfy the canonical commutation relations (Propositions 19.27 and 19.39). We illustrate this on a number of examples including the species of rooted trees and the species of elements.

**Colored Fock functors** (Chapter 20)**.** There is a higher dimensional generalization of the preceding theory in which species are replaced by colored species, and graded vector spaces by multigraded vector spaces. Recall from *Graded vector spaces* and *Colored species and $Q$-Hopf monoids* that for each square matrix $Q$ of size $r$, one can define a braiding on $r$-colored species as well as on $\mathbb{N}^r$-graded vector spaces. We construct bilax monoidal functors $\mathcal{K}_Q$ and $\mathcal{K}_Q^\vee$ from the category of $r$-colored species to the category of $\mathbb{N}^r$-graded vector spaces (Theorem 20.1). The Fock functors as well as their $q$-deformations occur as special cases when $Q = [q]$ is a matrix of size 1. The braidings are to be chosen as follows. If the square matrix $P$ is used for $r$-colored species, then the matrix $P \times Q$ (the Hadamard product of $P$ and $Q$) is to be used for $\mathbb{N}^r$-graded vector spaces. It follows that these functors take $P$-Hopf monoids to $(P \times Q)$-Hopf algebras. The constructions of these $\mathbb{N}^r$-graded Hopf algebras with respect to nontrivial braidings are now considerably more general than those of Chapter 15.

Recall that the braiding $\beta_Q$ is a symmetry if and only if $Q$ is log-antisymmetric. In this situation, the symmetric groups act via the braiding on appropriate components of a colored species (Proposition 20.3). By taking invariants and coinvariants with respect to this action, one obtains bistrong functors

$$\overline{\mathcal{K}}_Q \quad \text{and} \quad \overline{\mathcal{K}}_Q^\vee.$$

The bosonic and fermionic Fock functors correspond to the log-antisymmetric matrices $Q = [1]$ and $Q = [-1]$ respectively.

We construct a higher dimensional version of the norm transformation

$$\kappa_Q \colon \mathcal{K}_Q \Rightarrow \mathcal{K}_Q^\vee$$

and show that it is a morphism of bilax functors (Proposition 20.9). The image of $\kappa_Q$ is a new bilax monoidal functor $\Im_Q$ from $r$-colored species to $\mathbb{N}^r$-graded vector spaces. This functor is the multivariate version of the anyonic Fock functor. If the characteristic of the field is 0 and $Q$ is log-antisymmetric, then $\Im_Q$ coincides with $\overline{\mathcal{K}}_Q$ and $\overline{\mathcal{K}}_Q^\vee$. If $Q$ has generic entries, then the norm $\kappa_Q$ is an isomorphism (Theorem 20.11). This is again an application of Varchenko's result.

As an example, we consider a colored analog of the Hopf monoid $\mathbf{E}$ which we denote by $\mathbf{E}_{(r)}$. The functors $\mathcal{K}_Q$ and $\mathcal{K}_Q^\vee$ applied to $\mathbf{E}_{(r)}$ yield the Hopf algebra of noncommutative polynomials in $r$ variables and the quantum shuffle algebra respectively. The Hopf algebra $\Im_Q(\mathbf{E}_{(r)})$, on the other hand, is the *quantum symmetric algebra* associated to the matrix $Q$. The terminology *Nichols algebra of diagonal type* is used for this object in the theory of abstract Hopf algebras. Further, special choices of $Q$ lead to Manin's quantum linear spaces or to the nilpotent part of quantum enveloping algebras. The explicit calculation of Nichols algebras is widely regarded as a difficult problem, intimately related to the classification of pointed Hopf algebras. One may therefore expect the calculation of explicit values of the functor $\Im_Q$ to be similarly challenging and interesting.

The question of what Hopf algebras may arise when other colored species are considered is a completely open avenue. As another example, we mention that the functor $\mathcal{K}_Q$ applied to the colored linear order species yields a $Q$-Hopf algebra indexed by $r$-signed permutations.

**Yang–Baxter deformation of decorated Fock functors** (Chapters 19 and 20)**.** Let $R$ be a Yang–Baxter operator on the space of decorations $V$. In this setting, one can define functors $\mathcal{K}_{V,R}$ and $\mathcal{K}^\vee_{V,R}$ along with a norm transformation between them. The image of the norm yields a functor denoted $\Im_{V,R}$. These functors are not bilax in the usual sense. Just as bilax functors are the functorial analogues of bialgebras, these are the functorial analogues of braided bialgebras [356, Definition 5.1]. The Yang–Baxter operator plays a role in the lax and colax structures of these functors as well as in the braiding axiom. Applying $\Im_{V,R}$ to the exponential species $\mathbf{E}$ yields the *Nichols algebra* associated to $R$ (also known as the *quantum symmetric algebra*).

By letting $R$ to be the operator which switches the two tensor factors, one recovers the decorated Fock functors $\mathcal{K}_V$, $\mathcal{K}^\vee_V$ and $\Im_V$. By fixing a scalar $q$ and letting $R$ to be the operator

$$v \otimes w \mapsto q\, w \otimes v,$$

one obtains one-parameter deformations of the decorated Fock functors. It turns out that these are bilax in the usual sense. Thus we have $\mathcal{K}_{V,q}$ and $\mathcal{K}^\vee_{V,q}$, which deform the decorated full Fock functors, a decorated $q$-norm

$$\kappa_q \colon \mathcal{K}_{V,q} \Rightarrow \mathcal{K}^\vee_{V,q}$$

which is a morphism of bilax monoidal functors, and $\Im_{V,q}$, which is the image of $\kappa_q$. By letting $V = \Bbbk$, we recover the deformed Fock functors $\mathcal{K}_q$, $\mathcal{K}^\vee_q$ and $\Im_q$.

Let $Q$ be a square matrix of size $r$, where $r$ is the dimension of $V$. Fix a basis $x_1, x_2, \ldots, x_r$ of $V$, and consider the Yang–Baxter $R_Q$ operator on $V$:

$$V \otimes V \to V \otimes V, \qquad x_i \otimes x_j \mapsto q_{ji}\, x_j \otimes x_i$$

where $i$ and $j$ vary between 1 and $r$, and $q_{ji}$ denotes the entries of the matrix $Q$. The functors $\mathcal{K}_{V,R_Q}$, $\mathcal{K}^\vee_{V,R_Q}$ and $\Im_{V,R_Q}$ are closely related to the colored Fock functors $\mathcal{K}_Q$, $\mathcal{K}^\vee_Q$ and $\Im_Q$. The precise relation is given in Theorem 20.19.

The theory of creation-annihilation operators can also be developed in the setting of Yang–Baxter operators. They satisfy appropriately deformed commutation relations (Propositions 19.41 and 19.48).

## Appendices

Four appendices supplement the text. Appendix A reviews some basic notions from category theory, including adjunctions, equivalences, and colimits. The contents of Appendices B, C and D are summarized below.

**Operads** (Appendix B)**.** Operads are monoids in the monoidal category of species under substitution. These objects have been at the focus of intense activity in recent times, though not often from this point of view.

We have mentioned a variety of tensor products on species, centering primarily on the Cauchy product. Of these, the substitution product is the most subtle, and a definition in full generality requires some care. When the species do not vanish on the empty set, two different versions of substitution arise. One notion of substitution (B.9) gives rise to the general notion of operad and the other (B.15) to the general notion of cooperad. We provide a complete proof of associativity for the former version of substitution (Lemma B.14) and also describe its internal Hom (Proposition B.26). Further, we explain how a proper understanding of the latter version of substitution requires a more general setup, which is that of lax monoidal categories.

The main use for operads in this monograph occurs in Chapter 4, as already mentioned. To each operad corresponds a type of monoid (in a monoidal category) and a type of monoidal functor (between monoidal categories). Types of monoids may also be understood in terms of modules over operads. The two notions are equivalent (Proposition B.27).

The substitution and Hadamard products define a 2-monoidal structure on the category of species (Propositions B.31 and B.35). This provides a context for the notion of Hopf operad.

**Pseudomonoids and the looping principle** (Appendix C)**.** The notion of pseudomonoid is a 2-dimensional analogue of the notion of monoid in a monoidal category. We provide a complete definition, following work of Day, McCrudden, and Street among others. The context is that of monoidal 2-categories (not to be confused with 2-monoidal categories). A pseudomonoid possesses a product that is associative up to a 2-cell. A monoidal category is an example of a pseudomonoid (in the 2-category $\mathsf{Cat}$, which is monoidal under Cartesian product). Our main interest in pseudomonoids stems from a result we prove in Proposition 6.73, which states that a 2-monoidal category can be viewed as a pseudomonoid in two different monoidal 2-categories. These are the 2-categories $\mathsf{lCat}$ and $\mathsf{cCat}$ whose objects are monoidal categories and whose arrows are respectively lax and colax monoidal functors. This and other examples of pseudomonoids are summarized in Table C.1.

The passage from $\mathsf{Cat}$ to $\mathsf{lCat}$ and $\mathsf{cCat}$ is an instance of the *lax and colax constructions*. They are discussed in Section C.2.3. They play an important role in connection to the notion of higher monoidal categories, as already mentioned.

The set of endomorphisms of an object in a category is an ordinary monoid under composition. This is a first instance of the *looping principle* which is the subject of Section C.4. We are mainly interested in a 2-dimensional version of the principle which relates pseudomonoids (in a monoidal 2-category) to bicategories (enriched in the same monoidal 2-category). We arrive at this in Section C.4.4, after discussing simpler instances of the looping principle. We also discuss how some important examples of 2-monoidal categories arise in this manner, as loops in bicategories enriched by either $\mathsf{lCat}$ or $\mathsf{cCat}$.

**Monoids and the simplicial category** (Appendix D)**.** We discuss two generalizations of the notion of monoid in a monoidal category: lax monoids and homotopy monoids. They are due to Day and Street [94] and Leinster [229], respectively. We are mainly interested in two special instances of these notions: lax monoidal categories, and a particular homotopy monoid that we construct in the context of natural transformations between monoidal functors.

The key notion on which the generalizations are based is Mac Lane's simplicial category. This category plays a universal role in connection to monoids (Proposition D.2). Relaxing the conditions in this result leads to the notions of lax monoids and homotopy monoids.

We explain the notion of a lax monoidal category in some detail (Definition D.3). This is an example of a lax monoid. It, however, plays only a minor role in this monograph. It is required for a proper understanding of the second substitution product on species, as already mentioned.

We explore a notion of convolution for natural transformations from a colax monoidal functor $\mathcal{F}$ to a lax monoidal functor $\mathcal{G}$. This may be regarded as an analogue of the convolution operation on the set $\mathrm{Hom}(C, A)$ of maps from a comonoid $C$ to a monoid $A$. More precisely: The role of the set $\mathrm{Hom}(C, A)$ is played by a certain contravariant functor $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ on Mac Lane's simplicial category, that is, by an augmented simplicial set. An $n$-simplex in this simplicial set is a natural transformation from $\mathcal{F}_n$ to $\mathcal{G}_n$, where

$$\mathcal{F}_n(A_1, \ldots, A_n) := \mathcal{F}(A_1 \bullet \cdots \bullet A_n)$$

is a functor from the $n$-fold Cartesian product of the source category of $\mathcal{F}$ with itself to its target category (with $\bullet$ denoting the tensor product of the source category). Convolution of natural transformations turns $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ into a lax monoidal functor (Theorem D.9). This is an example of a homotopy monoid. We apply these ideas to Hopf lax functors, see Proposition D.12 and the discussion following it.

## Related work

Several references to related work in the literature are given in the text. Of these, Joyal's work on braided monoidal categories and on species [181, 184] has been the most influential. We would like to view our work as a contribution to his ideas.

We would also like to highlight the work of the following authors. Schmitt and Stover, independently and at about the same time, were the first to describe constructions of Hopf algebras from Hopf monoids in species [322, 346]. The connection to combinatorial Hopf algebras was brought forth by Patras and Reutenauer [291]. Patras and Schocker [292, 293], Patras and Livernet [234] and Livernet [233] have further advanced the subject.

Our interest in Hopf monoids in species developed from a lecture by Sagan in Montréal in 2001 on the algebra $\Pi$. In trying to understand the Hopf algebras $\Pi$ and $\mathrm{Q}\Pi$ from the point of view of universal properties, as had been done for $\mathrm{Q}\Lambda$ in [10], we were led to the consideration of species.

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# Part I

# Monoidal Categories

CHAPTER 1

# Monoidal Categories

In this chapter we review some basic notions related to monoidal categories. These include braided monoidal categories, bimonoids and Hopf monoids, Lie monoids, unbracketed and unordered tensor products, and the internal Hom functor. We provide a number of results, particularly on antipodes, which require some care when the braiding is not assumed to be a symmetry.

The exposition makes use of some standard notions from category theory such as products, limits and equivalence of categories. These are reviewed in Appendix A.

## 1.1. Braided monoidal categories

A detailed discussion of braided and symmetric monoidal categories can be found in the paper of Joyal and Street [184] and in the books by Kassel [191, Chapters XI and XIII], Kock [203, Chapter 3], Leinster [226, Chapter 3], Mac Lane [250, Chapters VII and XI], Street [348], and Yetter [379, Chapters 3 and 5].

**1.1.1. Monoidal categories.** Some of the earliest papers on monoidal categories are those of Bénabou [36], Mac Lane [248], and Kelly [194].

**Definition 1.1.** A *monoidal category* $(\mathsf{C}, \bullet)$ is a category $\mathsf{C}$ with a functor

$$\bullet \colon \mathsf{C} \times \mathsf{C} \to \mathsf{C},$$

together with a natural isomorphism

$$\alpha_{A,B,C} \colon (A \bullet B) \bullet C \xrightarrow{\cong} A \bullet (B \bullet C)$$

which satisfies the pentagon axiom:

$$(1.1)\qquad \begin{tikzcd}[column sep=small] & & (A \bullet B) \bullet (C \bullet D) \arrow[drr, "\alpha_{A,B,C\bullet D}"] & & \\ ((A \bullet B) \bullet C) \bullet D \arrow[urr, "\alpha_{A\bullet B,C,D}"] \arrow[dr, "\alpha_{A,B,C}\bullet \mathrm{id}_D"'] & & & & A \bullet \big(B \bullet (C \bullet D)\big) \\ & \big(A \bullet (B \bullet C)\big) \bullet D \arrow[rr, "\alpha_{A,B\bullet C,D}"'] & & A \bullet \big((B \bullet C) \bullet D\big). \arrow[ur, "\mathrm{id}_A \bullet \alpha_{B,C,D}"'] & \end{tikzcd}$$

Further, $\mathsf{C}$ has a distinguished object $I$ with natural isomorphisms

$$\lambda_A \colon A \to I \bullet A, \quad \text{and} \quad \rho_A \colon A \to A \bullet I,$$

which satisfy the triangle axiom:

$$\begin{tikzcd}
(A \bullet I) \bullet B \arrow[rr, "\alpha_{A,I,B}"] & & A \bullet (I \bullet B) \\
& A \bullet B. \arrow[ul, "\rho_A \bullet \mathrm{id}_B"] \arrow[ur, "\mathrm{id}_A \bullet \lambda_B"'] &
\end{tikzcd} \tag{1.2}$$

It follows [184, Proposition 1.1] that

$$\lambda_I = \rho_I, \tag{1.3}$$

and the following diagrams commute.

$$\begin{tikzcd}
(I \bullet A) \bullet B \arrow[rr, "\alpha_{I,A,B}"] & & I \bullet (A \bullet B) \\
& A \bullet B \arrow[ul, "\lambda_A \bullet \mathrm{id}_B"] \arrow[ur, "\lambda_{A \bullet B}"'] &
\end{tikzcd} \qquad
\begin{tikzcd}
(A \bullet B) \bullet I \arrow[rr, "\alpha_{A,B,I}"] & & A \bullet (B \bullet I) \\
& A \bullet B \arrow[ul, "\rho_{A \bullet B}"] \arrow[ur, "\mathrm{id}_A \bullet \rho_B"'] &
\end{tikzcd} \tag{1.4}$$

We refer to $A \bullet B$ as the *tensor product* of $A$ and $B$ and to $I$ as the *unit object* of $\mathsf{C}$. The natural isomorphism $\alpha$ above is called the *associativity constraint* and $\lambda$ and $\rho$ are called the *unit constraints*. A monoidal category is called *strict*, if the associativity and unit constraints are identities.

We often omit $\alpha$ from the notation and identify the objects $(A \bullet B) \bullet C$ and $A \bullet (B \bullet C)$. The identification is denoted $A \bullet B \bullet C$ and referred to as the unbracketed tensor product of $A$, $B$ and $C$; details are provided in Section 1.4. We often omit the subindexes $A$, $B$, $C$ from $\alpha$, $\lambda$ and $\rho$ if they can be told from the context.

Sometimes we write $\mathcal{M}$ for the functor $\bullet \colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, so that

$$\mathcal{M}(A, B) = A \bullet B.$$

We call it the tensor product functor.

Let $\mathsf{C}^{\mathrm{op}}$ denote the *opposite category* of $\mathsf{C}$; these categories have the same objects, and

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A).$$

If $(\mathsf{C}, \bullet)$ is monoidal, then so is $(\mathsf{C}^{\mathrm{op}}, \bullet)$.

The *transpose* of $(\mathsf{C}, \bullet)$ is the monoidal category $(\mathsf{C}, \tilde{\bullet})$ where

$$A \mathbin{\tilde{\bullet}} B := B \bullet A.$$

If $\mathsf{C}$ and $\mathsf{C}'$ are monoidal categories, then so is $\mathsf{C} \times \mathsf{C}'$ with tensor product

$$(A, A') \bullet (B, B') := (A \bullet B, A' \bullet B').$$

**1.1.2. Braided monoidal categories.** Braided monoidal categories were introduced by Joyal and Street [183, 184]. They generalize symmetric monoidal categories which were introduced by Mac Lane [248]. Some of the first papers on the subject are by Bénabou [37], Kelly [194], and Eilenberg and Kelly [117, Chapter III].

**Definition 1.2.** A *braided monoidal category* is a monoidal category $(\mathsf{C}, \bullet)$ together with a natural isomorphism

$$\beta_{A,B} \colon A \bullet B \to B \bullet A,$$

which satisfies the axioms

$$
\begin{tikzcd}
 & B \bullet A \bullet C \arrow[dr, "\mathrm{id}_B \bullet \beta_{A,C}"] & \\
A \bullet B \bullet C \arrow[ur, "\beta_{A,B} \bullet \mathrm{id}_C"] \arrow[rr, "\beta_{A,B \bullet C}"'] & & B \bullet C \bullet A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \bullet C \bullet B \arrow[dr, "\beta_{A,C} \bullet \mathrm{id}_B"] & \\
A \bullet B \bullet C \arrow[ur, "\mathrm{id}_A \bullet \beta_{B,C}"] \arrow[rr, "\beta_{A \bullet B,C}"'] & & C \bullet A \bullet B.
\end{tikzcd}
\tag{1.5}
$$

The natural isomorphism $\beta$ is called a *braiding*. We often omit the subindexes $A$ and $B$ from $\beta$ if they can be told from the context.

A monoidal category is *symmetric* if it is equipped with a braiding $\beta$ which satisfies $\beta^2 = \mathrm{id}$. In this case $\beta$ is called a *symmetry*.

It follows from the axioms that the following diagrams commute [184, Proposition 2.1].

$$
\begin{tikzcd}
A \bullet B \bullet C \arrow[r, "\mathrm{id}_A \bullet \beta_{B,C}"] \arrow[d, "\beta_{A,B} \bullet \mathrm{id}_C"'] & A \bullet C \bullet B \arrow[r, "\beta_{A,C} \bullet \mathrm{id}_B"] & C \bullet A \bullet B \arrow[d, "\mathrm{id}_C \bullet \beta_{A,B}"] \\
B \bullet A \bullet C \arrow[r, "\mathrm{id}_B \bullet \beta_{A,C}"'] & B \bullet C \bullet A \arrow[r, "\beta_{B,C} \bullet \mathrm{id}_A"'] & C \bullet B \bullet A
\end{tikzcd}
\tag{1.6}
$$

$$
\begin{tikzcd}
A \bullet I \arrow[rr, "\beta_{A,I}"] & & I \bullet A \\
 & A \arrow[ul, "\rho_A"] \arrow[ur, "\lambda_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \bullet A \arrow[rr, "\beta_{I,A}"] & & A \bullet I \\
 & A \arrow[ul, "\lambda_A"] \arrow[ur, "\rho_A"'] &
\end{tikzcd}
\tag{1.7}
$$

It follows from (1.3) and (1.7) that

$$
\beta_{I,I} = \mathrm{id}_{I \bullet I}. \tag{1.8}
$$

If $\beta$ is a braiding for $(\mathsf{C}, \bullet)$, then so is its inverse $\beta^{-1}$ defined by

$$
(\beta^{-1})_{A,B} := (\beta_{B,A})^{-1} \colon A \bullet B \to B \bullet A.
$$

The monoidal category $(\mathsf{C}^{\mathrm{op}}, \bullet)$ is braided, with the *opposite braiding* defined by

$$
(\beta^{\mathrm{op}})_{A,B} := \beta_{B,A},
$$

as is the category $(\mathsf{C}, \tilde{\bullet})$, with the *transpose braiding*

$$
(\beta^{t})_{A,B} := \beta_{B,A}.
$$

If $\mathsf{C}$ and $\mathsf{C}'$ are braided monoidal categories with braidings $\beta$ and $\beta'$, then so is $\mathsf{C} \times \mathsf{C}'$ with braiding

$$
\begin{tikzcd}
(A, A') \bullet (B, B') \arrow[r, dashed] \arrow[d, equal] & (B, B') \bullet (A, A') \arrow[d, equal] \\
(A \bullet B, A' \bullet B') \arrow[r, "(\beta_{A,B}, \beta'_{A',B'})"'] & (B \bullet A, B' \bullet A').
\end{tikzcd}
$$

**Example 1.3.** The smallest symmetric monoidal category is the *one-arrow category* $(\mathsf{I}, \bullet)$. It has one object and one morphism with the tensor product and braiding defined in the obvious manner. This example plays a useful role in the general theory.

Other basic examples of symmetric monoidal categories are $(\mathsf{Set}, \times)$, the category of sets under Cartesian product, and $(\mathsf{Vec}, \otimes)$, the category of vector spaces under tensor product. For the unit objects we choose the one-element set $\{\emptyset\}$ and the base field $\Bbbk$, respectively. The braiding is given by switching the tensor factors in both cases.

The following symmetric monoidal categories play a central role in this monograph: the category $\mathsf{gVec}$ of graded vector spaces equipped with the graded tensor product, also called the Cauchy product, and the category $\mathsf{Sp}$ of species equipped with the Cauchy product. These categories admit a number of other interesting tensor products as well; details are given in Sections 2.1 and 8.1.

**Example 1.4.** Let $\mathsf{C}$ be a category with finite products (Section A.1.1). Then $(\mathsf{C}, \times, J)$ is a symmetric monoidal category: $A \times B$ is a chosen product of $A$ and $B$ and $J$ is a chosen terminal object in $\mathsf{C}$. The associativity, unit constraints, and the braiding are defined via the universal property for products. We say in this case that the monoidal category $\mathsf{C}$ is *cartesian*. This example is also discussed in [250, Proposition III.5.1, Exercise III.5.2] and [226, Example 1.2.7].

Similarly, any category with finite coproducts (Section A.1.2) is symmetric monoidal, with the monoidal structure given by a choice of coproduct and the unit given by a chosen initial object. Such monoidal categories $(\mathsf{C}, \amalg, I)$ are called *cocartesian*.

Finally, suppose that $\mathsf{C}$ has finite biproducts (Section A.1.3). In this situation, the monoidal categories $(\mathsf{C}, \amalg, I)$ and $(\mathsf{C}, \times, J)$ are isomorphic by means of the canonical map (A.1). We may thus identify them and write $(\mathsf{C}, \oplus, Z)$, where $\oplus$ stands for either $\amalg$ or $\times$, and $Z$ for either $I$ or $J$. Such monoidal categories are called *bicartesian*.

**1.1.3. Lax braided monoidal categories.** In deriving the commutativity of diagrams (1.7) from Definition 1.2, the invertibility of $\beta$ is used. There is a generalization of the notion of braided monoidal category in which the natural transformation $\beta$ is not required to be an isomorphism, but the commutativity of (1.7) is. It is as follows.

**Definition 1.5.** A *lax braided monoidal category* is a monoidal category $(\mathsf{C}, \bullet)$ together with a natural transformation

$$\beta_{A,B} \colon A \bullet B \to B \bullet A,$$

which satisfies the axioms (1.5) and (1.7). The natural transformation $\beta$ is called a *lax braiding*.

Such $\beta$ have been considered in the literature: briefly by Yetter in [378, Definition 1.12], who calls them *pre-braidings*, and in more depth by Day, Panchadcharam, and Street in [91, Section 1], whose terminology we follow. They also appear in the work of Kapranov and Voevodsky [188, Definition 3.1].

**1.1.4. Linear and abelian monoidal categories.** Let $\Bbbk$ be a commutative ring.

**Definition 1.6.** A category $\mathsf{C}$ is $\Bbbk$-*linear* if each set $\mathrm{Hom}(A, B)$ is a $\Bbbk$-module and composition of arrows

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

is $\Bbbk$-bilinear. A functor between such categories is $\Bbbk$-*linear* if it induces morphisms of $\Bbbk$-modules on each Hom set.

A $\Bbbk$-linear category is the same thing as a category enriched by the monoidal category of $\Bbbk$-modules $(\mathsf{Mod}_\Bbbk, \otimes_\Bbbk)$ [277, Section 11]. We discuss enriched categories in Section C.3.

Mac Lane calls $\mathbb{Z}$-linear categories *preadditive* or *Ab-categories* [250, Section I.8], while Mitchell calls them *additive* [277, Section 1]. We do not employ either terminology. When the base ring $\Bbbk$ is understood, we speak simply of linear categories.

**Proposition 1.7.** *If* $\mathsf{C}$ *and* $\mathsf{C}'$ *are* $\Bbbk$*-linear, then so is* $\mathsf{C} \times \mathsf{C}'$ *by*

$$(f, f') + (g, g') := (f + g, f' + g') \qquad \textit{and} \qquad c(f, f') := (cf, cf').$$

A monoidal category $(\mathsf{C}, \bullet)$ is $\Bbbk$-*linear* if the category $\mathsf{C}$ is $\Bbbk$-linear and in addition tensoring is a bilinear operation over $\Bbbk$

$$\mathrm{Hom}(A_1, A_2) \times \mathrm{Hom}(B_1, B_2) \to \mathrm{Hom}(A_1 \bullet B_1, A_2 \bullet B_2).$$

A $\Bbbk$-linear monoidal category is *symmetric* if the underlying monoidal category is symmetric.

For the definition of *abelian category*, see [250, Section VIII.3].

**Definition 1.8.** We say that a monoidal category $(\mathsf{D}, \bullet)$ is *abelian* if the category $\mathsf{D}$ is abelian and for each object $A$ the functors

$$A \bullet (-) \quad \text{and} \quad (-) \bullet A$$

are exact.

## 1.2. Hopf monoids

A monoidal category allows one to define monoids and comonoids; if the category is braided, then one can also define bimonoids and Hopf monoids as well as different types of monoids. These objects along with their notations are summarized in Table 1.1.

### 1.2.1. Monoids and comonoids.

**Definition 1.9.** A *monoid* in a monoidal category $(\mathsf{C}, \bullet)$ is a triple $(A, \mu, \iota)$ where

$$\mu \colon A \bullet A \to A \quad \text{and} \quad \iota \colon I \to A$$

TABLE 1.1. Categories of "monoids" in monoidal categories.

| Category | Description | Category | Description |
|---|---|---|---|
| $\mathsf{Mon}(\mathsf{C})$ | Monoids | $\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$ | Com. bimonoids |
| $\mathsf{Comon}(\mathsf{C})$ | Comonoids | ${}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C})$ | Cocom. bimonoids |
| $\mathsf{Bimon}(\mathsf{C})$ | Bimonoids | ${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$ | Com. & cocom. bimonoids |
| $\mathsf{Hopf}(\mathsf{C})$ | Hopf monoids | $\mathsf{Hopf}^{\mathsf{co}}(\mathsf{C})$ | Com. Hopf monoids |
| $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$ | Com. monoids | ${}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{C})$ | Cocom. Hopf monoids |
| ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})$ | Cocom. comonoids | ${}^{\mathsf{co}}\mathsf{Hopf}^{\mathsf{co}}(\mathsf{C})$ | Com. & cocom. Hopf monoids |

(the product and the unit) satisfy the associativity and unit axioms, which state that the following diagrams commute.

$$\begin{tikzcd} A \bullet A \bullet A \arrow[r, "\mathrm{id}\bullet\mu"] \arrow[d, "\mu\bullet\mathrm{id}"'] & A \bullet A \arrow[d, "\mu"] \\ A \bullet A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} I \bullet A \arrow[r, "\iota\bullet\mathrm{id}"] & A \bullet A \arrow[d, "\mu"] & A \bullet I \arrow[l, "\mathrm{id}\bullet\iota"'] \\ & A \arrow[ul, "\lambda", "\cong"'] \arrow[ur, "\rho"', "\cong"] & \end{tikzcd}$$

A morphism $(A, \mu, \iota) \to (A', \mu', \iota')$ of monoids is a map $A \to A'$ which commutes with $\mu$ and $\mu'$, and $\iota$ and $\iota'$.

Similarly a *comonoid* in a monoidal category $(\mathsf{C}, \bullet)$ is a triple $(C, \Delta, \epsilon)$ where

$$\Delta\colon C \to C \bullet C \quad \text{and} \quad \epsilon\colon C \to I$$

(the coproduct and the counit) satisfy the coassociativity and counit axioms. These are obtained from the monoid axioms by replacing $\mu$ by $\Delta$ and $\iota$ by $\epsilon$, and reversing the arrows with those labels. A morphism $(C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ of comonoids is a map $C \to C'$ which commutes with $\Delta$ and $\Delta'$, and $\epsilon$ and $\epsilon'$.

We denote the categories of monoids and of comonoids in $(\mathsf{C}, \bullet)$ by $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ respectively. The notions of monoid and comonoid are dual in the sense that $\mathsf{Mon}(\mathsf{C})$ is equivalent to $\mathsf{Comon}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$.

A *nonunital monoid* is defined as in Definition 1.9, but omitting all references to the unit object $I$ of $\mathsf{C}$. *Noncounital comonoids* are the dual notion. These notions can be defined in any *nonunital monoidal category* (omit all references to $I$ in Definition 1.1). These objects are encountered in a small number of occasions in this monograph.

### 1.2.2. Bimonoids.

**Definition 1.10.** A *bimonoid* in a lax braided monoidal category $(\mathsf{C}, \bullet, \beta)$ is a quintuple $(H, \mu, \iota, \Delta, \epsilon)$ where $(H, \mu, \iota)$ is a monoid, $(H, \Delta, \epsilon)$ is a comonoid, and the two structures are compatible in the sense that the following four diagrams commute.

$$\begin{tikzcd} H \bullet H \bullet H \bullet H \arrow[rr, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"] & & H \bullet H \bullet H \bullet H \arrow[d, "\mu\bullet\mu"] \\ H \bullet H \arrow[u, "\Delta\bullet\Delta"] \arrow[r, "\mu"'] & H \arrow[r, "\Delta"'] & H \bullet H \end{tikzcd} \tag{1.9}$$

$$\begin{tikzcd} H \bullet H \arrow[r, "\epsilon\bullet\epsilon"] \arrow[d, "\mu"'] & I \bullet I \arrow[d, "\lambda_I^{-1}"] \\ H \arrow[r, "\epsilon"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\iota"] \arrow[d, "\lambda_I"'] & H \arrow[d, "\Delta"] \\ I \bullet I \arrow[r, "\iota\bullet\iota"'] & H \bullet H \end{tikzcd} \tag{1.10}$$

$$\begin{tikzcd} & H \arrow[dr, "\epsilon"] & \\ I \arrow[ur, "\iota"] \arrow[rr, equal] & & I \end{tikzcd} \tag{1.11}$$

A morphism of bimonoids is a morphism of the underlying monoids and comonoids.

In a braided monoidal category $(\mathsf{C}, \bullet, \beta)$, if $(A_1, \mu_1)$ and $(A_2, \mu_2)$ are monoids, then so is $A_1 \bullet A_2$, with structure maps

$$\begin{gathered} A_1 \bullet A_2 \bullet A_1 \bullet A_2 \xrightarrow{\mathrm{id}\bullet\beta\bullet\mathrm{id}} A_1 \bullet A_1 \bullet A_2 \bullet A_2 \xrightarrow{\mu_1\bullet\mu_2} A_1 \bullet A_2 \\ I \xrightarrow{\lambda_I=\rho_I} I \bullet I \xrightarrow{\iota_1\bullet\iota_2} A_1 \bullet A_2. \end{gathered} \tag{1.12}$$

Dually, if $(C_1, \Delta_1)$ and $(C_2, \Delta_2)$ are comonoids, then so is $C_1 \bullet C_2$.

In this manner, $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ are monoidal categories. One can then give the following alternative description for bimonoids.

**Proposition 1.11.** *A bimonoid is an object $H$ in a braided monoidal category with maps*

$$\mu\colon H \bullet H \to H \qquad \Delta\colon H \to H \bullet H$$
$$\iota\colon I \to H \qquad \epsilon\colon H \to I$$

*such that $(H, \mu, \iota)$ is a monoid, $(H, \Delta, \epsilon)$ is a comonoid, and $\mu$ and $\iota$ are morphisms of comonoids, or equivalently, $\Delta$ and $\epsilon$ are morphisms of monoids.*

We denote the category of bimonoids in $(\mathsf{C}, \bullet, \beta)$ by $\mathsf{Bimon}(\mathsf{C})$.

### 1.2.3. Modules and comodules.

**Definition 1.12.** Let $(A, \mu, \iota)$ be a monoid in $(\mathsf{C}, \bullet)$. A *left $A$-module* is a pair $(M, \chi)$ where

$$\chi\colon A \bullet M \to M$$

satisfies the usual associativity and unit axioms. A *right $A$-module* is defined similarly in terms of a structure map $M \bullet A \to M$.

A morphism $(M, \chi) \to (M', \chi')$ of left $A$-modules is a map $M \to M'$ which commutes with $\chi$ and $\chi'$. We denote the category of left $A$-modules in $(\mathsf{C}, \bullet)$ by $\mathsf{Mod}_A(\mathsf{C})$.

Given two monoids $A_1$ and $A_2$ in $(\mathsf{C}, \bullet)$, an $A_1$-$A_2$-*bimodule* is a triple $(M, \chi_1, \chi_2)$ where $(M, \chi_1)$ is a left $A_1$-module, $(M, \chi_2)$ is a right $A_2$-module, and in addition

$$\begin{tikzcd} A_1 \bullet M \bullet A_2 \arrow[r, "\chi_1\bullet\mathrm{id}"] \arrow[d, "\mathrm{id}\bullet\chi_2"'] & M \bullet A_2 \arrow[d, "\chi_2"] \\ A_1 \bullet M \arrow[r, "\chi_1"'] & M \end{tikzcd}$$

commutes. We refer to $A$-$A$-bimodules simply as $A$-bimodules.

Assume now that $(\mathsf{C}, \bullet, \beta)$ is a braided monoidal category. Let $A_i$ be a monoid in $(\mathsf{C}, \bullet)$, $i = 1, 2$, and consider the monoid $A_1 \bullet A_2$ defined in (1.12). If $(M_i, \chi_i)$ is a left $A_i$-module, then $M_1 \bullet M_2$ is a left $A_1 \bullet A_2$-module with structure map

$$A_1 \bullet A_2 \bullet M_1 \bullet M_2 \xrightarrow{\mathrm{id}\bullet\beta\bullet\mathrm{id}} A_1 \bullet M_1 \bullet A_2 \bullet M_2 \xrightarrow{\chi_1\bullet\chi_2} M_1 \bullet M_2.$$

It follows that if $H$ is a bimonoid, and $(M_1, \chi_1)$ and $(M_2, \chi_2)$ are left $H$-modules, then so is $M_1 \bullet M_2$, with structure map

$$H \bullet M_1 \bullet M_2 \xrightarrow{\Delta \bullet \mathrm{id} \bullet \mathrm{id}} H \bullet H \bullet M_1 \bullet M_2 \xrightarrow{\mathrm{id} \bullet \beta \bullet \mathrm{id}} H \bullet M_1 \bullet H \bullet M_2 \xrightarrow{\chi_1 \bullet \chi_2} M_1 \bullet M_2.$$

In addition, the unit object $I$ is a left $H$-module with structure map

$$H \bullet I \xrightarrow{\rho_H^{-1}} H \xrightarrow{\epsilon} I.$$

In this manner, the category $\mathsf{Mod}_H(\mathsf{C})$ is monoidal. A monoid in $\mathsf{Mod}_H(\mathsf{C})$ is called an *$H$-module-monoid*. A comonoid in $\mathsf{Mod}_H(\mathsf{C})$ is called an *$H$-module-comonoid*.

Let $C$ be a comonoid. Dualizing the above definitions, we obtain the notions of left $C$-comodule, right $C$-comodule, and $C$-$D$-bicomodule, where $D$ is another comonoid. Let $\mathsf{Comod}^C(\mathsf{C})$ be the category of left $C$-comodules. If $H$ is a bimonoid, $\mathsf{Comod}^H(\mathsf{C})$ is monoidal, and one has the notions of *$H$-comodule-monoid* and *$H$-comodule-comonoid*.

### 1.2.4. Convolution monoids.

Let $(\mathsf{C}, \bullet)$ be a monoidal category.

**Definition 1.13.** For $C$ a comonoid and $A$ a monoid in $\mathsf{C}$, define the *convolution monoid* as the set $\mathrm{Hom}(C, A)$ of all maps in $\mathsf{C}$ from $C$ to $A$ with the following product.

For $f, g \in \mathrm{Hom}(C, A)$, we let the product $f * g$ be the composite morphism

$$C \xrightarrow{\Delta} C \bullet C \xrightarrow{f \bullet g} A \bullet A \xrightarrow{\mu} A.$$

This is an associative product called *convolution*. The map $\iota\epsilon \colon C \to A$ serves as the unit for this product and is called the *convolution unit*. The set $\mathrm{Hom}(C, A)$ is thus an ordinary monoid, that is, a monoid in the monoidal category $(\mathsf{Set}, \times)$ of sets with Cartesian product.

If $\mathsf{C}$ is a linear monoidal category, then $\mathrm{Hom}(C, A)$ has the structure of an (associative) algebra. This is called the *convolution algebra*.

**Proposition 1.14.** *Let $C$ and $C'$ be comonoids and $A$ and $A'$ be monoids in $\mathsf{C}$. Let*

$$j \colon C' \to C \quad \textit{and} \quad k \colon A \to A'$$

*be a morphism of comonoids and a morphism of monoids, respectively. Then the maps*

$$\mathrm{Hom}(C, A) \to \mathrm{Hom}(C', A), \qquad f \mapsto fj$$

*and*

$$\mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A'), \qquad f \mapsto kf$$

*are morphisms of convolution monoids.*

The proof is straightforward.

**1.2.5. Hopf monoids.** Let $(\mathsf{C}, \bullet, \beta)$ be a lax braided monoidal category. For a bimonoid $H$ in $\mathsf{C}$, consider the convolution monoid $\mathrm{End}(H) := \mathrm{Hom}(H, H)$.

**Definition 1.15.** A *Hopf monoid* in $\mathsf{C}$ is a bimonoid $H$ for which the identity map $\mathrm{id}\colon H \to H$ is invertible in the convolution monoid $\mathrm{End}(H)$. Explicitly, there must exist a map $\mathrm{s}\colon H \to H$ such that

$$\begin{tikzcd} H \bullet H \arrow[rr, "\mathrm{id} \bullet \mathrm{s}"] & & H \bullet H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\epsilon"'] & I \arrow[r, "\iota"'] & H \end{tikzcd} \qquad \begin{tikzcd} H \bullet H \arrow[rr, "\mathrm{s} \bullet \mathrm{id}"] & & H \bullet H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\epsilon"'] & I \arrow[r, "\iota"'] & H \end{tikzcd} \tag{1.13}$$

commute. The map $\mathrm{s}$ is the *antipode* of $H$.

The antipode of a bimonoid $H$ may exist or not, but if it does, then it is unique (and $H$ is a Hopf monoid).

**Proposition 1.16.** *Let $H$ and $H'$ be Hopf monoids. A morphism of bimonoids $H \to H'$ necessarily commutes with the antipodes and the convolution units.*

PROOF. We prove the first claim. Let $k\colon H \to H'$ be a morphism of bimonoids. According to Proposition 1.14, we have morphisms of convolution monoids

$$\mathrm{Hom}(H', H') \to \mathrm{Hom}(H, H'), \qquad f \mapsto fk$$

and

$$\mathrm{Hom}(H, H) \to \mathrm{Hom}(H, H'), \qquad f \mapsto kf.$$

It follows that both $\mathrm{s}'\,k$ and $k\,\mathrm{s}$ are the inverse of $k$ in $\mathrm{Hom}(H, H')$, so they must coincide. □

A morphism of Hopf monoids $H \to H'$ is defined to be a morphism of the underlying bimonoids. In view of Proposition 1.16, such morphisms preserve the extra structure present in a Hopf monoid.

We denote the category of Hopf monoids in $(\mathsf{C}, \bullet, \beta)$ by $\mathsf{Hopf}(\mathsf{C})$.

**1.2.6. Commutative monoids.** In addition to playing a role in the definition of bimonoids, the braiding in a braided monoidal category is related to another aspect: the possibility of defining different types of monoids. Presently, we discuss the well-known example of commutative monoids. Lie monoids are discussed in Section 1.2.10.

**Definition 1.17.** A *commutative monoid* (resp. *cocommutative comonoid*) in a braided monoidal category $(\mathsf{C}, \bullet, \beta)$ is a monoid $A$ (resp. comonoid $C$) such that the left-hand (resp. right-hand) diagram below commutes.

$$\begin{tikzcd} A \bullet A \arrow[rr, "\beta"] \arrow[dr, "\mu"'] & & A \bullet A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} C \bullet C \arrow[rr, "\beta"] & & C \bullet C \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

A morphism of commutative monoids (resp. cocommutative comonoids) is a morphism of the underlying monoids (resp. comonoids).

We denote the category of commutative monoids and cocommutative comonoids in $(\mathsf{C}, \bullet, \beta)$ by $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$ and ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})$ respectively. The definition implies that they are full subcategories of $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ respectively.

We say that a bimonoid or Hopf monoid is *(co)commutative* if its underlying (co)monoid is (co)commutative. Following the above notation, this defines categories $\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$, ${}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C})$ and ${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$ and three more with bimonoids replaced by Hopf monoids as shown in Table 1.1.

**1.2.7. Iterations of the monoid and comonoid constructions.** Let $(\mathsf{C}, \bullet, \beta)$ be a braided monoidal category. As mentioned in Section 1.2.2, the categories $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ are themselves monoidal, and so we may consider monoids and comonoids therein. We have

$$\begin{gathered}\mathsf{Mon}(\mathsf{Comon}(\mathsf{C})) \cong \mathsf{Bimon}(\mathsf{C}) \cong \mathsf{Comon}(\mathsf{Mon}(\mathsf{C})),\\ \mathsf{Mon}(\mathsf{Mon}(\mathsf{C})) \cong \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}),\\ \mathsf{Comon}(\mathsf{Comon}(\mathsf{C})) \cong {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C}).\end{gathered} \tag{1.14}$$

The equivalences on the first row follow from Proposition 1.11. The other two follow from the *Eckmann–Hilton argument*. We prove more general results later (Propositions 6.29 and 6.36). This provides alternative descriptions of bimonoids and (co)commutative (co)monoids.

However, the monoidal categories $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ may fail to be braided, and the category $\mathsf{Bimon}(\mathsf{C})$ may fail to be monoidal.

On the other hand, suppose that $\beta$ is a symmetry. In this case, if $A$ and $B$ are monoids, then $\beta_{A,B} \colon A \bullet B \to B \bullet A$ is a morphism of monoids with respect to the monoid structure (1.12). It follows that $\mathsf{Mon}(\mathsf{C})$ is a symmetric monoidal category. Dually, $\mathsf{Comon}(\mathsf{C})$ is a symmetric monoidal category. Iterating these results and applying (1.14), we deduce that $\mathsf{Bimon}(\mathsf{C})$, $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$, and ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})$ are symmetric monoidal categories as well. In particular, the tensor product of two bimonoids is again a bimonoid, and the tensor product of two (co)commutative (co)monoids is again (co)commutative. These statements can also be deduced from later results on monoidal properties of the tensor product functor (Propositions 3.74 and 3.75) plus Propositions 3.31 and 3.37.

For a nonsymmetric braiding $\beta$ on $\mathsf{C}$, these assertions fail in general. In the symmetric case the monoid and comonoid constructions can be further iterated, but no new categories are obtained beyond those of (co)commutative (co, bi)monoids.

We now extend the above considerations to Hopf monoids. Let $(\mathsf{C}, \bullet, \beta)$ be a braided monoidal category. In general, $\mathsf{Hopf}(\mathsf{C})$ fails to be a monoidal category. However, if $\beta$ is a symmetry, then the tensor product $A \bullet B$ of two Hopf monoids $A$ and $B$ s another Hopf monoid. The bimonoid structure is as in Section 1.2.2 and the antipode is $\mathrm{s}_A \bullet \mathrm{s}_B$. This follows from later results on monoidal properties of the tensor product functor (Propositions 3.74 and 3.75) plus Proposition 3.50, but it can also be checked directly. It follows that if $\mathsf{C}$ is a symmetric monoidal category, then so is $\mathsf{Hopf}(\mathsf{C})$.

**1.2.8. Examples.** This monograph contains many examples of Hopf monoids. Here we list a few basic examples with pointers to more.

**Example 1.18.** Consider the symmetric monoidal categories in Example 1.3. A monoid in $(\mathsf{Set}, \times)$ is a monoid (as defined in any elementary algebra class). A monoid in $(\mathsf{Vec}, \otimes)$ is an algebra. The category of graded vector spaces and (co, bi)monoids therein along with related categories are discussed in Chapter 2. The analogous discussion for species is given in Chapter 8.

**Example 1.19.** Let $(\mathsf{C}, \times, J)$ be a cartesian monoidal category, as in Example 1.4. It is easy to see that every object $C$ of $\mathsf{C}$ has a unique comonoid structure with respect to $\times$. Indeed, the counit $\epsilon\colon C \to J$ is the unique map to the terminal object $J$, and the coproduct $\Delta\colon C \to C \times C$ is the *diagonal*: in the notation of Section A.1.1,

$$\Delta = (\mathrm{id}_C, \mathrm{id}_C).$$

Moreover, $(C, \Delta, \epsilon)$ is cocommutative.

Dually, any object in a cocartesian monoidal category $(\mathsf{C}, \amalg, I)$ has a unique monoid structure, and any object in a bicartesian monoidal category $(\mathsf{C}, \oplus, Z)$ has a unique bimonoid structure.

This yields equivalences of categories

$$\begin{gathered}
{}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C}, \times) \cong \mathsf{Comon}(\mathsf{C}, \times) \cong \mathsf{C}, \\
{}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C}, \times) \cong \mathsf{Bimon}(\mathsf{C}, \times) \cong \mathsf{Mon}(\mathsf{C}, \times), \\
\mathsf{Mon}^{\mathsf{co}}(\mathsf{C}, \amalg) \cong \mathsf{Mon}(\mathsf{C}, \amalg) \cong \mathsf{C}, \\
\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}, \amalg) \cong \mathsf{Bimon}(\mathsf{C}, \amalg) \cong \mathsf{Comon}(\mathsf{C}, \amalg), \\
{}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}, \oplus) \cong \mathsf{Bimon}(\mathsf{C}, \oplus) \cong \mathsf{C}.
\end{gathered}$$

**1.2.9. The opposite monoid.** We now show how a braiding can be used to twist the (co)product of a (co)monoid. This provides a more general context for the preceding discussion on commutativity.

**Proposition 1.20.** *Let $(\mathsf{C}, \bullet, \beta)$ be a braided monoidal category. If $A = (A, \mu, \iota)$ is a monoid in $(\mathsf{C}, \bullet)$, then so are*

$$A^{\mathrm{op}} := (A, \mu\beta, \iota) \qquad \text{and} \qquad {}^{\mathrm{op}}A := (A, \mu\beta^{-1}, \iota).$$

*Similarly, if $C = (C, \Delta, \epsilon)$ is a comonoid in $(\mathsf{C}, \bullet)$, then so are*

$$C^{\mathrm{cop}} := (C, \beta^{-1}\Delta, \epsilon) \qquad \text{and} \qquad {}^{\mathrm{cop}}C := (C, \beta\Delta, \epsilon).$$

One verifies that $(A, \mu\beta, \iota)$ is a monoid directly. The other assertions then follow by passing to the braided monoidal categories

$$(\mathsf{C}, \bullet, \beta^{-1}), \quad (\mathsf{C}^{\mathrm{op}}, \bullet, (\beta^{-1})^{\mathrm{op}}), \quad \text{or} \quad (\mathsf{C}^{\mathrm{op}}, \bullet, \beta^{\mathrm{op}}).$$

The four statements in each set below are clearly equivalent.

$$\begin{aligned}
&A \text{ is a commutative monoid;} \\
&\mathrm{id}\colon A \to A^{\mathrm{op}} \text{ is a morphism of monoids;} \\
&\mathrm{id}\colon A \to {}^{\mathrm{op}}A \text{ is a morphism of monoids;} \\
&\mu = \mu\beta.
\end{aligned} \tag{1.15}$$

$$\begin{aligned}
&C \text{ is a cocommutative comonoid;} \\
&\mathrm{id}\colon C \to C^{\mathrm{cop}} \text{ is a morphism of comonoids;} \\
&\mathrm{id}\colon C \to {}^{\mathrm{cop}}C \text{ is a morphism of comonoids;} \\
&\Delta = \beta\Delta.
\end{aligned} \tag{1.16}$$

In the context of bimonoids, these constructions can be combined as follows.

**Proposition 1.21.** *Let* $H = (H, \mu, \iota, \Delta, \epsilon)$ *be a bimonoid in* $(\mathsf{C}, \bullet, \beta)$*. Then*

$$H^{\mathrm{cop}} := (H, \mu, \iota, \beta^{-1}\Delta, \epsilon) \qquad \textit{and} \qquad {}^{\mathrm{op}}H := (H, \mu\beta^{-1}, \iota, \Delta, \epsilon)$$

*are bimonoids in* $(\mathsf{C}, \bullet, \beta^{-1})$*, and*

$$H^{\mathrm{op,cop}} := (H, \mu\beta, \iota, \beta^{-1}\Delta, \epsilon) \qquad \textit{and} \qquad {}^{\mathrm{op,cop}}H := (H, \mu\beta^{-1}, \iota, \beta\Delta, \epsilon)$$

*are bimonoids in* $(\mathsf{C}, \bullet, \beta)$*.*

We refer to these collectively as the *op* and *cop constructions*. Note that applying the ${}^{\mathrm{op}}(-)$ construction to the bimonoid $H^{\mathrm{cop}}$ in $(\mathsf{C}, \bullet, \beta^{-1})$ yields $H^{\mathrm{op,cop}}$. Thus, formally,

$$H^{\mathrm{op,cop}} = {}^{\mathrm{op}}(H^{\mathrm{cop}}) \qquad \text{and} \qquad {}^{\mathrm{op,cop}}H = ({}^{\mathrm{op}}H)^{\mathrm{cop}}.$$

If $\beta$ is a symmetry, then

$${}^{\mathrm{op,cop}}H = H^{\mathrm{op,cop}};$$

this is the case most often considered in the literature [191, Proposition III.2.3].

**Proposition 1.22.** *Let* $H$ *be a bimonoid in* $(\mathsf{C}, \bullet, \beta)$*.*

(i) *All three convolution products on the space*

$$\mathrm{End}(H) = \mathrm{End}(H^{\mathrm{op,cop}})^{\mathrm{op}} = \mathrm{End}({}^{\mathrm{op,cop}}H)^{\mathrm{op}}$$

*coincide.*

(ii) *If one of* $H$*,* $H^{\mathrm{op,cop}}$*, or* ${}^{\mathrm{op,cop}}H$ *is a Hopf monoid, then so are the other two, and all three share the same antipode.*

(iii) *Assume* $H$ *is a Hopf monoid. Then the antipode is a morphism of Hopf monoids in two ways:* $\mathrm{s}\colon H \to H^{\mathrm{op,cop}}$ *and* $\mathrm{s}\colon {}^{\mathrm{op,cop}}H \to H$*.*

Proof. The first statement is straightforward, and the second follows. For the third statement, we check below that $\mathrm{s}\colon H \to H^{\mathrm{op}}$ is a morphism of monoids. From here, passing to the category $(\mathsf{C}^{\mathrm{op}}, \bullet, \beta^{\mathrm{op}})$ we deduce that $\mathrm{s}\colon {}^{\mathrm{cop}}H \to H$ is a morphism of comonoids. Then, the naturality of $\beta$ allows us to deduce that $\mathrm{s}\colon {}^{\mathrm{op}}H \to H$ is a morphism of monoids and $\mathrm{s}\colon H \to H^{\mathrm{cop}}$ is a morphism of comonoids. This will complete the proof.

We now prove that $\mathrm{s}\colon H \to H^{\mathrm{op}}$ is a morphism of monoids. We need to show that

$$\mathrm{s}\,\mu = \mu\beta(\mathrm{s} \bullet \mathrm{s}) \qquad \text{and} \qquad \mathrm{s}\,\iota = \iota.$$

By Proposition 1.11, $\mu\colon H \bullet H \to H$ and $\iota\colon I \to H$ are morphisms of comonoids. Hence, by Proposition 1.14,

$$\mathrm{Hom}(H, H) \to \mathrm{Hom}(H \bullet H, H), \qquad f \mapsto f\mu,$$

and

$$\mathrm{Hom}(H, H) \to \mathrm{Hom}(I, H), \qquad f \mapsto f\iota,$$

are morphisms of convolution monoids. Therefore, $\mathrm{s}\,\mu$ is the convolution inverse of $\mu$ in $\mathrm{Hom}(H \bullet H, H)$, and $\mathrm{s}\,\iota$ is the convolution inverse of $\iota$ in $\mathrm{Hom}(I, H)$. The latter statement implies $\mathrm{s}\,\iota = \iota$, since $\iota$ is the unit element of $\mathrm{Hom}(I, H)$, and hence is its own inverse. The former implies that to obtain $\mathrm{s}\,\mu = \mu\beta(\mathrm{s} \bullet \mathrm{s})$, it suffices to check that

$$\big(\mu\beta(\mathrm{s} \bullet \mathrm{s})\big) * \mu = \iota(\epsilon \bullet \epsilon).$$

This follows from the commutativity of the following diagram.

$$
\begin{tikzcd}
 & H^4 \arrow[drr, "\beta\mathrm{id}^2"] & & & & \\
H^4 \arrow[ur, "\mathrm{s}^2\,\mathrm{id}^2"] \arrow[r, "\beta\mathrm{id}^2"] & H^4 \arrow[rr, "\mathrm{s}^2\,\mathrm{id}^2"] \arrow[dr, "\mathrm{s}\,\mathrm{id}^3"] & & H^4 \arrow[drr, "\mu^2"] \arrow[dr, "\mathrm{id}\mu\mathrm{id}"] & & \\
H^4 \arrow[u, "\mathrm{id}\beta\mathrm{id}"] \arrow[ur, "\beta_{H^2,H}\mathrm{id}"] & & H^4 \arrow[ur, "\mathrm{id}\,\mathrm{s}\,\mathrm{id}^2"] & & H^3 \arrow[dd, "\mu\mathrm{id}"] & H^2 \arrow[ddd, "\mu"] \\
 & H^3 \arrow[uu, "\mathrm{id}\Delta\mathrm{id}"] \arrow[r, "\mathrm{s}\,\mathrm{id}^2"] & H^3 \arrow[u, "\mathrm{id}\Delta\mathrm{id}"] \arrow[r, "\mathrm{id}\epsilon\mathrm{id}"] & HIH \arrow[ur, "\mathrm{id}\iota\mathrm{id}"] \arrow[dr, "\rho^{-1}\mathrm{id}"] & & \\
H^3 \arrow[uu, "\Delta\mathrm{id}^2"] \arrow[ur, "\beta\mathrm{id}"] \arrow[r, "\mathrm{id}\,\mathrm{s}\,\mathrm{id}"] & H^3 \arrow[ur, "\beta\mathrm{id}"] \arrow[r, "\epsilon\mathrm{id}^2"] \arrow[dr, "\mathrm{id}\mu"] & IHH \arrow[ur, "\beta\mathrm{id}"] \arrow[rr, "\lambda^{-1}\mathrm{id}=\lambda^{-1}"] \arrow[dr, "\mathrm{id}\mu"] & & H^2 \arrow[dr, "\mu"] & \\
H^2 \arrow[u, "\mathrm{id}\Delta"] \arrow[r, "\mathrm{id}\epsilon"] \arrow[drr, "\epsilon^2"'] & HI \arrow[r, "\mathrm{id}\iota"] \arrow[dr, "\epsilon\mathrm{id}"] & H^2 \arrow[r, "\epsilon\mathrm{id}"] & IH \arrow[rr, "\lambda^{-1}"] & & H \\
 & & II \arrow[ur, "\mathrm{id}\iota"] \arrow[rr, "\lambda^{-1}"'] & & I \arrow[ur, "\iota"'] &
\end{tikzcd}
$$

Above, the tensor product has been omitted from the notation. Thus, $H^3$ stands for $H \bullet H \bullet H$, and similarly for the rest. For the commutativity of the various smaller diagrams, we employ the associativity (twice) and unitality of $\mu$ and $\iota$ (Definition 1.9), one of the antipode axioms (the second diagram in (1.13), twice), part of the braiding axioms (the first diagram in (1.5) and the second in (1.7)), the first diagram in (1.4), plus naturality of the braiding and functoriality of the tensor product. This completes the proof. $\square$

**Proposition 1.23.** *Suppose both bimonoids $H$ and ${}^{\mathrm{op}}H$ are Hopf monoids. Let* $\mathrm{s}$ *and* $\overline{\mathrm{s}}$ *be the respective antipodes. Then* $\mathrm{s}$ *and* $\overline{\mathrm{s}}$ *are inverse maps* (*with respect to composition*):

$$\mathrm{s}\,\overline{\mathrm{s}} = \mathrm{id} = \overline{\mathrm{s}}\,\mathrm{s}\,.$$

*The same result holds replacing ${}^{\mathrm{op}}H$ with $H^{\mathrm{cop}}$.*

Proof. By Proposition 1.22, $\mathrm{s}\colon {}^{\mathrm{op}}H \to H$ is a morphism of monoids. Hence, by Proposition 1.14, the map

$$\mathrm{Hom}({}^{\mathrm{op}}H, {}^{\mathrm{op}}H) \to \mathrm{Hom}({}^{\mathrm{op}}H, H), \qquad f \mapsto \mathrm{s}\,f$$

is a morphism of convolution monoids. This map sends $\mathrm{id}_{{}^{\mathrm{op}}H}$ to $\mathrm{s}$ and $\overline{\mathrm{s}}$ to $\mathrm{s}\,\overline{\mathrm{s}}$. Therefore, $\mathrm{s}$ and $\mathrm{s}\,\overline{\mathrm{s}}$ are convolution inverses in $\mathrm{Hom}({}^{\mathrm{op}}H, H)$. But ${}^{\mathrm{op}}H = H$ as comonoids, so $\mathrm{Hom}({}^{\mathrm{op}}H, H) = \mathrm{Hom}(H, H)$ as convolution monoids. Thus, $\mathrm{s}\,\overline{\mathrm{s}}$ is the convolution inverse of $\mathrm{s}$ in $\mathrm{Hom}(H, H)$, and so

$$\mathrm{s}\,\overline{\mathrm{s}} = \mathrm{id}.$$

Using that ${}^{\mathrm{op}}({}^{\mathrm{op}}H) = H$, we deduce $\overline{\mathrm{s}}\,\mathrm{s} = \mathrm{id}$. $\square$

**Corollary 1.24.** *Let $H$ be a Hopf monoid which is either commutative or cocommutative. Then the antipode* $\mathrm{s}$ *is an involution with respect to composition*: $\mathrm{s}\,\mathrm{s} = \mathrm{id}$.

PROOF. Suppose $H$ is commutative. Then $H = {}^{\mathrm{op}}H$, so ${}^{\mathrm{op}}H$ is a Hopf monoid with antipode $\overline{\mathrm{s}} = \mathrm{s}$. By Proposition 1.23, $\mathrm{s}$ is its own inverse.

The cocommutative case follows similarly, using $H^{\mathrm{cop}}$. $\square$

**1.2.10. Lie monoids.** After monoids and commutative monoids, we turn our attention to Lie monoids. The words commutative, associative and Lie go hand in hand. A simple connection between the three is given by (1.17) below. Another well-known connection is provided by the Cartier–Milnor–Moore theorem (Section 11.9).

**Definition 1.25.** Let $(\mathsf{C}, \bullet, \beta)$ be a linear symmetric monoidal category (possibly without a unit). A *Lie monoid* in $(\mathsf{C}, \bullet, \beta)$ is a pair $(L, \gamma)$ where

$$\gamma \colon L \bullet L \to L$$

satisfies

$$\gamma + \gamma\beta_{L,L} = 0 \qquad \text{and} \qquad \gamma(\gamma \bullet \mathrm{id})(\mathrm{id} + \xi + \xi^2) = 0$$

where $\xi$ denotes the composite

$$L \bullet L \bullet L \xrightarrow{\mathrm{id}\bullet\beta_{L,L}} L \bullet L \bullet L \xrightarrow{\beta_{L,L}\bullet\mathrm{id}} L \bullet L \bullet L.$$

A morphism $(L, \gamma) \to (L', \gamma')$ is a map $L \to L'$ which commutes with $\gamma$ and $\gamma'$.

Let $\mathsf{Lie}(\mathsf{C})$ denote the category of Lie monoids in $\mathsf{C}$. Note that $\mathsf{C}$ is assumed to be symmetric.

**Proposition 1.26.** *Let $(\mathsf{C}, \bullet, \beta)$ be as before and let $(A, \mu)$ be a monoid (not necessarily unital). Let*

$$\gamma := \mu - \mu\beta_{A,A} \colon A \bullet A \to A.$$

*Then $(A, \gamma)$ is a Lie monoid.*

More precisely, one has the diagram of functors

$$\mathsf{Mon}^{\mathsf{co}}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{C}) \to \mathsf{Lie}(\mathsf{C}). \tag{1.17}$$

In particular, every commutative monoid is a monoid and every monoid is a Lie monoid.

**Remark 1.27.** Definition 1.25 can be stated in any linear braided monoidal category, not necessarily symmetric. Note that Proposition 1.26 fails at this level of generality. For this and other reasons, we restrict the consideration of Lie monoids to the context of linear symmetric monoidal categories.

## 1.3. The internal Hom functor

Let $(\mathsf{C}, \bullet)$ be a monoidal category. An *internal Hom* for $(\mathsf{C}, \bullet)$ is a functor

$$\mathcal{H}^\bullet \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$$

such that for any objects $A$, $B$, and $C$ in $\mathsf{C}$, there is a natural bijection

$$\mathrm{Hom}_{\mathsf{C}}(A \bullet B, C) \cong \mathrm{Hom}_{\mathsf{C}}\big(A, \mathcal{H}^\bullet(B, C)\big). \tag{1.18}$$

If we let $A$ be the unit object, then we obtain

$$\mathrm{Hom}_{\mathsf{C}}(B, C) \cong \mathrm{Hom}_{\mathsf{C}}\big(I, \mathcal{H}^\bullet(B, C)\big).$$

This explains the motivation behind the internal Hom terminology. Thus in many cases, the left hand side above leads to a description of the internal Hom.

The following is a list of examples considered in this monograph. The first two illustrate the preceding point.

- For the category $(\mathsf{Set}, \times)$, the internal Hom $\mathcal{H}^{\times}(X, Y)$ is the set of all maps from $X$ to $Y$.
- For the category $(\mathsf{Vec}, \otimes)$, the internal Hom $\mathcal{H}^{\otimes}(V, W)$ is the space of all linear maps from $V$ to $W$. This follows from (A.6).
- Internal Hom for monoidal categories related to graded vector spaces are discussed in Section 2.1.5.
- Internal Hom for monoidal categories related to species are discussed in Sections 8.11.2, 8.13.3 and B.5.2.

**Proposition 1.28.** *For any $M$, the object*

$$\mathcal{E}^{\bullet}(M) := \mathcal{H}^{\bullet}(M, M)$$

*is a monoid in $(\mathsf{C}, \bullet)$. If $A$ is a monoid, then a $A$-module structure on an object $M$ is equivalent to a morphism of monoids*

$$A \to \mathcal{E}^{\bullet}(M).$$

Monoids and modules over a monoid are defined in Section 1.2.1. The proof of this proposition is standard [259, pp. 26–27]. Some details are given below.

Proof. By letting $A = \mathcal{E}^{\bullet}(M)$ and $B = C = M$ in (1.18), the identity morphism on $\mathcal{E}^{\bullet}(M)$ yields a morphism

$$\mathcal{E}^{\bullet}(M) \bullet M \to M.$$

By tensoring on the left by $\mathcal{E}^{\bullet}(M)$, we obtain

$$\mathcal{E}^{\bullet}(M) \bullet \mathcal{E}^{\bullet}(M) \bullet M \to \mathcal{E}^{\bullet}(M) \bullet M \to M.$$

Now applying (1.18) with $A = \mathcal{E}^{\bullet}(M) \bullet \mathcal{E}^{\bullet}(M)$ and $B = C = M$, we obtain

$$\mathcal{E}^{\bullet}(M) \bullet \mathcal{E}^{\bullet}(M) \to \mathcal{E}^{\bullet}(M).$$

The unit constraint $I \bullet M \to M$, by applying (1.18), yields a morphism

$$I \to \mathcal{E}^{\bullet}(M).$$

One checks that the above structure maps turn $\mathcal{E}^{\bullet}(M)$ into a monoid.

By letting $B = C = M$ in (1.18), we see that there is a correspondence

$$A \bullet M \to M \qquad \longleftrightarrow \qquad A \to \mathcal{E}^{\bullet}(M).$$

The second claim can be verified using this correspondence. □

## 1.4. Coherence

Let $(\mathsf{D}, \bullet)$ be a monoidal category and $V_1, V_2, \ldots, V_k$ a sequence of $k$ objects in $\mathsf{D}$. Their *unbracketed tensor product*, denoted

$$V_1 \bullet V_2 \bullet \cdots \bullet V_k,$$

is an object in $\mathsf{D}$ that is canonically isomorphic to any object obtained by first bracketing the $k$ objects in the sequence in a meaningful manner and then tensoring them together. Moreover, if two such bracketed tensor products are related by applications of the associativity constraint, the corresponding isomorphisms must be likewise related. For instance, the unbracketed tensor product of 4 objects fits in a commutative cone based on the pentagon (1.1), as follows:

$$
\begin{tikzcd}
 & & (V_1 \bullet V_2) \bullet (V_3 \bullet V_4) \arrow[d, "\cong"] \arrow[drr, "\alpha_{V_1, V_2, V_3 \bullet V_4}"] & & \\
((V_1 \bullet V_2) \bullet V_3) \bullet V_4 \arrow[urr, "\alpha_{V_1 \bullet V_2, V_3, V_4}"] \arrow[rr, "\cong"] \arrow[dr, "\alpha_{V_1, V_2, V_3} \bullet \mathrm{id}_{V_4}"'] & & V_1 \bullet V_2 \bullet V_3 \bullet V_4 & & V_1 \bullet \big(V_2 \bullet (V_3 \bullet V_4)\big) \arrow[ll, "\cong"'] \\
 & \big(V_1 \bullet (V_2 \bullet V_3)\big) \bullet V_4 \arrow[ur, "\cong"] \arrow[rr, "\alpha_{V_1, V_2 \bullet V_3, V_4}"'] & & V_1 \bullet ((V_2 \bullet V_3) \bullet V_4). \arrow[ul, "\cong"'] \arrow[ur, "\mathrm{id}_{V_1} \bullet \alpha_{V_2, V_3, V_4}"'] &
\end{tikzcd}
$$

We explain this in more detail next.

Consider the following graph. There is one vertex for each meaningful bracketing of $k$ ordered variables. There is an edge between two vertices if their bracketings differ by a single application of the transformation

$$(A \bullet B) \bullet C \rightsquigarrow A \bullet (B \bullet C), \tag{1.19}$$

where $A$, $B$, and $C$ are bracketed substrings of variables. The resulting graph is connected. It is the 1-skeleton of a polytope of dimension $k-2$ known as the *associahedron*. For $k = 4$, this yields the pentagon of (1.1).

Let $\mathsf{K}$ be the indiscrete category on the vertices of the associahedron of dimension $k-2$ (Section A.3.2). In other words, the objects are the vertices and there is a unique morphism between any two objects. It follows that every morphism is an isomorphism.

Define a functor $\mathsf{K} \to \mathsf{D}$ as follows. The functor sends a vertex to the corresponding bracketed tensor product of $V_1, V_2, \dots, V_k$. To define it on the unique morphism from vertex $a$ to vertex $b$, choose an arbitrary path from $a$ to $b$ on the 1-skeleton of the associahedron. Going from $a$ to $b$ along this path, each edge is traversed either in the direction of the transformation (1.19) or in the opposite direction. In the first case, we label the edge by a bracketed tensor product of identity maps and the associativity constraint $\alpha_{A,B,C}$. In the other case, we label it with the inverse map. On the unique morphism from $a$ to $b$, the value of the functor is the composite in $\mathsf{D}$ of the labels of the edges along this path, in the order the edges are traversed. The resulting map is unique in view of Mac Lane's coherence theorem [250, Section VII.2], [184, Corollary 1.4], so the functor is well-defined.

The unbracketed tensor product of $V_1, V_2, \dots, V_k$ is defined to be the colimit of this functor. Since $\mathsf{K}$ is an indiscrete category, the colimit exists, and is isomorphic to the value of the functor on any particular object.

**Notation 1.29.** Whenever we write a tensor product of three or more objects without specifying brackets, it stands for the unbracketed tensor product of the objects. The first instance of this can be found in Definition 1.2.

Now let $(\mathsf{D}, \bullet, \beta)$ be a symmetric monoidal category. For any permutation $\sigma$ on $k$ letters, there is a well-defined isomorphism

$$V_1 \bullet V_2 \bullet \cdots \bullet V_k \xrightarrow{\cong} V_{\sigma(1)} \bullet V_{\sigma(2)} \bullet \cdots \bullet V_{\sigma(k)}$$

between unbracketed tensor products, defined using the symmetry $\beta$. Let $I$ be a finite set (not necessarily ordered) and $\{V_i\}_{i\in I}$ a family of objects in $\mathsf{D}$ indexed by $I$. Their *unordered tensor product*, denoted

$$\underset{i\in I}{\bullet} V_i,$$

is an object in $\mathsf{D}$ such that for any linear order $i_1, i_2, \ldots, i_k$ on the elements of $I$, there is an isomorphism

$$V_{i_1} \bullet V_{i_2} \bullet \cdots \bullet V_{i_k} \xrightarrow{\cong} \underset{i\in I}{\bullet} V_i$$

which commutes with the isomorphisms relating different choices of linear orders on $I$.

More precisely, the unordered tensor product on $I$ is the colimit of a functor whose source category has linear orders on $I$ for its objects and a unique morphism between any two objects. This morphism can be interpreted as the unique bijection on $I$ which takes one linear order to the other. The functor sends each linear order to the corresponding unbracketed tensor product. Since the source category is indiscrete, the colimit exists.

Note that any unordered tensor product is a fortiori unbracketed.

A bijection between sets $I$ and $J$ induces an isomorphism

$$\underset{i\in I}{\bullet} V_i \xrightarrow{\cong} \underset{j\in J}{\bullet} V_j.$$

By convention, the unbracketed or unordered tensor product over the empty set is the unit object in $\mathsf{D}$.

**Example 1.30.** Unbracketed and unordered tensor products are frequently used, though often only implicitly. On $(\mathsf{Vec}, \otimes)$ we deal with such products with the aid of the following universal property, just as we do with the familiar tensor product $V_1 \otimes V_2$ of two vector spaces.

Let

$$\mathrm{Hom}_{\mathsf{Vec}}(V, W)$$

denote the space of linear functions from $V$ to $W$. Given a family of vector spaces $\{V_i\}_{i\in I}$, consider the Cartesian product

$$\prod_{i\in I} V_i := \left\{ f\colon I \to \bigcup_{i\in I} V_i \;\middle|\; f(i) \in V_i \text{ for all } i \in I \right\}.$$

Note this involves no ordering or bracketing. A function $T\colon \prod_{i\in I} V_i \to W$ to another vector space $W$ is multilinear if for each $f \in \prod_{i\in I} V_i$ and each $j \in I$, the function

$$V_j \xrightarrow{f_j} \prod_{i\in I} V_i \xrightarrow{T} W$$

is linear, where for each $v \in V_j$ and $i \in I$,

$$f_j(v)(i) := \begin{cases} f(i) & \text{if } i \neq j, \\ v & \text{if } i = j. \end{cases}$$

Let

$$\mathrm{Mul}_{\mathsf{Vec}}(\{V_i\}_{i\in I}, W)$$

denote the set of such multilinear functions. Then $\bigotimes_{i\in I} V_i$ is characterized by the existence of a bijection

$$\mathrm{Hom}_{\mathsf{Vec}}\Big(\bigotimes_{i\in I} V_i, W\Big) \cong \mathrm{Mul}_{\mathsf{Vec}}(\{V_i\}_{i\in I}, W)$$

natural in $W$.

CHAPTER 2

# Graded Vector Spaces

The category of graded vector spaces serves to illustrate the theory of Chapter 1. In this chapter, we discuss this category along with related ones, such as the category of chain complexes. Each of these categories carries a monoidal structure which we call the Cauchy product. (Co, Bi)monoids with respect to this structure are familiar objects such as graded (co, bi)algebras, or variations on them.

Section 2.1 is basic and deals purely with graded vector spaces; the ideas and notations introduced here are frequently used in the later parts of this monograph. In Section 2.2 we introduce the Schubert statistic and related combinatorial notions such as the $q$-binomial coefficients. The Schubert statistic plays important roles in Chapters 9, 10, 12, and 14.

In Section 2.3 we continue to deal with graded vector spaces; however we deform the braiding by a parameter $q$. This leads to the notion of $q$-Hopf algebras. Higher dimensional versions of these notions are discussed in Section 2.4.

Section 2.5 is of a different nature. It discusses basic notions from group representation theory with special emphasis on the norm map, which relates coinvarians to invariants.

Section 2.6 explains the constructions of the tensor, shuffle, and symmetric algebras. They are related by symmetrization, which is an instance of the norm map. We also discuss universal properties and $q$-deformations of these objects. Several ideas in this section play an important role in later chapters. For example, the approach to tensor, shuffle, and symmetric algebras and their deformations is paralleled by the theory of Fock functors in Chapters 15 and 16, while the universal properties are paralleled by the constructions of Chapter 11 in the context of species.

Sections 2.7, 2.8 and 2.9 deal with graded vector spaces with the added structure of boundary maps. They are needed mainly in Chapters 5 and 19, and their reading may be safely postponed if desired.

## 2.1. Graded vector spaces

In this section, apart from the usual Cauchy product, we discuss two other tensor products on graded vector spaces, namely, the Hadamard and the substitution product. Monoids with respect to the substitution product, for example, are less familiar and known as nonsymmetric operads. We also discuss the internal Homs for each of these tensor products.

**2.1.1. Monoidal structures.** Let $\mathsf{gVec}$ be the category of graded vector spaces over a field $\Bbbk$. An object in this category is a sequence $V = (V_n)_{n \geq 0}$ of vectors spaces $V_n$ over $\Bbbk$. A morphism $f\colon V \to W$ is a sequence of linear maps $f_n\colon V_n \to W_n$.

We refer to $V_n$ as the *component of degree* $n$ of $V$. We often identify

$$V \longleftrightarrow \bigoplus_{n \geq 0} V_n \qquad \text{and} \qquad f \longleftrightarrow \bigoplus_{n \geq 0} f_n. \tag{2.1}$$

In addition, we often assume that each component is finite-dimensional, particularly in any discussion involving duality.

*Warning.* We make no notational distinction between the category of all graded vector spaces and the full subcategory of graded spaces with finite-dimensional components. The notation $\mathsf{gVec}$ refers to either one or the other depending on the context.

Let $\otimes$ denote the usual tensor product of vector spaces (over the base field).

**Definition 2.1.** Given graded vector spaces $V$ and $W$, new graded vector spaces $V \cdot W$, $V \times W$ and $V \circ W$ are defined by

$$(V \cdot W)_n := \bigoplus_{i=0}^{n} V_i \otimes W_{n-i} \tag{2.2}$$

$$(V \times W)_n := V_n \otimes W_n \tag{2.3}$$

$$(V \circ W)_n := \bigoplus_{k \geq 0} V_k \otimes \Big( \bigoplus_{i_1 + \cdots + i_k = n} W_{i_1} \otimes \cdots \otimes W_{i_k} \Big). \tag{2.4}$$

In (2.4), the sums are over all $k \geq 0$ and all sequences $(i_1, \ldots, i_k)$ of nonnegative integers whose sum is $n$. In particular,

$$(V \circ W)_0 = \bigoplus_{k \geq 0} V_k \otimes W_0^{\otimes k}.$$

More generally, it follows from (2.2) and (2.4) that

$$(V \circ W)_n = \bigoplus_{k \geq 0} V_k \otimes (W^{\cdot k})_n, \tag{2.5}$$

where $W^{\cdot k}$ is the Cauchy product of $W$ with itself $k$ times. Explicitly,

$$(W^{\cdot k})_n = \bigoplus_{i_1 + \cdots + i_k = n} W_{i_1} \otimes W_{i_2} \otimes \cdots \otimes W_{i_k}, \tag{2.6}$$

where the sum is over all sequences $(i_1, \ldots, i_k)$ of nonnegative integers of length $k$ whose sum is $n$.

We refer to the operations $V \cdot W$, $V \times W$ and $V \circ W$ as the Cauchy, Hadamard, and substitution products of graded vector spaces respectively.

Table 2.1. Monoidal structures on graded vector spaces.

| Name | Tensor product | Unit |
|---|---|---|
| Cauchy | $\cdot$ | $1$ |
| Hadamard | $\times$ | $E$ |
| Substitution | $\circ$ | $X$ |

The Cauchy, Hadamard, and substitution products give rise to three monoidal categories $(\mathsf{gVec}, \cdot)$, $(\mathsf{gVec}, \times)$, and $(\mathsf{gVec}, \circ)$, as shown in Table 2.1. The unit objects are 1, $E$, and $X$ respectively, where

$$1_n := \begin{cases} \Bbbk & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad E_n := \Bbbk, \qquad X_n := \begin{cases} \Bbbk & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Note that under the identification in (2.1), $V \cdot W$ agrees with the usual tensor product of vector spaces and the unit object 1 with the base field $\Bbbk$. The unit object $E$ can be identified with the space $\Bbbk[x]$ of polynomials in one variable. In this manner, the unit object $X$ is identified with $\Bbbk x$, the one-dimensional space spanned by the variable $x$ inside the space of polynomials.

The monoidal categories $(\mathsf{gVec}, \cdot)$ and $(\mathsf{gVec}, \times)$ are symmetric. For the braiding $\beta\colon V \cdot W \to W \cdot V$, we choose the map

$$v \otimes w \mapsto w \otimes v \tag{2.8}$$

which interchanges the tensor factors. This braiding is a symmetry. The braiding for the Hadamard product is defined similarly.

The tensor products defined above also interact with one another in many interesting ways. These interactions belong to the realm of 2-monoidal categories and are discussed in Chapter 6; see Examples 6.22 and 6.23.

**2.1.2. Generating functions.** To each graded vector space $V$ one can associate a formal power series; namely, the generating function for the dimensions of the components $V_n$ (assuming they are finite-dimensional):

$$f_V(x) := \sum_{n \geq 0} \dim(V_n) x^n.$$

Under the association $V \mapsto f_V(x)$, the products of Definition 2.1 correspond to familiar operations among formal power series: the usual product, the Hadamard or componentwise product, and the substitution of power series. These are respectively defined as follows. Given

$$f(x) = \sum_{n \geq 0} a_n x^n \quad \text{and} \quad g(x) = \sum_{n \geq 0} b_n x^n,$$

we have

$$\begin{aligned} (f \cdot g)(x) &:= \sum_{n \geq 0} \Big( \sum_{i=0}^{n} a_i b_{n-i} \Big) x^n \\ (f \times g)(x) &:= \sum_{n \geq 0} (a_n b_n) x^n \\ (f \circ g)(x) &:= \sum_{n \geq 0} \Big( \sum_{k=0}^{n} a_k \Big( \sum_{i_1 + \cdots + i_k = n} b_{i_1} \cdots b_{i_k} \Big) \Big) x^n. \end{aligned}$$

The last operation is only defined if $b_0 = 0$ (and the inner sum is over all sequences $(i_1, \ldots, i_k)$ of positive integers whose sum is $n$).

These facts motivate the notation and terminology employed in Section 2.1.1.

TABLE 2.2. Categories of "monoids" in (graded) vector spaces.

| Category | Description | Category | Description |
|---|---|---|---|
| $\mathsf{Alg}$ | Algebras | $\mathsf{gAlg}$ | Graded algebras |
| $\mathsf{Coalg}$ | Coalgebras | $\mathsf{gCoalg}$ | Graded coalgebras |
| $\mathsf{Lie}$ | Lie algebras | $\mathsf{gLie}$ | Graded Lie algebras |
| $\mathsf{Hopf}$ | Hopf algebras | $\mathsf{gHopf}$ | Graded Hopf algebras |
| $\mathsf{Alg}^{\mathsf{co}}$ | Comm. algebras | $\mathsf{gAlg}^{\mathsf{co}}$ | Graded comm. algebras |

**2.1.3. Graded Hopf algebras.** We now consider the various types of monoids with respect to each of the above tensor products.

Let us begin with the Hadamard product. A (co, bi, Hopf) monoid in $(\mathsf{gVec}, \times, \beta)$ is a graded vector space $(V_n)_{n\geq 0}$ such that each graded component $V_n$ is a (co, bi, Hopf) algebra. Similarly, a (commutative, Lie) monoid is a graded vector space such that each graded component is a (commutative, Lie) algebra.

The monoidal category $(\mathsf{gVec}, \circ)$ is not braided. Therefore, it makes sense to consider monoids (or comonoids) in this category but not commutative monoids or bimonoids. Monoids are called nonsymmetric operads; these objects are briefly discussed in Section B.7.

We now turn to the Cauchy product. A (co, bi, Hopf) monoid in $(\mathsf{gVec}, \cdot, \beta)$ is a graded (co, bi, Hopf) algebra. Similarly, a commutative monoid is a graded commutative algebra, in the sense that all elements commute regardless of their degrees, and dually for comonoids. A Lie monoid in $(\mathsf{gVec}, \cdot, \beta)$ is a graded Lie algebra (in which the Lie algebra axioms are not affected by the degree of the elements). Table 2.2 shows the notations that we will employ to denote the corresponding categories.

For information on the history of the notion of Hopf algebras and its origins in connection to the work of Heinz Hopf [172], see [21].

**2.1.4. Duality.** Recall the notion of monoidal category with (left) duals [191, Chapter XIV]). A symmetric monoidal category with duals is called *compact closed* in the work of Kelly and Laplaza [198]. In the latter context (which is of interest to us), left duals imply right duals and viceversa.

The *contragredient* or *dual* $V^*$ of a graded vector space $V$ is defined by

$$(V^*)_n := (V_n)^*.$$

Let $\mathsf{gVec}^{\mathrm{op}}$ denote the opposite category of $\mathsf{gVec}$. Duality is a functor

$$(-)^*\colon \mathsf{gVec}^{\mathrm{op}} \to \mathsf{gVec}.$$

For the rest of this section, we assume that all graded vector spaces have finite-dimensional components. In this situation, there are canonical morphisms of graded vector spaces

$$E \to V \times V^* \qquad \text{and} \qquad V^* \times V \to E$$

that turn $(\mathsf{gVec}, \times, *)$ into a monoidal category with duals.

**Proposition 2.2.** *For any graded vector spaces $V$ and $W$, there are natural isomorphisms*

$$(V \times W)^* \cong V^* \times W^* \qquad \textit{and} \qquad (V^*)^* \cong V.$$

The duality functor also behaves well with respect to the Cauchy product, namely, we have canonical isomorphisms

$$(V \cdot W)^* \cong V^* \cdot W^*. \tag{2.9}$$

Further, these isomorphisms commute with the braiding. In the language of monoidal functors (which is developed in detail in Chapter 3), this is equivalent to saying that (2.9) turns

$$(-)^* \colon (\mathsf{gVec}^{\mathrm{op}}, \cdot, \beta^{\mathrm{op}}) \to (\mathsf{gVec}, \cdot, \beta) \tag{2.10}$$

into a bistrong monoidal functor.

This allows us to conclude that $(-)^*$ maps graded algebras to graded coalgebras and viceversa, and graded Hopf algebras to graded Hopf algebras preserving antipodes. If $V$ is a Hopf algebra, the resulting Hopf algebra $V^*$ is called the dual of $V$.

**Example 2.3.** Consider the polynomial algebra $\Bbbk[x]$ in the variable $x$ with coproduct

$$\Delta(x^n) = \sum_{s=0}^{n} \binom{n}{s} x^s \otimes x^{n-s}.$$

This is the Hopf algebra of polynomials in one variable. Its dual, denoted $\Bbbk\{x\}$ is called the divided power Hopf algebra; see [283, Sections III and XI], [1, Example 2.6], or [279, Example 5.6.8]. We recall it below. It has a linear basis consisting of symbols $x^{(n)}$, $n \geq 0$, and the structure is

$$x^{(s)} \cdot x^{(t)} := \binom{s+t}{s} x^{(s+t)}, \qquad \Delta(x^{(n)}) := \sum_{s+t=n} x^{(s)} \otimes x^{(t)}.$$

The element $x^{(n)}$ has degree $n$ and $\Bbbk\{x\}$ and $\Bbbk[x]$ are dual as graded Hopf algebras via $\langle x^{(n)}, x^m \rangle = \delta_{n,m}$. Further, the map

$$\Bbbk[x] \to \Bbbk\{x\}, \qquad x^n \mapsto n! x^{(n)}, \tag{2.11}$$

is a morphism of Hopf algebras. Note that it is an isomorphism if and only if the field has characteristic sero. It follows that under this hypothesis, $\Bbbk[x]$ is a self-dual Hopf algebra.

In characteristic $p$, the image of the above map is spanned by $x^{(i)}$, for $i$ ranging from 0 to $p-1$. This image is isomorphic to the quotient by the kernel which is

$$\Bbbk[x]/(x^p).$$

It follows that this is a $p$-dimensional self-dual Hopf algebra.

**2.1.5. Internal Homs.** The notion of internal Hom in monoidal categories was discussed in Section 1.3. Let

$$\mathcal{H}^{\cdot}, \mathcal{H}^{\times}, \mathcal{H}^{\circ} \colon \mathsf{gVec}^{\mathrm{op}} \times \mathsf{gVec} \to \mathsf{gVec}$$

be the functors defined by

$$\begin{aligned}
\mathcal{H}^{\cdot}(V, W)_k &:= \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathsf{Vec}}(V_n, W_{k+n}), \\
\mathcal{H}^{\times}(V, W)_k &:= \mathrm{Hom}_{\mathsf{Vec}}(V_k, W_k), \\
\mathcal{H}^{\circ}(V, W)_k &:= \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathsf{Vec}}\big((V^{\cdot k})_n, W_n\big),
\end{aligned}$$

where $(V^{\cdot k})_n$ is as in (2.6). The following result says that these functors are the internal Homs for the Cauchy, Hadamard and substitution products respectively.

**Proposition 2.4.** *For any graded vector spaces $U$, $V$ and $W$, there are natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathsf{gVec}}(U\cdot V, W) &\cong \mathrm{Hom}_{\mathsf{gVec}}\big(U, \mathcal{H}^{\cdot}(V,W)\big)\\ \mathrm{Hom}_{\mathsf{gVec}}(U\times V, W) &\cong \mathrm{Hom}_{\mathsf{gVec}}\big(U, \mathcal{H}^{\times}(V,W)\big)\\ \mathrm{Hom}_{\mathsf{gVec}}(U\circ V, W) &\cong \mathrm{Hom}_{\mathsf{gVec}}\big(U, \mathcal{H}^{\circ}(V,W)\big).\end{aligned}$$

Proof. The first two claims follow directly from the definitions, while the third claim follows from (2.5). □

If $V$ is finite-dimensional, there is a natural isomorphism

$$\mathcal{H}^{\times}(V,W) \cong V^* \times W.$$

For the monoidal properties of the functor $\mathcal{H}^{\times}$ with respect to the Cauchy product, see Remark 8.65.

## 2.2. The Schubert statistic

In this section, we study the Schubert statistic which, given a subset $S$ of a linearly ordered set (list), counts the number of minimum adjacent transpositions required to bring the elements of $S$ to the beginning of the list. We also discuss its relation with the Schubert decomposition of the Grassmannian. This is our motivation for the chosen terminology. An interpretation of the statistic in terms of the gallery metric on the Coxeter complex of type $A$ is given in Section 10.13.

The Schubert statistic plays an important role in deformation theory. It is used in Part III in the construction of the deformed Fock functors, and their decorated and colored versions. Consequently, it appears in the definitions of many interesting deformed Hopf algebras, some of which are discussed later in this chapter.

The Schubert statistic also plays a role in Part II where it is regarded as a 2-cocycle and then used to construct deformed Hopf monoids in species.

**Notation 2.5.** We write $|S|$ for the cardinality of the set $S$. We write $[s]$ for the set $\{1,\dots,s\}$ and $[s+1,s+t]$ for the set $\{s+1,\dots,s+t\}$. There is a unique order-preserving map between two $n$-sets of integers. We denote this map by cano, as a short form for "canonical". For example, the map $\mathrm{cano}\colon [t]\to[s+1,s+t]$ shifts the entries in $[t]$ by $s$; while if $S$ is a set of integers, the map $\mathrm{cano}\colon S\to[|S|]$ standardizes the entries of $S$ to the initial segment $[1,|S|]$. We sometimes refer to these two instances of the cano maps as *shifting* and *standardization.*

**2.2.1. Definition.** Given a subset $S$ of $[n]$, let

$$\mathrm{Sch}_n(S) := \{(i,j)\in[n]^2 \mid i\in S, j\in T, i>j\}. \tag{2.12}$$

The *Schubert statistic* is the cardinality of this set:

$$\mathrm{sch}_n(S) := |\mathrm{Sch}_n(S)|. \tag{2.13}$$

Write $S=\{i_1,\dots,i_s\}\subseteq[n]$. We have the explicit formula

$$\mathrm{sch}_n(S) = \sum_{j=1}^{s}(i_j - j) = \sum_{i\in S} i - \frac{s(s+1)}{2}.$$

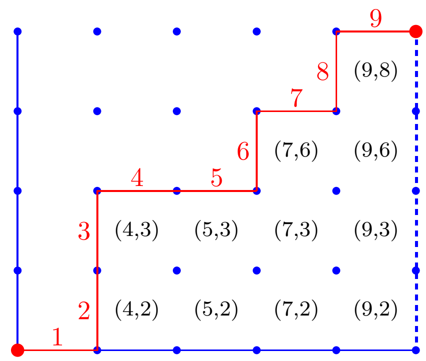



FIGURE 2.1. The Schubert statistic as the area under a lattice path.

Consider paths in the integer lattice based at the origin and consisting of unit steps which are either horizontal or vertical. If we represent $S$ as the path whose $i$-th step is horizontal if and only if $i \in S$, then $\mathrm{Sch}_n(S)$ is in bijection with the set of unit squares that lie between the path and the $x$-axis. Thus, $\mathrm{sch}_n(S)$ is the area of this region. This is illustrated in Figure 2.1, where $\mathrm{sch}_9(\{1, 4, 5, 7, 9\}) = 11$.

**2.2.2. Elementary properties.** The Schubert statistic satisfies the following properties.

$$\mathrm{sch}_n(\emptyset) = \mathrm{sch}_n([n]) = 0. \tag{2.14}$$

Let $S \sqcup T = [n]$ be a decomposition with cardinalities $s$ and $t$. That is, $S$ and $T$ are disjoint subsets of $[n]$ with cardinalities $s$ and $t$ respectively whose union is $[n]$. Then

$$\mathrm{sch}_n(S) + \mathrm{sch}_n(T) = st. \tag{2.15}$$

Let $\omega_n$ be the permutation on $n$ letters which sends $i$ to $n+1-i$ for each $i$. If $S'$ denotes the image of $S$ under the map $\omega_n$, then

$$\mathrm{sch}_n(S') = \mathrm{sch}_n(T). \tag{2.16}$$

Let $R \sqcup S \sqcup T = [n]$ be a decomposition with cardinalities $r$, $s$, and $t$. Let $\bar{R}$ and $\bar{S}$ be the images of $R$ and $S$ under the maps

$$\mathrm{cano}\colon R \sqcup S \to [r+s] \quad \text{and} \quad \mathrm{cano}\colon S \sqcup T \to [s+t],$$

respectively. Then

$$\mathrm{sch}_n(R \sqcup S) + \mathrm{sch}_{r+s}(\bar{R}) = \mathrm{sch}_n(R) + \mathrm{sch}_{s+t}(\bar{S}). \tag{2.17}$$

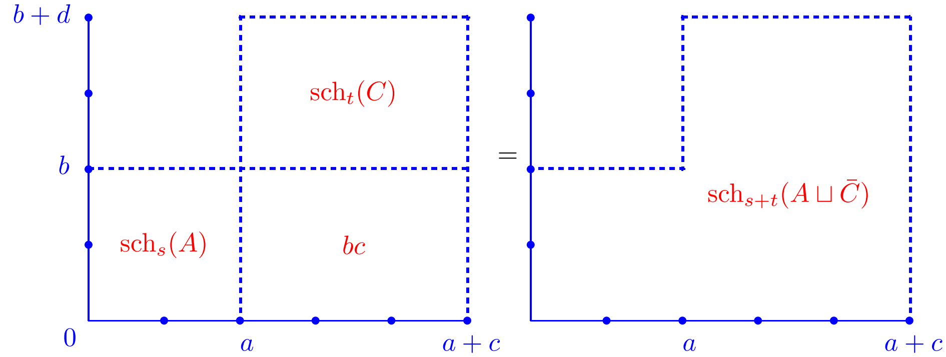
Let $A \sqcup B = [s]$ and $C \sqcup D = [t]$ be two decompositions with cardinalities $a$, $b$, $c$, and $d$. Let $\bar{C}$ be the image of $C$ under the map

$$\mathrm{cano}\colon C \sqcup D \to [s+1, s+t].$$

Then

$$\mathrm{sch}_s(A) + \mathrm{sch}_t(C) + bc = \mathrm{sch}_{s+t}(A \sqcup \bar{C}). \tag{2.18}$$

These assertions are well-known. Many can be proved using the description of the statistic in terms of areas. For instance, formula (2.18) has the following graphic proof.




**2.2.3. Inversions of a permutation and shuffles.** The Schubert statistic can also be interpreted as the number of inversions of a permutation. Details follow. Let $\mathrm{S}_n$ denote the symmetric group on $n$ letters. The *inversion set* of a permutation $\sigma \in \mathrm{S}_n$ is

$$\mathrm{Inv}(\sigma) := \{(i,j) \mid 1 \le i < j \le n, \sigma(i) > \sigma(j)\}. \tag{2.19}$$

Let

$$\mathrm{inv}(\sigma) := |\mathrm{Inv}(\sigma)| \tag{2.20}$$

denote the number of inversions of $\sigma$. This is also the *length* of $\sigma$, denoted $l(\sigma)$, which is the minimum number of elementary transpositions required to express $\sigma$.

Let $s + t = n$. Define the set of $(s,t)$-shuffle permutations to be

$$\mathrm{Sh}\,(s,t) := \{\zeta \in \mathrm{S}_n \mid \zeta(1) < \zeta(2) < \cdots < \zeta(s), \zeta(s+1) < \cdots < \zeta(s+t)\}. \tag{2.21}$$

Recall that these are coset representatives for $\mathrm{S}_s \times \mathrm{S}_t$ as a subgroup of $\mathrm{S}_n$, so that for any $\rho \in \mathrm{S}_n$ there are unique permutations $\sigma \in \mathrm{S}_s$, $\tau \in \mathrm{S}_t$, and $\zeta \in \mathrm{Sh}\,(s,t)$ such that

$$\rho = \zeta \cdot (\sigma \times \tau), \tag{2.22}$$

where $\sigma \times \tau \in \mathrm{S}_n$ is defined by

$$(\sigma \times \tau)(i) := \begin{cases} \sigma(i) & \text{if } 1 \le i \le s, \\ s + \tau(i-s) & \text{if } s+1 \le i \le s+t. \end{cases} \tag{2.23}$$

It then follows that

$$\mathrm{inv}(\rho) = \mathrm{inv}(\zeta) + \mathrm{inv}(\sigma) + \mathrm{inv}(\tau). \tag{2.24}$$

Also, for any $\sigma \in \mathrm{S}_n$,

$$\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1}). \tag{2.25}$$

For more precise statements involving inversion sets, see [14, Lemmas 2.4 and 2.6].

Given a decomposition $S \sqcup T = [n]$ with $|S| = s$ and $|T| = t$, let $\zeta \in \mathrm{S}_n$ be the unique permutation which sends $[s]$ to $S$ and $[s+1, s+t]$ to $T$ in an order-preserving manner. It is in fact an $(s,t)$-shuffle permutation. Then

$$\mathrm{sch}_n(S) = \mathrm{inv}(\zeta) = l(\zeta). \tag{2.26}$$

This follows from the definitions.

**2.2.4. $q$-binomial coefficients.** Let $\Bbbk$ be a commutative ring and let $q \in \Bbbk$ be a scalar. The *$q$-binomial coefficients* are the scalars defined by

$$\binom{n}{s}_q := \sum_{S \subseteq [n], |S|=s} q^{\mathrm{sch}_n(S)}. \tag{2.27}$$

These coefficients satisfy *Pascal's recursion* [16, Section 1.6] or [191, Proposition IV.2.1]:

$$\binom{n}{s}_q = \binom{n-1}{s-1}_q + q^s \binom{n-1}{s}_q \qquad \binom{n}{n}_q = \binom{n}{0}_q = 1. \tag{2.28}$$

There is also a formula for the $q$-binomial coefficients in terms of $q$-factorials: Let

$$(n)_q! := \sum_{\sigma \in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)} = \prod_{i=1}^{n-1} (1 + q + \cdots + q^i) \tag{2.29}$$

be the $q$-factorials. Then

$$(n)_q! = \binom{n}{s}_q (s)_q!(n-s)_q!.$$

This formula implies:

**Lemma 2.6.** *If $\Bbbk$ is an integral domain and $q$ is a primitive $n$-th root of unity, then $\binom{n}{s}_q = 0$ for $s = 1, \ldots, n-1$.*

**Remark 2.7.** The above result is not true for general commutative rings. For example, suppose $\Bbbk = \mathbb{Z}_8$ (integers modulo 8) and $q = 3$. Then $q^2 = 1$ but $\binom{2}{1}_q = 1 + q = 4 \neq 0$.

Suppose $\mathbb{Z} \subseteq \Bbbk$. The numbers $\binom{n}{s}_1$ are the usual binomial coefficients:

$$\binom{n}{s}_1 = |\{S \subseteq [n], |S| = s\}|.$$

The numbers $\binom{n}{s}_{-1}$ are also nonnegative; indeed one has

$$\binom{n}{s}_{-1} = \begin{cases} 0 & \text{if } n \text{ is even and } s \text{ is odd,} \\ \binom{\lfloor n/2 \rfloor}{\lfloor s/2 \rfloor}_1 & \text{otherwise,} \end{cases} \tag{2.30}$$

where $\lfloor x \rfloor$ denotes the biggest integer smaller than or equal to $x$. One may derive this formula by invoking Pascal's recursion twice, which leads to

$$\binom{n}{s}_{-1} = \binom{n-2}{s-2}_{-1} + \binom{n-2}{s}_{-1}.$$

The Grassmannian of $s$-planes in $n$-space admits a CW-complex structure in which for each subset $S$ of $[n]$ of cardinality $s$ there is a cell of dimension equal to $\mathrm{sch}_n(S)$. These are the Schubert cells. More details can be found in Example 13.16 and [275, §6]. It follows from here that $\binom{n}{s}_q$ is the Poincaré polynomial of the complex (which counts cells according to their dimension). In particular, $\binom{n}{s}_{-1}$ is the Euler characteristic of the real Grassmannian, and if $q$ is a prime power, then

$\binom{n}{s}_q$ is the cardinality of the Grassmannian over the finite field $\mathbb{F}_q$ (the number of $s$-dimensional subspaces of $\mathbb{F}_q^n$).

Formula (2.30) is a special case of known formulas for the Euler characteristics of more general Grassmann manifolds, as given in [276, p. 393]. The number $\binom{n}{s}_{-1}$ is also the signature of the complex Grassmannian viewed as a Kähler manifold, and as such formula (2.30) is a special case of the formula given in [310, Theorem 1].

We thank Ryan Budney, Allen Hatcher, and Vic Reiner for help with these references.

**2.2.5. Square matrices.** We set up some terminology to deal with square matrices of size $r$. A general square matrix of size $r$ is denoted:

$$Q := (q_{ij})_{1\leq i,j\leq r}.$$

Let $0_{r,r}$ be the matrix all of whose entries are 0, $1_{r,r}$ be the matrix all of whose entries are 1, and $I_{r,r}$ be the identity matrix. For $r = s + t$, let

$$1(s,t) := \begin{pmatrix} 1_{s,s} & 1_{s,t} \\ 1_{t,s} & -1_{t,t} \end{pmatrix} \tag{2.31}$$

where $1_{s,t}$ stands for the $s \times t$ matrix all of whose entries are 1.

Let $P$ and $Q$ be square matrices of size $r$. Let $P \times Q$ be the matrix obtained by multiplying the corresponding entries of $P$ and $Q$. This is the Hadamard product on matrices [173]. The unit element for this product is the matrix $1_{r,r}$. A matrix $Q$ is invertible with respect to this product if and only if each entry of $Q$ is nonzero. In this case, the inverse if obtained by inverting each entry of $Q$. We denote the inverse by $Q^-$.

Let $Q^t$ denote the transpose of $Q$ and let $Q^{-t} := (Q^t)^-$ be its inverse with respect to the Hadamard product. Explicitly, it is obtained by taking the transpose of $Q$ and inverting each entry.

We say $Q$ is *log-antisymmetric* if

$$q_{ij}q_{ji} = 1 \quad \text{for } 1 \leq i, j \leq r. \tag{2.32}$$

Equivalently, $Q$ is log-antisymmetric if $Q = Q^{-t}$, or equivalenly, if $Q$ and $Q^t$ are inverses with respect to the Hadamard product. In particular, the diagonal entries of a log-antisymmetric matrix are either 1 or $-1$. A log-antisymmetric matrix is symmetric if and only if the matrix entries are either 1 or $-1$. An explicit example of a log-antisymmetric matrix is the matrix $1(s,t)$ of (2.31).

Let $A$ be an integer square matrix of size $r$ and let $q$ be an invertible scalar. Define $Q$ by

$$q_{ij} := q^{a_{ij}} \tag{2.33}$$

where $q_{ij}$ and $a_{ij}$ refer to the $ij$-th entries of $Q$ and $A$ respectively. If $A$ is antisymmetric, then the resulting $Q$ is log-antisymmetric (and all diagonal entries are 1). If $q$ is not a root of unity, then the converse holds as well.

**2.2.6. The weighted Schubert statistic.** Fix any square matrix $A$ of size $r$:

$$A := (a_{ij})_{1\leq i,j\leq r},$$

and a function $f \colon [n] \to [r]$. We say that $f(i)$ is the color of $i$. Now list the elements of $[n]$ in their canonical linear order. Given a subset $S$ of $[n]$, the weighted (additive) Schubert statistic is a weighted sum indexed by the set of minimum adjacent transpositions required so that the elements of $S$ appear at the beginning

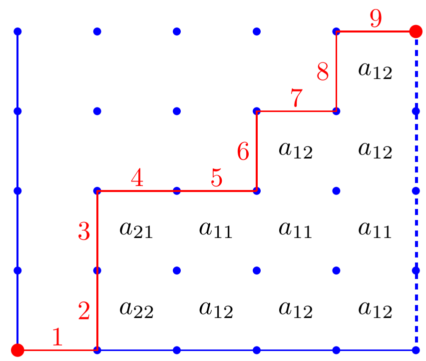



FIGURE 2.2. The weighted Schubert statistic as a weighted area.

of the list. The weight is calculated by looking at the colors of the elements that are being switched and using the matrix $A$. More precisely:

Let $A$ be a matrix of size $r$, $f\colon [n] \to [r]$ be a function, and $S$ be a subset of $[n]$. The *weighted additive Schubert statistic* is

$$\mathrm{sch}_n^A(S, f) := \sum_{(i,j)\in \mathrm{Sch}_n(S)} a_{f(i)f(j)} \tag{2.34}$$

where $\mathrm{Sch}_n(S)$ is as in (2.12).

If all the entries of $A$ are 1, then

$$\mathrm{sch}_n^A(S, f) = \mathrm{sch}_n(S),$$

where the latter is the Schubert statistic. In particular, this holds if $r = 1$ and $A = [1]$.

Recall that the Schubert statistic $\mathrm{sch}_n(S)$ is the area of the region under a lattice path. Similarly, the weighted Schubert statistic $\mathrm{sch}_n^A(S, f)$ can be interpreted as an area, in which the unit square labeled $(i, j)$ has area $a_{f(i)f(j)}$. For example, for $S = \{1, 4, 5, 7, 9\}$ and $f : [9] \to [2]$ which sends odd numbers to 1 and even numbers to 2, the squares along with their weights are shown in Figures 2.1 and 2.2. It follows that

$$\mathrm{sch}_9^A(\{1, 4, 5, 7, 9\}, f) = 3a_{11} + 6a_{12} + a_{21} + a_{22}.$$

One can define a multiplicative version of this statistic by replacing addition by multiplication. To keep the notation distinct from the additive case, we denote the fixed matrix by $Q$.

Let $Q$ be a matrix of size $r$, $f\colon [n] \to [r]$ be a function, and $S$ be a subset of $[n]$. The *weighted multiplicative Schubert statistic* is

$$\mathrm{sch}_n^Q(S, f) := \prod_{(i,j)\in \mathrm{Sch}_n(S)} q_{f(i)f(j)}. \tag{2.35}$$

If $A$ and $Q$ are related as in (2.33), then

$$\mathrm{sch}_n^Q(S, f) = q^{\mathrm{sch}_n^A(S,f)}.$$

Continuing with the above example,

$$\mathrm{sch}_9^Q(\{1, 4, 5, 7, 9\}, f) = q_{11}^3\, q_{12}^6\, q_{21}\, q_{22}.$$

**Convention 2.8.** It is to be understood that $\mathrm{sch}_n^A(S,f)$ denotes the weighted (additive) Schubert statistic, while $\mathrm{sch}_n^Q(S,f)$ denotes the weighted multiplicative Schubert statistic. No further distinction is made in the two notations. We follow the same convention in all related contexts (for example, for the weighted inversion statistic of Section 2.2.9).

**2.2.7. The braid coefficients.** We now introduce the braid coefficients which are closely related to the present discussion. The motivation for this terminology is made clear in Section 2.4, where we use these coefficients to construct braidings on multigraded vector spaces.

Let $\mathsf{d} = (d^1, \dots, d^r)$ and $\mathsf{e} = (e^1, \dots, e^r)$ be two $r$-tuples of nonnegative integers. Define

$$\mathrm{brd}^A_{\mathsf{d},\mathsf{e}} := \sum_{1\le i,j\le r} a_{ij} d^j e^i \qquad \text{and} \qquad \mathrm{brd}^Q_{\mathsf{d},\mathsf{e}} := \prod_{1\le i,j\le r} (q_{ij})^{d^j e^i}. \tag{2.36}$$

We refer to these as the additive and multiplicative braid coefficients. If $A$ and $Q$ are related by (2.33), then

$$\mathrm{brd}^Q_{\mathsf{d},\mathsf{e}} = q^{\mathrm{brd}^A_{\mathsf{d},\mathsf{e}}}.$$

It also follows that

$$\mathrm{brd}^A_{\mathsf{d},\mathsf{e}} = \mathrm{brd}^{A^t}_{\mathsf{e},\mathsf{d}} \qquad \text{and} \qquad \mathrm{brd}^Q_{\mathsf{d},\mathsf{e}} = \mathrm{brd}^{Q^t}_{\mathsf{e},\mathsf{d}}. \tag{2.37}$$

Suppose $r = 1$, $A = [1]$, $Q = [q]$, $\mathsf{d} = (s)$ and $\mathsf{e} = (t)$. Then

$$\mathrm{brd}^A_{\mathsf{d},\mathsf{e}} = st \qquad \text{and} \qquad \mathrm{brd}^Q_{\mathsf{d},\mathsf{e}} = q^{st}.$$

**2.2.8. Properties of the weighted Schubert statistic.** Let $I$ be any finite set. Given a function $f\colon I \to [r]$, let

$$\mathsf{d}(f) := (|f^{-1}(1)|, \dots, |f^{-1}(r)|) \tag{2.38}$$

be the sequence of cardinalities of its fibers.

The weighted analogues of (2.14)–(2.18) can be established along similar lines and are given below. We point out that the braid coefficients appear in two of the identities. This was also true in the unweighted case; however, due to their simplicity in dimension one, we did not have to confront them explicitly then.

Let $A$ and $Q$ be matrices of size $r$ and let $f\colon [n] \to [r]$.

$$\begin{aligned} \mathrm{sch}^A_n(\emptyset, f) &= \mathrm{sch}^A_n([n], f) = 0,\\ \mathrm{sch}^Q_n(\emptyset, f) &= \mathrm{sch}^Q_n([n], f) = 1. \end{aligned} \tag{2.39}$$

Let $S \sqcup T = [n]$ be a decomposition. Then

$$\begin{aligned} \mathrm{sch}^{A^t}_n(S, f) + \mathrm{sch}^A_n(T, f) &= \mathrm{brd}^A_{\mathsf{d}(f|_S)\mathsf{d}(f|_T)},\\ \mathrm{sch}^{Q^t}_n(S, f)\,\mathrm{sch}^Q_n(T, f) &= \mathrm{brd}^Q_{\mathsf{d}(f|_S)\mathsf{d}(f|_T)}, \end{aligned} \tag{2.40}$$

where $f|_S$ denotes the restriction of $f$ to the subset $S$.

Let $\omega_n$ be the permutation which sends $i$ to $n+1-i$ for each $i$. If $S'$ denotes the image of $S$ under the map $\omega_n$, then

$$\begin{aligned} \mathrm{sch}^A_n(S', f) &= \mathrm{sch}^{A^t}_n(T, f\omega_n),\\ \mathrm{sch}^Q_n(S', f) &= \mathrm{sch}^{Q^t}_n(T, f\omega_n). \end{aligned} \tag{2.41}$$

Let $R \sqcup S \sqcup T = [n]$ be a decomposition with cardinalities $r$, $s$, and $t$. Let $\bar{R}$ and $\bar{S}$ be the images of $R$ and $S$ under the maps

$$\mathrm{cano}\colon R \sqcup S \to [r+s] \quad \text{and} \quad \mathrm{cano}\colon S \sqcup T \to [s+t],$$

respectively. Then

$$\begin{aligned} \mathrm{sch}^A_n(R \sqcup S, f) + \mathrm{sch}^A_{r+s}(\overline{R}, \overline{f|_{R\sqcup S}}) &= \mathrm{sch}^A_n(R, f) + \mathrm{sch}^A_{s+t}(\overline{S}, \overline{f|_{S\sqcup T}}), \\ \mathrm{sch}^Q_n(R \sqcup S, f)\, \mathrm{sch}^Q_{r+s}(\overline{R}, \overline{f|_{R\sqcup S}}) &= \mathrm{sch}^Q_n(R, f)\, \mathrm{sch}^Q_{s+t}(\overline{S}, \overline{f|_{S\sqcup T}}), \end{aligned} \tag{2.42}$$

where

$$\overline{f|_{R\sqcup S}} = f|_{R\sqcup S}\, \mathrm{cano}^{-1} \qquad \text{and} \qquad \overline{f|_{S\sqcup T}} = f|_{S\sqcup T}\, \mathrm{cano}^{-1}.$$

Let $A \sqcup B = [s]$ and $C \sqcup D = [t]$ be two decompositions with cardinalities $a$, $b$, $c$, and $d$. Let $\overline{C}$ be the image of $C$ under the map

$$\mathrm{cano}\colon C \sqcup D \to [s+1, s+t].$$

In addition, let $g\colon [s] \to [r]$ and $h\colon [t] \to [r]$ be functions. Then

$$\begin{aligned} \mathrm{sch}^A_s(A, g) + \mathrm{sch}^A_t(C, h) + \mathrm{brd}^A_{\mathsf{d}(g|_B)\mathsf{d}(h|_C)} &= \mathrm{sch}^A_{s+t}(A \sqcup \overline{C}, g \sqcup \bar{h}), \\ \mathrm{sch}^Q_s(A, g)\, \mathrm{sch}^Q_t(C, h)\, \mathrm{brd}^Q_{\mathsf{d}(g|_B)\mathsf{d}(h|_C)} &= \mathrm{sch}^Q_{s+t}(A \sqcup \overline{C}, g \sqcup \bar{h}), \end{aligned} \tag{2.43}$$

where $\bar{h} = h\, \mathrm{cano}^{-1}$ and $g \sqcup \bar{h}$ is the function on $[s+t]$ whose restriction to $[s]$ is $g$, and whose restriction to $[s+1, s+t]$ is $\bar{h}$.

**2.2.9. The weighted inversion statistic.** Let $\mathrm{Inv}(\sigma)$ be the inversion set of a permutation $\sigma \in \mathrm{S}_n$ as defined in (2.19). For a fixed matrix $A$ and a function $f\colon [n] \to [r]$, we assign an additive statistic to this set:

$$\mathrm{inv}^A_f(\sigma) := \sum_{(i,j)\in \mathrm{Inv}(\sigma)} a_{f(j)f(i)}. \tag{2.44}$$

Similarly, for a fixed matrix $Q$ of size $r$ and a function $f\colon [n] \to [r]$, we assign a multiplicative statistic to this set:

$$\mathrm{inv}^Q_f(\sigma) := \prod_{(i,j)\in \mathrm{Inv}(\sigma)} q_{f(j)f(i)}. \tag{2.45}$$

We refer to these as the weighted inversion statistics, the first being the additive version and the second being the multiplicative version. If $Q$ and $A$ are related by (2.33), then

$$\mathrm{inv}^Q_f(\sigma) = q^{\mathrm{inv}^A_f(\sigma)}.$$

In the special case, when all entries of $A$ are 1 and all entries of $Q$ are $q$, we have

$$\mathrm{inv}^A_f(\sigma) = \mathrm{inv}(\sigma) \qquad \text{and} \qquad \mathrm{inv}^Q_f(\sigma) = q^{\mathrm{inv}(\sigma)}.$$

Given a decomposition $S \sqcup T = [n]$ with $|S| = s$ and $|T| = t$, let $\zeta \in \mathrm{S}_n$ be the unique $(s,t)$-shuffle permutation such that $\zeta([s]) = S$ and (hence) $\zeta([s+1, s+t]) = T$. Then

$$\mathrm{inv}^A_f(\zeta^{-1}) = \mathrm{sch}^A_n(S, f) \qquad \text{and} \qquad \mathrm{inv}^Q_f(\zeta^{-1}) = \mathrm{sch}^Q_n(S, f). \tag{2.46}$$

This generalizes (2.26). Similarly,

$$\mathrm{inv}^A_f(\sigma^{-1}) = \mathrm{inv}^{A^t}_{f\sigma}(\sigma) \qquad \text{and} \qquad \mathrm{inv}^Q_f(\sigma^{-1}) = \mathrm{inv}^{Q^t}_{f\sigma}(\sigma). \tag{2.47}$$

This generalizes (2.25).

Continuing with the same general setup as above, let $g = f|_S$ and $h = f|_T$. Define $\bar{g}$ and $\bar{h}$ by

$$\begin{tikzcd} S \arrow[rr, "\mathrm{cano}"] \arrow[dr, "g"'] & & {[s]} \arrow[dl, "\bar{g}"] \\ & {[r]} & \end{tikzcd} \qquad \begin{tikzcd} T \arrow[rr, "\mathrm{cano}"] \arrow[dr, "h"'] & & {[t]} \arrow[dl, "\bar{h}"] \\ & {[r]} & \end{tikzcd} \tag{2.48}$$

Let $\rho$, $\sigma$, $\tau$ and $\zeta$ be as in (2.22). Then

$$\begin{aligned} \mathrm{inv}^A_f(\rho^{-1}) &= \mathrm{inv}^A_f(\zeta^{-1}) + \mathrm{inv}^A_{\bar{g}}(\sigma^{-1}) + \mathrm{inv}^A_{\bar{h}}(\tau^{-1}), \\ \mathrm{inv}^Q_f(\rho^{-1}) &= \mathrm{inv}^Q_f(\zeta^{-1})\, \mathrm{inv}^Q_{\bar{g}}(\sigma^{-1})\, \mathrm{inv}^Q_{\bar{h}}(\tau^{-1}). \end{aligned} \tag{2.49}$$

This generalizes (2.24). It is important that in the above identities we use the inversion statistic for the inverse of the permutations. This was not crucial in the one-dimensional theory since $\sigma$ and $\sigma^{-1}$ have the same number of inversions. In the higher dimensional theory, the relation between the inversion statistic for $\sigma$ and $\sigma^{-1}$ is more complicated and is given by (2.47).

## 2.3. $q$-Hopf algebras

One can perform a one-parameter deformation of the braidings on the category of graded vector spaces equipped with the Cauchy product (2.2). In this section, we discuss $q$-Hopf algebras, which are Hopf monoids in this deformed monoidal category. We also discuss more specialized notions such as connected and positive $q$-Hopf algebras.

In this section, $\Bbbk$ is a field and $q \in \Bbbk$ denotes a fixed scalar, possibly zero.

**2.3.1. A deformation of the braiding and $q$-Hopf algebras.** We endow the monoidal category $(\mathsf{gVec}, \cdot)$ of graded vector spaces with a twist map that depends on $q$. Let $\beta_q \colon V \cdot W \to W \cdot V$ be the map

$$v \otimes w \mapsto q^{st} w \otimes v, \tag{2.50}$$

where $v \in V$ and $w \in W$ are homogeneous elements of degrees $s$ and $t$. We have $\beta_1 = \beta$, as defined in (2.8). Note that $\beta_0$ is not invertible, hence it is not a braiding; however it is a lax braiding. If $q$ is nonzero, then $\beta_q$ is indeed a braiding. The inverse braiding is $\beta_{q^{-1}}$, so $\beta_q$ is a symmetry if and only if $q = \pm 1$.

Now consider the lax braided monoidal category $(\mathsf{gVec}, \cdot, \beta_q)$. We write lax braided instead of braided to include the case $q = 0$. (Co)monoids in this category are graded (co)algebras as before since these notions do not depend on the braiding. Bimonoids and Hopf monoids in this category are known as *$q$-bialgebras* and *$q$-Hopf algebras* respectively.

The isomorphisms (2.9) continue to commute with the braidings $\beta_q$. In other words, the duality functor on the category of graded vector spaces with finite-dimensional components

$$(-)^* \colon (\mathsf{gVec}^{\mathrm{op}}, \cdot, \beta_q^{\mathrm{op}}) \to (\mathsf{gVec}, \cdot, \beta_q) \tag{2.51}$$

is a bistrong monoidal functor. This generalizes (2.10). Hence the duality functor maps $q$-Hopf algebras to $q$-Hopf algebras preserving antipodes.

**Example 2.9.** Consider the polynomial algebra in the variable $x$ with coproduct

$$\Delta(x^n) = \sum_{s=0}^{n} \binom{n}{s}_q x^s \otimes x^{n-s},$$

where the coefficient in front is the $q$-binomial coefficient (2.27). This is the *Eulerian* $q$-Hopf algebra $\Bbbk_q[x]$ of Joni and Rota [179, Section VIII].

The classical cases are $q = \pm 1$ and $q = 0$. The $q = 1$ case was treated in Example 2.3. Some information about the binomial coefficients at $q = -1$ is given in Section 2.2.4. In particular, using (2.30), one can deduce that $\Bbbk_{-1}[x]$ is cocommutative. For $q = 0$, the binomial coefficients are all 1 and the above coproduct becomes the deconcatenation coproduct. This yields a 0-Hopf algebra, which in our notation is denoted $\Bbbk_0[x]$.

The dual of the Eulerian $q$-Hopf algebra is a $q$-analogue of the divided power Hopf algebra. We denote it by $\Bbbk_q\{x\}$. It has a linear basis consisting of symbols $x^{(n)}$, $n \geq 0$, and the structure is

$$x^{(s)} \cdot x^{(t)} := \binom{s+t}{s}_q x^{(s+t)}, \qquad \Delta(x^{(n)}) := \sum_{s+t=n} x^{(s)} \otimes x^{(t)}.$$

The element $x^{(n)}$ has degree $n$ and $\Bbbk_q\{x\}$ and $\Bbbk_q[x]$ are dual as $q$-Hopf algebras via $\langle x^{(n)}, x^m \rangle = \delta_{n,m}$. Further, the map

$$\Bbbk_q[x] \to \Bbbk_q\{x\}, \qquad x^n \mapsto (n)_q!\, x^{(n)}, \tag{2.52}$$

where $(n)_q!$ is the $q$-factorial (2.29), is a morphism of $q$-Hopf algebras. It is an isomorphism if $q$ is not a root of unity. It follows that under this hypothesis, $\Bbbk_q[x]$ is a self-dual $q$-Hopf algebra.

Let us now apply the op and cop constructions (Section 1.2.9) to the Eulerian $q$-Hopf algebra. Applying the cop construction, we obtain a $q^{-1}$-Hopf algebra which we claim is:

$$\Bbbk_q[x]^{\mathrm{cop}} = \Bbbk_{q^{-1}}[x]. \tag{2.53}$$

This follows from the following computation.

$$\Delta^{\mathrm{cop}}(x^n) = \sum_{s=0}^{n} q^{-s(n-s)} \binom{n}{s}_q x^{n-s} \otimes x^s = \sum_{s=0}^{n} \binom{n}{s}_{q^{-1}} x^s \otimes x^{n-s}.$$

Now applying the op construction yields $\Bbbk_q[x]^{\mathrm{op,cop}}$ which is a $q$-Hopf algebra: the coproduct is $\Delta^{\mathrm{cop}}$ while the product is given by

$$\mu^{\mathrm{op}}(x^i \otimes x^j) = q^{ij}\, x^{i+j}.$$

One can check that if $q \neq 0$, then

$$\Bbbk_q[x] \to \Bbbk_q[x]^{\mathrm{op,cop}}, \qquad x^n \mapsto q^{\binom{n}{2}} x^n, \tag{2.54}$$

is an isomorphism of $q$-Hopf algebras.

Note that the op and cop constructions are not interesting for $q = 1$ since $\Bbbk[x]$ is both commutative and cocommutative.

**2.3.2. Connected $q$-bialgebras.** A *connected graded vector space* is a graded vector space with a specified isomorphism from the degree zero component to the base field. A morphism of connected graded vector spaces is a map of graded vector spaces which commutes with the specified isomorphisms in degree zero. We denote the category of connected graded vector spaces by $\mathsf{gVec}^{\circ}$. The Cauchy product and the lax braiding $\beta_q$ defined as before turn it into a lax braided monoidal category $(\mathsf{gVec}^{\circ}, \cdot, \beta_q)$.

A *connected $q$-bialgebra* is defined to be a bialgebra in $(\mathsf{gVec}^{\circ}, \cdot, \beta_q)$. Now let $H$ be a connected $q$-bialgebra. The axioms imply that the unit and counit of $H$ define inverse isomorphisms $H_0 \cong \Bbbk$ which further agree with the specified isomorphism. It follows from here that a connected $q$-bialgebra is equivalent to a $q$-bialgebra whose component of degree zero is of dimension 1,

A graded Hopf algebra $H$ is equivalent to a graded bialgebra $H$ for which $H_0$ is a Hopf algebra. This result follows from a result of Takeuchi [354, Lemma 14]. More generally, a $q$-Hopf algebra $H$ is equivalent to a $q$-bialgebra $H$ for which $H_0$ is a Hopf algebra. In particular, a connected $q$-bialgebra is always a $q$-Hopf algebra. The latter result is due to Milnor and Moore [274, Proposition 8.2] (in the case $q = -1$). We expand on these results next.

**2.3.3. Antipode formulas of Takeuchi and of Milnor and Moore.** We first discuss a general formula for the antipode of a graded connected Hopf algebra due to Takeuchi (see the proof of [354, Lemma 14] or [279, Lemma 5.2.10]). The same formula is valid for a connected $q$-Hopf algebra, so we directly work in this setting.

Let $H$ be a connected $q$-Hopf algebra. Let $H_+$ be the part of positive degree of $H$. It is an ideal of $H$; let $\mu_+ : H_+ \cdot H_+ \to H_+$ be the restriction of the product $\mu$, and let $\Delta_+ \colon H_+ \to H_+ \cdot H_+$ be

$$\Delta_+(x) := \Delta(x) - 1 \otimes x - x \otimes 1.$$

On $H_+$, the antipode is given by the formula

$$\mathrm{S} = \sum_{k \geq 0} (-1)^{k+1} \mu_+^{(k)} \Delta_+^{(k)}, \tag{2.55}$$

where $\mu_+^{(k)}$ and $\Delta_+^{(k)}$ are obtained from $\mu_+$ and $\Delta_+$ by iteration, with $\mu_+^{(0)} = \Delta_+^{(0)} = \mathrm{id}$.

We emphasize that, in general, the interest is in finding an explicit formula for the structure constants of the antipode in a given basis. This requires further work since many cancellations often take place in Takeuchi's formula.

**Remark 2.10.** Note that according to our definition, a connected $q$-bialgebra is in particular graded. Takeuchi's formula holds for a more general class of connected bialgebras which are not necessarily graded. We do not define this class in this monograph.

There is also a recursive expression for the antipode of a connected $q$-Hopf algebra, due to Milnor and Moore [274, Proposition 8.2] (their work is in the setting $q = -1$, but the same result holds for general $q$).

Let $H$ be a connected $q$-bialgebra. Define maps $\mathrm{S}$ and $\mathrm{S}'$ by induction on the degree $n$ of an element $h$ as follows. Let

$$\mathrm{S}(1) = \mathrm{S}'(1) = 1,$$

and for $n > 0$,

$$\begin{aligned} \mathrm{s}(h) &:= -h - \mu_+(\mathrm{id} \otimes \mathrm{s})\Delta_+(h), \\ \mathrm{s}'(h) &:= -h - \mu_+(\mathrm{s}' \otimes \mathrm{id})\Delta_+(h). \end{aligned} \tag{2.56}$$

Then

$$\mathrm{s} = \mathrm{s}'$$

and this map is the antipode of $H$.

**2.3.4. Positive $q$-bialgebras.** A *positively graded vector space* is a graded vector space whose degree zero component is zero. The full subcategory of $\mathsf{gVec}$ consisting of positively graded vector spaces is denoted $\mathsf{gVec}_+$. The Cauchy product of two positively graded vector spaces is again positively graded. This yields a nonunital monoidal category $(\mathsf{gVec}_+, \cdot)$. The following modified Cauchy product

$$V \odot W := V \oplus W \oplus V \cdot W$$

turns $\mathsf{gVec}_+$ into a monoidal category. The zero space serves as the unit object. We observe that a nonunital (co)monoid in $(\mathsf{gVec}_+, \cdot)$ is equivalent to a (co)monoid in $(\mathsf{gVec}_+, \odot)$.

We proceed with $(\mathsf{gVec}_+, \odot)$. The map

$$\beta_q \colon V \oplus W \oplus V \cdot W \to W \oplus V \oplus W \cdot V$$

which interchanges the first two terms and uses the twist map $\beta_q$ for the Cauchy product on the third term, is a lax braiding. This yields a lax braided monoidal category $(\mathsf{gVec}_+, \odot, \beta_q)$. A *positive $q$-bialgebra* is defined to be a bialgebra in $(\mathsf{gVec}_+, \odot, \beta_q)$.

**2.3.5. Interaction between connected and positive $q$-bialgebras.** Consider the functor $(-)^{\mathrm{o}} \colon \mathsf{gVec}_+ \to \mathsf{gVec}^{\mathsf{o}}$ which sends $W$ to $W^{\mathrm{o}}$ where

$$(W^{\mathrm{o}})_n := \begin{cases} \Bbbk & \text{if } n = 0, \\ W_n & \text{otherwise,} \end{cases} \tag{2.57}$$

with the zero component identified with $\Bbbk$ via the identity. In other words, $W^{\mathrm{o}} = 1 \oplus W$.

Consider the functor $(-)_+ \colon \mathsf{gVec}^{\mathsf{o}} \to \mathsf{gVec}_+$ which sends $W$ to $W_+$ where

$$(W_+)_n := \begin{cases} 0 & \text{if } n = 0, \\ W_n & \text{otherwise.} \end{cases} \tag{2.58}$$

One can check that $(-)^{\mathrm{o}}$ and $(-)_+$ define equivalences between $\mathsf{gVec}_+$ and $\mathsf{gVec}^{\mathsf{o}}$. Further, they respect the monoidal structures, that is, there are natural isomorphisms

$$(V \cdot W)_+ \cong V_+ \odot W_+ \quad \text{and} \quad (1 \oplus V) \cdot (1 \oplus W) \cong 1 \oplus (V \odot W)$$

which commute with the braidings. It follows that there is an equivalence of categories

$$\mathsf{Bimon}(\mathsf{gVec}^{\mathsf{o}}, \cdot, \beta_q) \cong \mathsf{Bimon}(\mathsf{gVec}_+, \odot, \beta_q).$$

In particular, we note that a connected $q$-bialgebra is equivalent to a positive $q$-bialgebra. The same statement holds with Hopf algebras instead of bialgebras.

**2.3.6. 0-Hopf algebras.** We now look at the case $q = 0$ in more detail. We have already seen an example of a 0-Hopf algebra, namely, the polynomial algebra $\Bbbk_0[x]$ in one variable with the deconcatenation coproduct (Example 2.9). We now give explicit descriptions for connected 0-bialgebras and positive 0-bialgebras (these are equivalent notions) and state a rigidity result due to Loday and Ronco.

**Proposition 2.11.** *A positive* 0*-bialgebra is a triple* $(V, \mu, \Delta)$ *such that* $V$ *is a positively graded vector space,* $(V, \mu)$ *is a nonunital algebra,* $(V, \Delta)$ *is a noncounital coalgebra* (*with respect to the Cauchy product*) *and the following compatibility holds*

$$\Delta(ab) = ab_{(1)} \otimes b_{(2)} + a_{(1)} \otimes a_{(2)}b + a \otimes b.$$

*In the above equation, the product* $\mu(a, b)$ *is denoted by* $ab$ *and Sweedler's notation is used for the coproduct:* $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

This result is given in [9, Remark A.3].

**Proposition 2.12.** *A connected* 0*-bialgebra is a quintuple* $(V, \mu, \iota, \Delta, \epsilon)$ *such that* $(V, \mu, \iota)$ *is a algebra,* $(V, \Delta, \epsilon)$ *is a coalgebra,* $\dim V_0 = 1$ *and* $\iota$ *and* $\epsilon$ *define inverse isomorphisms of bialgebras* $\Bbbk \cong V_0$*, and the following compatibility holds*

$$\Delta(ab) = ab_{(1)} \otimes b_{(2)} + a_{(1)} \otimes a_{(2)}b - a \otimes b$$

*with* $a$ *and* $b$ *as in Proposition 2.11.*

The notion of 0-Hopf algebras is closely related to the notion studied in [239, 244, 241]. In the terminology of Loday [241, Section 4.2.1], a connected unital infinitesimal bialgebra in $\mathsf{gVec}$ is the same as a connected 0-bialgebra in our sense, and a nonunital infinitesimal bialgebra in $\mathsf{gVec}_+$ is the same as a positive 0-bialgebra in our sense. In this monograph, we reserve the infinitesimal terminology for the objects to be discussed in Example 6.47.

**Theorem 2.13 (Loday–Ronco).** *Any connected* 0*-bialgebra is free as a graded algebra and cofree as a graded coalgebra.*

The above rigidity result is the graded version of [244, Theorem 2.6] and follows from it. In Theorem 11.49 we provide an analogous result for species.

**2.3.7. Graded commutative and Lie algebras.** A commutative monoid in $(\mathsf{gVec}, \cdot, \beta_q)$ is a graded algebra $A$ where

$$ab = q^{st}ba$$

for $a \in A_s$ and $b \in A_t$. We briefly discuss the case $q = -1$. A commutative monoid in $(\mathsf{gVec}, \cdot, \beta_{-1})$ is a graded algebra in which pairs of elements of odd degrees anti-commute and all other pairs of elements commute (these objects are also sometimes called graded commutative algebras).

A Lie monoid in this category is a graded Lie algebra in which the Lie algebra axioms are twisted by a sign according to the degree of the elements, as in [274, Proposition 5.2], or [157, Section 2.2], or [149, Definition 5.22], or [357, Example 3.7, p. 327].

**2.3.8. Super vector spaces.** A *super vector space* is a $\mathbb{Z}_2$-graded vector space. Let $\mathsf{sVec}$ denote the category of super vector spaces. There is an obvious functor from graded to super vector spaces: Lump the even degree pieces in one part and the odd degree pieces in the other part.

Let $V$ and $W$ be super vector spaces. Their Cauchy product is the super vector space whose even and odd parts are

$$V_0 \otimes W_0 \oplus V_1 \otimes W_1 \qquad \text{and} \qquad V_1 \otimes W_0 \oplus V_0 \otimes W_1$$

respectively. This defines the monoidal category $(\mathsf{sVec}, \cdot)$. However, in contrast to graded vector spaces, only $\beta$ and $\beta_{-1}$ yield braidings.

A monoid in $\mathsf{sVec}$ is a $\mathbb{Z}_2$-graded algebra, which is commonly known as a super algebra. Commutative and Lie monoids in $(\mathsf{sVec}, \cdot, \beta_{-1})$ are known as super commutative and super Lie algebras respectively. They have essentially the same descriptions as their graded counterparts discussed above.

## 2.4. Multigraded vector spaces and $Q$-Hopf algebras

In this section, we briefly touch upon a generalization of the preceding theory to higher dimensions.

**2.4.1. Multigraded vector spaces.** Let $\mathbb{N}$ denote the monoid of nonnegative integers under addition, and

$$\mathbb{N}^r := \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{r}.$$

This is a monoid under coordinatewise addition (the free commutative monoid on $r$ generators). A typical element of $\mathbb{N}^r$ is denoted by $\mathsf{d} = (d^1, \ldots, d^r)$.

Let $\mathsf{gVec}^{(r)}$ be the category of $\mathbb{N}^r$-graded vector spaces over the field $\Bbbk$. An object in this category is a sequence $V = (V_\mathsf{d})_{\mathsf{d} \in \mathbb{N}^r}$ of vectors spaces $V_\mathsf{d}$ over $\Bbbk$ and a morphism $f \colon V \to W$ is a sequence of linear maps $f_\mathsf{d} \colon V_\mathsf{d} \to W_\mathsf{d}$. We often identify

$$V \longleftrightarrow \bigoplus_{\mathsf{d} \in \mathbb{N}^r} V_\mathsf{d} \qquad \text{and} \qquad f \longleftrightarrow \bigoplus_{\mathsf{d} \in \mathbb{N}^r} f_\mathsf{d}.$$

If $v \in V_\mathsf{d}$ we may write $|v| = \mathsf{d}$ and say that $v$ is homogeneous of multidegree $\mathsf{d}$. An $\mathbb{N}^r$-graded vector space has an underlying $\mathbb{N}$-grading for which an element of multidegree $(d^1, \ldots, d^r)$ has degree $d^1 + \cdots + d^r$. More formally, there is a functor

$$\mathsf{gVec}^{(r)} \to \mathsf{gVec}.$$

**2.4.2. The Cauchy product for multigraded vector spaces.** The Cauchy product $V \cdot W$ of two $\mathbb{N}^r$-graded vector spaces is defined by

$$(V \cdot W)_\mathsf{d} := \bigoplus_{\mathsf{d}_1 + \mathsf{d}_2 = \mathsf{d}} V_{\mathsf{d}_1} \otimes W_{\mathsf{d}_2}, \tag{2.59}$$

where $\otimes$ denotes the usual tensor product of spaces. The unit object $I$ is defined by

$$I_\mathsf{d} = \begin{cases} \Bbbk & \text{if } \mathsf{d} = \mathsf{0}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathsf{0} = (0, \ldots, 0)$ denotes the unit element of the monoid $\mathbb{N}^r$.

Note that for $r = 1$, one recovers the Cauchy product of graded vector spaces given in (2.2).

**2.4.3. Braidings for the Cauchy product.** We now make use of the braid coefficients introduced in Section 2.2.7 to define braidings on multigraded vector spaces.

Let $Q$ be a square matrix of size $r$. Define a map $\beta_Q\colon V \cdot W \to W \cdot V$ by

$$v \otimes w \mapsto \mathrm{brd}^{Q}_{\mathsf{d},\mathsf{e}}\; w \otimes v, \tag{2.60}$$

where $v$ and $w$ are homogeneous elements with multidegrees $\mathsf{d}$ and $\mathsf{e}$ respectively, and $\mathrm{brd}^{Q}_{\mathsf{d},\mathsf{e}}$ is the braid coeffcient (2.36).

Similarly, for an integer square matrix $A$ and a nonzero scalar $q$, define a map $\beta_{A,q}\colon V \cdot W \to W \cdot V$ by

$$v \otimes w \mapsto q^{\mathrm{brd}^{A}_{\mathsf{d},\mathsf{e}}}\, w \otimes v. \tag{2.61}$$

Since the multiplicative case includes the additive case via (2.33), we proceed with the multiplicative case.

The map $\beta_Q$ is a lax braiding (Definition 1.5). It is a braiding precisely if all entries of $Q$ are nonzero. In this case,

$$(\beta_Q)^{-1} = \beta_{Q^{-t}}, \tag{2.62}$$

with notations as in Section 2.2.5. It follows that $\beta_Q$ is a symmetry precisely if $Q$ is log-antisymmetric (2.32).

Note that for $r = 1$ and $Q = [q]$, the braiding $\beta_Q$ coincides with the braiding $\beta_q$ on graded vector spaces given in (2.50).

**2.4.4. The duality functor.** One can define the dual of any multigraded vector space in the usual manner: take the dual of each component. This gives rise to the multivariate version of the duality functor on graded vector spaces.

The duality functor on the category of multigraded vector spaces with finite-dimensional components

$$(-)^{*}\colon ((\mathsf{gVec}^{(r)})^{\mathrm{op}}, \cdot, \beta_Q^{\mathrm{op}}) \to (\mathsf{gVec}^{(r)}, \cdot, \beta_{Q^t}) \tag{2.63}$$

is bistrong. This generalizes the assertion made for (2.10) and (2.51). Note that in the above result one uses $Q$ for one lax braiding and $Q^t$ for the other lax braiding. This feature is not visible in the one-dimensional theory since matrices of size 1 are always symmetric.

**2.4.5. *Q*-Hopf algebras.** We continue to assume that $Q$ is a square matrix of size $r$. Consider the lax braided monoidal category $(\mathsf{gVec}^{(r)}, \cdot, \beta_Q)$. (We write lax braided instead of braided to include the case when one or more entries of $Q$ are zero.) Bimonoids and Hopf monoids in this category are known as *$Q$-bialgebras* and *$Q$-Hopf algebras* respectively.

As for graded bialgebras (Section 2.3.2), a $Q$-bialgebra for which the component of degree $\mathsf{0}$ is a Hopf algebra is automatically a $Q$-Hopf algebra. In particular, a connected $Q$-bialgebra (defined in the obvious manner) is always a $Q$-Hopf algebra. In this case, the antipode is still given by (2.55).

The notion of $Q$-Hopf algebras is well-known, although the terminology is not standard. They are a special kind of *braided Hopf algebras* [356, Definition 5.1]. Certain $Q$-Hopf algebras known as *Nichols algebras of diagonal type* are central to the construction by Lusztig and by Rosso of quantum enveloping algebras (quantum groups) [246, 316], and play a key role in the classification of pointed Hopf algebras by Andruskiewitsch and Schneider [20, 22, 23, 24, 25]. We say more about these objects in Sections 19.9 and 20.5.

We now discuss some elementary examples.

**Example 2.14.** Consider the free associative algebra on $r$ generators:

$$\Bbbk\langle x_1, \dots, x_r\rangle.$$

It is $\mathbb{N}^r$-graded: A monomial in which $x_i$ occurs $d^i$ times, $i = 1, \dots, r$, has multidegree $(d^1, \dots, d^r)$.

We proceed to turn it into a $Q$-Hopf algebra. Note that a monomial of length $n$ is equivalent to a function $f\colon [n] \to [r]$. More precisely, $f$ corresponds to the monomial

$$x_f := x_{f(1)} \cdots x_{f(n)}.$$

The product of two monomials is given by concatenation. This is the same as the product in the free algebra. The coproduct is given by

$$\Delta(x_f) = \sum_{S \sqcup T = [n]} \mathrm{sch}_n^Q(S, f)\, x_{f|_S} \otimes x_{f|_T},$$

where $\mathrm{sch}_n^Q(S, f)$ is the weighted Schubert statistic (2.35), and

$$x_{f|_S} := x_{f(i_1)} \cdots x_{f(i_k)}$$

where $S = \{i_1, \dots, i_k\}$ and $i_1 < \cdots < i_k$. For example,

$$\begin{aligned} \Delta(x_1x_3x_2) = 1 \otimes x_1x_3x_2 + x_1 \otimes x_3x_2 + q_{31}\, x_3 \otimes x_1x_2 + q_{21}q_{23}\, x_2 \otimes x_1x_3 \\ + q_{31}q_{21}\, x_3x_2 \otimes x_1 + q_{23}\, x_1x_2 \otimes x_3 + x_1x_3 \otimes x_2 + x_1x_3x_2 \otimes 1. \end{aligned}$$

**Example 2.15.** Let $Q$ be a log-antisymmetric matrix (2.32). Consider the following quotient of the free associative algebra.

$$\Bbbk\langle x_1, \dots, x_r\rangle / (x_ix_j - q_{ji}x_jx_i),$$

where $1 \le i, j \le r$. This is known as Manin's *quantum linear space* [258]. Note that $i$ and $j$ can be equal, in which case we obtain the relation $x_i^2 = 0$ if $q_{ii} = -1$, and no relation if $q_{ii} = 1$. The canonical quotient map from the free associative algebra (viewed as a $Q$-Hopf algebra as in the previous example) to the quantum linear space turns the latter into a $Q$-Hopf algebra.

For $r = s + t$, let $1(s, t)$ be as in (2.31). Observe that for $Q = 1(s, t)$, the quantum linear space is

$$\Bbbk[x_1, \dots, x_s] \otimes \Bbbk\{x_{s+1}, \dots, x_{s+t}\},$$

namely, the tensor product of the symmetric algebra on $x_1, \dots, x_s$ (free commutative algebra) and the exterior algebra on $x_{s+1}, \dots, x_{s+t}$. For $s = t$, this is the algebra of differential forms.

Let us now go back to the general case. Let $x_f$ be a monomial in the free associative algebra and let $x_{[f]}$ denote its image under the canonical quotient map. It is convenient to write $x_{[f]}$ as a wedge of variables:

$$x_{[f]} := x_{f(1)} \wedge \cdots \wedge x_{f(n)}.$$

Interchanging two adjacent variables in the wedge incurs a scalar. For example, for $r = 2$,

$$x_1 \wedge x_2 \wedge x_1 = q_{12}\, x_1 \wedge x_1 \wedge x_2.$$

This is an element of multidegree $(2,1)$. We provide examples to show how the product and coproduct works in the wedge notation.

$$(x_2 \wedge x_1) \otimes x_1 \mapsto x_2 \wedge x_1 \wedge x_1.$$

$$x_2 \wedge x_1 \mapsto 1 \otimes (x_2 \wedge x_1) + x_2 \otimes x_1 + q_{12}\, x_1 \otimes x_2 + (x_2 \wedge x_1) \otimes 1.$$

## 2.5. The norm map

Let $\Bbbk$ be a commutative ring and $G$ be a finite group. In this section, we discuss the norm map on $\Bbbk G$-modules. We state the results under very general hypotheses, though we are mainly interested in the case when $\Bbbk$ is a field. We thank Ken Brown and Steve Chase for help with this discussion.

**2.5.1. Invariants and coinvariants.** Let $\Bbbk$ be a commutative ring, $G$ a finite group, and $V$ a $\Bbbk G$-module. Let

$$V^G := \{x \in V \mid gx = x \text{ for all } g \in G\}.$$

This is the space of *$G$-invariants* of $V$. Let $V_G$ be the quotient of $V$ modulo the subspace spanned by

$$\{x - gx \mid g \in G,\ x \in V\}.$$

This is the space of *$G$-coinvariants* of $V$.

A $\Bbbk G$ module is free if it is of the form $\Bbbk G \otimes V_0$ for some vector space $V_0$, with $G$ acting on the first coordinate.

**Lemma 2.16.** *Let $V = \Bbbk G \otimes V_0$ be a free $\Bbbk G$-module. Then,*

$$V_G \cong V_0 \cong V^G.$$

Proof. The following maps are bijective:

$$V_0 \to (\Bbbk G \otimes V_0)_G, \quad v \mapsto \overline{1 \otimes v};$$
$$V_0 \to (\Bbbk G \otimes V_0)^G, \quad v \mapsto \sum_{g \in G} g \otimes v. \qquad \square$$

In particular, if $V = \Bbbk G$, then $V_G$ is spanned by $\overline{g}$ for any $g \in G$, and $V^G$ is spanned by $\sum_{g \in G} g$.

**Lemma 2.17.** *Let $V$ and $W$ be two $\Bbbk G$-modules. Let $G$ act diagonally on $V \otimes W$. If one of the two is free, then so is $V \otimes W$.*

Proof. It suffices to prove the claim when the free module is of rank one, and we may assume this is the case for $V$. Let $W_t$ denote the vector space $W$ viewed as a trivial $\Bbbk G$-module. In this situation, the map

$$\Bbbk G \otimes W \to \Bbbk G \otimes W_t, \quad g \otimes w \mapsto g \otimes g^{-1} \cdot w$$

is an isomorphism of $\Bbbk G$-modules, with inverse $g \otimes w \mapsto g \otimes g \cdot w$. $\square$

The previous result does not need the hypothesis that $G$ be finite.

**Lemma 2.18.** *Let $V$ be a $\Bbbk G$-module. Let $G$ act diagonally on $\Bbbk G \otimes V$. Then,*

$$(\Bbbk G \otimes V)_G \cong V \cong (\Bbbk G \otimes V)^G.$$

Proof. Combine Lemmas 2.16 and 2.17. $\square$

Combining the proofs of the preceding lemmas we obtain the following explicit form for the isomorphisms of Lemma 2.18:

$$V \to (\Bbbk G \otimes V)_G, \quad v \mapsto \overline{1 \otimes v};$$
$$V \to (\Bbbk G \otimes V)^G, \quad v \mapsto \sum_{g \in G} g \otimes g \cdot v.$$

**2.5.2. The norm map.** We begin by recalling some facts about projective and flat modules. A projective module is flat but the converse is not true in general. However, if $\Bbbk$ is a field (of arbitrary characteristic), then the group ring $\Bbbk G$ satisfies the descending chain condition on principal (right) ideals and hence, by a theorem of Bass [218, Theorem 24.25], projective and flat (left) $\Bbbk G$-modules coincide. Further, if $\Bbbk$ is a field and the characteristic of $\Bbbk$ does not divide the order of $G$, then all modules over $\Bbbk G$ are projective, or equivalently, flat.

Let $V$ continue to denote a $\Bbbk G$-module. The map

$$N_V \colon V \to V, \qquad v \mapsto \sum_{g \in G} g \cdot v$$

is called the *norm map* [69, Section III.1, Example 2] (and sometimes also the *trace map* [221, Chapter XX, Exercises on finite groups]).

**Example 2.19.** Suppose $E \supseteq K$ is a Galois extension of fields. Let $G$ be the Galois group. Then $E$ is a vector space over $K$ (under addition) and a $KG$-module, and $N_E$ is the classical trace of Galois theory. In addition, $E^\times := E \setminus \{0\}$ is an abelian group (under multiplication) and a $\mathbb{Z}G$-module, and $N_{E^\times}$ is the classical norm of Galois theory. See [221, Chapter VI, §5].

The norm map factors through the quotient space $V_G$ of coinvariants and its image is contained in the subspace $V^G$ of invariants. The induced map

$$\overline{N}_V \colon V_G \to V^G, \qquad \overline{v} \mapsto \sum_{g \in G} g \cdot v,$$

is also called the norm map. There is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "N_V"] \arrow[d, two heads] & V \\ V_G \arrow[r, "\overline{N}_V"'] & V^G \arrow[u, hook] \end{tikzcd} \tag{2.64}$$

**Lemma 2.20.** *Consider the following hypotheses.*

(i) *The order of $G$ is invertible in the ring $\Bbbk$.*
(ii) *The $\Bbbk G$-module $V$ is flat.*

*Under either hypothesis, the norm map $\overline{N}_V$ is an isomorphism*

$$V_G \cong V^G.$$

If $\Bbbk$ is a field and the characteristic of $\Bbbk$ does not divide the order of $G$, then *all* the above hypotheses are satisfied. In particular, the norm map is an isomorphism in this case.

Proof. Let $|G|$ denote the order of $G$. If $|G|$ is invertible in $\Bbbk$, then the inverse of $\overline{N}_V$ is simply given by

$$v \mapsto \frac{1}{|G|}\overline{v}.$$

Consider the second hypothesis. By Lazard's theorem [61, Chapitre X, §1, N° 6, Théorème 1] or [221, Chapter XVI, Exercise 13], any flat module is a direct limit of free modules. Since coinvariants and invariants commute with direct limits (the former by [250, Theorem V.3.1] and the latter by [250, Theorem IX.2.1]) and the norm map is natural, it is enough to prove the result when $V$ is free. In this case, the result follows from Lemma 2.16. $\square$

We remark that in the projective case, the proof of Lemma 2.20 is simpler and does not require the use of Lazard's theorem. This applies if we are working over a field.

**2.5.3. The dual of the norm map.** Let $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ be the dual of $V$, with the usual $\Bbbk G$-module structure:

$$(g \cdot f)(v) := f(g^{-1} \cdot v)$$

for $g \in G$, $f \in V^*$, $v \in V$.

**Lemma 2.21.** *Let $\Bbbk$, $G$, and $V$ be as above.*

(a) *There is a canonical isomorphism*

$$(V_G)^* \cong (V^*)^G.$$

(b) *Consider the following hypotheses.*
   - (i) *The order of $G$ is invertible in the ring $\Bbbk$.*
   - (ii) *The ring $\Bbbk$ is self-injective (that is, $\Bbbk$ is injective as a $\Bbbk$-module).*
   - (iii) *The $\Bbbk G$-modules $V$ and $V^*$ are flat.*

   *Under either hypothesis, there is a canonical isomorphism*

$$(V^*)_G \cong (V^G)^*.$$

If $\Bbbk$ is a field of arbitrary characteristic, then (ii) holds. In particular, the above map is an isomorphism in this case.

Proof. Recall the tensor-Hom adjunction [318, Theorem 2.11]:

$$\mathrm{Hom}_S(L \otimes_R M, N) \cong \mathrm{Hom}_R\big(L, \mathrm{Hom}_S(M, N)\big),$$

where $L$ is a right $R$-module, $M$ is an $R$-$S$-bimodule, and $N$ is a right $S$-module. Choosing $S = \Bbbk$, $R = \Bbbk G$, $L = \Bbbk$ (as a trivial $R$-module), $M = V$, $N = \Bbbk$, we obtain the isomorphism in (a).

There is a canonical map

$$\mathrm{Hom}_S(L, M) \otimes_R N \to \mathrm{Hom}_S(\mathrm{Hom}_R(N, L), M),$$

where $L$ is an $R$-$S$-bimodule, $M$ is a right $S$-module, and $N$ is a left $R$-module. Rotman [318, Lemmas 3.59 and 3.60] gives certain conditions under which this map is an isomorphism. Let us choose $S = \Bbbk$, $R = \Bbbk G$, $L = V$, $M = \Bbbk$, and $N = \Bbbk$ (as trivial $R$-module). The above isomorphism, when valid, becomes the isomorphism in (b). Now let us relate the hypotheses in (b) to the conditions in Rotman.

If (i) holds, then the augmentation $\epsilon\colon \Bbbk G \to \Bbbk$, $\epsilon(g) = 1$ for all $g \in G$, splits. Hence, $\Bbbk$ is projective as $\Bbbk G$-module, and [318, Lemma 3.59] applies.

If (ii) holds, then [318, Lemma 3.60] applies.

It remains to discuss hypothesis (iii). In this case, both $\overline{N}_{V^*}$ and $(\overline{N}_V)^*$ are isomorphisms, by Lemma 2.20. Since the canonical map $(V^*)_G \to (V^G)^*$ and the isomorphism $(V_G)^* \cong (V^*)^G$ fit in the following commutative diagram,

$$\begin{tikzcd} (V^*)_G \arrow[r, "\overline{N}_{V^*}"] \arrow[d] & (V^*)^G \\ (V^G)^* \arrow[r, "(\overline{N}_V)^*"'] & (V_G)^* \arrow[u, "\cong"'] \end{tikzcd}$$

it follows that $(V^*)_G \to (V^G)^*$ is an isomorphism as well. □

Consider the dual of diagram (2.64). The dual of $N_V : V \to V$ is $N_{V^*} : V^* \to V^*$. The proof of Lemma 2.21 shows that, under any of the hypotheses in (b), we may also identify $\overline{N}_{V^*}$ with $(\overline{N}_V)^*$. Hence we obtain the following result.

**Lemma 2.22.** *Under any of hypotheses* (i), (ii) *or* (iii) *of Lemma* 2.21, *the dual of diagram* (2.64) *for* $V$ *is diagram* (2.64) *for* $V^*$.

In particular, if $\Bbbk$ is a field, then the conclusion of the lemma holds.

## 2.6. The tensor algebra and its relatives

This section focuses on a number of important Hopf algebras associated to a vector space, including the tensor algebra, the shuffle algebra, and the symmetric algebra. The latter are related through symmetrization, a special instance of the norm map of Section 2.5, as shown in (2.66). This diagram provides the guiding philosophy for the universal constructions on species in Chapter 11 and also for the construction of the Fock functors in Chapter 15.

We present these Hopf algebras explicitly and briefly discuss certain universal properties that characterize them. We also discuss a number of variants including $q$-deformations and the quasi-shuffle product.

**2.6.1. The tensor, shuffle and symmetric algebras.** Let $V$ be a vector space over the field $\Bbbk$. The tensor algebra of $V$ is

$$\mathcal{T}(V) := \bigoplus_{k\geq 0} V^{\otimes k}.$$

The product is concatenation of tensors. Consider the left action of the symmetric group $\mathrm{S}_k$ on $V^{\otimes k}$: For $\sigma \in \mathrm{S}_k$,

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

The symmetric algebra of $V$ is

$$\mathcal{S}(V) := \bigoplus_{k\geq 0} (V^{\otimes k})_{\mathrm{S}_k},$$

where $(-)_{\mathrm{S}_k}$ denotes the space of $\mathrm{S}_k$-coinvariants (Section 2.5.1). It is a quotient algebra of $\mathcal{T}(V)$. Both $\mathcal{T}(V)$ and $\mathcal{S}(V)$ are graded connected Hopf algebras: the coproduct is determined by declaring that the elements of $V$ are primitive. In more explicit terms, it is given by deshuffling [191, Theorem III.2.4]. If the characteristic of $\Bbbk$ is 0, then the space of primitive elements of $\mathcal{T}(V)$ is $\mathcal{L}ie(V)$, the free Lie algebra on $V$ [311, Section 1.3]. The space of primitive elements of $\mathcal{S}(V)$ is $V$.

A related construction is that of the *shuffle Hopf algebra* of a vector space,

$$\mathcal{T}^\vee(V) := \bigoplus_{k\geq 0} V^{\otimes k}.$$

It has the same underlying vector space as $\mathcal{T}(V)$, but a different Hopf algebra structure: the product is given by shuffling and the coproduct by deconcatenation. Related references are [244, Section 1.3], [307], [311, Section 1.4], or [350, Chapter XII]. The counterpart of the symmetric algebra is

$$\mathcal{S}^\vee(V) := \bigoplus_{k\geq 0} (V^{\otimes k})^{\mathrm{S}_k},$$

where $(-)^{\mathrm{S}_k}$ denotes the space of $\mathrm{S}_k$-invariants (Section 2.5.1). It is a Hopf subalgebra of $\mathcal{T}^\vee(V)$. Both $\mathcal{T}^\vee(V)$ and $\mathcal{S}^\vee(V)$ are graded connected Hopf algebras.

The norm map (Section 2.5) corresponding to the action of $\mathrm{S}_k$ on $V^{\otimes k}$ is the *symmetrization*

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto \sum_{\sigma \in S_k} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

Adding over all $k \geq 0$, we obtain a map

$$\kappa \colon \mathcal{T}(V) \to \mathcal{T}^\vee(V) \tag{2.65}$$

which turns out to be a morphism of graded Hopf algebras. It is far from being an isomorphism; in fact, it factors through invariants and coinvariants to yield the following commutative diagram of graded Hopf algebras.

$$\begin{tikzcd} \mathcal{T}(V) \arrow[r, "\kappa"] \arrow[d, two heads] & \mathcal{T}^\vee(V) \\ \mathcal{S}(V) \arrow[r, "\bar{\kappa}"'] & \mathcal{S}^\vee(V) \arrow[u, hook] \end{tikzcd} \tag{2.66}$$

If the characteristic of $\Bbbk$ is 0, then $\bar{\kappa}$ is an isomorphism. This follows from Lemma 2.20. If $V = \Bbbk$, then diagram (2.66) reduces to the map (2.11).

**2.6.2. The deformed tensor and shuffle algebras and the exterior algebra.** The tensor and shuffle Hopf algebras can be deformed using a parameter $q$: $\mathcal{T}_q(V)$ is the usual tensor algebra on $V$ endowed with a $q$-version of the deshuffle coproduct constructed using the Schubert statistic, while $\mathcal{T}_q^\vee(V)$ is the *$q$-shuffle algebra* of Duchamp, Klyachko, Krob, and Thibon [108, Section 4.1] (a special case of the *quantum shuffle algebras* as defined by Green [152] and Rosso [316, Proposition 9]) with the deconcatenation coproduct.

The symmetrization map $\kappa$ (2.65) can be deformed as well to obtain the *$q$-symmetrization*

$$\kappa_q \colon \mathcal{T}_q(V) \to \mathcal{T}_q^\vee(V), \quad v_1 \otimes \cdots \otimes v_k \mapsto \sum_{\sigma \in S_k} q^{\operatorname{inv}\sigma} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}. \tag{2.67}$$

The deformed objects $\mathcal{T}_q(V)$ and $\mathcal{T}_q^\vee(V)$ are $q$-Hopf algebras (Section 2.1.3), while $\kappa_q$ is a morphism of $q$-Hopf algebras. If the characteristic of $\Bbbk$ is 0 and $q$ is not a root of unity, then $\kappa_q$ is in fact an isomorphism. This result appears in [108, Proposition 4.5], where it is deduced from Zagier's formula [380, Theorem 2]. It is greatly generalized in Theorem 16.19; also see Example 16.31.

If $q = 0$, then $\mathcal{T}_0(V) = \mathcal{T}_0^\vee(V)$, with the product given by concatenation and the coproduct given by deconcatenation. Further, note that the 0-symmetrization $\kappa_0$ is the identity.

If $q$ is a root of unity, then the image of $\kappa_q$ may be considered as a deformation of the symmetric algebra. These objects are special instances of *Nichols algebras*. (We say more about Nichols algebras in Sections 19.9 and 20.5.) The case $q = -1$ yields the *exterior algebra* $\Lambda(V)$, which we describe next.

The symmetric algebra $\mathcal{S}(V)$ has a signed analogue called the exterior algebra, denoted $\Lambda(V)$. It is obtained by taking $\mathrm{S}_k$-coinvariants with respect to the action:

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto (-1)^{\operatorname{inv}\sigma} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

This is the usual action tensored with the sign representation.

The exterior algebra is a $(-1)$-Hopf algebra; the coproduct is determined by declaring that the elements of $V$ are primitive. There is a similar signed analogue of $\mathcal{S}^\vee(V)$ which we denote by $\Lambda^\vee(V)$. The following is the signed analogue of (2.66).

$$\begin{tikzcd} \mathcal{T}_{-1}(V) \arrow[r, "\kappa_{-1}"] \arrow[d, two heads] & \mathcal{T}_{-1}^\vee(V) \\ \Lambda(V) \arrow[r, "\bar{\kappa}_{-1}"'] & \Lambda^\vee(V) \arrow[u, hook] \end{tikzcd} \tag{2.68}$$

The map $\kappa_{-1}$ is *antisymmetrization*. If the characteristic of $\Bbbk$ is 0, then $\bar{\kappa}_{-1}$ is an isomorphism.

**2.6.3. Duality.** Let $V$ be a finite-dimensional vector space, and let $V^*$ denote its dual. Then $(V^*)^* \cong V$ canonically. This induces isomorphisms

$$\mathcal{T}_q(V)^* \cong \mathcal{T}_q^\vee(V^*), \qquad \mathcal{S}(V)^* \cong \mathcal{S}^\vee(V^*), \qquad \text{and} \qquad \Lambda(V)^* \cong \Lambda^\vee(V^*), \tag{2.69}$$

where the duals on the right refer to the graded duals. The isomorphisms are of $q$-Hopf algebras, Hopf algebras and $(-1)$-Hopf algebras respectively. We say that $\mathcal{T}_q^\vee$ is the contragredient (dual) of $\mathcal{T}_q$, and so forth. Further, if the characteristic of $\Bbbk$ is 0 and $q$ is not a root of unity, then

$$\mathcal{T}_q \cong \mathcal{T}_q^\vee, \qquad \mathcal{S} \cong \mathcal{S}^\vee \qquad \text{and} \qquad \Lambda \cong \Lambda^\vee.$$

The condition on $q$ is only relevant to the first isomorphism. These isomorphisms are induced by the $(q)$ symmetrization. Further, $\mathcal{T}_0 \cong \mathcal{T}_0^\vee$ regardless of the characteristic. Since these functors are isomorphic to their contragredients, we say that they are self-dual (under the stated conditions).

Now choose an isomorphism $V \cong V^*$, and let $\mathcal{F}$ stand for either $\mathcal{T}_q$, or $\mathcal{S}$ or $\Lambda$. Then

$$\mathcal{F}(V)^* \cong \mathcal{F}^\vee(V^*) \cong \mathcal{F}(V^*) \cong \mathcal{F}(V).$$

The first isomorphism follows from the contragredient property, the second follows from self-duality of the functor, and the third follows from the chosen isomorphism between $V$ and its dual. This shows that $\mathcal{F}(V)$ is self-dual (isomorphic to its dual). The self-duality is noncanonical because it depends on a choice.

To summarize: if the characteristic of $\Bbbk$ is 0 and $q$ is not a root of unity, then $\mathcal{T}_q(V)$ is a self-dual $q$-Hopf algebra; if the characteristic of $\Bbbk$ is 0, then $\mathcal{S}(V)$ is a self-dual Hopf algebra and $\Lambda(V)$ is a self-dual $(-1)$-Hopf algebra. Finally, $\mathcal{T}_0(V)$ is a self-dual 0-Hopf algebra, regardless of the characteristic.

**2.6.4. Freeness and cofreeness.** The tensor algebra $\mathcal{T}(V)$ is the free algebra on $V$. In other words, given an algebra $A$ and a linear map $\zeta\colon V \to A$, there is a unique morphism of algebras $\hat{\zeta}\colon \mathcal{T}(V) \to A$ such that

$$\begin{tikzcd} \mathcal{T}(V) \arrow[rr, "\hat{\zeta}"] & & A \\ & V \arrow[ul] \arrow[ur, "\zeta"'] & \end{tikzcd}$$

commutes, where the map on the left is the canonical inclusion. Explicitly,

$$\hat{\zeta}(v_1 \cdots v_n) = \zeta(v_1) \cdots \zeta(v_n).$$

This is an instance of the general construction of Section 6.10.1.

The coalgebra $\mathcal{T}^\vee(V)$ satisfies the following universal property. Let $(C, \Delta, \epsilon)$ be a coalgebra and $\zeta\colon C \to V$ a linear map such that given $c \in C$, there exists $k \geq 1$ with

$$\zeta^{\otimes k} \Delta^{(k-1)}(c) = 0.$$

In this situation, there exists a unique morphism of coalgebras $\hat{\zeta}\colon C \to \mathcal{T}^\vee(V)$ such that

$$\begin{tikzcd} C \arrow[rr, "\hat{\zeta}"] \arrow[dr, "\zeta"'] & & \mathcal{T}^\vee(V) \arrow[dl] \\ & V & \end{tikzcd}$$

commutes, where the map on the right is the canonical projection. Explicitly,

$$\hat{\zeta}(c) = \epsilon(c) + \sum_{k \geq 1} \zeta^{\otimes k} \Delta^{(k-1)}(c).$$

The coalgebra $\mathcal{T}^\vee(V)$ is at times mistaken with the cofree coalgebra on (the projection to) $V$. The latter is a more complicated object; see [55, 133, 162].

Similarly, $\mathcal{S}(V)$ is the free commutative algebra on $V$, and $\mathcal{S}^\vee(V)$ satisfies an analogous property to that of $\mathcal{T}^\vee(V)$ but among cocommutative coalgebras.

It can also be shown that $\mathcal{S}^\vee(V)$ is the free algebra *with divided powers* on $V$. For information on algebras with divided powers, see [136, Section 1.2.2] and [137, Proposition 1.2.15]; this structure is not essential for our purposes.

**Remark 2.23.** The symmetrization map (2.65) can be obtained from either the universal property of $\mathcal{T}(V)$ or that of $\mathcal{T}^\vee(V)$. Namely, it is the unique morphism of algebras (coalgebras) which makes the left (right) diagram below commute.

$$\begin{tikzcd} \mathcal{T}(V) \arrow[rr, dashed, "\kappa"] & & \mathcal{T}^\vee(V) \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}(V) \arrow[rr, dashed, "\kappa"] \arrow[dr] & & \mathcal{T}^\vee(V) \arrow[dl] \\ & V & \end{tikzcd}$$

**2.6.5. The tensor algebra on a coalgebra.** Let $(C, \Delta, \epsilon)$ be a coalgebra. We may use freeness of the tensor algebra $\mathcal{T}(C)$ to extend the structure maps of $C$

$$C \xrightarrow{\Delta} C \otimes C \subseteq \mathcal{T}(C) \otimes \mathcal{T}(C) \qquad \text{and} \qquad C \xrightarrow{\epsilon} \Bbbk$$

to morphisms of algebras

$$\widehat{\Delta} \colon \mathcal{T}(C) \to \mathcal{T}(C) \otimes \mathcal{T}(C) \qquad \text{and} \qquad \widehat{\epsilon} \colon \mathcal{T}(C) \to \Bbbk.$$

The map $\widehat{\Delta}$ coincides with $\Delta$ on the space $C$ which generates $\mathcal{T}(C)$, hence $\widehat{\Delta}$ is coassociative. For a similar reason, it is counital with respect to $\widehat{\epsilon}$. This turns the tensor algebra $\mathcal{T}(C)$ into a bialgebra.

The bialgebra $\mathcal{T}(C)$ satisfies the following universal property. Given a bialgebra $B$ and a morphism of coalgebras $\zeta \colon C \to B$, there exists a unique morphism of bialgebras $\hat{\zeta} \colon \mathcal{T}(C) \to B$ such that

$$\begin{tikzcd}
\mathcal{T}(C) \arrow[rr, "\hat{\zeta}"] & & B \\
& C \arrow[ul, hook] \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

commutes. In other words, $\mathcal{T}(C)$ is the free bialgebra on the coalgebra $C$. This is an instance of the general construction of Section 6.10.2.

The bialgebra $\mathcal{T}(C)$ is not a Hopf algebra in general. The free Hopf algebra on $C$ is a more complicated object; see [354].

The usual grading of $\mathcal{T}(C)$ is not compatible with the above coalgebra structure. On the other hand, if $C = \oplus_{k \geq 0} C_k$ is a graded coalgebra (Section 2.1.3), then $\mathcal{T}(C)$ inherits a grading in which the elements of $C_{k_1} \otimes \cdots \otimes C_{k_n}$ have degree $k_1 + \cdots + k_n$, and which makes it a graded bialgebra.

There is a variant of the above construction that allows us to turn $\mathcal{T}(C)$ into a bialgebra in a different manner. The coproduct is defined by the universal property as before, but applying it now to the map

$$C \to (\Bbbk \otimes C) \oplus (C \otimes C) \oplus (C \otimes \Bbbk) \subseteq \mathcal{T}(C) \otimes \mathcal{T}(C), \quad c \mapsto 1 \otimes c + \Delta(c) + c \otimes 1.$$

The counit is the canonical projection

$$\mathcal{T}(C) = \bigoplus_{k \geq 0} C^{\otimes k} \twoheadrightarrow \Bbbk$$

onto the first component. This construction yields a bialgebra structure on $\mathcal{T}(C)$ for any coalgebra $(C, \Delta)$, not necessarily counital. This bialgebra satisfies a universal property analogous to the preceding one, in which $B$ is a (unital and counital) bialgebra and the map $\zeta$ is a morphism of noncounital coalgebras.

If $C$ is a graded coalgebra, then $\mathcal{T}(C)$ is a graded bialgebra, as above. If in addition $C_0 = 0$, then $\mathcal{T}(C)$ is a graded connected bialgebra, and hence a Hopf algebra.

If $V$ is a vector space, we may endow it with the trivial coproduct (the zero map $V \to V \otimes V$). In this case, the elements of $V$ are primitive in $\mathcal{T}(V)$, and we recover the graded Hopf algebra structure mentioned in Section 2.6.1.

**2.6.6. The quasi-shuffle bialgebra.** We are interested in a construction dual to the last construction of Section 2.6.5.

Let $A$ be an algebra, not necessarily unital. Consider the map

$$(\Bbbk \otimes A) \oplus (A \otimes A) \oplus (A \otimes \Bbbk) \to A, \quad x \otimes y \mapsto xy.$$

Composing with the canonical projection we obtain a map

$$\mathcal{T}^\vee(A) \otimes \mathcal{T}^\vee(A) \twoheadrightarrow (\Bbbk \otimes A) \oplus (A \otimes A) \oplus (A \otimes \Bbbk) \to A$$

which vanishes on $A^{\otimes n} \otimes A^{\otimes m}$ if $n = m = 0$ or if $n > 1$ or $m > 1$. It is possible to apply the universal property of the coalgebra $\mathcal{T}^\vee(A)$ to this map (Section 2.6.4), to obtain a morphism of coalgebras

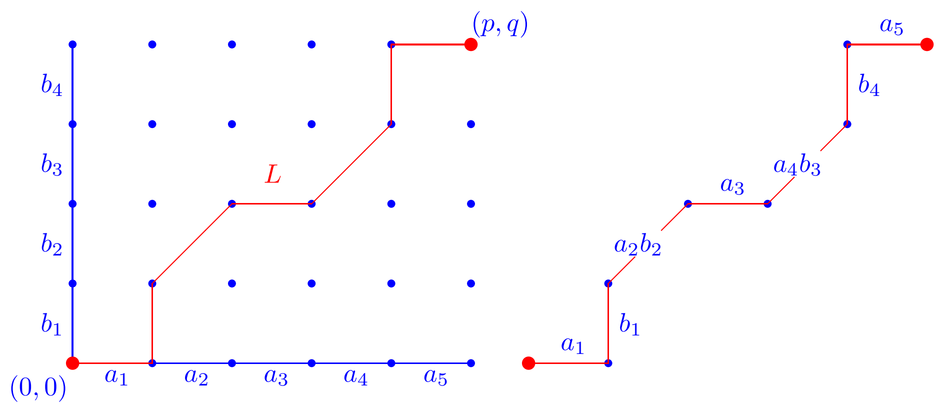
$$\mathcal{T}^\vee(A) \otimes \mathcal{T}^\vee(A) \to \mathcal{T}^\vee(A).$$

It can be shown that this product turns $\mathcal{T}^\vee(A)$ into a bialgebra with unit $1 \in \Bbbk \subseteq \mathcal{T}^\vee(A)$. This construction is incorrectly formulated in [240, Proposition 1.3].

Endowed with this structure, $\mathcal{T}^\vee(A)$ is known as the *quasi-shuffle bialgebra*. The quasi-shuffle product admits the following explicit description.

Given nonnegative integers $p$ and $q$, consider the set $\mathcal{L}(p,q)$ of lattice paths from $(0,0)$ to $(p,q)$ consisting of unit steps which are either horizontal, vertical, or diagonal (sometimes called *Delannoy paths*). An element of $\mathcal{L}(p,q)$ is thus a sequence $L = (\ell_1, \dots, \ell_s)$ such that each $\ell_i$ is either $(1,0)$, $(0,1)$, or $(1,1)$, and $\sum \ell_i = (p,q)$.

Given tensors $\alpha = a_1 \otimes \cdots \otimes a_p \in A^{\otimes p}$, $\beta = b_1 \otimes \cdots \otimes b_q \in A^{\otimes q}$, and a path $L \in \mathcal{L}(p,q)$, we label each step of $L$ according to its horizontal and vertical projections, as indicated in the example below ($p = 5$, $q = 4$):




Then we obtain a tensor $\gamma_L(\alpha,\beta)$ by reading off the labels along the path $L$ in order. When a step is diagonal, we read off the product of the labels in the algebra $A$. In the example above,

$$\gamma_L(\alpha,\beta) = a_1 \otimes b_1 \otimes (a_2b_2) \otimes a_3 \otimes (a_4b_3) \otimes b_4 \otimes a_5.$$

The tensor $\gamma_L(\alpha,\beta)$ is the quasi-shuffle of $\alpha$ and $\beta$ corresponding to $L$. If $L$ does not involve diagonal steps, then $\gamma_L(\alpha,\beta)$ is an ordinary shuffle.

The quasi-shuffle product of the tensors $\alpha$ and $\beta$ is given by

$$\alpha \cdot \beta = \sum_{L \in \mathcal{L}(p,q)} \gamma_L(\alpha,\beta).$$

The quasi-shuffle product appears in less explicit forms in the work of Hazewinkel [161] and Hoffman [169], with some precedent in the work of Cartier [76]. The above description in terms of lattice paths is due to Fares [124].

The bialgebra $\mathcal{T}^\vee(A)$ satisfies the following universal property. Let $B$ be a (unital and counital) bialgebra and $\zeta\colon B \to A$ a morphism of nonunital algebras such that given $b \in B$, there exists $k \geq 1$ with

$$\zeta^{\otimes k}\Delta^{(k-1)}(b) = 0.$$

Then there exists a unique morphism of bialgebras $\hat{\zeta}\colon B \to \mathcal{T}^\vee(A)$ such that

$$\begin{tikzcd} B \arrow[rr, "\hat{\zeta}"] \arrow[dr, "\zeta"'] & & \mathcal{T}^\vee(A) \arrow[dl] \\ & A & \end{tikzcd}$$

commutes.

If $A = \oplus_{k\geq 0} A_k$ is a graded algebra (Section 2.1.3), then $\mathcal{T}^\vee(A)$ inherits a grading in which the elements of $A_{k_1} \otimes \cdots \otimes A_{k_n}$ have degree $k_1 + \cdots + k_n$, and which makes it a graded bialgebra. If $A_0 = 0$, then $\mathcal{T}^\vee(A)$ is a graded connected Hopf algebra.

If $V$ is a vector space, we may endow it with the trivial product (the zero map $V \otimes V \to V$). In this case we recover the shuffle Hopf algebra $\mathcal{T}^\vee(V)$ of Section 2.6.1.

## 2.7. Chain complexes

In this section, we review (co)chain complexes, which constitute the basic objects of homological algebra. Our goal is to recall that familiar objects such as differential graded algebras are monoids in the category of chain complexes equipped with the Cauchy product. This discussion is taken up and developed further in Chapter 5.

**2.7.1. Homogeneous maps between graded vector spaces.** In order to define chain complexes, we need to allow more morphisms among graded vector spaces than the degree-preserving maps.

Let $K$ and $L$ be graded vector spaces, and $k \in \mathbb{Z}$. A *homogeneous map* $f\colon K \to L$ *of degree* $k$ is a family of linear maps

$$f_n\colon K_n \to L_{n+k},$$

one for each $n \geq 0$. We make the convention that $K_n = 0$ for any $n < 0$.

Note that $k$ is independent of $n$. Thus homogeneous maps of degree $k$ alter the degree by an arbitrary (but constant) amount $k \in \mathbb{Z}$. Homogeneous maps of degree 0 are morphisms of graded vector spaces.

The Cauchy product of a homogeneous map $f\colon K \to L$ of degree $k$ with a homogeneous map $g\colon M \to N$ of degree $h$ is the homogeneous map $K \cdot M \to L \cdot N$ of degree $k + h$ whose $n$-component is the sum of the maps

$$K_i \otimes M_j \xrightarrow{f_i \otimes g_j} L_{i+k} \otimes N_{j+h}$$

for $i + j = n$.

The dual of a homogeneous map $f\colon K \to L$ of degree $k$ is the homogeneous map $f^*\colon L^* \to K^*$ of degree $-k$ with components

$$(f^*)_n\colon (L^*)_n \xrightarrow{(f_{n-k})^*} (K^*)_{n-k}.$$

**2.7.2. Chain complexes.** A *chain complex* is a graded vector space $K = (K_n)_{n\geq 0}$ equipped with a homogeneous map of degree $-1$

$$\partial\colon K \to K$$

such that $\partial^2 = 0$. In other words, there are linear maps

$$\partial_n\colon K_n \to K_{n-1} \quad \text{such that } \partial_{n-1}\partial_n = 0$$

for every $n \geq 1$. The maps $\partial_n$ are called *boundary maps*. We often use the same notation $\partial$ to denote the boundary maps of different chain complexes. We also define $\partial_0 := 0$, for convenience.

A *cochain complex* is a graded vector space $K = (K^n)_{n\geq 0}$ equipped with a homogeneous map of degree $+1$

$$d\colon K \to K$$

such that $d^2 = 0$. In other words, there are linear maps

$$d^n\colon K^n \to K^{n+1} \quad \text{such that } d^{n+1}d^n = 0$$

for every $n \geq 0$. The maps $d^n$ are called *coboundary maps*.

A morphism of chain complexes $f\colon K \to L$ is a map of the underlying graded vector spaces commuting with the boundary maps, that is such that

$$\begin{tikzcd} K_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & K_{n-1} \arrow[d, "f_{n-1}"] \\ L_n \arrow[r, "\partial_n"'] & L_{n-1} \end{tikzcd}$$

commutes for every $n \geq 1$. This defines the category $\mathsf{dgVec_a}$ of chain complexes. The category $\mathsf{dgVec^c}$ of cochain complexes is defined similarly.

The maps $\partial$ or $d$ are also referred to as the differential. Consequently (co)chain complexes are also called *differential graded vector spaces*; hence the notation. The subscript and superscript are used to distinguish between the two cases. They stand for annihilation and creation, as in Section 2.8.

**2.7.3. The Cauchy product.** The Cauchy product of two chain complexes $K$ and $L$ is the chain complex, whose underlying graded vector space is $K \cdot L$ and whose boundary maps are defined by

$$\partial_n(a \otimes b) := \partial_i(a) \otimes b + (-1)^i\, a \otimes \partial_j(b) \tag{2.70}$$

for $a \in K_i$, $b \in L_j$. The unit object is 1, that is, the graded vector space $\Bbbk$ concentrated on degree 0, equipped with the zero map. The symmetry $\beta_{-1}$ consists of the maps

$$K_i \otimes L_j \to L_j \otimes K_i, \qquad a \otimes b \mapsto (-1)^{ij}\, b \otimes a. \tag{2.71}$$

In other words, the braiding is as in (2.50) with $q = -1$. This turns $\mathsf{dgVec_a}$ into a symmetric monoidal category.

The same construction as above applies to $\mathsf{dgVec^c}$.

**2.7.4. Monoids and comonoids.** A monoid in $(\mathsf{dgVec_a}, \cdot)$ is a *differential graded algebra* with a differential of degree $-1$. More explicitly, it is a graded algebra $(K, \mu, \iota)$ equipped with a graded derivation $\partial\colon K \to K$ of degree $-1$. In other words, $\partial$ is a homogeneous map of degree $-1$ and the following diagram commutes.

$$
\begin{tikzcd}
K_i \otimes K_j \arrow[r, "\partial_i \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes \partial_j"] \arrow[d, "\mu"'] & K_{i-1} \otimes K_j \oplus K_i \otimes K_{j-1} \arrow[d, "\mu"] \\
K_{i+j} \arrow[r, "\partial_{i+j}"'] & K_{i+j-1}
\end{tikzcd}
\tag{2.72}
$$

Similarly, a comonoid in $(\mathsf{dgVec_a}, \cdot)$ is a *differential graded coalgebra* with a differential of degree $-1$, that is, a graded coalgebra with a graded coderivation of degree $-1$. A bimonoid in $(\mathsf{dgVec_a}, \cdot, \beta_{-1})$ is a *differential graded bialgebra* with a differential of degree $-1$. The differential is a derivation with respect to the algebra structure and a coderivation with respect to the coalgebra structure.

(Co, Bi)monoids in $(\mathsf{dgVec^c}, \cdot)$ have a similar description with the degree of the differential being $+1$.

If $K$ is a chain complex, its *dual* $K^*$ is the cochain complex with

$$(K^*)^n := \mathrm{Hom}_{\Bbbk}(K_n, \Bbbk) \qquad \text{and} \qquad d^n := \mathrm{Hom}_{\Bbbk}(\partial_{n+1}, \Bbbk).$$

Duality exchanges monoids and comonoids as usual.

**2.7.5. Homotopy and homology.** We recall the notions of chain homotopy, homology, and cohomology.

Two morphisms of complexes $f, g\colon K \to L$ are said to be *chain homotopic* if there exists a sequence of maps $s_n\colon K_n \to L_{n+1}$ such that

$$f_n - g_n = \partial_{n+1} s_n + s_{n-1} \partial_n$$

$$
\begin{tikzcd}
K_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & K_n \arrow[r, "\partial_n"] \arrow[dl, "s_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & K_{n-1} \arrow[dl, "s_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \\
L_{n+1} \arrow[r, "\partial_{n+1}"'] & L_n \arrow[r, "\partial_n"'] & L_{n-1}
\end{tikzcd}
$$

Chain homotopy is an equivalence relation on the set of morphisms from $K$ to $L$ in $\mathsf{dgVec_a}$ and it is compatible with the category structure. The homotopy category $\overline{\mathsf{dgVec_a}}$ of chain complexes has the same objects as the category $\mathsf{dgVec_a}$ (chain complexes) and a morphism from $K$ to $L$ in $\overline{\mathsf{dgVec_a}}$ is a chain homotopy class of morphisms from $K$ to $L$ in $\mathsf{dgVec_a}$. It is also a symmetric monoidal category via (2.70) and (2.71). The same construction applies to $\mathsf{dgVec^c}$. This yields functors

$$\mathsf{dgVec_a} \to \overline{\mathsf{dgVec_a}} \qquad \text{and} \qquad \mathsf{dgVec^c} \to \overline{\mathsf{dgVec^c}} \tag{2.73}$$

which are the identity on objects and which send a morphism to its chain homotopy class.

The *homology* of a chain complex $K$ is the graded $\Bbbk$-module with components

$$\mathcal{H}_n(K) := \ker(\partial_n\colon K_n \to K_{n-1}) / \mathrm{im}(\partial_{n+1}\colon K_{n+1} \to K_n).$$

Similarly, the *cohomology* of a cochain complex $K$ is the graded $\Bbbk$-module $\mathcal{H}^\bullet(K)$ with components

$$\mathcal{H}^n(K) := \ker(d^n\colon K^n \to K^{n+1})/\mathrm{im}(d^{n-1}\colon K^{n-1} \to K^n).$$

A morphism of chain complexes $f\colon K \to L$ gives rise to a map

$$\mathcal{H}_n(f)\colon \mathcal{H}_n(K) \to \mathcal{H}_n(L),$$

and two chain homotopic morphisms induce the same maps in homology. The same statement holds for cochain complexes. This defines functors

$$\mathcal{H}_\bullet\colon \overline{\mathsf{dgVec_a}} \to \mathsf{gVec} \quad \text{and} \quad \mathcal{H}^\bullet\colon \overline{\mathsf{dgVec^c}} \to \mathsf{gVec}. \tag{2.74}$$

We call them the homology and cohomology functors respectively. Their monoidal properties are discussed in Section 5.5.3.

**2.7.6. Cohomology of algebras and coalgebras.** We conclude by recalling the notions of Hochschild cohomology for algebras and the dual version for coalgebras.

Let $A$ be a $\Bbbk$-algebra and $M$ an $A$-bimodule (Section 1.2.3). For each $n \geq 0$, let

$$C^n(A, M) := \mathrm{Hom}_\Bbbk(A^{\otimes n}, M).$$

We identify it with the space of all multilinear maps from $A^{\times n}$ to $M$. In particular, $C^0(A, M) = M$.

Define also

$$d^n\colon C^n(A, M) \to C^{n+1}(A, M)$$

by

$$\begin{aligned} d^n(f)(a_1, \dots, a_{n+1}) := {}& a_1 \cdot f(a_2, \dots, a_n) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_1, \dots, a_n) \cdot a_{n+1}, \end{aligned}$$

for $f\colon A^{\times n} \to M$ and $a_i \in A$. The bimodule structure is used in the first and last terms, and the algebra structure in the middle terms.

Then $d^2 = 0$ and the graded vector space $C(A, M)$ is a cochain complex equipped with the differential $d$. The cohomology of this complex is the *Hochschild cohomology* of the algebra $A$ with coefficients in the bimodule $A$. It was introduced by Hochschild [168]. For modern treatments, see [237] or [372].

Let $C$ be a $\Bbbk$-coalgebra and $M$ a $C$-bicomodule with structure maps

$$\chi^1\colon M \to C \otimes M \quad \text{and} \quad \chi^2\colon M \to M \otimes C$$

(Section 1.2.3). For each $n \geq 0$, let

$$C^n(C, M) := \mathrm{Hom}_\Bbbk(M, C^{\otimes n}).$$

Define

$$d^n\colon C^n(C, M) \to C^{n+1}(C, M)$$

by

$$d^n(f) := (\mathrm{id} \otimes f) \circ \chi^1 + \sum_{i=1}^{n} (-1)^i (\mathrm{id}^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(n-i)}) \circ f + (-1)^{n+1} (f \otimes \mathrm{id}) \circ \chi^2,$$

for $f\colon M \to C^{\otimes n}$. We obtain a cochain complex $C(C, M)$ with differential $d$. The resulting coalgebra cohomology, or related notions, appear in work of Adams [4] and Cartier [75]. A more recent treatment is given by Doi [100].

## 2.8. Graded vector spaces with creation-annihilation operators

We now discuss a variation of (co)chain complexes where the nilpotency condition on the (co)boundary maps is dropped. Such objects are known as graded vector spaces with creation or annihilation operators. This terminology comes from physics. The present discussion is continued and related to species in Chapter 19. The motivation for our terminology is made clearer at that point.

**2.8.1. Creation-annihilation operators.** A graded vector space with *creation operators* is a graded vector space $V$ with a homogeneous map of degree $+1$

$$c\colon V \to V.$$

A graded vector space with *annihilation operators* is a graded vector space $V$ with a homogeneous map of degree $-1$

$$a\colon V \to V.$$

A morphism of graded vector spaces with creation operators $(V, c) \to (W, d)$ is a morphism of graded vector spaces $V \to W$ that intertwines $c$ with $d$. This defines the category $\mathsf{gVec}^{\mathsf{c}}$ of graded vector spaces with creation operators. The categories $\mathsf{gVec}_{\mathsf{a}}$ of graded vector spaces with annihilation operators and $\mathsf{gVec}^{\mathsf{c}}_{\mathsf{a}}$ of graded vector spaces with creation-annihilation operators are defined similarly.

**2.8.2. The Cauchy product.** Define the Cauchy product of two graded vector spaces with creation operators $(V, c)$ and $(W, d)$ to be $(V \cdot W, e)$ where

$$e\colon V \cdot W \xrightarrow{c\cdot \mathrm{id}+\mathrm{id}\cdot d} V \cdot W. \tag{2.75}$$

Note that $e$ is a homogeneous map of degree $+1$.

This turns $\mathsf{gVec}^{\mathsf{c}}$ into a symmetric monoidal category which we denote by $(\mathsf{gVec}^{\mathsf{c}}, \cdot, \beta)$. The unit object is $1$, that is, the graded vector space $\Bbbk$ concentrated on degree 0, equipped with the zero map. The symmetry $\beta$ is given by interchanging the tensor factors.

Let $q \in \Bbbk$ be a fixed scalar. The $q$-deformed Cauchy product of $(V, c)$ and $(W, d)$ is defined to be $(V \cdot W, e)$ where $e$ is given by the formula

$$e(v \otimes w) := c(v) \otimes w + q^i v \otimes d(w)$$

for $v \in V_i$, $w \in W_j$. Define also a map $\tau\colon V \to V$ by

$$\tau(v) := q^i v \tag{2.76}$$

for every $v \in V_i$. The definition of the creation operator of $V \cdot W$ can be rewritten as:

$$e = c \cdot \mathrm{id} + \tau \cdot d. \tag{2.77}$$

To emphasize the dependence on $q$, we write $\cdot_q$ for the deformed Cauchy product. This defines a monoidal category $(\mathsf{gVec}^{\mathsf{c}}, \cdot_q)$.

We point out that we do not have a braided monoidal category in this generality. This point is discussed in more detail in the next section. From this point of view, along with $q = 1$, the case $q = -1$ is of interest. Here we do obtain a symmetric monoidal category which we denote by $(\mathsf{gVec}^{\mathsf{c}}, \cdot_{-1}, \beta_{-1})$, with braiding as in (2.71).

The above constructions apply to $\mathsf{gVec_a}$ and $\mathsf{gVec_a^c}$ as well.

**2.8.3. Monoids and comonoids.** A monoid in $(\mathsf{gVec^c}, \cdot)$ is a graded algebra $(A, \mu, \iota)$ equipped with a derivation $c\colon A \to A$ of degree $+1$. In other words, $c$ is a homogeneous map of degree $+1$ and the following diagram commutes.

$$\begin{tikzcd} A \cdot A \arrow[r, "c\cdot \mathrm{id} + \mathrm{id}\cdot c"] \arrow[d, "\mu"'] & A \cdot A \arrow[d, "\mu"] \\ A \arrow[r, "c"'] & A \end{tikzcd} \tag{2.78}$$

It follows from (2.78) plus unitality of $\mu$ that $c(1) = 0$.

A comonoid in $(\mathsf{gVec^c}, \cdot)$ is a graded coalgebra $(C, \Delta, \epsilon)$ equipped with a coderivation $c\colon C \to C$ of degree $+1$.

For describing (co)monoids in $(\mathsf{gVec_a}, \cdot)$, one uses (co)derivations of degree $-1$, all other things being same.

For describing (co)monoids with respect to the tensor product $\cdot_q$, one needs a $q$-version of (co)derivations. These are obtained by replacing the map

$$c \cdot \mathrm{id} + \mathrm{id} \cdot c \quad \text{by} \quad c \cdot \mathrm{id} + \tau \cdot c$$

in the usual definition, with $\tau$ as in (2.76). We have already seen the $q = -1$ case before in (2.72). It is customary to call $q$-(co)derivations as skewed (co)derivations and $(-1)$-(co)derivations as graded (co)derivations.

The dual $(V^*, c^*)$ of a graded vector space $(V, c)$ with creation operators is a graded vector space with annihilation operators, and viceversa. Duality exchanges monoids and comonoids as usual.

## 2.9. *N*-complexes

In this section, we discuss $N$-complexes. These generalize chain complexes in a natural way and provide the basic objects for a $q$-analogue of homological algebra. These ideas can be traced to Mayer [263] and Spanier [334]. More recent work on $N$-complexes can be found in the papers of Kapranov [187], Dubois-Violette [104, 105], Kassel and Wambst [192], and references therein.

We return to $N$-complexes only in Chapter 5. However, they do serve a purpose at this juncture as well. Namely, they provide a unifying framework for graded vector spaces, chain complexes, as well as graded vector spaces with annihilation operators. This serves to clarify the similarities as well as the differences between the three situations discussed in the previous sections.

**2.9.1. *N*-complexes.** Let $N$ be a positive integer or $\infty$. An $N$-*complex* [187] is a graded vector space $K$ equipped with a homogeneous map of degree $-1$

$$\partial\colon K \to K$$

that is nilpotent of order at most $N$:

$$\partial^N = 0.$$

When $N = \infty$, we interpret this condition as vacuous.

A morphism of $N$-complexes is a map of the underlying graded vector spaces commuting with the boundary maps. Let $\mathsf{dgVec}_N$ denote the resulting category.

TABLE 2.3. Monoidal categories related to graded vector spaces.

| Present notation | Alternative notation | Description |
|---|---|---|
| $\mathsf{dgVec}^1 = \mathsf{dgVec}_1$ | $\mathsf{gVec}$ | Graded vector spaces |
| $\mathsf{dgVec}_2$ | $\mathsf{dgVec}_\mathsf{a}$ | Chain complexes |
| $\mathsf{dgVec}^2$ | $\mathsf{dgVec}^\mathsf{c}$ | Cochain complexes |
| $\mathsf{dgVec}_\infty$ | $\mathsf{gVec}_\mathsf{a}$ | Annihilation operators |
| $\mathsf{dgVec}^\infty$ | $\mathsf{gVec}^\mathsf{c}$ | Creation operators |

The category $\mathsf{dgVec}^N$ is defined similarly using maps of degree 1. For simplicity of exposition, we only deal with $\mathsf{dgVec}_N$.

Note that a 1-complex is simply a graded vector space and a 2-complex is a chain complex. At the opposite end, an $\infty$-complex is a graded vector space equipped a homogeneous map of degree $-1$, without further requirements. This is precisely, a graded vector space with annihilation operators. Thus, $N$-complexes provide a unified framework for the objects considered earlier in this chapter. These observations are summarized in Table 2.3.

**2.9.2. The Cauchy product.** Given two $\infty$-complexes $K$ and $L$, on the tensor product $K \cdot L$ of the underlying graded vector spaces we define a new boundary map by the formula

$$\partial_n(a \otimes b) := \partial_i(a) \otimes b + q^i \, a \otimes \partial_j(b)$$

for $a \in K_i$, $b \in L_j$, where $q \in \Bbbk$ is the parameter that has been fixed from the start. The definition of the boundary map of $K \cdot L$ can be rewritten as follows:

$$\partial_{K \cdot L} = \partial_K \cdot \mathrm{id}_L + \tau_K \cdot \partial_L, \tag{2.79}$$

where $\tau$ is as defined in (2.76). Since the boundary map is of degree $-1$, we have

$$\partial_K \tau_K = q \, \tau_K \partial_K.$$

It follows from the quantum binomial theorem [191, Proposition IV.2.2] that

$$(\partial_{K \cdot L})^N = \sum_{i=0}^{N} \binom{N}{i}_q (\tau_K)^i (\partial_K)^{N-i} \cdot (\partial_L)^i,$$

where $\binom{N}{i}_q$ is the $q$-binomial coefficient (2.27).

Suppose now that the following hypotheses are satisfied:

$$2 \le N < \infty \text{ and } q \in \Bbbk \text{ is a primitive } N\text{-th root of unity.} \tag{2.80}$$

In this case Lemma 2.6 says that $\binom{N}{i}_q = 0$ for all $i = 1, \ldots, N-1$. As a result,

$$(\partial_{K \cdot L})^N = (\partial_K)^N \cdot \mathrm{id} + (\tau_K)^N \cdot (\partial_L)^N.$$

It follows that the the tensor product of two $N$-complexes is another $N$-complex. This assertion is clearly true if $N = \infty$ or $N = 1$, for any $q$. For $N = 1$, the boundary maps are always zero and the tensor product is independent of $q$.

When we need to emphasize the role of $q$ in the above definition, we use $K \cdot_q L$ to denote the tensor product of two $N$-complexes $K$ and $L$. To summarize:

**Proposition 2.24.** *For $N = \infty$ or $N = 1$ and $q$ arbitrary, or for $2 \leq N < \infty$ and $q$ as in (2.80),*

$$(\mathsf{dgVec}_N, \cdot_q, 1)$$

*is a monoidal category. The unit object 1 is defined as for chain complexes.*

**Remark 2.25.** We have been working under the assumption that $\Bbbk$ is a field. However, it is useful to note that the above result is true in greater generality: for $N = \infty$ or $N = 1$, if $\Bbbk$ is any commutative ring, and for $2 \leq N < \infty$, if $\Bbbk$ is an integral domain.

**2.9.3. Braided monoidal categories of $N$-complexes.** We saw above conditions under which $\mathsf{dgVec}_N$ is a monoidal category. Now we would like to view it as a braided monoidal category under the morphism of graded vector spaces $\beta_q$ of (2.50). However, this is not possible. In this regard, we have the following result.

**Proposition 2.26.** *Suppose that $N = \infty$ or $N = 1$, or else that hypotheses (2.80) are satisfied, so that $\mathsf{dgVec}_N$ is a monoidal category by Proposition 2.24. Let $K$ and $L$ be two $N$-complexes. Suppose $q$ is invertible in $\Bbbk$. The map*

$$\beta_q \colon K \cdot_q L \to L \cdot_{q^{-1}} K$$

*is an isomorphism of $N$-complexes.*

Proof. We have to check that $\beta_q$ commutes with the boundary maps. This follows from the commutativity of the following two diagrams, in view of (2.79).

$$\begin{tikzcd} K_i \otimes L_j \arrow[r, "\partial\otimes\mathrm{id}"] \arrow[d, "\beta_q"'] & K_{i-1} \otimes L_j \arrow[d, "\beta_q"] \\ L_j \otimes K_i \arrow[r, "\tau^{-1}\otimes\partial"'] & L_j \otimes K_{i-1} \end{tikzcd} \qquad \begin{tikzcd} K_i \otimes L_j \arrow[r, "\tau\otimes\partial"] \arrow[d, "\beta_q"'] & K_i \otimes L_{j-1} \arrow[d, "\beta_q"] \\ L_j \otimes K_i \arrow[r, "\partial\otimes\mathrm{id}"'] & L_{j-1} \otimes K_i \end{tikzcd}$$

Take $a \otimes b \in K_i \otimes L_j$. Both sides of the first diagram lead to

$$q^{(i-1)j}\, b \otimes \partial(a)$$

and both sides of the second diagram lead to

$$q^{ij}\, \partial(b) \otimes a.$$

Thus, both diagrams commute. □

**Proposition 2.27.** *For $(\mathsf{dgVec}_N, \cdot_q, \beta_q)$ to be a braided monoidal category, it has to be one of*

$$(\mathsf{gVec}, \cdot, \beta_q), \quad (\mathsf{dgVec}_{\mathsf{a}}, \cdot, \beta_{-1}), \quad (\mathsf{gVec}_{\mathsf{a}}, \cdot, \beta), \quad \text{or} \quad (\mathsf{gVec}_{\mathsf{a}}, \cdot_{-1}, \beta_{-1}).$$

Proof. For $N = 1$, the tensor product is independent of $q$, thus from Proposition 2.26 we see that $\beta_q$ is a candidate for braiding for all $q$. The braiding axioms (1.5) are easily verified and one obtains the braided monoidal category $(\mathsf{gVec}, \cdot, \beta_q)$.

In the remaining cases, $q$ must satisfy $q^2 = 1$; thus $q = \pm 1$. For $N = \infty$, both choices work and one obtains $(\mathsf{gVec}_{\mathsf{a}}, \cdot, \beta)$ and $(\mathsf{gVec}_{\mathsf{a}}, \cdot_{-1}, \beta_{-1})$. For $2 \leq N < \infty$, $q$ must be a primitive $N$-th root of unity, which forces $N = 2$. This yields $(\mathsf{dgVec}_{\mathsf{a}}, \cdot, \beta_{-1})$. □

All of these examples have been considered in earlier sections, so we do not get anything new; however, we do see how they fit together in the theory of $N$-complexes.

**Remark 2.28.** Let us now see how Proposition 2.27 works for more general scalars. This complements the discussion in Remark 2.25.

- The $N = 1$ case works the same way even if $\Bbbk$ is a commutative ring.
- The $N = \infty$ case may be different: If $\Bbbk$ is a commutative ring, then $q^2 = 1$ may have solutions other than $\pm 1$. Any such $q$ yields a symmetric monoidal category $(\mathsf{gVec_a}, \cdot_q, \beta_q)$.
- The $2 \leq N < \infty$ case works the same way: If $\Bbbk$ is an integral domain, then $q^2 = 1$ implies $q = \pm 1$.

**Remark 2.29.** The calculations in [329] indicate that if $q^2 \neq 1$, one cannot hope to turn $\mathsf{dgVec}_N$ into a braided monoidal category with tensor product $\cdot_q$, not even a 2-monoidal category with tensor products $\cdot_q$ and $\cdot_{q^{-1}}$. The latter concept generalizes a braided monoidal category and is discussed in Chapter 6.

CHAPTER 3

# Monoidal Functors

The main goal of this chapter is to study appropriate notions of functors between monoidal categories.

Complementing the classical notions of lax and colax monoidal functors, we provide a definition of bilax monoidal functors between braided monoidal categories (Section 3.1). The Fock functors which occupy us throughout Part III of this monograph are all examples of bilax monoidal functors. Another very important example, this one of a classical nature, is the object of Chapter 5. A summary of these examples is given in Section 3.2. This section also discusses some other interesting examples.

An important property of bilax monoidal functors is that they preserve bimonoids. More generally, the composite of two bilax monoidal functors is again bilax monoidal. These properties are discussed in Sections 3.3 and 3.4. Additional properties of bilax monoidal functors are studied in Section 3.5; these involve a normalization condition with interesting consequences. Monoidal functors admit a strong version, in which the structure transformations are required to be invertible. These are studied in Section 3.6. Section 3.7 deals with Hopf lax functors. These are bilax monoidal functors with additional properties that ensure that they preserve Hopf monoids. We know that a bimonoid can be viewed as a monoid in a category of comonoids and viceversa. A similar interpretation for bilax monoidal functors is given in Section 3.8. We study adjunctions in the context of monoidal categories in Section 3.9. Section 3.10 discusses a construction which, in a certain context, allows us to dualize monoidal categories and functors.

In an abelian category, any morphism admits a monic-epi factorization. This idea can be expanded in various ways in the context of abelian monoidal categories. In particular, a morphism between (co, bi) lax monoidal functors can be similarly factorized. This gives rise to a new monoidal functor, the *image* of the given morphism. This important construction is the object of Section 3.11.

## 3.1. Bilax monoidal functors

Two kinds of morphisms one can consider between monoidal categories are the lax and colax monoidal functors introduced by Bénabou [36]. We follow the terminology of Kelly and Street [199, pp. 83–84]; see also Leinster [226, Definition 1.2.10] and Yetter [379, Definition 3.11]. We recall these notions below. Moreover, if the categories are braided, then one can define the notion of bilax monoidal functor, which appropriately combines the notions of lax and colax monoidal functors. This concept is very natural; however, it seems hard to find a reference where it is discussed. It is of central importance to our work in the rest of the monograph. We provide a definition in this section. The rest of the chapter is devoted to the study of basic properties of lax, colax and bilax monoidal functors.

Lax, colax and bilax monoidal functors may be regarded as analogues of monoids, comonoids and bimonoids, respectively. We make this analogy precise in Section 3.4.

**3.1.1. Lax, colax, and bilax monoidal functors.** Let $(\mathsf{C}, \bullet)$ and $(\mathsf{D}, \bullet)$ be two monoidal categories and $\mathcal{F}$ be a functor from $\mathsf{C}$ to $\mathsf{D}$. We denote the unit object in both categories by $I$ and write $\mathcal{M}$ for the tensor product functors. Let

$$\mathcal{F}^2 := \mathcal{M} \circ (\mathcal{F} \times \mathcal{F}) \quad \text{and} \quad \mathcal{F}_2 := \mathcal{F} \circ \mathcal{M}; \tag{3.1}$$

they are functors from $\mathsf{C} \times \mathsf{C}$ to $\mathsf{D}$. Let $\mathsf{I}$ be the one-arrow category and let

$$\mathcal{F}^0 \colon \mathsf{I} \to \mathsf{D} \quad \text{and} \quad \mathcal{F}_0 \colon \mathsf{I} \to \mathsf{D} \tag{3.2}$$

be the functors that send the unique object of $\mathsf{I}$ to $I$ and $\mathcal{F}(I)$ respectively.

**Definition 3.1.** We say that a functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ is *lax monoidal* if there is a natural transformation

$$\mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\varphi_{A,B}} \mathcal{F}(A \bullet B) \tag{3.3}$$

from the functor $\mathcal{F}^2$ to the functor $\mathcal{F}_2$ and a map

$$\varphi_0 \colon I \to \mathcal{F}(I) \tag{3.4}$$

in $\mathsf{D}$ such that the conditions below are satisfied. Observe that one may view $\varphi_0$ as a natural transformation between $\mathcal{F}^0$ and $\mathcal{F}_0$.

**Associativity.** The transformation $\varphi$ is associative, in the sense that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \arrow[r, "\mathrm{id} \bullet \varphi_{B,C}"] \arrow[d, "\varphi_{A,B} \bullet \mathrm{id}"'] & \mathcal{F}(A) \bullet \mathcal{F}(B \bullet C) \arrow[d, "\varphi_{A,B \bullet C}"] \\
\mathcal{F}(A \bullet B) \bullet \mathcal{F}(C) \arrow[r, "\varphi_{A \bullet B,C}"'] & \mathcal{F}(A \bullet B \bullet C)
\end{tikzcd} \tag{3.5}$$

**Unitality.** The transformation $\varphi$ is left and right unital, in the sense that the following diagrams commute.

$$\begin{tikzcd}
I \bullet \mathcal{F}(A) \arrow[d, "\varphi_0 \bullet \mathrm{id}"'] & \mathcal{F}(A) \arrow[l, "\lambda_{\mathcal{F}(A)}"'] \arrow[d, "\mathcal{F}(\lambda_A)"] \\
\mathcal{F}(I) \bullet \mathcal{F}(A) \arrow[r, "\varphi_{I,A}"'] & \mathcal{F}(I \bullet A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(A) \bullet I \arrow[d, "\mathrm{id} \bullet \varphi_0"'] & \mathcal{F}(A) \arrow[l, "\rho_{\mathcal{F}(A)}"'] \arrow[d, "\mathcal{F}(\rho_A)"] \\
\mathcal{F}(A) \bullet \mathcal{F}(I) \arrow[r, "\varphi_{A,I}"'] & \mathcal{F}(A \bullet I)
\end{tikzcd} \tag{3.6}$$

The above three diagrams are the analogues of the associativity and unit axioms for a monoid.

**Definition 3.2.** We say that a functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ is *colax monoidal* if there is a natural transformation

$$\mathcal{F}(A \bullet B) \xrightarrow{\psi_{A,B}} \mathcal{F}(A) \bullet \mathcal{F}(B) \tag{3.7}$$

and a map

$$\psi_0 \colon \mathcal{F}(I) \to I \tag{3.8}$$

satisfying axioms dual to those in Definition 3.1. Namely, one replaces $\varphi$ by $\psi$ and reverses the arrows with those labels in diagrams (3.5) and (3.6).

**Definition 3.3.** Let $(\mathsf{C}, \bullet, \beta)$ and $(\mathsf{D}, \bullet, \beta)$ be two braided monoidal categories. We say that a functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is *bilax monoidal* if there are natural transformations $\varphi$ and $\psi$,

$$\begin{tikzcd} \mathcal{F}(A) \bullet \mathcal{F}(B) \arrow[r, bend left, "\varphi_{A,B}"] & \mathcal{F}(A \bullet B), \arrow[l, bend left, "\psi_{A,B}"] \end{tikzcd} \tag{3.9}$$

between the functors $\mathcal{F}^2$ and $\mathcal{F}_2$ defined in (3.1), and morphisms

$$\varphi_0\colon I \to \mathcal{F}(I) \quad \text{and} \quad \psi_0\colon \mathcal{F}(I) \to I \tag{3.10}$$

in $\mathsf{D}$ such that $(\mathcal{F}, \varphi)$ is lax, $(\mathcal{F}, \psi)$ is colax and the conditions below are satisfied. Note that $\varphi_0$ and $\psi_0$ are natural transformations between the functors $\mathcal{F}^0$ and $\mathcal{F}_0$ defined in (3.2).

**Braiding.** The following hexagon commutes.

$$\begin{tikzcd}[column sep=small] & \mathcal{F}(A \bullet B) \bullet \mathcal{F}(C \bullet D) \arrow[dl, "\varphi_{A\bullet B, C\bullet D}"'] \arrow[dr, "\psi_{A,B} \bullet \psi_{C,D}"] & \\ \mathcal{F}(A \bullet B \bullet C \bullet D) \arrow[d, "\mathcal{F}(\mathrm{id}\bullet\beta\bullet\mathrm{id})"'] & & \mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \bullet \mathcal{F}(D) \arrow[d, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"] \\ \mathcal{F}(A \bullet C \bullet B \bullet D) \arrow[dr, "\psi_{A\bullet C, B\bullet D}"'] & & \mathcal{F}(A) \bullet \mathcal{F}(C) \bullet \mathcal{F}(B) \bullet \mathcal{F}(D) \arrow[dl, "\varphi_{A,C} \bullet \varphi_{B,D}"] \\ & \mathcal{F}(A \bullet C) \bullet \mathcal{F}(B \bullet D) & \end{tikzcd} \tag{3.11}$$

where $\beta$ denotes the braiding in either category.

**Unitality.** The following diagrams commute.

$$\begin{tikzcd} I \arrow[r, "\varphi_0"] \arrow[d, "\lambda_I"'] & \mathcal{F}(I) \arrow[r, "\mathcal{F}(\lambda_I)"] & \mathcal{F}(I \bullet I) \arrow[d, "\psi_{I,I}"] \\ I \bullet I \arrow[rr, "\varphi_0 \bullet \varphi_0"'] & & \mathcal{F}(I) \bullet \mathcal{F}(I) \end{tikzcd} \qquad \begin{tikzcd} I & \mathcal{F}(I) \arrow[l, "\psi_0"'] & \mathcal{F}(I \bullet I) \arrow[l, "\mathcal{F}(\lambda_I^{-1})"'] \\ I \bullet I \arrow[u, "\lambda_I^{-1}"] & & \mathcal{F}(I) \bullet \mathcal{F}(I) \arrow[ll, "\psi_0 \bullet \psi_0"] \arrow[u, "\varphi_{I,I}"'] \end{tikzcd} \tag{3.12}$$

$$\begin{tikzcd} & \mathcal{F}(I) \arrow[dr, "\psi_0"] & \\ I \arrow[ur, "\varphi_0"] \arrow[rr, equal] & & I \end{tikzcd} \tag{3.13}$$

The above four diagrams are the analogues of the compatibility axioms for a bimonoid given in Definition 1.10. In (3.12), we may write $\rho_I$ instead of $\lambda_I$, in view of (1.3).

**Notation 3.4.** For the sake of brevity we may sometimes omit the word "monoidal" and refer to the above classes of functors simply as lax, colax, or bilax. Suppose that $\mathcal{F}$ is a lax functor with structure maps $\varphi_{A,B}$ and $\varphi_0$, as in Definition 3.1. In order to denote this lax functor we may use $(\mathcal{F}, \varphi, \varphi_0)$, or $(\mathcal{F}, \varphi)$, or simply $\mathcal{F}$, if the structure maps are understood. A similar convention applies to colax and bilax functors.

**Definition 3.5.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor, $\varphi$ and $\psi$ be transformations as in (3.3) and (3.7), and $\varphi_0$, and $\psi_0$ maps as in (3.4) and (3.8). We say that $(\mathcal{F}, \varphi)$ is *strong* if it is lax and $\varphi$ and $\varphi_0$ are invertible. We say that $(\mathcal{F}, \psi)$ is *costrong* if it is colax and $\psi$ and $\psi_0$ are invertible. We say that $(\mathcal{F}, \varphi, \psi)$ is *bistrong* if it is bilax and $\varphi$, $\psi$, $\varphi_0$ and $\psi_0$ are all invertible.

Note that $(\mathcal{F}, \varphi)$ is strong if and only if $(\mathcal{F}, \varphi^{-1})$ is costrong. Strong functors are studied in more depth in Section 3.6. In Proposition 3.45 we show that if $(\mathcal{F}, \varphi, \psi)$ is bistrong, then $\varphi = \psi^{-1}$.

We turn to basic constructions involving monoidal functors.

**Proposition 3.6.** *If $(\mathcal{F}, \varphi)\colon \mathsf{C} \to \mathsf{D}$ and $(\mathcal{F}', \varphi')\colon \mathsf{C}' \to \mathsf{D}'$ are lax (resp. colax), then so is*

$$\big(\mathcal{F} \times \mathcal{F}', \varphi \times \varphi'\big)\colon \mathsf{C} \times \mathsf{C}' \to \mathsf{D} \times \mathsf{D}'.$$

*Further, if $(\mathcal{F}, \varphi, \psi)\colon \mathsf{C} \to \mathsf{D}$ and $(\mathcal{F}', \varphi', \psi')\colon \mathsf{C}' \to \mathsf{D}'$ are bilax, then so is*

$$\big(\mathcal{F} \times \mathcal{F}', \varphi \times \varphi', \psi \times \psi'\big)\colon \mathsf{C} \times \mathsf{C}' \to \mathsf{D} \times \mathsf{D}'.$$

The above result is a straightforward consequence of the definitions.

**Proposition 3.7.** *Let $(\mathsf{C}^{\mathrm{op}}, \bullet, \beta^{\mathrm{op}})$ denote the opposite category of $(\mathsf{C}, \bullet, \beta)$. If $(\mathcal{F}, \varphi)\colon \mathsf{C} \to \mathsf{D}$ is lax (resp. colax) monoidal, then $(\mathcal{F}, \varphi)\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$ is colax (resp. lax) monoidal. Further, if $(\mathcal{F}, \varphi, \psi)\colon \mathsf{C} \to \mathsf{D}$ is bilax monoidal, then so is $(\mathcal{F}, \psi, \varphi)\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$.*

Proof. For the first assertion, observe that reversing the arrows labeled $\varphi$ in the diagrams for a lax functor yield the diagrams for a colax functor and viceversa. The diagrams for a bilax functor are preserved by switching $\varphi$ with $\psi$, and reversing the arrows with those labels. This proves the second assertion. $\square$

Thus passing to the opposite categories transforms a lax functor to a colax functor and viceversa, and preserves bilax functors.

### 3.1.2. Morphisms between monoidal functors.

**Definition 3.8.** Let $(\mathsf{C}, \bullet)$ and $(\mathsf{D}, \bullet)$ be two monoidal categories, and $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \gamma)$ be lax monoidal functors from $\mathsf{C}$ to $\mathsf{D}$. A morphism from $\mathcal{F}$ to $\mathcal{G}$ of lax monoidal functors is a natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ such that both diagrams below commute.

$$
\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\theta_A \bullet \theta_B"'] & \mathcal{F}(A \bullet B) \arrow[d, "\theta_{A \bullet B}"] \\
\mathcal{G}(A) \bullet \mathcal{G}(B) \arrow[r, "\gamma_{A,B}"'] & \mathcal{G}(A \bullet B)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{F}(I) \arrow[dd, "\theta_I"] \\
I \arrow[ur, "\varphi_0"] \arrow[dr, "\gamma_0"'] & \\
& \mathcal{G}(I)
\end{tikzcd}
\tag{3.14}
$$

Let $(\mathcal{F}, \psi)$ and $(\mathcal{G}, \delta)$ be colax monoidal functors from $\mathsf{C}$ to $\mathsf{D}$. A morphism from $\mathcal{F}$ to $\mathcal{G}$ of colax monoidal functors is a natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ such that both diagrams below commute.

$$
\begin{tikzcd}
\mathcal{F}(A \bullet B) \arrow[r, "\psi_{A,B}"] \arrow[d, "\theta_{A \bullet B}"'] & \mathcal{F}(A) \bullet \mathcal{F}(B) \arrow[d, "\theta_A \bullet \theta_B"] \\
\mathcal{G}(A \bullet B) \arrow[r, "\delta_{A,B}"'] & \mathcal{G}(A) \bullet \mathcal{G}(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(I) \arrow[dr, "\psi_0"] \arrow[dd, "\theta_I"'] & \\
& I \\
\mathcal{G}(I) \arrow[ur, "\delta_0"'] &
\end{tikzcd}
\tag{3.15}
$$

A morphism of (co)strong monoidal functors is a morphism of the underlying (co)lax monoidal functors.

**Definition 3.9.** Let $(\mathsf{C}, \bullet, \beta)$ and $(\mathsf{D}, \bullet, \beta)$ be two braided monoidal categories, and $\mathcal{F}, \mathcal{G}\colon \mathsf{C} \to \mathsf{D}$ be bilax monoidal functors. A morphism from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ such that diagrams (3.14) and (3.15) commute. In other words, a morphism of bilax functors is a morphism of the underlying lax and colax functors.

A morphism of bistrong monoidal functors is a morphism of the underlying bilax monoidal functors.

The following is straightforward.

**Proposition 3.10.** *The composite of two morphisms of lax* (*colax*, *bilax*) *monoidal functors is again a morphism of lax* (*colax*, *bilax*) *monoidal functors.*

Thus for two fixed monoidal categories, we have the categories of lax and colax functors between them. Similarly for two fixed braided monoidal categories, we have the category of bilax functors between them. We elaborate on this point in Section 3.3.3.

**3.1.3. Braided bilax monoidal functors.** We now define the notion that plays the role of commutativity for a bilax monoidal functor.

**Definition 3.11.** A lax (resp. colax) monoidal functor $(\mathcal{F}, \varphi)$ between two braided monoidal categories (resp. $(\mathcal{F}, \psi)$) is *braided* if the right-hand (resp. left-hand) diagram below commutes.

$$
\begin{tikzcd}
\mathcal{F}(A \bullet B) \arrow[r, "\psi_{A,B}"] \arrow[d, "\mathcal{F}(\beta)"'] & \mathcal{F}(A) \bullet \mathcal{F}(B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\beta"] & \mathcal{F}(A \bullet B) \arrow[d, "\mathcal{F}(\beta)"] \\
\mathcal{F}(B \bullet A) \arrow[r, "\psi_{B,A}"'] & \mathcal{F}(B) \bullet \mathcal{F}(A) \arrow[r, "\varphi_{B,A}"'] & \mathcal{F}(B \bullet A)
\end{tikzcd}
\tag{3.16}
$$

A morphism of braided lax (colax) monoidal functors is a morphism of the underlying lax (colax) monoidal functors.

**Example 3.12.** The functor Hom in Example 3.17 is a braided lax monoidal functor from $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ to $\mathsf{Set}$, where the former category is endowed with the braiding $\big((\beta^{-1})^{\mathrm{op}}, \beta\big)$.

**Definition 3.13.** A bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$ is *braided* if both the diagrams in (3.16) commute, or equivalently, if $(\mathcal{F}, \varphi)$ and $(\mathcal{F}, \psi)$ are braided lax and colax monoidal functors, respectively.

A morphism of braided bilax monoidal functors is a morphism of the underlying bilax monoidal functors.

Thus for two fixed braided monoidal categories, we have the categories of braided lax, braided colax and braided bilax functors between them. They are full subcategories respectively of the categories of lax, colax and bilax functors.

**Definition 3.14.** Let $(\mathsf{C}, \bullet, \beta)$ and $(\mathsf{D}, \bullet, \beta)$ be two braided monoidal categories and $\varphi$ and $\psi$ be as in (3.9). Define natural transformations $\varphi^b$, ${}^b\varphi$, $\psi^b$ and ${}^b\psi$ as the following composites.

$$\varphi^b\colon \mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\beta} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{B,A}} \mathcal{F}(B \bullet A) \xrightarrow{\mathcal{F}(\beta^{-1})} \mathcal{F}(A \bullet B),$$

$${}^b\varphi\colon \mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\beta^{-1}} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{B,A}} \mathcal{F}(B \bullet A) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(A \bullet B),$$

$$\psi^b\colon \mathcal{F}(A \bullet B) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(B \bullet A) \xrightarrow{\psi_{B,A}} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\beta^{-1}} \mathcal{F}(A) \bullet \mathcal{F}(B),$$

$${}^b\psi\colon \mathcal{F}(A \bullet B) \xrightarrow{\mathcal{F}(\beta^{-1})} \mathcal{F}(B \bullet A) \xrightarrow{\psi_{B,A}} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\beta} \mathcal{F}(A) \bullet \mathcal{F}(B).$$

We state analogues to Propositions 1.20 and 1.21. The proofs are straightforward.

**Proposition 3.15.** *If $(\mathcal{F}, \varphi)$ (resp. $(\mathcal{F}, \psi)$) is a lax (resp. colax) monoidal functor from $(\mathsf{C}, \bullet)$ to $(\mathsf{D}, \bullet)$, then so are $(\mathcal{F}, \varphi^b)$ and $(\mathcal{F}, {}^b\varphi)$ (resp. $(\mathcal{F}, \psi^b)$ and $(\mathcal{F}, {}^b\psi)$).*

**Proposition 3.16.** *Let $(\mathcal{F}, \varphi, \psi)$ be a bilax monoidal functor from $(\mathsf{C}, \bullet, \beta)$ to $(\mathsf{D}, \bullet, \beta)$. Then*

$$(\mathcal{F}, \varphi, \psi^b) \qquad \textit{and} \qquad (\mathcal{F}, {}^b\varphi, \psi)$$

*are bilax monoidal functors from $(\mathsf{C}, \bullet, \beta^{-1})$ to $(\mathsf{D}, \bullet, \beta^{-1})$. Therefore,*

$$(\mathcal{F}, \varphi^b, \psi^b) \qquad \textit{and} \qquad (\mathcal{F}, {}^b\varphi, {}^b\psi)$$

*are bilax monoidal functors from $(\mathsf{C}, \bullet, \beta)$ to $(\mathsf{D}, \bullet, \beta)$.*

In analogy with (1.15)–(1.16), we have equivalences among the four statements in each set below.

(3.17)
- $(\mathcal{F}, \varphi)$ is a braided lax monoidal functor;
- $\mathrm{id}\colon (\mathcal{F}, \varphi) \Rightarrow (\mathcal{F}, \varphi^b)$ is a morphism of lax monoidal functors;
- $\mathrm{id}\colon (\mathcal{F}, \varphi) \Rightarrow (\mathcal{F}, {}^b\varphi)$ is a morphism of lax monoidal functors;
- $\varphi = \varphi^b$.

(3.18)
- $(\mathcal{F}, \psi)$ is a braided colax monoidal functor;
- $\mathrm{id}\colon (\mathcal{F}, \psi) \Rightarrow (\mathcal{F}, \psi^b)$ is a morphism of colax monoidal functors;
- $\mathrm{id}\colon (\mathcal{F}, \psi) \Rightarrow (\mathcal{F}, {}^b\psi)$ is a morphism of colax monoidal functors;
- $\psi = \psi^b$.

**3.1.4. The convolution comma category.** Let $\mathsf{A}, \mathsf{B}$ and $\mathsf{C}$ be monoidal categories,

$$(\mathcal{F}, \psi) : (\mathsf{A}, \bullet) \to (\mathsf{C}, \bullet)$$

a colax monoidal functor, and

$$(\mathcal{G}, \varphi) : (\mathsf{B}, \bullet) \to (\mathsf{C}, \bullet)$$

a lax monoidal functor. Consider the comma category $\mathcal{F} \downarrow \mathcal{G}$, as in Section A.5; its objects are triples $(A, \gamma, B)$ with

$$\gamma : \mathcal{F}(A) \to \mathcal{G}(B)$$

an arrow in $\mathsf{C}$.

Given arrows $\gamma_i \colon \mathcal{F}(A_i) \to \mathcal{G}(B_i)$ in $\mathsf{C}$, $i = 1, 2$, we may form the composite

$$\mathcal{F}(A_1 \bullet A_2) \xrightarrow{\psi_{A_1,A_2}} \mathcal{F}(A_1) \bullet \mathcal{F}(A_2) \xrightarrow{\gamma_1 \bullet \gamma_2} \mathcal{G}(B_1) \bullet \mathcal{G}(B_2) \xrightarrow{\varphi_{B_1,B_2}} \mathcal{G}(B_1 \bullet B_2).$$

We may also consider the composite

$$\mathcal{F}(I_{\mathsf{A}}) \xrightarrow{\psi_0} I_{\mathsf{C}} \xrightarrow{\varphi_0} \mathcal{G}(I_{\mathsf{B}}),$$

where $I_{\mathsf{A}}, I_{\mathsf{B}}$ and $I_{\mathsf{C}}$ are the unit objects of each category. This allows us to turn $\mathcal{F} \downarrow \mathcal{G}$ into a monoidal category, as follows. The tensor product on objects is

$$(A_1, \gamma_1, B_1) \bullet (A_2, \gamma_2, B_2) := (A_1 \bullet A_2,\ \varphi_{B_1,B_2}(\gamma_1 \bullet \gamma_2)\psi_{A_1,A_2},\ B_1 \bullet B_2).$$

It is defined similarly on morphisms. The unit object is $(I_{\mathsf{A}},\ \varphi_0\psi_0,\ I_{\mathsf{B}})$. Associativity follows from (3.5) and unitality from (3.6) (and the dual diagrams).

Suppose all the given data is braided (the monoidal categories $\mathsf{A}, \mathsf{B}$ and $\mathsf{C}$, the colax monoidal functor $(\mathcal{F}, \psi)$ and the lax monoidal functor $(\mathcal{G}, \varphi)$). It then follows from (3.16) that the pair $(\beta_{A_1,A_2},\ \beta_{B_1,B_2})$ defines a morphism from

$$(A_1, \gamma_1, B_1) \bullet (A_2, \gamma_2, B_2) \to (A_2, \gamma_2, B_2) \bullet (A_1, \gamma_1, B_1)$$

in the comma category $\mathcal{F} \downarrow \mathcal{G}$. It follows that in this situation the monoidal category $(\mathcal{F} \downarrow \mathcal{G}, \bullet)$ is braided.

## 3.2. Examples of bilax monoidal functors

In this section, we provide pointers to the main examples of bilax and bistrong monoidal functors and morphisms between them which are discussed in this monograph. We also provide some other basic examples.

**3.2.1. Classical example from homological algebra.** In Chapter 5, we discuss what may be the most classical bilax monoidal functor: the chain complex functor from simplicial modules to chain complexes. In this example, the transformations $\varphi$ and $\psi$ are the Eilenberg–Zilber and Alexander–Whitney maps. It turns out that the associated chain complex functor from simplicial modules to the homotopy category of chain complexes is bistrong.

The interested reader may enjoy going over Chapter 5 at this point; the discussion there uses some of the terminology developed so far and some results from later sections in this chapter.

**3.2.2. The image of a morphism of bilax monoidal functors.** A general procedure to factorize a morphism of bilax functors from a braided monoidal category to an abelian braided monoidal category is given in Section 3.11. In particular, this yields a new bilax functor, which is the image of the morphism. Our method makes use of the existence of monic-epi factorizations in abelian categories and a related bistrong functor called the image functor.

**3.2.3. Examples related to species and Fock functors.** The main examples of bilax monoidal functors in this monograph are the Fock functors from species to graded vector spaces. They are the object of study of Chapters 15 and 16. Decorated and colored versions of these functors are discussed in Chapters 19 and 20. Other examples of bilax monoidal functors include the Hadamard functor and the Hom functor on species. A summary is provided in Table 3.1.

The main examples of bistrong monoidal functors are summarized in Table 3.2. They include the duality functor on species and on graded vector spaces, the signature functor on species, and the bosonic and fermionic Fock functors from species to graded vector spaces.

Several morphisms of bilax monoidal functors play an important role in this monograph. The main ones are the morphisms $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee} \Rightarrow \mathcal{K}^{\vee}$ relating the full Fock functors with the bosonic Fock functors, similar morphisms with fermionic replacing bosonic, and the norm and half-twist transformations that relate the full Fock functors. These are summarized in Table 3.3. Generalizations of these morphisms are discussed in Chapters 19 and 20.

Table 3.1. Bilax monoidal functors.

| Bilax monoidal functors | |
|---|---|
| Chain complex functor | Section 5.4 |
| Hadamard functor and the Hom functor | Sections 8.13 and 9.4 |
| Full Fock functors | Section 15.1 |
| Deformed full Fock functors | Section 16.1 |
| Decorated Fock functors | Table 19.1 |
| Colored Fock functors | Table 20.1 |

Table 3.2. Bistrong monoidal functors.

| Bistrong monoidal functors | Section |
|---|---|
| Image functor | 3.11.4 |
| Chain complex (up to homotopy) functors | 5.5 |
| Duality functor on species | 8.6 |
| Signature functor | 9.4 |
| Bosonic Fock functors | 15.1 |
| Fermionic Fock functors | 16.3 |

TABLE 3.3. Morphisms between bilax monoidal functors.

| Morphism | Sections |
| --- | --- |
| Monic-epi factorization | 3.11.5 |
| Between full Fock and bosonic Fock functors | 15.1 |
| Between full Fock and fermionic Fock functors | 16.3 |
| Norm transformation | 15.4 and 16.2 |
| Half-twist transformation | 15.5 and 16.4 |

**3.2.4. Some other examples.** In the remainder of this section, we discuss some other interesting examples.

**Example 3.17.** Let $(\mathsf{Set}, \times)$ be the symmetric monoidal category of sets under Cartesian product, as in Example 1.3. The unit object is the one-element set $\{\emptyset\}$. For any monoidal category $(\mathsf{C}, \bullet)$, the functor

$$\mathrm{Hom}\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$$

is a lax monoidal functor with the map

$$\varphi_{(A,C),(B,D)}\colon \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, D) \to \mathrm{Hom}(A \bullet B, C \bullet D)$$

which sends $(f, g)$ to $f \bullet g$, and the map

$$\varphi_0\colon \{\emptyset\} \to \mathrm{Hom}(I, I)$$

which sends $\emptyset$ to the identity morphism from the unit object $I$ to itelf.

If $(\mathsf{C}, \bullet)$ is a linear monoidal category (Definition 1.6), then one similarly obtains a lax monoidal functor

$$(\mathrm{Hom}, \varphi)\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Vec}.$$

If $\mathsf{C}$ is the category of finite-dimensional vector spaces under ordinary tensor product, then the map $\varphi$ is an isomorphism. By letting $\psi = \varphi^{-1}$, one obtains a colax functor $(\mathrm{Hom}, \psi)$. Moreover, the two structures are compatible and we obtain a bilax functor $(\mathrm{Hom}, \varphi, \psi)$, which is in fact bistrong by construction.

The situation concerning Hom is more delicate for graded vector spaces and for species; see Proposition 8.58, Proposition 8.64, and Remark 8.65.

**Example 3.18.** Let $\mathsf{C}_{\mathbb{N}}$ be the category whose objects are nonnegative integers and whose morphisms are

$$\mathrm{Hom}_{\mathsf{C}_{\mathbb{N}}}(n, m) := \begin{cases} \mathrm{id}_n & \text{if } n = m, \\ \emptyset & \text{if } n \neq m. \end{cases}$$

This is the *discrete category* on $\mathbb{N}$. It is a symmetric monoidal category under

$$n \otimes m := n + m \quad \text{and} \quad \beta_{n,m} = \mathrm{id}_{n+m}.$$

In addition, $\mathsf{C}_{\mathbb{N}}$ is strict, that is, the associative and unit constraints are all identities.

Given a functor $\mathcal{F}\colon \mathsf{C}_{\mathbb{N}} \to \mathsf{Vec}$, let

$$|\mathcal{F}| := \bigoplus_{n \in \mathbb{N}} \mathcal{F}(n).$$

Then $|\mathcal{F}|$ is a graded vector space. If $x \in \mathcal{F}(n)$ we write $|x| = n$.

Suppose $\mathcal{F}$ is a lax monoidal functor with structure maps

$$\varphi_{n,m} \colon \mathcal{F}(n) \otimes \mathcal{F}(m) \to \mathcal{F}(n+m).$$

Defining

$$x \cdot y := \varphi_{|x|,|y|}(x \otimes y)$$

on homogeneous elements $x$ and $y$ endows $|\mathcal{F}|$ with the structure of a graded algebra. The unit element is $\varphi_0(1) \in \mathcal{F}(0)$.

If $\mathcal{F}$ is a colax monoidal functor, then we may similarly use the structure maps

$$\psi_{n,m} \colon \mathcal{F}(n+m) \to \mathcal{F}(n) \otimes \mathcal{F}(m)$$

to turn $|\mathcal{F}|$ into a graded coalgebra; the coproduct is

$$\Delta(x) = \sum_{n+m=|x|} \psi_{n,m}(x)$$

on homogeneous elements $x \in |\mathcal{F}|$ and the counit is defined in terms of $\psi_0$.

The above constructions define equivalences between the category of functors (resp. lax monoidal functors, colax monoidal functors) $\mathsf{C}_{\mathbb{N}} \to \mathsf{Vec}$ and the category of graded vector spaces (resp. graded algebras, graded coalgebras).

One may expect a similar result for bilax monoidal functors and graded bialgebras, but it turns out that these two notions are not equivalent. Indeed, if $\mathcal{F} \colon \mathsf{C}_{\mathbb{N}} \to \mathsf{Vec}$ is a bilax monoidal functor and we view $|\mathcal{F}|$ as an algebra and as a coalgebra as above, then diagram (3.11) leads to the following relation between the product and coproduct of $|\mathcal{F}|$:

$$\psi_{n,m}(x \cdot y) = \psi_{a,b}(x)\psi_{c,d}(y) \tag{3.19}$$

*whenever* $a$, $b$, $c$, and $d$ are related to $n$, $m$, and the degrees of $x$ and $y$ by

$$a + c = n, \quad b + d = m, \quad a + b = |x|, \quad c + d = |y|, \quad n + m = |x| + |y|. \tag{3.20}$$

On the other hand, the definition of graded bialgebra would instead require the following compatibility condition between the product and coproduct of $|\mathcal{F}|$:

$$\psi_{n,m}(x \cdot y) = \sum_{a,b,c,d} \psi_{a,b}(x)\psi_{c,d}(y), \tag{3.21}$$

the sum being over *all* $a$, $b$, $c$, and $d$ subject to the conditions (3.20). Conditions (3.19) and (3.21) are distinct in general. Specifically, consider the case of the graded bialgebra of polynomials $\Bbbk[t]$, where

$$\Delta(t^n) = \sum_k \binom{n}{k} t^k \otimes t^{n-k}.$$

Take $x = t^i$ and $y = t^j$. The right-hand side of (3.19) is

$$\binom{i}{a}\binom{j}{c} t^{a+c} \otimes t^{b+d},$$

while the left-hand side of (3.19) is

$$\binom{i+j}{n} t^n \otimes t^m.$$

(This agrees with the right-hand side of (3.21) by Vandermonde's identity for binomial coefficients.)

In summary, a bilax monoidal functor $\mathsf{C}_{\mathbb{N}} \to \mathsf{Vec}$ is equivalent to a graded vector space endowed with a structure of graded algebra and of graded coalgebra, linked by (3.19) (plus standard conditions involving the unit and counit), and this is not a graded bialgebra.

This flaw is rectified if one replaces graded vector spaces by species. In Proposition 8.35 we show how an analogous construction for species does lead to an equivalence between bimonoids in species and certain bilax monoidal functors.

There is a different way in which graded bialgebras may be seen as bilax monoidal functors. This in fact holds for bimonoids in an arbitrary braided monoidal category and is explained in Section 3.4.1.

**Example 3.19.** This example was proposed by George Janelidze. Throughout this discussion, we employ the terminology and notations of Section A.1.

Let $\mathsf{C}$ and $\mathsf{D}$ be categories with finite products. Consider the corresponding cartesian monoidal categories $(\mathsf{C}, \times, J)$ and $(\mathsf{D}, \times, J)$ as in Example 1.4.

Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be an arbitrary functor. Given objects $A$ and $B$ of $\mathsf{C}$, let

$$\psi_{A,B} := \big(\mathcal{F}(\pi_A), \mathcal{F}(\pi_B)\big)\colon \mathcal{F}(A \times B) \to \mathcal{F}(A) \times \mathcal{F}(B),$$

and let

$$\psi_0\colon \mathcal{F}(J) \to J$$

be the unique such arrow in $\mathsf{D}$. Then $(\mathcal{F}, \psi, \psi_0)$ is a colax monoidal functor. Indeed, both composites in the dual of diagram (3.5) coincide with the arrow

$$\big(\mathcal{F}(\pi_A), \mathcal{F}(\pi_B), \mathcal{F}(\pi_C)\big).$$

The other diagrams in Definition 3.1 can be verified similarly. Note that the functor is costrong if and only if $\mathcal{F}$ preserves products.

Thus, any functor between categories with finite products carries a canonical colax monoidal structure. Moreover, this structure is braided (Definition 3.11).

Dually, any functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ between cocartesian monoidal categories carries a canonical braided lax monoidal structure. The structure maps are

$$\varphi_{A,B} := \begin{pmatrix} \mathcal{F}(\iota_A) \\ \mathcal{F}(\iota_B) \end{pmatrix}\colon \mathcal{F}(A) \amalg \mathcal{F}(B) \to \mathcal{F}(A \amalg B)$$

and

$$\varphi_0\colon I \to \mathcal{F}(I).$$

Suppose now that $\mathsf{C}$ has finite biproducts. Consider the corresponding bicartesian monoidal category $(\mathsf{C}, \oplus, Z)$ of Example 1.4. Let $\mathsf{D}$ be another such category. By the above, any functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ carries a canonical lax structure $\varphi$ and a canonical colax structure $\psi$. It turns out that

$$(\mathcal{F}, \varphi, \psi)\colon (\mathsf{C}, \oplus, Z) \to (\mathsf{D}, \oplus, Z)$$

is bilax monoidal. Indeed, the composites along both sides of diagram (3.11) are equal to

$$\begin{pmatrix} \mathcal{F}\big(\iota_A^{A\oplus C}\pi_A^{A\oplus B}\big) & \mathcal{F}\big(\iota_B^{B\oplus D}\pi_B^{A\oplus B}\big) \\ \mathcal{F}\big(\iota_C^{A\oplus C}\pi_C^{C\oplus D}\big) & \mathcal{F}\big(\iota_D^{B\oplus D}\pi_D^{C\oplus D}\big) \end{pmatrix} : \mathcal{F}(A\oplus B)\oplus\mathcal{F}(C\oplus D) \to \mathcal{F}(A\oplus C)\oplus\mathcal{F}(B\oplus D)$$

and so the braiding axiom is satisfied. The other axioms can be verified similarly.

Thus, any functor between bicartesian monoidal categories carries a canonical braided bilax monoidal structure. As an example, we may choose $\mathsf{C} = \mathsf{D} = \mathsf{Vec}$, the category of vector spaces (under direct sum), and

$$\mathcal{F}(V) := V \otimes V.$$

This functor is not bistrong.

## 3.3. Composites of bilax monoidal functors

Monoidal functors exhibit an interesting feature which is not visible for monoids. Namely, it is meaningful to ask whether the composite of monoidal functors is again monoidal. The first result in this direction is by Bénabou who showed that the composite of lax monoidal functors is again lax monoidal [36, Proposition 5].

In this section, we show that the same assertion also holds for bilax monoidal functors. This includes Bénabou's result. We also briefly explain how this leads to 2-categories based on (co,bi)lax monoidal functors.

### 3.3.1. Composites of bilax monoidal functors.

**Definition 3.20.** Let $(\mathcal{F}, \varphi)\colon (\mathsf{C}, \bullet) \to (\mathsf{D}, \bullet)$, where $\varphi\colon \mathcal{F}^2 \Rightarrow \mathcal{F}_2$ and $\varphi_0\colon I \to \mathcal{F}(I)$ are as in (3.3) and (3.4). Similarly, let $(\mathcal{G}, \gamma)\colon (\mathsf{D}, \bullet) \to (\mathsf{E}, \bullet)$ with $\gamma\colon \mathcal{G}^2 \Rightarrow \mathcal{G}_2$ and $\gamma_0\colon I \to \mathcal{G}(I)$. Now let

$$(\mathcal{G}\mathcal{F}, \varphi\gamma)\colon (\mathsf{C}, \bullet) \to (\mathsf{E}, \bullet),$$

where the functor $\mathcal{G}\mathcal{F}\colon \mathsf{C} \to \mathsf{E}$ is the composite of $\mathcal{F}$ and $\mathcal{G}$, and the transformations

$$\varphi\gamma\colon (\mathcal{G}\mathcal{F})^2 \Rightarrow (\mathcal{G}\mathcal{F})_2 \quad \text{and} \quad (\varphi\gamma)_0\colon I \to \mathcal{G}\mathcal{F}(I)$$

are defined as follows.

$$\begin{tikzcd}[column sep=small]
\mathcal{G}\mathcal{F}(A) \bullet \mathcal{G}\mathcal{F}(B) \arrow[rr, dashed, "(\varphi\gamma)_{A,B}"] \arrow[dr, "\gamma_{\mathcal{F}(A),\mathcal{F}(B)}"'] & & \mathcal{G}\mathcal{F}(A \bullet B) & I \arrow[rr, dashed, "(\varphi\gamma)_0"] \arrow[dr, "\gamma_0"'] & & \mathcal{G}\mathcal{F}(I) \\
& \mathcal{G}\big(\mathcal{F}(A) \bullet \mathcal{F}(B)\big) \arrow[ur, "\mathcal{G}(\varphi_{A,B})"'] & & & \mathcal{G}(I) \arrow[ur, "\mathcal{G}(\varphi_0)"'] &
\end{tikzcd}$$

Similarly for the dual situation, given $(\mathcal{F}, \psi)$ and $(\mathcal{G}, \delta)$, we define $(\mathcal{G}\mathcal{F}, \delta\psi)$, where $\delta\psi$ and $(\delta\psi)_0$ are obtained from the above by switching $\varphi$ with $\psi$, and $\gamma$ with $\delta$, and reversing the arrows. Combining the two situations, given $(\mathcal{F}, \varphi, \psi)$ and $(\mathcal{G}, \gamma, \delta)$, we define $(\mathcal{G}\mathcal{F}, \varphi\gamma, \delta\psi)$.

**Theorem 3.21.** *If $(\mathcal{F}, \varphi)\colon \mathsf{C} \to \mathsf{D}$ and $(\mathcal{G}, \gamma)\colon \mathsf{D} \to \mathsf{E}$ are lax monoidal, then the functor $(\mathcal{G}\mathcal{F}, \varphi\gamma)\colon \mathsf{C} \to \mathsf{E}$ is lax monoidal. Similarly, if $(\mathcal{F}, \psi)$ and $(\mathcal{G}, \delta)$ are colax, then so is $(\mathcal{G}\mathcal{F}, \delta\psi)$.*

*If $\mathcal{F} \Rightarrow \mathcal{F}'$ is a morphism of (co)lax monoidal functors, then the induced natural transformation $\mathcal{G}\mathcal{F} \Rightarrow \mathcal{G}\mathcal{F}'$ is also a morphism of (co)lax monoidal functors.*

*If $\mathcal{G} \Rightarrow \mathcal{G}'$ is a morphism of (co)lax monoidal functors, then the induced natural transformation $\mathcal{G}\mathcal{F} \Rightarrow \mathcal{G}'\mathcal{F}$ is also a morphism of (co)lax monoidal functors.*

Proof. We prove the first statement (for lax functors). The statement for colax follows by passing to the opposite categories (Proposition 3.7) and the remaining assertions can be shown similarly.

The associativity axiom for $(\mathcal{GF}, \varphi\gamma)$ follows by the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{GF}A \bullet \mathcal{GF}B \bullet \mathcal{GF}C \arrow[r, "\mathrm{id}\bullet\gamma_{\mathcal{F}B,\mathcal{F}C}"] \arrow[d, "\gamma_{\mathcal{F}A,\mathcal{F}B}\bullet\mathrm{id}"'] & \mathcal{GF}A \bullet \mathcal{G}(\mathcal{F}B \bullet \mathcal{F}C) \arrow[r, "\mathrm{id}\bullet\mathcal{G}(\varphi_{B,C})"] \arrow[d, "\gamma_{\mathcal{F}A,\mathcal{F}B\bullet\mathcal{F}C}"'] & \mathcal{GF}A \bullet \mathcal{GF}(B \bullet C) \arrow[d, "\gamma_{\mathcal{F}A,\mathcal{F}(B\bullet C)}"] \\
\mathcal{G}(\mathcal{F}A \bullet \mathcal{F}B) \bullet \mathcal{GF}C \arrow[r, "\gamma_{\mathcal{F}A\bullet\mathcal{F}B,\mathcal{F}C}"] \arrow[d, "\mathcal{G}(\varphi_{A,B})\bullet\mathrm{id}"'] & \mathcal{G}(\mathcal{F}A \bullet \mathcal{F}B \bullet \mathcal{F}C) \arrow[r, "\mathcal{G}(\mathrm{id}\bullet\varphi_{B,C})"] \arrow[d, "\mathcal{G}(\varphi_{A,B}\bullet\mathrm{id})"'] & \mathcal{G}\big(\mathcal{F}A \bullet \mathcal{F}(B \bullet C)\big) \arrow[d, "\mathcal{G}(\varphi_{A,B\bullet C})"] \\
\mathcal{GF}(A \bullet B) \bullet \mathcal{GF}C \arrow[r, "\gamma_{\mathcal{F}(A\bullet B),\mathcal{F}C}"'] & \mathcal{G}(\mathcal{F}(A \bullet B) \bullet \mathcal{F}C) \arrow[r, "\mathcal{G}(\varphi_{A\bullet B,C})"'] & \mathcal{GF}(A \bullet B \bullet C)
\end{tikzcd}$$

The top left diagram is the associativity of $\gamma$, while the bottom right diagram is the functor $\mathcal{G}$ applied to the associativity of $\varphi$. The remaining two diagrams commute by the naturality of $\gamma$.

The left unitality axiom for $(\mathcal{GF}, \varphi\gamma)$ follows by the commutativity of the following diagram.

$$\begin{tikzcd}
 & I \bullet \mathcal{GF}(A) \arrow[dl, "\gamma_0\bullet\mathrm{id}"'] & & \mathcal{GF}(A) \arrow[ll, "\lambda_{\mathcal{GF}(A)}"'] \arrow[dl, "\mathcal{G}(\lambda_{\mathcal{F}(A)})"] \arrow[dd, "\mathcal{GF}(\lambda_A)"] \\
\mathcal{G}(I) \bullet \mathcal{GF}(A) \arrow[rr, "\gamma_{I,\mathcal{F}(A)}"'] \arrow[dd, "\mathcal{G}(\varphi_0)\bullet\mathrm{id}"'] & & \mathcal{G}\big(I \bullet \mathcal{F}(A)\big) \arrow[dd, "\mathcal{G}(\varphi_0\bullet\mathrm{id})"'] & \\
 & & & \mathcal{GF}(I \bullet A) \\
\mathcal{GF}(I) \bullet \mathcal{GF}(A) \arrow[rr, "\gamma_{\mathcal{F}(I),\mathcal{F}(A)}"'] & & \mathcal{G}\big(\mathcal{F}(I) \bullet \mathcal{F}(A)\big) \arrow[ur, "\mathcal{G}(\varphi_{I,A})"'] &
\end{tikzcd}$$

The oblique squares commute by the left unitality of $\gamma$ and the functor $\mathcal{G}$ applied to the left unitality of $\varphi$, while the third square commutes by the naturality of $\gamma$.

The verification of the right unitality axiom is similar. This proves that the composite functor $(\mathcal{GF}, \varphi\gamma)$ is lax. $\square$

**Theorem 3.22.** *If* $(\mathcal{F}, \varphi, \psi)\colon \mathsf{C} \to \mathsf{D}$ *and* $(\mathcal{G}, \gamma, \delta)\colon \mathsf{D} \to \mathsf{E}$ *are bilax monoidal, then so is* $(\mathcal{GF}, \varphi\gamma, \delta\psi)$.

*In addition, pre or post composing by a bilax monoidal functor preserves morphisms between bilax monoidal functors.*

PROOF. In view of Theorem 3.21, one only needs to prove the commutativity of diagrams (3.11), (3.12) and (3.13) for $(\mathcal{GF}, \varphi\gamma, \delta\psi)$.

The commutativity of (3.11) follows from that of the following diagram. For simplicity of notation, the tensor product symbol has been suppressed.

$$
\begin{tikzcd}
\mathcal{G}\mathcal{F}(AB)\mathcal{G}\mathcal{F}(CD) \arrow[r, "\mathcal{G}(\psi_{A,B})\mathcal{G}(\psi_{C,D})"] \arrow[d, "\gamma_{\mathcal{F}(AB),\mathcal{F}(CD)}"] & \mathcal{G}(\mathcal{F}A\mathcal{F}B)\mathcal{G}(\mathcal{F}C\mathcal{F}D) \arrow[r, "\delta_{\mathcal{F}A,\mathcal{F}B}\delta_{\mathcal{F}C,\mathcal{F}D}"] \arrow[d, "\gamma_{\mathcal{F}A\mathcal{F}B,\mathcal{F}C\mathcal{F}D}"'] & (\mathcal{G}\mathcal{F}A)(\mathcal{G}\mathcal{F}B)(\mathcal{G}\mathcal{F}C)(\mathcal{G}\mathcal{F}D) \arrow[d, "\mathrm{id}\,\beta\,\mathrm{id}"'] \\
\mathcal{G}(\mathcal{F}(AB)\mathcal{F}(CD)) \arrow[r, "\mathcal{G}(\psi_{A,B}\psi_{C,D})"] \arrow[d, "\mathcal{G}(\varphi_{AB,CD})"] & \mathcal{G}(\mathcal{F}A\mathcal{F}B\mathcal{F}C\mathcal{F}D) \arrow[d, "\mathcal{G}(\mathrm{id}\,\beta\,\mathrm{id})"] & (\mathcal{G}\mathcal{F}A)(\mathcal{G}\mathcal{F}C)(\mathcal{G}\mathcal{F}B)(\mathcal{G}\mathcal{F}D) \arrow[d, "\gamma_{\mathcal{F}A,\mathcal{F}C}\gamma_{\mathcal{F}B,\mathcal{F}D}"'] \\
\mathcal{G}\mathcal{F}(ABCD) \arrow[d, "\mathcal{G}\mathcal{F}(\mathrm{id}\,\beta\,\mathrm{id})"] & \mathcal{G}(\mathcal{F}A\mathcal{F}C\mathcal{F}B\mathcal{F}D) \arrow[r, "\delta_{\mathcal{F}A\mathcal{F}C,\mathcal{F}B\mathcal{F}D}"'] \arrow[d, "\mathcal{G}(\varphi_{A,C}\varphi_{B,D})"] & \mathcal{G}(\mathcal{F}A\mathcal{F}C)\mathcal{G}(\mathcal{F}B\mathcal{F}D) \arrow[d, "\mathcal{G}(\varphi_{A,C})\mathcal{G}(\varphi_{B,D})"'] \\
\mathcal{G}\mathcal{F}(ACBD) \arrow[r, "\mathcal{G}(\psi_{AC,BD})"'] & \mathcal{G}(\mathcal{F}(AC)\mathcal{F}(BD)) \arrow[r, "\delta_{\mathcal{F}(AC),\mathcal{F}(BD)}"'] & \mathcal{G}\mathcal{F}(AC)\mathcal{G}\mathcal{F}(BD)
\end{tikzcd}
$$

The squares commute by the naturality of $\gamma$ and $\delta$, while the hexagons commute by the braiding axiom (3.11) for $(\mathcal{G}, \gamma, \delta)$ and the functor $\mathcal{G}$ applied to the same axiom for $(\mathcal{F}, \varphi, \psi)$.

The first axiom in (3.12) follows from the commutativity of the following diagram.

$$
\begin{tikzcd}
I \arrow[r, "\gamma_0"] \arrow[dd, "\lambda_I"'] & \mathcal{G}(I) \arrow[r, "\mathcal{G}(\varphi_0)"] \arrow[dr, "\mathcal{G}(\lambda_I)"'] & \mathcal{G}\mathcal{F}(I) \arrow[r, "\mathcal{G}\mathcal{F}(\lambda_I)"] & \mathcal{G}\mathcal{F}(I \bullet I) \arrow[dr, "\mathcal{G}(\varphi_{I,I})"] & \\
& & \mathcal{G}(I \bullet I) \arrow[rr, "\mathcal{G}(\varphi_0 \bullet \varphi_0)"'] \arrow[d, "\delta_{I,I}"'] & & \mathcal{G}\big(\mathcal{F}(I) \bullet \mathcal{F}(I)\big) \arrow[d, "\delta_{\mathcal{F}(I),\mathcal{F}(I)}"] \\
I \bullet I \arrow[rr, "\gamma_0 \bullet \gamma_0"'] & & \mathcal{G}(I) \bullet \mathcal{G}(I) \arrow[rr, "\mathcal{G}(\varphi_0) \bullet \mathcal{G}(\varphi_0)"'] & & \mathcal{G}\mathcal{F}(I) \bullet \mathcal{G}\mathcal{F}(I)
\end{tikzcd}
$$

The pentagons commute by the first axiom in (3.12) for $(\mathcal{G}, \gamma, \delta)$ and the functor $\mathcal{G}$ applied to the same axiom for $(\mathcal{F}, \varphi, \psi)$. The square commutes by the naturality of $\delta$.

The proof for the second axiom in (3.12) can be obtained from the above by reversing the appropriate arrows. Axiom (3.13) follows directly. $\square$

An alternative proof of Theorem 3.22 is given in Remark 3.78. It is also clear that if $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are composable bilax monoidal functors, then

$$\mathcal{H}(\mathcal{G}\mathcal{F}) \cong (\mathcal{H}\mathcal{G})\mathcal{F}$$

as bilax monoidal functors.

**3.3.2. Composites of braided bilax monoidal functors.** We now turn our attention to the interaction between composites and the constructions of Definition 3.14.

**Proposition 3.23.** *We have*

$$(\varphi\gamma)^b = \varphi^b\gamma^b \qquad {}^b(\varphi\gamma) = ({}^b\varphi)({}^b\gamma) \qquad (\delta\psi)^b = \delta^b\psi^b \qquad {}^b(\delta\psi) = ({}^b\delta)({}^b\psi).$$

This is an easy consequence of the definitions. Thus, the composition of lax, colax, and bilax monoidal functors is compatible with conjugation by the braidings, and hence compatible with the constructions in Propositions 3.15 and 3.16. Combining Proposition 3.23 with (3.17) and (3.18) one obtains the following result.

**Proposition 3.24.** *The composite of two braided lax* (*colax*, *bilax*) *monoidal functors is again braided lax* (*colax*, *bilax*) *monoidal.*

*Further, pre or post composing by a braided lax (colax, bilax) monoidal functor preserves morphisms between braided lax (colax, bilax) monoidal functors.*

**3.3.3. 2-categories arising from monoidal functors.** The preceding results can be succintly expressed using the notion of 2-category (Section C.1.1). Let $\mathsf{Cat}$ be the 2-category whose 0-cells, 1-cells, and 2-cells are respectively categories, functors, and natural transformations. Together with Proposition 3.10, the preceding results say that lax monoidal functors are the 1-cells of a 2-category whose objects are monoidal categories and whose 2-cells are morphisms of lax monoidal functors. We call this 2-category $\mathsf{lCat}$. The same construction with colax replacing lax yields $\mathsf{cCat}$. Similarly, there is a 2-category whose 0-cells, 1-cells, and 2-cells are respectively braided monoidal categories, bilax monoidal functors, and their morphisms.

Further, there are braided versions of all these: Braided lax (colax, bilax) monoidal functors are the 1-cells of a 2-category whose objects are braided monoidal categories and whose 2-cells are morphisms of braided lax (colax, bilax) monoidal functors. We return to these ideas in Sections 6.11 and 7.9.

## 3.4. A comparison of bimonoids and bilax monoidal functors

The discussion in Section 3.1 reveals a parallel between the notions of monoid, comonoid, bimonoid, and those of lax, colax, and bilax monoidal functor, respectively. These notions are connected in this section in two different ways. First we show that any bimonoid may be seen as a special case of a bilax monoidal functor, then we deduce that the image of a bimonoid under a bilax monoidal functor is again a bimonoid. Similar results connect monoids to lax monoidal functors and comonoids to colax monoidal functors.

**3.4.1. Bimonoids as bilax monoidal functors.** Monoids may be viewed as lax monoidal functors. We recall this construction of Bénabou [38, Section 5.4.1] and then give the corresponding result for bimonoids.

Let $(A, \mu, \iota)$ be a monoid in a monoidal category $(\mathsf{C}, \bullet)$ with unit object $I$. Let $(\mathsf{I}, \bullet)$ be the one-arrow category and let

$$\mathcal{F}_A : \mathsf{I} \to \mathsf{C}$$

be the functor that sends the unique object $*$ of $\mathsf{I}$ to $A$. Next, we define a transformation $\varphi$ and a map $\varphi_0$ in order to turn $\mathcal{F}_A$ into a lax monoidal functor (Definition 3.1). Since there is only one object and one morphism in the category $\mathsf{I}$, $\varphi$

consists of only one map, which is $\varphi_{*,*}$; also, $\varphi_0$ is a map $I \to \mathcal{F}(*)$. We let

$$\mathcal{F}_A(*) \bullet \mathcal{F}_A(*) \xrightarrow{\varphi_{*,*} := \mu} \mathcal{F}_A(* \bullet *) \quad \text{and} \quad I \xrightarrow{\varphi_0 := \iota} \mathcal{F}_A(*).$$

Then

$$(\mathcal{F}_A, \varphi, \varphi_0) \colon (\mathsf{I}, \bullet) \to (\mathsf{C}, \bullet) \tag{3.22}$$

is a lax monoidal functor. Associativity of $\mu$ translates into associativity of $\varphi$ (3.5) and similarly for unitality (3.6).

Similarly, given a comonoid $(C, \Delta, \epsilon)$ in $(\mathsf{C}, \bullet)$, define a colax monoidal functor

$$(\mathcal{F}_C, \psi, \psi_0)$$

by $\mathcal{F}_C(*) := C$,

$$\mathcal{F}_C(* \bullet *) \xrightarrow{\psi_{*,*} := \Delta} \mathcal{F}_C(*) \bullet \mathcal{F}_C(*) \quad \text{and} \quad \mathcal{F}_C(*) \xrightarrow{\psi_0 := \epsilon} I.$$

**Proposition 3.25.** *The above construction defines an equivalence from the category of (co)monoids in $(\mathsf{C}, \bullet)$ to the category of (co)lax monoidal functors from $(\mathsf{I}, \bullet)$ to $(\mathsf{C}, \bullet)$.*

Proof. Given a monoid $(A, \mu, \iota)$, we send it to the lax functor $(\mathcal{F}_A, \varphi, \varphi_0)$ defined above. Conversely, given a lax functor $(\mathcal{F}, \varphi, \varphi_0) \colon (\mathsf{I}, \bullet) \to (\mathsf{C}, \bullet)$, we send it to $(\mathcal{F}(*), \varphi_{*,*}, \varphi_0)$. One can check directly that this is a monoid. These correspondences define the equivalence. The case of comonoids is similar. □

Combining the two situations above, given a bimonoid $(H, \mu, \iota, \Delta, \epsilon)$ we construct

$$(\mathcal{F}_H, \varphi, \varphi_0, \psi, \psi_0). \tag{3.23}$$

This is a bilax monoidal functor: the four compatibility diagrams for a bimonoid (Definition 1.10) correspond to the four compatibility diagrams for a bilax monoidal functor (Definition 3.3).

**Proposition 3.26.** *Let $(\mathsf{C}, \bullet, \beta)$ be a braided monoidal category. The above construction defines an equivalence from the category of bimonoids in $(\mathsf{C}, \bullet, \beta)$ to the category of bilax monoidal functors from $(\mathsf{I}, \bullet, \beta)$ to $(\mathsf{C}, \bullet, \beta)$.*

Proof. As for Proposition 3.25. □

This discussion shows that a bilax monoidal functor need not be braided and a braided (co)lax monoidal functor need not be bilax.

**3.4.2. Commutative monoids as braided lax monoidal functors.** Recall that the (co)product of a (co)monoid can be twisted by the braiding to yield its opposite (co)monoid (Section 1.2.9). Recall that in much the same way, the structure of a (co)lax monoidal functor can be twisted to yield its conjugate (co)lax monoidal functor (Definition 3.14). We now make the analogy between these two constructions precise using the preceding discussion.

If the monoid $A = (A, \mu, \iota)$ corresponds to the lax monoidal functor $(\mathcal{F}_A, \varphi)$, then the opposite monoid $A^{\mathrm{op}} = (A, \mu\beta, \iota)$ corresponds to the conjugate lax monoidal functor $(\mathcal{F}_A, \varphi^b)$, and similarly, ${}^{\mathrm{op}}A = (A, \mu\beta^{-1}, \iota)$ corresponds to $(\mathcal{F}_A, {}^b\varphi)$. This is clear from the definitions.

Similarly, if the comonoid $C = (C, \Delta, \epsilon)$ corresponds to the colax monoidal functor $(\mathcal{F}_C, \psi)$, then the opposite comonoid $C^{\mathrm{cop}} = (C, \beta^{-1}\Delta, \epsilon)$ corresponds to the conjugate colax monoidal functor $(\mathcal{F}_C, \psi^b)$, and similarly, ${}^{\mathrm{cop}}C = (C, \beta\Delta, \epsilon)$ corresponds to $(\mathcal{F}_C, {}^b\psi)$.

In addition, it is clear from (1.15) and (1.16), and (3.17) and (3.18) that

$$A \text{ is a commutative monoid} \iff (\mathcal{F}_A, \varphi) \text{ is a braided lax monoidal functor}$$
$$C \text{ is a cocommutative comonoid} \iff (\mathcal{F}_C, \psi) \text{ is a braided colax monoidal functor.}$$

The preceding statements can be phrased as follows.

**Proposition 3.27.** *The category of (co)commutative (co)monoids in $(\mathsf{C}, \bullet, \beta)$ is equivalent to the category of braided (co)lax monoidal functors from $(\mathsf{I}, \bullet, \beta)$ to $(\mathsf{C}, \bullet, \beta)$.*

**3.4.3. Bilax monoidal functors preserve bimonoids.** A significant property of lax, colax and bilax monoidal functors is that they preserve monoids, comonoids and bimonoids respectively. The assertions for lax (and colax) monoidal functors appear in [38, Proposition 6.1] and are also given below (Proposition 3.29).

**Definition 3.28.** Let $(\mathcal{F}, \varphi)\colon (\mathsf{C}, \bullet) \to (\mathsf{D}, \bullet)$, where $\varphi\colon \mathcal{F}^2 \Rightarrow \mathcal{F}_2$ and $\varphi_0\colon I \to \mathcal{F}(I)$ are as in (3.3) and (3.4). Also consider $(A, \mu, \iota)$ where $A$ is an object and $\mu\colon A \bullet A \to A$ and $\iota\colon I \to A$ are morphisms in $\mathsf{C}$. Then define the triple

$$(\mathcal{F}(A), \mu\varphi, \iota\varphi_0),$$

where $\mu\varphi$ and $\iota\varphi_0$ are given by the following composites.

$$\mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{A,A}} \mathcal{F}(A \bullet A) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(A)$$
$$I \xrightarrow{\varphi_0} \mathcal{F}(I) \xrightarrow{\mathcal{F}(\iota)} \mathcal{F}(A)$$

Similarly for the dual situation, given $(\mathcal{F}, \psi)$ and $(C, \Delta, \epsilon)$, we define the triple

$$(\mathcal{F}(C), \psi\Delta, \psi_0\epsilon),$$

where $\psi\Delta$ and $\psi_0\epsilon$ are given by the following composites.

$$\mathcal{F}(C) \bullet \mathcal{F}(C) \xleftarrow{\psi_{C,C}} \mathcal{F}(C \bullet C) \xleftarrow{\mathcal{F}(\Delta)} \mathcal{F}(C)$$
$$I \xleftarrow{\psi_0} \mathcal{F}(I) \xleftarrow{\mathcal{F}(\epsilon)} \mathcal{F}(C)$$

Combining the two situations, given $(\mathcal{F}, \varphi, \psi)$ and $(H, \mu, \iota, \Delta, \epsilon)$, we can consider the quintuple

$$(\mathcal{F}(H), \mu\varphi, \iota\varphi_0, \psi\Delta, \psi_0\epsilon).$$

**Proposition 3.29.** *If $\mathcal{F}$ is a (co)lax monoidal functor from $(\mathsf{C}, \bullet)$ to $(\mathsf{D}, \bullet)$ and $H$ is a (co)monoid in $(\mathsf{C}, \bullet)$, then $\mathcal{F}(H)$ is a (co)monoid in $(\mathsf{D}, \bullet)$ with the (co)product and (co)unit as in Definition 3.28.*

*Moreover, if $f\colon H \to H'$ is a morphism of (co)monoids in $(\mathsf{C}, \bullet)$, then the induced morphism $\mathcal{F}(f)\colon \mathcal{F}(H) \to \mathcal{F}(H')$ is a morphism of (co)monoids in $(\mathsf{D}, \bullet)$.*

Proof. We explain the case of monoids. Recall that associated to a monoid $H$ there is the lax monoidal functor $\mathcal{F}_H$ of (3.22). We have the following commutative diagram of functors.

$$\begin{tikzcd} \mathsf{I} \arrow[rr, "\mathcal{F}_{\mathcal{F}(H)}"] \arrow[dr, "\mathcal{F}_H"'] & & \mathsf{D} \\ & \mathsf{C} \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

Since $\mathcal{F}$ and $\mathcal{F}_H$ are lax monoidal functors, so is $\mathcal{F}_{\mathcal{F}(H)}$, by Theorem 3.21. Hence, by Proposition 3.25, $\mathcal{F}(H)$ is a monoid in $(\mathsf{D}, \bullet)$, and further this monoid structure coincides with that in Definition 3.28. The assertion about morphisms follows similarly. □

**Proposition 3.30.** *A morphism of (co)lax monoidal functors $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ yields a morphism of (co)monoids $\theta_H\colon \mathcal{F}(H) \to \mathcal{G}(H)$ in $(\mathsf{D}, \bullet)$, when $H$ is a (co)monoid in $(\mathsf{C}, \bullet)$.*

PROOF. We explain the case of monoids. Let $\mathcal{F}_H$ be the lax monoidal functor of (3.22). By precompsing $\theta$ with $\mathcal{F}_H$, as shown below

$$\begin{tikzcd} \mathsf{I} \arrow[r, "\mathcal{F}_H"] & \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] & \mathsf{D}, \end{tikzcd}$$

and applying Theorem 3.21, we obtain a morphism $\mathcal{F}_{\mathcal{F}(H)} \to \mathcal{F}_{\mathcal{G}(H)}$ of lax monoidal functors. Equivalently, from Proposition 3.25, this yields the morphism $\theta_H\colon \mathcal{F}(H) \to \mathcal{G}(H)$ of monoids. □

The above results imply that lax and colax monoidal functors from $\mathsf{C}$ to $\mathsf{D}$ induce functors

$$\mathsf{Mon}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{D}) \qquad \text{and} \qquad \mathsf{Comon}(\mathsf{C}) \to \mathsf{Comon}(\mathsf{D})$$

respectively, and that a morphism between two (co)lax monoidal functors yields a natural transformation between the induced functors on (co)monoids.

**Proposition 3.31.** *If $(\mathcal{F}, \varphi, \psi)$ is a bilax monoidal functor from $(\mathsf{C}, \bullet, \beta)$ to $(\mathsf{D}, \bullet, \beta)$ and $H$ is a bimonoid in $(\mathsf{C}, \bullet, \beta)$, then $\mathcal{F}(H)$ is a bimonoid in $(\mathsf{D}, \bullet, \beta)$ with structure maps as in Definition 3.28.*

*Moreover, if $f\colon H \to H'$ is a morphism of bimonoids in $(\mathsf{C}, \bullet, \beta)$, then the induced morphism $\mathcal{F}(f)\colon \mathcal{F}(H) \to \mathcal{F}(H')$ is a morphism of bimonoids in $(\mathsf{D}, \bullet, \beta)$.*

PROOF. Argue as in the proof of Proposition 3.29, using Theorem 3.22 and Proposition 3.26. □

**Proposition 3.32.** *A morphism of bilax monoidal functors from $\theta\colon \mathcal{F} \to \mathcal{G}$ yields a morphism of bimonoids $\theta_H\colon \mathcal{F}(H) \to \mathcal{G}(H)$ in $(\mathsf{D}, \bullet)$ when $H$ is a bimonoid in $(\mathsf{C}, \bullet)$.*

PROOF. This follows from Proposition 3.30. □

**3.4.4. Braided lax functors preserve commutative monoids.** Braided lax monoidal functors preserve commutative monoids, and there are dual results for braided colax monoidal functors. These and related results are discussed next.

Recall the discussion in Section 3.4.2 which relates the opposite construction on (co)monoids to the conjugate construction on (co)lax monoidal functors. This used in conjunction with Proposition 3.23 yields the following.

**Proposition 3.33.** *Let $(\mathcal{F}, \varphi)$ be a lax monoidal functor. The images of a monoid $A$ under the lax monoidal functors $(\mathcal{F}, \varphi^b)$ and $(\mathcal{F}, {}^b\varphi)$ are respectively*

$$\mathcal{F}({}^{\mathrm{op}}A)^{\mathrm{op}} \qquad \text{and} \qquad {}^{\mathrm{op}}\mathcal{F}(A^{\mathrm{op}}).$$

*Let $(\mathcal{F}, \psi)$ be a colax monoidal functor. The images of a comonoid $C$ under the colax monoidal functors $(\mathcal{F}, \psi^b)$ and $(\mathcal{F}, {}^b\psi)$ are respectively*

$$\mathcal{F}({}^{\mathrm{cop}}C)^{\mathrm{cop}} \qquad \text{and} \qquad {}^{\mathrm{cop}}\mathcal{F}(C^{\mathrm{cop}}).$$

**Proposition 3.34.** *Let $(\mathcal{F}, \varphi, \psi)$ be a bilax monoidal functor $(\mathsf{C}, \bullet, \beta) \to (\mathsf{D}, \bullet, \beta)$ and let $H$ be a bimonoid in $(\mathsf{C}, \bullet, \beta)$. The image of $H^{\mathrm{cop}}$ under the bilax monoidal functor $(\mathcal{F}, \varphi, \psi^b)$ is*

$$\mathcal{F}(H)^{\mathrm{cop}}$$

*and the image of ${}^{\mathrm{op}}H$ under the bilax monoidal functor $(\mathcal{F}, {}^b\varphi, \psi)$ is*

$${}^{\mathrm{op}}\mathcal{F}(H).$$

*The images of $H$ under the bilax monoidal functors $(\mathcal{F}, \varphi^b, \psi^b)$ and $(\mathcal{F}, {}^b\varphi, {}^b\psi)$ are respectively*

$$\mathcal{F}({}^{\mathrm{op,cop}}H)^{\mathrm{op,cop}} \qquad \text{and} \qquad {}^{\mathrm{op,cop}}\mathcal{F}(H^{\mathrm{op,cop}}).$$

Proof. This follows from Proposition 3.33. Alternatively, it may also be deduced directly from Proposition 3.23. ◻

We emphasize a small point here. For the bilax functor $(\mathcal{F}, \varphi, \psi^b)$, the correct braiding to use on $\mathsf{C}$ is $\beta^{-1}$ rather than $\beta$. Hence in the statement above, this functor is applied to $H^{\mathrm{cop}}$ and not $H$. A similar remark applies to the bilax functor $(\mathcal{F}, {}^b\varphi, \psi)$.

**Proposition 3.35.** *Let $\mathcal{F}$ be a braided lax (resp. colax) monoidal functor. Then for $A$ a monoid (resp. $C$ a comonoid), we have*

$$\mathcal{F}({}^{\mathrm{op}}A) = {}^{\mathrm{op}}\mathcal{F}(A) \qquad \text{and} \qquad \mathcal{F}(A^{\mathrm{op}}) = \mathcal{F}(A)^{\mathrm{op}}$$

*as monoids (resp.*

$$\mathcal{F}({}^{\mathrm{cop}}C) = {}^{\mathrm{cop}}\mathcal{F}(C) \qquad \text{and} \qquad \mathcal{F}(C^{\mathrm{cop}}) = \mathcal{F}(C)^{\mathrm{cop}}$$

*as comonoids).*

**Proposition 3.36.** *Let $\mathcal{F}$ be a braided bilax monoidal functor. Then for $H$ a bimonoid, we have*

$$\mathcal{F}(H^{\mathrm{cop}}) = \mathcal{F}(H)^{\mathrm{cop}}, \qquad \mathcal{F}({}^{\mathrm{op}}H) = {}^{\mathrm{op}}\mathcal{F}(H),$$
$$\mathcal{F}({}^{\mathrm{op,cop}}H) = {}^{\mathrm{op,cop}}\mathcal{F}(H) \qquad \text{and} \qquad \mathcal{F}(H^{\mathrm{op,cop}}) = \mathcal{F}(H)^{\mathrm{op,cop}}$$

*as bimonoids.*

The above results follow from Propositions 3.33 and 3.34.

**Proposition 3.37.** *A braided (co)lax monoidal functor preserves (co)commutativity of (co)monoids and morphisms between (co)commutative (co)monoids.*

Proof. The first assertion follows by combining Proposition 3.35 with (1.15) and (1.16). It may also be viewed as a special case of Proposition 3.24. The second assertion follows from the fact that the category of (co)commutative (co)monoids is a full subcategory of the category of (co)monoids. ◻

The above result says that braided lax and braided colax monoidal functors induce functors

$$\mathsf{Mon}^{\mathsf{co}}(\mathsf{C}) \to \mathsf{Mon}^{\mathsf{co}}(\mathsf{D}) \qquad \text{and} \qquad {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C}) \to {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{D})$$

respectively.

**Proposition 3.38.** *A morphism of braided (co)lax monoidal functors $\theta\colon \mathcal{F} \to \mathcal{G}$ yields a morphism of (co)commutative (co)monoids $\theta_H\colon \mathcal{F}(H) \to \mathcal{G}(H)$ in $(\mathsf{D}, \bullet)$ when $H$ is a (co)commutative (co)monoid in $(\mathsf{C}, \bullet)$.*

PROOF. This follows from Proposition 3.30. □

**3.4.5. The convolution monoid revisited.** Recall the construction of the convolution monoid from Definition 1.13. This construction can be understood in terms of monoidal functors as follows.

A monoid $(C, A)$ in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ is the same as a comonoid $C$ and a monoid $A$ in $\mathsf{C}$. The convolution monoid $\mathrm{Hom}(C, A)$ then arises as the image of the monoid $(C, A)$ under the lax monoidal functor $\mathrm{Hom}\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$ of Example 3.17. In fact, we saw in Example 3.12 that this functor is braided; so it preserves commutativity. A commutative monoid $(C, A)$ in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ is the same as a cocommutative comonoid $C$ and a commutative monoid $A$ in $\mathsf{C}$. Thus, in this case the convolution monoid $\mathrm{Hom}(C, A)$ is commutative.

The convolution monoid arises in yet another manner. Consider the functors

$$\mathcal{F}_C\colon \mathsf{I} \to \mathsf{C} \quad \text{and} \quad \mathcal{F}_A\colon \mathsf{I} \to \mathsf{C},$$

as in Section 3.4.1. The former is colax monoidal and the latter is lax monoidal. We may thus consider the convolution comma category $\mathcal{F}_C \downarrow \mathcal{F}_A$ of Section 3.1.4. This is a monoidal category. The objects are arrows $C \to A$ in $\mathsf{C}$, and the only morphisms are identities. In other words, it is the discrete category corresponding to the set $\mathrm{Hom}(C, A)$. Further, the monoidal structure of $\mathcal{F}_C \downarrow \mathcal{F}_A$ boils down to the monoid structure of $\mathrm{Hom}(C, A)$.

## 3.5. Normal bilax monoidal functors

In this section, we discuss normal bilax monoidal functors. The terminology is motivated by the example of the normalized chain complex functor, which is discussed in Section 5.4.

**Definition 3.39.** A bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$ is called *normal* if

$$\varphi_0 \psi_0 = \mathrm{id}. \tag{3.24}$$

Note that a bilax functor always satisfies the condition $\psi_0 \varphi_0 = \mathrm{id}$, by (3.13). Thus, if it is normal, then $\varphi_0$ and $\psi_0$ are inverse maps. The Fock functors we consider in Part III are normal. However, not every bilax functor is normal. An example of a class of bilax functors that are not normal is given below.

**Remark 3.40.** Let $H$ be a bimonoid and $\mathcal{F}_H$ be the corresponding bilax functor of (3.23). The bilax functor $\mathcal{F}_H$ is normal if and only if $\iota\colon I \to H$ and $\epsilon\colon H \to I$ are inverse maps. Thus, $\mathcal{F}_H$ is normal if and only if $H$ is the trivial bimonoid. In particular, not every bilax functor is normal.

Normal bilax functors satisfy some interesting properties which we discuss next.

**Proposition 3.41.** *Let $(\mathcal{F}, \varphi, \psi)$ be a normal bilax monoidal functor. Then the following properties hold for any objects $A, B, C$.*

(1) *The maps $\varphi_{A,I}$ and $\psi_{A,I}$ are inverse, and so are the maps $\varphi_{I,A}$ and $\psi_{I,A}$.*
(2) $\psi_{A,B}\varphi_{A,B} = \mathrm{id}$.
(3) *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\beta"'] & \mathcal{F}(A \bullet B) \arrow[d, "\mathcal{F}(\beta)"] \\
\mathcal{F}(B) \bullet \mathcal{F}(A) & \mathcal{F}(B \bullet A) \arrow[l, "\psi_{B,A}"]
\end{tikzcd} \tag{3.25}$$

*This is equivalent to $\psi({}^b\varphi) = \mathrm{id}$ and also to $(\psi^b)\varphi = \mathrm{id}$.*
(4) *The following diagrams commute.*

$$\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(B \bullet C) \arrow[r, "\mathrm{id} \bullet \psi_{B,C}"] \arrow[d, "\varphi_{A,B\bullet C}"'] & \mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \arrow[d, "\varphi_{A,B} \bullet \mathrm{id}"] \\
\mathcal{F}(A \bullet B \bullet C) \arrow[r, "\psi_{A\bullet B,C}"'] & \mathcal{F}(A \bullet B) \bullet \mathcal{F}(C)
\end{tikzcd} \tag{3.26}$$

$$\begin{tikzcd}
\mathcal{F}(A \bullet B) \bullet \mathcal{F}(C) \arrow[r, "\psi_{A,B} \bullet \mathrm{id}"] \arrow[d, "\varphi_{A\bullet B,C}"'] & \mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \arrow[d, "\mathrm{id} \bullet \varphi_{B,C}"] \\
\mathcal{F}(A \bullet B \bullet C) \arrow[r, "\psi_{A,B\bullet C}"'] & \mathcal{F}(A) \bullet \mathcal{F}(B \bullet C)
\end{tikzcd} \tag{3.27}$$

Proof. The right diagram in (3.6) tells us that

$$\varphi_{A,I} = \mathcal{F}(\rho_A)\rho^{-1}_{\mathcal{F}(A)}(\mathrm{id} \bullet \varphi_0^{-1}),$$

while the dual diagram tells us that

$$\psi_{A,I} = (\mathrm{id} \bullet \psi_0^{-1})\rho_{\mathcal{F}(A)}\mathcal{F}(\rho_A)^{-1}$$

Since $\varphi_0 = \psi_0^{-1}$, we have $\varphi_{A,I} = \psi^{-1}_{A,I}$. One checks similarly that $\varphi_{I,A}$ and $\psi_{I,A}$ are inverse maps. This proves the first property.

The second property follows by the commutativity of the diagram below. For simplicity of notation, we omit the tensor product symbols. We write $I$ for the unit object in both the source and the target category. The composite map on the top side of the big square is $\psi_{A,B}\varphi_{A,B}$ and the composite obtained by following the other three sides is the identity, since $\beta_{I,I} = \mathrm{id}$.

$$
\begin{tikzcd}
\mathcal{F}A\mathcal{F}B \arrow[rr, "\varphi_{A,B}"] \arrow[ddr, "\mathcal{F}(\rho_A)\mathcal{F}(\lambda_B)"'] \arrow[dddd, "\rho_{\mathcal{F}A}\lambda_{\mathcal{F}B}"'] & & \mathcal{F}(AB) \arrow[r, "\psi_{A,B}"] \arrow[dl, "\mathcal{F}(\rho_A\lambda_B)"'] \arrow[d, "\mathcal{F}(\rho_A\lambda_B)"] & \mathcal{F}A\mathcal{F}B \\
& \mathcal{F}(AIIB) \arrow[r, "\mathrm{id}\,\beta_{I,I}\,\mathrm{id}"'] & \mathcal{F}(AIIB) \arrow[dr, "\psi_{AI,IB}"] & \\
& \mathcal{F}(AI)\mathcal{F}(IB) \arrow[u, "\varphi_{AI,IB}"'] \arrow[d, "\psi_{A,I}\psi_{I,B}"] & & \mathcal{F}(AI)\mathcal{F}(IB) \arrow[uu, "\mathcal{F}(\rho_A^{-1})\mathcal{F}(\lambda_B^{-1})"] \\
& \mathcal{F}A\mathcal{F}I\mathcal{F}I\mathcal{F}B \arrow[r, "\mathrm{id}\,\beta_{\mathcal{F}I,\mathcal{F}I}\,\mathrm{id}"'] \arrow[dl, "\mathrm{id}\,\psi_0\psi_0\,\mathrm{id}"'] & \mathcal{F}A\mathcal{F}I\mathcal{F}I\mathcal{F}B \arrow[ur, "\varphi_{A,I}\varphi_{I,B}"] \arrow[d, bend right, "\mathrm{id}\,\psi_0\psi_0\,\mathrm{id}"'] & \\
(\mathcal{F}A)II(\mathcal{F}B) \arrow[rr, "\mathrm{id}\,\beta_{I,I}\,\mathrm{id}"'] & & (\mathcal{F}A)II(\mathcal{F}B) \arrow[r, "\rho^{-1}_{\mathcal{F}A}\lambda^{-1}_{\mathcal{F}B}"'] \arrow[u, bend right, "\mathrm{id}\,\varphi_0\varphi_0\,\mathrm{id}"'] & \mathcal{F}A\mathcal{F}B \arrow[uu, "\mathcal{F}(\rho_A)\mathcal{F}(\lambda_B)"]
\end{tikzcd}
$$

The hexagon in the center commutes since $\mathcal{F}$ satisfies the braiding axiom (3.11). The other smaller diagrams, starting from the top left corner and going in counterclockwise direction, commute by the naturality of $\varphi$, the unitality of $\psi$, the naturality of $\beta$, the hypothesis $\varphi_0\psi_0 = \mathrm{id}$, the unitality of $\varphi$ and the naturality of $\psi$ respectively.

For the third property, we may proceed directly as in the above proof. Alternatively, we may use Proposition 3.16 to first deduce that $(\mathcal{F}, {}^b\varphi, \psi)$ and $(\mathcal{F}, \varphi, \psi^b)$ are bilax monoidal functors. Note that $\varphi_0$ and $\psi_0$ do not change during this construction. Now applying the second property to each of these functors, we obtain $\psi({}^b\varphi) = \mathrm{id}$ and $(\psi^b)\varphi = \mathrm{id}$, which are both equivalent to the commutativity of (3.25).

For the fourth property, diagram (3.26) commutes by the commutativity of the diagram below. One can then use symmetry to deduce that diagram (3.27) commutes as well.

$$\begin{tikzcd}
\mathcal{F}A\mathcal{F}(BC) \arrow[rrrr, "\mathrm{id}\,\psi_{B,C}"] \arrow[ddddd, "\varphi_{A,BC}"'] \arrow[ddr, "\mathcal{F}(\rho_A)\,\mathrm{id}"] & & & & \mathcal{F}A\mathcal{F}B\mathcal{F}C \arrow[ddd, "\mathrm{id}"] \arrow[ddl, "\mathrm{id}\,\lambda_{\mathcal{F}B}\,\mathrm{id}"'] \\
& & \mathcal{F}(AI)\mathcal{F}B\mathcal{F}C \arrow[urr, "\mathcal{F}(\rho_A^{-1})\,\mathrm{id}\,\mathrm{id}"] \arrow[d, "\psi_{A,I}\,\mathrm{id}\,\mathrm{id}"] & & \\
& \mathcal{F}(AI)\mathcal{F}(BC) \arrow[ur, "\mathrm{id}\,\psi_{B,C}"] \arrow[r, "\psi_{A,I}\psi_{B,C}"] \arrow[d, "\varphi_{AI,BC}"'] & \mathcal{F}A\mathcal{F}I\mathcal{F}B\mathcal{F}C \arrow[r, "\mathrm{id}\,\psi_0\,\mathrm{id}\,\mathrm{id}"] \arrow[d, "\mathrm{id}\,\beta_{\mathcal{F}I,\mathcal{F}C}\,\mathrm{id}"'] & (\mathcal{F}A)I(\mathcal{F}B)\mathcal{F}C \arrow[d, "\mathrm{id}\,\beta_{I,\mathcal{F}C}\,\mathrm{id}"] & \\
& \mathcal{F}(AIBC) \arrow[d, "\mathcal{F}(\mathrm{id}\,\beta_{I,B}\,\mathrm{id})"] & \mathcal{F}A\mathcal{F}B\mathcal{F}I\mathcal{F}C \arrow[r, shift left, "\mathrm{id}\,\mathrm{id}\,\psi_0\,\mathrm{id}"] \arrow[d, "\varphi_{A,B}\varphi_{I,C}"'] \arrow[dr, "\mathrm{id}\,\mathrm{id}\,\varphi_{I,C}"] & \mathcal{F}A(\mathcal{F}B)I(\mathcal{F}C) \arrow[l, shift left, "\mathrm{id}\,\mathrm{id}\,\varphi_0\,\mathrm{id}"] \arrow[r, "\mathrm{id}\,\rho^{-1}_{\mathcal{F}B}\,\mathrm{id}"] & \mathcal{F}A\mathcal{F}B\mathcal{F}C \arrow[dl, "\mathrm{id}\,\mathrm{id}\,\mathcal{F}(\lambda_C)"] \arrow[dd, "\varphi_{A,C}\,\mathrm{id}"'] \\
& \mathcal{F}(ABIC) \arrow[r, "\psi_{AB,IC}"'] & \mathcal{F}(AB)\mathcal{F}(IC) \arrow[drr, "\mathrm{id}\,\mathcal{F}(\lambda_C^{-1})"'] & \mathcal{F}A\mathcal{F}B\mathcal{F}(IC) \arrow[l, "\varphi_{A,C}\,\mathrm{id}"] & \\
\mathcal{F}(ABC) \arrow[uur, "\mathcal{F}(\rho_A\,\mathrm{id}\,\mathrm{id})"] \arrow[ur, "\mathcal{F}(\mathrm{id}\,\mathrm{id}\,\lambda_C)"'] \arrow[rrrr, "\psi_{AB,C}"'] & & & & \mathcal{F}(AB)\mathcal{F}C
\end{tikzcd}$$

The outside square is the diagram we want. The hexagon in the center commutes since $\mathcal{F}$ satisfies the braiding axiom (3.11). The other smaller diagrams commute by the naturality and unitality of $\varphi$ and $\psi$, the naturality of $\beta$ and its compatibility with the unit (1.7), and the hypothesis $\varphi_0\psi_0 = \mathrm{id}$. $\square$

We give an example below which shows that the converse to Proposition 3.41 is false. For a related result, see Proposition 3.46.

**Example 3.42.** Let $G$ be a finite group and let $\mathsf{Mod}_G$ be the symmetric monoidal category of left $G$-modules. Consider the functor

$$(-)^G\colon \mathsf{Mod}_G \to \mathsf{Vec}$$

which sends a module $M$ to $M^G$, the space of $G$-invariants of $M$ (Section 2.5.1).

Define natural transformations $\varphi$ and $\psi$ as in (3.9) to be

$$\begin{tikzcd} M^G \otimes N^G \arrow[r, bend left, "\varphi_{M,N}"] & (M \otimes N)^G, \arrow[l, bend left, "\psi_{M,N}"] \end{tikzcd}$$

where $\varphi$ is the natural inclusion and $\psi$ is given by either of the following expressions

$$\begin{aligned}\psi\left(\sum_i m_i \otimes n_i\right) &= \frac{1}{|G|}\sum_{g\in G}\sum_i g\cdot m_i \otimes n_i\\ &= \frac{1}{|G|}\sum_{h\in G}\sum_i m_i \otimes h\cdot n_i\\ &= \frac{1}{|G|^2}\sum_{g,h\in G}\sum_i g\cdot m_i \otimes h\cdot n_i.\end{aligned}$$

Define the morphisms $\varphi_0$ and $\psi_0$ as in (3.10) to be the identity maps.

One has $\varphi_0\psi_0 = \mathrm{id}$ and can show that $\big((-)^G, \varphi, \psi\big)$ satisfies all the properties in Proposition 3.41; however, it is not bilax. In fact, one may check that the braiding axiom (3.11) does not hold.

**Remark 3.43.** Functors $(\mathcal{F}, \varphi, \psi)$ satisfying (3.26) and (3.27) have been considered in the literature; they are called *Frobenius monoidal functors* [92, 265, 352, 353]. A Frobenius functor is said to be *separable* if in addition it satisfies

$$\varphi_{A,B}\psi_{A,B} = \mathrm{id}_{\mathcal{F}(A\bullet B)}.$$

Note the difference with condition (ii) in Proposition 3.41. McCurdy and Street [265, Proposition 3.10] show that a separable Frobenius functor necessarily satisfies the braiding axiom (3.11). By contrast, the conditions in Proposition 3.41 do not suffice to imply (3.11), as Example 3.42 shows.

## 3.6. Bistrong monoidal functors

Strong, costrong and bistrong monoidal functors were introduced in Definition 3.5. In this section, we study these notions in more detail.

Recall that a strong monoidal functor is a lax monoidal functor $(\mathcal{F}, \varphi)$ for which the transformation $\varphi$ is invertible. In this case, the functor $(\mathcal{F}, \varphi^{-1})$ is colax, and so it is natural to wonder whether $(\mathcal{F}, \varphi, \varphi^{-1})$ may be bilax. In Proposition 3.46 we show that this is the case if and only if the lax monoidal functor $(\mathcal{F}, \varphi)$ is braided. This is an important difference with the general case, in which a braided lax monoidal functor may not be bilax, and a bilax monoidal functor may not be braided.

Recall that a bistrong monoidal functor is a bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$ for which $\varphi$ and $\psi$ are invertible. In Proposition 3.45 we show that in this case necessarily $\psi = \varphi^{-1}$. It then follows that bistrong and braided strong are equivalent notions.

Another significant property of bistrong monoidal functors is that they preserve Hopf monoids. This is shown in Proposition 3.50. We mention that a general setup for dealing with the problem of preservation of Hopf monoids is considered in Section 3.7; see Proposition 3.60 and Remark 3.71 for the relevance to the present discussion.

**3.6.1. Strong and bistrong monoidal functors.** Throughout this discussion, $\mathsf{C}$ and $\mathsf{D}$ are monoidal categories, $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is a functor between them, $\varphi\colon \mathcal{F}^2 \Rightarrow \mathcal{F}_2$ and $\psi\colon \mathcal{F}_2 \to \mathcal{F}^2$ are natural transformations as in (3.3) and (3.7), and finally $\varphi_0\colon I \to \mathcal{F}(I)$ and $\psi_0\colon \mathcal{F}(I) \to I$ are maps in $\mathsf{D}$.

The following result is immediate from Definition 3.5.

**Proposition 3.44.** *Assume that $\varphi$ and $\varphi_0$ are invertible. Define $\psi = \varphi^{-1}$ and $\psi_0 = \varphi_0^{-1}$. Then*

$$(\mathcal{F}, \varphi) \text{ is strong} \iff (\mathcal{F}, \psi) \text{ is costrong.}$$

*Thus, strong and costrong are equivalent notions.*

Recall the notion of normal bilax monoidal functors from Definition 3.39.

**Proposition 3.45.** *Let $(\mathcal{F}, \varphi, \psi)$ be a bistrong functor. Then $\psi_0 = \varphi_0^{-1}$ and $\psi = \varphi^{-1}$. In particular, $\mathcal{F}$ is normal. Conversely, if $(\mathcal{F}, \varphi, \psi)$ is a normal bilax monoidal functor such that $\varphi\psi = \mathrm{id}$, then $(\mathcal{F}, \varphi, \psi)$ is bistrong.*

Proof. From (3.13) we know that $\psi_0\varphi_0 = \mathrm{id}$. But since under the present hypothesis these maps are invertible (Definition 3.5), we have that $\psi_0 = \varphi_0^{-1}$, that is, $\mathcal{F}$ is normal. It follows from Proposition 3.41 that $\psi\varphi = \mathrm{id}$, and again since these maps are invertible we have that $\psi = \varphi^{-1}$.

Conversely, if $\mathcal{F}$ is normal, then $\psi_0$ and $\varphi_0$ are inverse maps. Further, Proposition 3.41 gives $\psi\varphi = \mathrm{id}$, which with the hypothesis says that $\psi$ and $\varphi$ are inverse maps. Hence $\mathcal{F}$ is bistrong. ☐

**Proposition 3.46.** *The following are equivalent.*

(i) *$(\mathcal{F}, \varphi, \psi)$ is bistrong.*
(ii) *$(\mathcal{F}, \varphi)$ is braided strong, $\psi = \varphi^{-1}$, and $\psi_0 = \varphi_0^{-1}$.*
(iii) *$(\mathcal{F}, \psi)$ is braided costrong, $\varphi = \psi^{-1}$, and $\varphi_0 = \psi_0^{-1}$.*

Proof. It is clear that braided strong is equivalent to braided costrong. The nontrivial part is to show the equivalence between bistrong and braided strong. Suppose $(\mathcal{F}, \varphi, \psi)$ is bistrong. By Proposition 3.45, it is normal and $\psi = \varphi^{-1}$. We may then use Proposition 3.41, part (iii), to deduce that diagram (3.25) commutes, which since $\psi = \varphi^{-1}$ is equivalent to $(\mathcal{F}, \varphi)$ being braided.

For the converse implication, we first note that for a braided strong functor $(\mathcal{F}, \varphi)$, diagram (3.16) and the associativity of $\varphi$ imply that diagrams (3.25), (3.26) and (3.27) commute. The braiding axiom (3.11) for $(\mathcal{F}, \varphi, \psi)$ then follows from the commutativity of the diagram below (in which tensor product symbols are omitted).

$$\begin{tikzcd}
\mathcal{F}(AB)\mathcal{F}(CD) \arrow[rrr, "\psi_{A,B}\psi_{C,D}"] \arrow[dr, "\mathrm{id}\,\psi_{C,D}"'] \arrow[ddd, "\varphi_{AB,CD}"'] & & & \mathcal{F}A\mathcal{F}B\mathcal{F}C\mathcal{F}D \arrow[dl, "\mathrm{id}\,\varphi_{B,C}\,\mathrm{id}"] \arrow[dr, "\mathrm{id}\,\beta_{B,C}\,\mathrm{id}"] & \\
& \mathcal{F}(AB)\mathcal{F}C\mathcal{F}D \arrow[urr, "\psi_{A,B}\,\mathrm{id}"] \arrow[d, "\varphi_{AB,C}\,\mathrm{id}"'] & \mathcal{F}A\mathcal{F}(BC)\mathcal{F}D \arrow[dr, "\mathrm{id}\,\mathcal{F}(\beta_{B,C})\,\mathrm{id}"'] & & \mathcal{F}A\mathcal{F}C\mathcal{F}B\mathcal{F}D \arrow[ddl, "\varphi_{A,C}\,\mathrm{id}"'] \arrow[ddd, "\varphi_{A,C}\varphi_{B,D}"] \\
& \mathcal{F}(ABC)\mathcal{F}D \arrow[ur, "\psi_{A,BC}\,\mathrm{id}"'] & & \mathcal{F}A\mathcal{F}(CB)\mathcal{F}D \arrow[ur, "\mathrm{id}\,\psi_{C,B}\,\mathrm{id}"] \arrow[dl, "\varphi_{A,CB}\,\mathrm{id}"'] & \\
\mathcal{F}(ABCD) \arrow[ur, "\psi_{ABC,D}"'] \arrow[dr, "\mathcal{F}(\mathrm{id}\,\beta_{B,C}\,\mathrm{id})"'] & & \mathcal{F}(ACB)\mathcal{F}D \arrow[r, "\psi_{AC,B}\,\mathrm{id}"'] \arrow[dl, "\varphi_{ACB,D}"'] & \mathcal{F}(AC)\mathcal{F}B\mathcal{F}D \arrow[dr, "\mathrm{id}\,\varphi_{B,D}"'] & \\
& \mathcal{F}(ACBD) \arrow[rrr, "\psi_{AC,BD}"'] & & & \mathcal{F}(AC)\mathcal{F}(BD)
\end{tikzcd}$$

The two triangles commute by construction, the hexagon commutes by the naturality of $\varphi$ or $\psi$ (here we use $\psi = \varphi^{-1}$), and the remaining five squares commute by diagrams (3.25), (3.26) and (3.27).

Since $\psi = \varphi^{-1}$, the two diagrams in (3.12) are essentially the same. Their commutativity follows by setting $A = I$ in the unitality axiom (3.6) for $(\mathcal{F}, \varphi)$. The commutativity of (3.13) follows since $\psi_0 = \varphi_0^{-1}$. This verifies the unitality axioms for $(\mathcal{F}, \varphi, \psi)$ and completes the proof. □

**Remark 3.47.** The same proof essentially as above, but with a new perspective, is given in Chapter 6. It works as follows. There is an equivalent way to define a braided (co)lax functor which is closer to the definition of a bilax functor. In fact, by reversing some arrows, one can pass back and forth between the two definitions. The above result then follows, since all arrows are invertible, For more details, see Example 6.64. To summarize, in the strong situation, the distinction between braided lax and bilax disappears.

**3.6.2. Bistrong functors preserve Hopf monoids and antipodes.** Below (Proposition 3.50) we show that the image of a Hopf monoid under a bistrong monoidal functor is again a Hopf monoid, and that the antipode of the former is the image of the antipode of the latter.

**Proposition 3.48.** *A bilax monoidal functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is bistrong if and only if the natural transformation*

$$\mathrm{Hom}_{\mathsf{C}}(-,-) \implies \mathrm{Hom}_{\mathsf{D}}\big(\mathcal{F}(-), \mathcal{F}(-)\big)$$

*which sends $f$ to $\mathcal{F}(f)$ is a morphism of lax monoidal functors.*

The second functor is the composite of the lax functors

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \xrightarrow{\mathcal{F}\times\mathcal{F}} \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \xrightarrow{\mathrm{Hom}} \mathsf{Set}.$$

For the lax structure on $\mathcal{F} \times \mathcal{F}$, one uses the colax structure of $\mathcal{F}$ on the first component and the lax structure on the second component. The lax structure of Hom is described in Example 3.17.

Proof. The natural transformation

$$\mathrm{Hom}_{\mathsf{C}}(-,-) \implies \mathrm{Hom}_{\mathsf{D}}\big(\mathcal{F}(-), \mathcal{F}(-)\big)$$

is a morphism of lax functors if and only if the following diagrams commute.

$$\begin{tikzcd}
\mathrm{Hom}(A,C) \times \mathrm{Hom}(B,D) \arrow[r] \arrow[dd] & \mathrm{Hom}\big(\mathcal{F}(A),\mathcal{F}(C)\big) \times \mathrm{Hom}\big(\mathcal{F}(B),\mathcal{F}(D)\big) \arrow[d] \\
 & \mathrm{Hom}\big(\mathcal{F}(A)\bullet\mathcal{F}(B), \mathcal{F}(C)\bullet\mathcal{F}(D)\big) \arrow[d, "\varphi_{C,D}\circ - \circ\psi_{A,B}"] \\
\mathrm{Hom}(A\bullet B, C\bullet D) \arrow[r] & \mathrm{Hom}\big(\mathcal{F}(A\bullet B), \mathcal{F}(C\bullet D)\big)
\end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Hom}(I,I) \arrow[d, "\varphi_0\circ - \circ\psi_0"] \\ \{\emptyset\} \arrow[ur] \arrow[dr] & \\ & \mathrm{Hom}\big(\mathcal{F}(I),\mathcal{F}(I)\big) \end{tikzcd}$$

This is equivalent to the commutativity of the diagrams

$$\begin{tikzcd} \mathcal{F}(A\bullet B) \arrow[r, "\mathcal{F}(f\bullet g)"] \arrow[d, "\psi_{A,B}"'] & \mathcal{F}(C\bullet D) \\ \mathcal{F}(A)\bullet\mathcal{F}(B) \arrow[r, "\mathcal{F}(f)\bullet\mathcal{F}(g)"'] & \mathcal{F}(C)\bullet\mathcal{F}(D) \arrow[u, "\varphi_{C,D}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(I) \arrow[r, "\mathrm{id}"] \arrow[d, "\psi_0"'] & \mathcal{F}(I) \\ I \arrow[r, "\mathrm{id}"'] & I \arrow[u, "\varphi_0"'] \end{tikzcd}$$

for any morphisms $f\colon A \to C$ and $g\colon B \to D$.

For the forward implication, we note that if $\mathcal{F}$ is bistrong, then the left diagram commutes by the naturality of $\psi$ or $\varphi$ (since $\varphi = \psi^{-1}$) and the right diagram commutes since $\varphi_0 = \psi_0^{-1}$.

For the backward implication, assume that the above diagrams commute. Setting $A = C$ and $B = D$ and $f$ and $g$ to be identities, we conclude that $\varphi\psi = \mathrm{id}$ and $\varphi_0\psi_0 = \mathrm{id}$. Then Proposition 3.45 implies that $\mathcal{F}$ is bistrong. $\square$

Let $A$ be a monoid and $C$ be a comonoid in $\mathsf{C}$. As mentioned in Section 3.4.5, the convolution monoid $\mathrm{Hom}(C, A)$ arises as the image of the monoid $(C, A)$ in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ under the lax monoidal functor $\mathrm{Hom}\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$. Therefore, Propositions 3.30 and 3.48 imply:

**Proposition 3.49.** *For $\mathcal{F}$ a bistrong monoidal functor from $(\mathsf{C},\bullet)$ to $(\mathsf{D},\bullet)$ and $C$ a comonoid and $A$ a monoid in $(\mathsf{C},\bullet)$, there is a morphism of convolution monoids*

$$\mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}\big(\mathcal{F}(C), \mathcal{F}(A)\big)$$

*which sends $f$ to $\mathcal{F}(f)$.*

It follows that a bistrong monoidal functor preserves antipodes. In more detail, we have:

**Proposition 3.50.** *If $\mathcal{F}$ is a bistrong monoidal functor from $(\mathsf{C},\bullet)$ to $(\mathsf{D},\bullet)$ and $H$ is a Hopf monoid in $(\mathsf{C},\bullet)$ with antipode $\mathrm{S}\colon H \to H$, then $\mathcal{F}(H)$ is a Hopf monoid in $(\mathsf{D},\bullet)$ with antipode $\mathcal{F}(\mathrm{S})\colon \mathcal{F}(H) \to \mathcal{F}(H)$.*

*In addition, if $f\colon H \to H'$ is a morphism of Hopf monoids in $(\mathsf{C},\bullet)$, then $\mathcal{F}(f)\colon \mathcal{F}(H) \to \mathcal{F}(H')$ is a morphism of Hopf monoids in $(\mathsf{D},\bullet)$.*

Since a morphism of Hopf monoids is a morphism of the underlying bimonoids, Proposition 3.32 implies:

**Proposition 3.51.** *A morphism of bistrong monoidal functors from $\mathcal{F}$ to $\mathcal{G}$ yields a morphism of Hopf monoids $\mathcal{F}(H) \to \mathcal{G}(H)$ in $(\mathsf{D},\bullet)$ when $H$ is a Hopf monoid in $(\mathsf{C},\bullet)$.*

**Example 3.52.** Let $\Bbbk$ be a commutative ring. Consider the *linearization functor*

$$\Bbbk(-)\colon (\mathsf{Set}, \times, \{*\}) \longrightarrow (\mathsf{Mod}_{\Bbbk}, \otimes, \Bbbk),$$

which sends a set to the free $\Bbbk$-module with basis the given set. This functor is bistrong. Below we discuss three implications of this statement.

- Every set $X$ carries a unique comonoid structure in $(\mathsf{Set}, \times, \{*\})$. The coproduct $\Delta : X \to X \times X$ is $\Delta(x) = (x, x)$ and the counit $\epsilon : X \to \{*\}$ is $\epsilon(x) = *$.

  It follows that $\Bbbk X$ is a coalgebra in which all elements of $X$ are group-like, that is, $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$ for $x \in X$. This is the *coalgebra of a set* [1, Example 2.1] or [350, Section 1.0, Example 1].
- If the set $X$ is a monoid (in the usual sense), then it is canonically a bimonoid in $(\mathsf{Set}, \times, \{*\})$, and hence $\Bbbk X$ is a bialgebra.
- A monoid $X$ is a Hopf monoid precisely if $X$ is a group. (The antipode sends an element to its inverse.) Hence for any group $X$, the group algebra $\Bbbk X$ is a Hopf algebra. Its antipode is the linearization of the map of $x \mapsto x^{-1}$. See [191, Section III.3, Example 2] or [350, Section 4.0, Example 1].

## 3.7. Hopf lax monoidal functors

We know that a bilax monoidal functor preserves bimonoids. In addition, we have seen that bistrong monoidal functors preserve Hopf monoids and antipodes (Proposition 3.50). However, an arbitrary bilax monoidal functor need not preserve Hopf monoids or antipodes. In other words: If $H$ is a Hopf monoid in $\mathsf{C}$ with antipode $\mathrm{s}\colon H \to H$, and if $\mathcal{F}$ is a bilax monoidal functor from $\mathsf{C}$ to $\mathsf{D}$, then $\mathcal{F}(H)$ need not be a Hopf monoid, and even if it is, the antipode of $\mathcal{F}(H)$ need not be $\mathcal{F}(\mathrm{s})$.

We provide a simple example. Consider the one-arrow category $\mathsf{I}$. Then its unique object $*$ is a Hopf monoid, whose antipode is the identity. Associated to any bimonoid $H$ there is the bilax monoidal functor $\mathcal{F}_H$ of (3.23), and $\mathcal{F}_H(*) = H$, which may be a Hopf monoid or not. Even when this is the case, the antipode of $H$ need not be the identity.

Numerous examples with these features appear in the later parts of the monograph. To give one concrete example, consider the Hopf monoid $\mathbf{L}^*$ of linear orders in the category of species (Example 8.24). Call its antipode $\mathrm{s}$. Applying the full Fock functor $\mathcal{K}$ to $\mathbf{L}^*$ yields the Hopf algebra of permutations $\mathrm{S}\Lambda$ (Example 15.17), whose antipode is not $\mathcal{K}(\mathrm{s})$.

The goal of this section is to show that there is an intermediate class of functors between bilax and bistrong that preserves Hopf monoids but modifies antipodes in a predictable manner, much as the rest of the structure is modified by a bilax monoidal functor. We call them *Hopf lax monoidal functors*. They have an interesting basic theory which we now present.

**Notation 3.53.** For $(\mathcal{F}, \varphi)$ lax, we write

$$\varphi_{A,B,C}\colon \mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \to \mathcal{F}(A \bullet B \bullet C)$$

for the map obtained by following the two directions in diagram (3.5). Note that we are not specifying brackets here; the objects are to be interpreted as the unbracketed tensor products.

Similarly for $(\mathcal{F}, \psi)$ colax, we write

$$\psi_{A,B,C}\colon \mathcal{F}(A \bullet B \bullet C) \to \mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C).$$

Suppose $(\mathcal{F}, \varphi, \psi)$ and $(\mathcal{G}, \gamma, \delta)$ are composable functors, as in Definition 3.20. Then,

$$\begin{aligned} (\varphi\gamma)_{A,B,C} &= \mathcal{G}(\varphi_{A,B,C})\gamma_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C} \\ (\delta\psi)_{A,B,C} &= \delta_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C}\mathcal{G}(\psi_{A,B,C}). \end{aligned} \tag{3.28}$$

These identities follow from the proof of Theorem 3.21.

### 3.7.1. Hopf lax monoidal functors.

Let $(\mathsf{C}, \bullet, \beta)$ and $(\mathsf{D}, \bullet, \beta)$ be braided monoidal categories.

**Definition 3.54.** A *Hopf lax monoidal functor* $(\mathcal{F}, \varphi, \psi, \Upsilon)$ consists of a bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$ from $\mathsf{C}$ to $\mathsf{D}$ and a natural transformation $\Upsilon\colon \mathcal{F} \Rightarrow \mathcal{F}$ such that the following diagrams commute.

$$\begin{tikzcd} \mathcal{F}(A)\bullet\mathcal{F}(B)\bullet\mathcal{F}(C) \arrow[r, "\mathrm{id}_A\bullet\Upsilon_B\bullet\mathrm{id}_C"] & \mathcal{F}(A)\bullet\mathcal{F}(B)\bullet\mathcal{F}(C) \arrow[d, "\varphi_{A,B,C}"] \\ \mathcal{F}(A\bullet B\bullet C) \arrow[u, "\psi_{A,B,C}"] \arrow[r, "\mathrm{id}_{A\bullet B\bullet C}"'] & \mathcal{F}(A\bullet B\bullet C) \end{tikzcd} \tag{3.29}$$

$$\begin{tikzcd} \mathcal{F}(A)\bullet\mathcal{F}(B)\bullet\mathcal{F}(C) \arrow[r, "\Upsilon_A\bullet\mathrm{id}_B\bullet\Upsilon_C"] & \mathcal{F}(A)\bullet\mathcal{F}(B)\bullet\mathcal{F}(C) \arrow[d, "\varphi_{A,B,C}"] \\ \mathcal{F}(A\bullet B\bullet C) \arrow[u, "\psi_{A,B,C}"] \arrow[r, "\Upsilon_{A\bullet B\bullet C}"'] & \mathcal{F}(A\bullet B\bullet C) \end{tikzcd} \tag{3.30}$$

$$\begin{tikzcd} \mathcal{F}(I)\bullet\mathcal{F}(I) \arrow[r, "\mathrm{id}_I\bullet\Upsilon_I"] & \mathcal{F}(I)\bullet\mathcal{F}(I) \arrow[d, "\varphi_{I,I}"] & \mathcal{F}(I)\bullet\mathcal{F}(I) \arrow[r, "\Upsilon_I\bullet\mathrm{id}_I"] & \mathcal{F}(I)\bullet\mathcal{F}(I) \arrow[d, "\varphi_{I,I}"] \\ \mathcal{F}(I\bullet I) \arrow[u, "\psi_{I,I}"] & \mathcal{F}(I\bullet I) \arrow[d, "\mathcal{F}(\rho_I^{-1})"] & \mathcal{F}(I\bullet I) \arrow[u, "\psi_{I,I}"] & \mathcal{F}(I\bullet I) \arrow[d, "\mathcal{F}(\rho_I^{-1})"] \\ \mathcal{F}(I) \arrow[u, "\mathcal{F}(\rho_I)"] \arrow[r, "\varphi_0\psi_0"'] & \mathcal{F}(I) & \mathcal{F}(I) \arrow[u, "\mathcal{F}(\rho_I)"] \arrow[r, "\varphi_0\psi_0"'] & \mathcal{F}(I). \end{tikzcd} \tag{3.31}$$

We say that $\mathcal{F}$ is a Hopf lax monoidal functor with antipode $\Upsilon$.

We give a reformulation of axiom (3.31). Recall that the unit object $I$ of a braided monoidal category $\mathsf{C}$ is a bimonoid. Suppose that $(\mathcal{F}, \varphi, \psi)$ is a bilax functor from $\mathsf{C}$ to $\mathsf{D}$ and $\Upsilon\colon \mathcal{F} \Rightarrow \mathcal{F}$ is a natural transformation. Then $\mathcal{F}$ preserves bimonoids by Proposition 3.31. By construction, the coproduct and product of $\mathcal{F}(I)$ are the composites of the vertical maps in (3.31) and the counit and unit are $\psi_0$ and $\varphi_0$ respectively. In this situation,

$$\mathcal{F} \text{ satisfies axiom (3.31)} \iff \mathcal{F}(I) \text{ is a Hopf monoid with antipode } \Upsilon_I. \tag{3.32}$$

In particular, if $(\mathcal{F}, \varphi, \psi, \Upsilon)$ is Hopf lax, then $\mathcal{F}(I)$ is a Hopf monoid with antipode $\Upsilon_I$.

**Lemma 3.55.** *The antipode of a Hopf lax functor $\mathcal{F}$ is determined by its value on the unit object, in the sense that the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}(I) \bullet \mathcal{F}(A) \bullet \mathcal{F}(I) \arrow[rr, "\Upsilon_I \bullet \mathrm{id}_A \bullet \Upsilon_I"] & & \mathcal{F}(I) \bullet \mathcal{F}(A) \bullet \mathcal{F}(I) \arrow[d, "\varphi_{I,A,I}"] \\
\mathcal{F}(I \bullet A \bullet I) \arrow[u, "\psi_{I,A,I}"] & & \mathcal{F}(I \bullet A \bullet I) \arrow[d, "\cong"] \\
\mathcal{F}(A) \arrow[u, "\cong"] \arrow[rr, "\Upsilon_A"'] & & \mathcal{F}(A).
\end{tikzcd} \tag{3.33}$$

Proof. The vertical maps near the bottom are the canonical isomorphisms constructed from the unit constraints. The diagram can be divided in two by inserting $\Upsilon_{I \bullet A \bullet I}$ in the middle. The top half commutes by (3.30) and the bottom half by naturality of $\Upsilon$. □

**Proposition 3.56.** *The antipode of a Hopf lax functor is unique.*

Proof. Let $\mathcal{F}$ be a Hopf lax functor. We know from (3.32) that $\mathcal{F}(I)$ is a Hopf monoid with antipode $\Upsilon_I$. Hence $\Upsilon_I$ is unique. Then $\Upsilon_A$ is determined by (3.33). □

**Definition 3.57.** Let $\mathcal{F}$ be a Hopf lax functor. Define natural transformations $\upsilon \colon \mathcal{F} \Rightarrow \mathcal{F}$ and $\upsilon' \colon \mathcal{F} \Rightarrow \mathcal{F}$ by

$$\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(I) \arrow[r, "\mathrm{id}_A \bullet \Upsilon_I"] & \mathcal{F}(A) \bullet \mathcal{F}(I) \arrow[d, "\varphi_{A,I}"] \\
\mathcal{F}(A \bullet I) \arrow[u, "\psi_{A,I}"] & \mathcal{F}(A \bullet I) \arrow[d, "\mathcal{F}(\rho_A^{-1})"] \\
\mathcal{F}(A) \arrow[u, "\mathcal{F}(\rho_A)"] \arrow[r, dashed, "\upsilon_A"'] & \mathcal{F}(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(I) \arrow[r, "\Upsilon_A \bullet \mathrm{id}_I"] & \mathcal{F}(A) \bullet \mathcal{F}(I) \arrow[d, "\varphi_{A,I}"] \\
\mathcal{F}(A \bullet I) \arrow[u, "\psi_{A,I}"] & \mathcal{F}(A \bullet I) \arrow[d, "\mathcal{F}(\rho_A^{-1})"] \\
\mathcal{F}(A) \arrow[u, "\mathcal{F}(\rho_A)"] \arrow[r, dashed, "\upsilon'_A"'] & \mathcal{F}(A).
\end{tikzcd} \tag{3.34}$$

We say that $\upsilon$ and $\upsilon'$ are the *convolution units* associated to $\mathcal{F}$.

There is an alternative way to define $\upsilon$ and $\upsilon'$; see Remark 3.67.

### 3.7.2. Morphisms of Hopf lax monoidal functors.

**Definition 3.58.** A morphism of Hopf lax monoidal functors is a morphism of the underlying bilax monoidal functors (Definition 3.9).

Next we show that such a morphism necessarily preserves antipodes and the associated convolution units, thus justifying the terminology.

**Proposition 3.59.** *Let $(\mathcal{F}, \varphi, \psi, \Upsilon)$ and $(\mathcal{G}, \gamma, \delta, \Omega)$ be Hopf lax functors from $\mathsf{C}$ to $\mathsf{D}$. Let $\upsilon$ and $\upsilon'$ be the convolution units associated to $\mathcal{F}$, as in (3.34), and $\omega$ and*

*$\omega'$ those associated to $\mathcal{G}$. Let $\theta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of bilax functors. Then the following diagrams commute, for any object $A$.*

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\Upsilon_A"] \arrow[d, "\theta_A"'] & \mathcal{F}(A) \arrow[d, "\theta_A"] \\
\mathcal{G}(A) \arrow[r, "\Omega_A"'] & \mathcal{G}(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\upsilon_A"] \arrow[d, "\theta_A"'] & \mathcal{F}(A) \arrow[d, "\theta_A"] \\
\mathcal{G}(A) \arrow[r, "\omega_A"'] & \mathcal{G}(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\upsilon'_A"] \arrow[d, "\theta_A"'] & \mathcal{F}(A) \arrow[d, "\theta_A"] \\
\mathcal{G}(A) \arrow[r, "\omega'_A"'] & \mathcal{G}(A).
\end{tikzcd}
\tag{3.35}$$

Proof. Applying the forward direction of (3.32), $\mathcal{F}(I)$ and $\mathcal{G}(I)$ are Hopf monoids with antipodes $\Upsilon_I$ and $\Omega_I$, respectively. By Proposition 3.32, $\theta_I\colon \mathcal{F}(I) \to \mathcal{G}(I)$ is a morphism of bimonoids, and hence, by Proposition 1.16, it preserves the antipodes. Thus, the first diagram in (3.35) commutes when $A = I$. The general case follows by using diagrams (3.14) and (3.15), since $\Upsilon_A$ is determined by $\Upsilon_I$ (3.33).

The commutativity of the other diagrams follows similarly, using that $\upsilon_A$ and $\upsilon'_A$ are determined by $\Upsilon$ by means of (3.34). $\square$

**3.7.3. Bistrong versus Hopf lax.** Bistrong monoidal functors (Definition 3.5) are always Hopf lax. The converse implication holds provided the functor is in addition normal (Definition 3.39). In other words:

**Proposition 3.60.** *Let $(\mathcal{F}, \varphi, \psi)$ be a bilax monoidal functor. Then*

$$\mathcal{F} \text{ is bistrong} \iff \mathcal{F} \text{ is Hopf lax and normal.}$$

*In this case, the antipode is $\Upsilon = \mathrm{id}$.*

Proof. If $\mathcal{F}$ is bistrong, the transformations $\varphi_{A,B}$ and $\psi_{A,B}$ are inverse. It follows that so are $\varphi_{A,B,C}$ and $\psi_{A,B,C}$. Hence diagrams (3.29)–(3.30) commute with $\Upsilon = \mathrm{id}$. Since $\varphi_0$ and $\psi_0$ are inverse, diagrams (3.31) commute too. Thus, $\mathcal{F}$ is Hopf lax and normal.

Assume now that $\mathcal{F}$ is Hopf lax and normal. Proposition 3.41 implies $\psi\varphi = \mathrm{id}$, so we only need to show $\varphi\psi = \mathrm{id}$.

Proposition 3.41 also tells us that $\varphi_{I,I}$ and $\psi_{I,I}$ are inverse maps. Therefore, diagrams (3.31) commute when $\Upsilon_I$ is replaced by $\mathrm{id}_I$. By uniqueness of antipodes for the Hopf monoid $\mathcal{F}(I)$, we must have $\Upsilon_I = \mathrm{id}_I$. Now considering diagram (3.29) with $B = I$ we deduce

$$\varphi_{A,I,C}\psi_{A,I,C} = \mathrm{id}.$$

But the diagram below shows that $\varphi_{A,I,C}$ identifies with $\varphi_{A,C}$ by composing with $\varphi_0$:

$$\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(I) \bullet \mathcal{F}(C) \arrow[r, "\mathrm{id}\bullet\varphi_{I,C}"] \arrow[rr, bend left, "\varphi_{A,I,C}"] & \mathcal{F}(A) \bullet \mathcal{F}(I \bullet C) \arrow[r, "\varphi_{A,I\bullet C}"] & \mathcal{F}(A \bullet I \bullet C) \\
\mathcal{F}(A) \bullet I \bullet \mathcal{F}(C) \arrow[u, "\mathrm{id}\bullet\varphi_0\bullet\mathrm{id}"] & \mathcal{F}(A) \bullet \mathcal{F}(C) \arrow[l, "\mathrm{id}\bullet\lambda_{\mathcal{F}(C)}"] \arrow[u, "\mathrm{id}\bullet\mathcal{F}(\lambda_C)"'] \arrow[r, "\varphi_{A,C}"'] & \mathcal{F}(A \bullet C). \arrow[u, "\mathcal{F}(\mathrm{id}\bullet\lambda_C)"']
\end{tikzcd}$$

The diagram commutes by (3.5), (3.6), and the naturality of $\lambda$. Similarly, $\psi_{A,I,C}$ identifies with $\psi_{A,C}$ by composing with $\psi_0$. Therefore,

$$\varphi_{A,C}\psi_{A,C} = \mathrm{id}. \qquad \square$$

The main examples of bilax functors discussed in this monograph that are not bistrong turn out not to be Hopf lax either. The reason is that they are normal.

**3.7.4. Convolution of natural transformations.** The antipode of a Hopf monoid is the inverse of the identity map in the convolution monoid (Definition 1.15). The situation is somewhat different for bilax monoidal functors; nevertheless, there is an operation between natural transformations that plays the role of convolution; it is defined in (3.37). The convolution identities in this context involve another operation that has no analogue for Hopf monoids; it is defined in (3.36).

Recall from (3.1) that if $\mathcal{F}$ is a functor between monoidal categories $\mathsf{C}$ and $\mathsf{D}$, then $\mathcal{F}_2 \colon \mathsf{C} \times \mathsf{C} \to \mathsf{D}$ is the functor

$$\mathcal{F}_2(A, B) := \mathcal{F}(A \bullet B).$$

Suppose that $\mathcal{G} \colon \mathsf{C} \to \mathsf{D}$ is another functor and $\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation. We let

$$\theta^{(2)} \colon \mathcal{F}_2 \Rightarrow \mathcal{G}_2$$

be the natural transformation

$$\theta^{(2)}_{A,B} \colon \mathcal{F}(A \bullet B) \xrightarrow{\theta_{A \bullet B}} \mathcal{G}(A \bullet B). \tag{3.36}$$

Now assume that $(\mathcal{F}, \psi)$ is a colax monoidal functor and $(\mathcal{G}, \gamma)$ is a lax monoidal functor. Given natural transformations $\sigma$ and $\tau \colon \mathcal{F} \Rightarrow \mathcal{G}$, we define their *convolution*

$$\sigma * \tau \colon \mathcal{F}_2 \Rightarrow \mathcal{G}_2$$

as the natural transformation

$$(\sigma*\tau)_{A,B} \colon \mathcal{F}(A\bullet B) \xrightarrow{\psi_{A,B}} \mathcal{F}(A)\bullet\mathcal{F}(B) \xrightarrow{\sigma_A \bullet \tau_B} \mathcal{G}(A)\bullet\mathcal{G}(B) \xrightarrow{\gamma_{A,B}} \mathcal{G}(A\bullet B). \tag{3.37}$$

We study the behavior of morphisms of lax and colax functors (Definition 3.8) with respect to convolution of natural transformations.

**Proposition 3.61.** *Let $(\mathcal{F}, \psi)$ and $(\mathcal{F}', \psi')$ be colax functors and $(\mathcal{G}, \gamma)$ and $(\mathcal{G}', \gamma')$ be lax functors, all from $\mathsf{C}$ to $\mathsf{D}$. Let*

$$\theta \colon (\mathcal{F}', \psi') \Rightarrow (\mathcal{F}, \psi) \quad \textit{and} \quad \kappa \colon (\mathcal{G}, \gamma) \Rightarrow (\mathcal{G}', \gamma')$$

*be a morphism of colax functors and a morphism of lax functors, respectively. Then, for any natural transformations $\sigma, \tau \colon \mathcal{F} \Rightarrow \mathcal{G}$, we have*

$$(\sigma\theta) * (\tau\theta) = (\sigma * \tau)\theta^{(2)} \qquad \textit{and} \qquad (\kappa\sigma) * (\kappa\tau) = \kappa^{(2)}(\sigma * \tau).$$

This is the analogue of Proposition 1.14. The proof is straightforward.

**3.7.5. Convolution identities.** We now establish some familiar convolution identities in the context of Hopf lax functors. The transformations that play the role of the convolution unit are defined in (3.34). Here we take the most direct approach to establishing these identities in order to quickly build up to Theorem 3.70. A more in-depth study of convolution of natural transformations is carried out in Section D.4.

Throughout this discussion, $(\mathcal{F}, \varphi, \psi, \Upsilon)$ denotes a Hopf lax functor.

**Proposition 3.62.** *We have* $\upsilon_I = \upsilon'_I = \varphi_0\psi_0$.

Proof. This follows from definition (3.34) and axiom (3.31). □

Let $\mathrm{id}_1$ denote the identity natural transformation of $\mathcal{F}$ and $\mathrm{id}_2$ that of $\mathcal{F}_2$.

**Proposition 3.63.** *We have* $\upsilon * \mathrm{id}_1 = \mathrm{id}_1 * \upsilon' = \mathrm{id}_2$.

Proof. The proof of the identity $\upsilon * \mathrm{id}_1 = \mathrm{id}_2$ follows from the commutativity of the diagram below. The proof of the other identity is similar.

$$\begin{tikzcd}
\mathcal{F}(A)\mathcal{F}(B) \arrow[rrrrrr, "\upsilon_A\,\mathrm{id}_B"] \arrow[dr, "\mathcal{F}(\rho_A)\,\mathrm{id}_B"] & & & & & & \mathcal{F}(A)\mathcal{F}(B) \arrow[dddd, "\varphi_{A,B}"] \\
& \mathcal{F}(AI)\mathcal{F}(B) \arrow[dr, "\psi_{A,I}\,\mathrm{id}_B"] & & & & \mathcal{F}(AI)\mathcal{F}(B) \arrow[ur, "\mathcal{F}(\rho_A^{-1})\,\mathrm{id}_B"] \arrow[dd, "\varphi_{AI,B}"] & \\
& & \mathcal{F}(A)\mathcal{F}(I)\mathcal{F}(B) \arrow[rr, "\mathrm{id}_A\,\Upsilon_I\,\mathrm{id}_B"] & & \mathcal{F}(A)\mathcal{F}(I)\mathcal{F}(B) \arrow[ur, "\varphi_{A,I}\,\mathrm{id}_B"] \arrow[dr, "\varphi_{A,I,B}"] & & \\
& \mathcal{F}(AIB) \arrow[uu, "\psi_{AI,B}"] \arrow[ur, "\psi_{A,I,B}"] \arrow[rrrr, "\mathrm{id}_{AIB}"'] & & & & \mathcal{F}(AIB) \arrow[dr, "\mathcal{F}(\rho_A^{-1}\,\mathrm{id})"] & \\
\mathcal{F}(AB) \arrow[uuuu, "\psi_{A,B}"] \arrow[ur, "\mathcal{F}(\rho_A\,\mathrm{id})"] \arrow[rrrrrr, "\mathrm{id}_{AB}"'] & & & & & & \mathcal{F}(AB)
\end{tikzcd}$$

The hexagon commutes by the definition of $\upsilon$, the square in the center commutes by axiom (3.29) and the remaining three squares commute by naturality of $\psi$, id, and $\varphi$. □

**Proposition 3.64.** *We have* $\Upsilon * \upsilon = \upsilon' * \Upsilon = \Upsilon^{(2)}$.

Proof. The proof of the identity $\upsilon' * \Upsilon = \Upsilon^{(2)}$ follows from the commutativity of the diagram below. The proof of the other identity is similar.

$$
\begin{tikzcd}
\mathcal{F}(A)\mathcal{F}(B) \arrow[rrrrr, "\upsilon'_A \Upsilon_B"] \arrow[dr, "\mathcal{F}(\rho_A)\,\mathrm{id}_B"] & & & & & \mathcal{F}(A)\mathcal{F}(B) \arrow[dddd, "\varphi_{A,B}"] \\
& \mathcal{F}(AI)\mathcal{F}(B) \arrow[dr, "\psi_{A,I}\,\mathrm{id}_B"] & & & \mathcal{F}(AI)\mathcal{F}(B) \arrow[ur, "\mathcal{F}(\rho_A^{-1})\,\mathrm{id}_B"] \arrow[dd, "\varphi_{AI,B}"] & \\
& & \mathcal{F}(A)\mathcal{F}(I)\mathcal{F}(B) \arrow[r, "\Upsilon_A\,\mathrm{id}_I\,\Upsilon_B"] & \mathcal{F}(A)\mathcal{F}(I)\mathcal{F}(B) \arrow[ur, "\varphi_{A,I}\,\mathrm{id}_B"] \arrow[dr, "\varphi_{A,I,B}"] & & \\
& \mathcal{F}(AIB) \arrow[uu, "\psi_{AI,B}"] \arrow[ur, "\psi_{A,I,B}"] \arrow[rrr, "\Upsilon_{AIB}"'] & & & \mathcal{F}(AIB) \arrow[dr, "\mathcal{F}(\rho_A^{-1}\,\mathrm{id})"] & \\
\mathcal{F}(AB) \arrow[uuuu, "\psi_{A,B}"] \arrow[ur, "\mathcal{F}(\rho_A\,\mathrm{id})"] \arrow[rrrrr, "\Upsilon_{AB}"'] & & & & & \mathcal{F}(AB)
\end{tikzcd}
$$

The hexagon commutes by the definition of $\upsilon'$, the square in the center commutes by axiom (3.30) and the remaining three squares commute by naturality of $\psi$, $\Upsilon$ and $\varphi$. $\square$

**Proposition 3.65.** *We have* $\mathrm{id}_1 * \Upsilon = \upsilon^{(2)}$ *and* $\Upsilon * \mathrm{id}_1 = \upsilon'^{(2)}$.

Proof. The proof of the identity $\mathrm{id}_1 * \Upsilon = \upsilon^{(2)}$ follows from the commutativity of the diagram below. The proof of the other identity is similar.

$$
\begin{tikzcd}
\mathcal{F}(A)\mathcal{F}(B) \arrow[rrrrr, "\mathrm{id}_A\,\Upsilon_B"] \arrow[dr, "\mathrm{id}_A\,\mathcal{F}(\rho_B)"] & & & & & \mathcal{F}(A)\mathcal{F}(B) \arrow[ddddd, "\varphi_{A,B}"] \\
& \mathcal{F}(A)\mathcal{F}(BI) \arrow[rrr, "\mathrm{id}_A\,\Upsilon_{BI}"] \arrow[dr, "\mathrm{id}_A\,\psi_{B,I}"] & & & \mathcal{F}(A)\mathcal{F}(BI) \arrow[ur, "\mathrm{id}_A\,\mathcal{F}(\rho_B^{-1})"] \arrow[ddd, "\varphi_{A,BI}"] & \\
& & \mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(I) \arrow[r, "\mathrm{id}_A\,\upsilon'_B\Upsilon_I"] & \mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(I) \arrow[ur, "\mathrm{id}_A\,\varphi_{B,I}"] \arrow[d, "\varphi_{A,B}\,\mathrm{id}_I"] & & \\
& & \mathcal{F}(AB)\mathcal{F}(I) \arrow[u, "\psi_{A,B}\,\mathrm{id}_I"] \arrow[r, "\mathrm{id}_{AB}\,\Upsilon_I"'] & \mathcal{F}(AB)\mathcal{F}(I) \arrow[dr, "\varphi_{AB,I}"] & & \\
& \mathcal{F}(ABI) \arrow[uuu, "\psi_{A,BI}"] \arrow[ur, "\psi_{AB,I}"] & & & \mathcal{F}(ABI) \arrow[dr, "\mathcal{F}(\mathrm{id}\,\rho_B^{-1})", "\mathcal{F}(\rho_{AB}^{-1})"'] & \\
\mathcal{F}(AB) \arrow[uuuuu, "\psi_{A,B}"] \arrow[ur, "\mathcal{F}(\mathrm{id}\,\rho_B)", "\mathcal{F}(\rho_{AB})"'] \arrow[rrrrr, "\upsilon_{AB}"'] & & & & & \mathcal{F}(AB)
\end{tikzcd}
$$

The outer squares commute by the naturality of $\psi$, $\Upsilon$ and $\varphi$. The hexagon commutes by the definition of $\upsilon$, the square in the center commutes by Proposition 3.63, the square above it commutes by Proposition 3.64, and the squares on its sides commute by the associativity of $\psi$ and $\varphi$. $\square$

**Proposition 3.66.** *The following diagrams commute.*

$$
\begin{tikzcd}
\mathcal{F}(I)\bullet\mathcal{F}(A) \arrow[r, "\mathrm{id}_I\bullet\Upsilon_A"] & \mathcal{F}(I)\bullet\mathcal{F}(A) \arrow[d, "\varphi_{I,A}"] \\
\mathcal{F}(I\bullet A) \arrow[u, "\psi_{I,A}"] & \mathcal{F}(I\bullet A) \arrow[d, "\mathcal{F}(\lambda_A^{-1})"] \\
\mathcal{F}(A) \arrow[u, "\mathcal{F}(\lambda_A)"] \arrow[r, "\upsilon_A"'] & \mathcal{F}(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(I)\bullet\mathcal{F}(A) \arrow[r, "\Upsilon_I\bullet\mathrm{id}_A"] & \mathcal{F}(I)\bullet\mathcal{F}(A) \arrow[d, "\varphi_{I,A}"] \\
\mathcal{F}(I\bullet A) \arrow[u, "\psi_{I,A}"] & \mathcal{F}(I\bullet A) \arrow[d, "\mathcal{F}(\lambda_A^{-1})"] \\
\mathcal{F}(A) \arrow[u, "\mathcal{F}(\lambda_A)"] \arrow[r, "\upsilon'_A"'] & \mathcal{F}(A)
\end{tikzcd}
\tag{3.38}
$$

PROOF. Proposition 3.65 implies

$$(\mathrm{id} * \Upsilon)_{I,A} = \upsilon^{(2)}_{I,A} = \upsilon_{I\bullet A}.$$

The naturality of $\upsilon$ then gives the result. The proof for $\upsilon'$ is similar. □

**Remark 3.67.** In defining $\upsilon$ and $\upsilon'$ by means of (3.34) we made an asymmetric choice: we decided to place the unit object $I$ on the right. In Proposition 3.66 we have arrived at the same diagrams with the unit object on the left. Thus, symmetry is recovered.

**Proposition 3.68.** *We have $\upsilon * \upsilon = \upsilon^{(2)}$ and $\upsilon' * \upsilon' = \upsilon'^{(2)}$.*

PROOF. This follows by a similar argument to those for Propositions 3.63, 3.64 and 3.65. □

**3.7.6. A comparison of Hopf monoids and Hopf lax monoidal functors.** We complement the results of Section 3.4 by showing that any Hopf monoid can be viewed as a special case of a Hopf lax functor, and that Hopf lax functors preserve Hopf monoids.

Let $(\mathsf{I}, \bullet, \beta)$ be the one-arrow category and let $*$ denote its unique object.

**Proposition 3.69.** *The category of Hopf monoids in $(\mathsf{C}, \bullet, \beta)$ is equivalent to the category of Hopf lax functors from $(\mathsf{I}, \bullet, \beta)$ to $(\mathsf{C}, \bullet, \beta)$.*

PROOF. Given a Hopf monoid $(H, \mu, \iota, \Delta, \epsilon, \mathrm{s})$ in $(\mathsf{C}, \bullet, \beta)$, define a Hopf lax functor

$$(\mathcal{F}_H, \varphi, \psi, \Upsilon)$$

from $(\mathsf{I}, \bullet, \beta)$ to $(\mathsf{C}, \bullet, \beta)$, where $(\mathcal{F}_H, \varphi, \psi)$ is defined as in (3.23) and $\Upsilon_*$ is defined to be $\mathrm{s}$. We know from Proposition 3.26 that $(\mathcal{F}_H, \varphi, \psi)$ is bilax. Since the antipode $\mathrm{s}$ is the inverse of the identity in the convolution monoid $\mathrm{Hom}(H, H)$, we have

$$\mathrm{id} * \mathrm{s} * \mathrm{id} = \mathrm{id}, \qquad \mathrm{s} * \mathrm{id} * \mathrm{s} = \mathrm{s} \qquad \text{and} \qquad \mathrm{id} * \mathrm{s} = \mathrm{s} * \mathrm{id} = \iota\epsilon.$$

Hence, axioms (3.29), (3.30) and (3.31) hold and $(\mathcal{F}_H, \varphi, \psi, \Upsilon)$ is Hopf lax.

Conversely, given a Hopf lax functor $(\mathcal{F}, \varphi, \psi, \Upsilon)$ from $\mathsf{I}$ to $\mathsf{C}$, the object $\mathcal{F}(*)$ is a Hopf monoid with antipode $\Upsilon_*$ by applying the forward direction of (3.32). □

Bilax functors preserve bimonoids (Proposition 3.31) and the bimonoid structure maps get twisted by the structure maps of the functor, as in Definition 3.28. Hopf lax functors act similarly on Hopf monoids and their antipodes.

**Theorem 3.70.** *Let $\mathcal{F}$ be a Hopf lax functor from $\mathsf{C}$ to $\mathsf{D}$ with antipode $\Upsilon$. If $H$ is a Hopf monoid in $\mathsf{C}$ with antipode $\mathrm{s}$, then $\mathcal{F}(H)$ is a Hopf monoid in $\mathsf{D}$ with antipode*

$$\Upsilon_H \mathcal{F}(\mathrm{s}) = \mathcal{F}(\mathrm{s})\Upsilon_H. \tag{3.39}$$

Proof. The equality in (3.39) holds by naturality of $\Upsilon$. We only need to show that this map satisfies axioms (1.13). The first of these follows from the commutativity of the diagram below; the second axiom can be checked similarly.

$$\begin{tikzcd}
\mathcal{F}(H)\mathcal{F}(H) \arrow[rr, "\mathrm{id}_H \mathcal{F}(\mathrm{s})"] & & \mathcal{F}(H)\mathcal{F}(H) \arrow[r, "\mathrm{id}_H \Upsilon_H"] & \mathcal{F}(H)\mathcal{F}(H) \arrow[d, "\varphi_{H,H}"] \\
\mathcal{F}(HH) \arrow[u, "\psi_{H,H}"] \arrow[rr, "\mathcal{F}(\mathrm{id}\,\mathrm{s})"] & & \mathcal{F}(HH) \arrow[u, "\psi_{H,H}"] \arrow[r, "\upsilon_{HH}"] \arrow[d, "\mathcal{F}(\mu)"] & \mathcal{F}(HH) \arrow[d, "\mathcal{F}(\mu)"] \\
\mathcal{F}(H) \arrow[u, "\mathcal{F}(\Delta)"] \arrow[r, "\mathcal{F}(\epsilon)"] & \mathcal{F}(I) \arrow[r, "\mathcal{F}(\iota)"] \arrow[d, "\psi_0"] \arrow[dr, "\upsilon_I"] & \mathcal{F}(H) \arrow[r, "\upsilon_H"] & \mathcal{F}(H) \\
& I \arrow[r, "\varphi_0"'] & \mathcal{F}(I) \arrow[ur, "\mathcal{F}(\iota)"'] &
\end{tikzcd}$$

The squares commute by the naturality of $\psi$ and $\upsilon$, the antipode axiom for $H$ and Proposition 3.65. The triangle commutes by Proposition 3.62. $\square$

**Remark 3.71.** Suppose $\mathcal{F}$ is a bistrong monoidal functor and $H$ is a Hopf monoid with antipode $\mathrm{s}$. By Proposition 3.60, $\mathcal{F}$ is a Hopf lax monoidal functor with antipode $\Upsilon = \mathrm{id}$. Therefore, by Theorem 3.70, $\mathcal{F}(H)$ is a Hopf monoid with antipode $\mathcal{F}(\mathrm{s})$. This gives another proof of Proposition 3.50.

**3.7.7. Composites of Hopf lax functors.** Consider two bilax monoidal functors $(\mathcal{F}, \varphi, \psi)\colon \mathsf{C} \to \mathsf{D}$ and $(\mathcal{G}, \gamma, \delta)\colon \mathsf{D} \to \mathsf{E}$. Their composite $(\mathcal{GF}, \varphi\gamma, \delta\psi)$ (Definition 3.20) is also bilax, by Theorem 3.22. If $\Upsilon\colon \mathcal{F} \Rightarrow \mathcal{F}$ and $\Omega\colon \mathcal{G} \Rightarrow \mathcal{G}$ are natural transformations, we may define a new natural transformation $\Omega\Upsilon\colon \mathcal{GF} \Rightarrow \mathcal{GF}$ by going around the diagram below.

$$\begin{tikzcd}
& \mathcal{GF}(A) \arrow[dr, "\Omega_{\mathcal{F}A}"] & \\
\mathcal{GF}(A) \arrow[ur, "\mathcal{G}(\Upsilon_A)"] \arrow[dr, "\Omega_{\mathcal{F}A}"'] & & \mathcal{GF}(A) \\
& \mathcal{GF}(A) \arrow[ur, "\mathcal{G}(\Upsilon_A)"'] &
\end{tikzcd}$$

The above diagram commutes by the naturality of $\Omega$.

**Theorem 3.72.** *If $(\mathcal{F}, \varphi, \psi, \Upsilon)\colon \mathsf{C} \to \mathsf{D}$ and $(\mathcal{G}, \gamma, \delta, \Omega)\colon \mathsf{D} \to \mathsf{E}$ are Hopf lax functors, then so is $(\mathcal{GF}, \varphi\gamma, \delta\psi, \Omega\Upsilon)$.*

PROOF. We only need to check that $\Omega\Upsilon$ is the antipode of $\mathcal{GF}$.

We first check axiom (3.31) for $\mathcal{GF}$. The forward direction of (3.32) applied to $\mathcal{F}$ says that $\mathcal{F}(I)$ is a Hopf monoid with antipode $\Upsilon_I$. This along with Theorem 3.70 says that $\mathcal{GF}(I)$ is a Hopf monoid with antipode

$$\mathcal{G}(\Upsilon_I)\Omega_{\mathcal{F}(I)} = \Omega_{\mathcal{F}(I)}\mathcal{G}(\Upsilon_I),$$

which by definition is $(\Omega\Upsilon)_I$. Further, the bimonoid structure on $\mathcal{GF}(I)$ comes from the bilax structure of $\mathcal{GF}$. Therefore by the backward direction of (3.32), $\mathcal{GF}$ satisfies axiom (3.31).

We now check axiom (3.30) for $\mathcal{GF}$. This follows from (3.28) and the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{GF}(A)\mathcal{GF}(B)\mathcal{GF}(C) \arrow[r, "\Omega_{\mathcal{F}A}\mathrm{id}_{\mathcal{F}B}\Omega_{\mathcal{F}C}"] & \mathcal{GF}(A)\mathcal{GF}(B)\mathcal{GF}(C) \arrow[r, "\mathcal{G}(\Upsilon_A)\mathcal{G}(\mathrm{id}_B)\mathcal{G}(\Upsilon_C)"] \arrow[d, "\gamma_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C}"] & \mathcal{GF}(A)\mathcal{GF}(B)\mathcal{GF}(C) \arrow[d, "\gamma_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C}"'] \\
\mathcal{G}(\mathcal{F}A\mathcal{F}B\mathcal{F}C) \arrow[u, "\delta_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C}"'] \arrow[r, "\Omega_{\mathcal{F}A\mathcal{F}B\mathcal{F}C}"] & \mathcal{G}(\mathcal{F}A\mathcal{F}B\mathcal{F}C) \arrow[r, "\mathcal{G}(\Upsilon_A\mathrm{id}_B\Upsilon_C)"] & \mathcal{G}(\mathcal{F}A\mathcal{F}B\mathcal{F}C) \arrow[d, "\mathcal{G}(\varphi_{A,B,C})"'] \\
\mathcal{GF}(ABC) \arrow[u, "\mathcal{G}(\psi_{A,B,C})"'] \arrow[r, "\Omega_{\mathcal{F}(ABC)}"'] & \mathcal{GF}(ABC) \arrow[u, "\mathcal{G}(\psi_{A,B,C})"] \arrow[r, "\mathcal{G}(\Upsilon_{ABC})"'] & \mathcal{GF}(ABC)
\end{tikzcd}$$

The four squares commute by the naturality of $\gamma$ and $\Omega$, and axiom (3.30) applied to $\mathcal{G}$ and $\mathcal{F}$.

The verification of axiom (3.29) for $\mathcal{GF}$ is similar. $\square$

Theorem 3.70 can be deduced from Theorem 3.72 by specializing $\mathsf{C}$ to the one-arrow category and using Proposition 3.69. The reason for writing these results in the opposite order is that we needed the former in the proof of the latter.

**Remark 3.73.** Theorem 3.72 can be used to supplement the discussion in Section 3.3.3: there is a 2-category whose 0-cells, 1-cells, and 2-cells are respectively braided monoidal categories, Hopf lax monoidal functors, and their morphisms.

## 3.8. An alternative description of bilax monoidal functors

We begin this section by studying the monoidal properties of the tensor product functor. This allows us to formulate an alternative description of bilax monoidal functors (Proposition 3.77). This result is the analogue of the description of a bimonoid as a monoid in a category of comonoids and viceversa.

**3.8.1. The tensor product as a monoidal functor.** Let $(\mathsf{C}, \bullet)$ be a monoidal category together with natural isomorphisms

$$\beta\colon A \bullet B \to B \bullet A.$$

We do not assume that $\beta$ is a braiding. Consider the tensor product functor

$$\mathcal{M}\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$$

which sends $(A, B)$ to $A \bullet B$. Define natural transformations $\varphi$ and $\psi$ as in (3.9) to be

$$\mathcal{M}(A,B) \bullet \mathcal{M}(C,D) \underset{\psi_{(A,B),(C,D)}}{\overset{\varphi_{(A,B),(C,D)}}{\rightleftarrows}} \mathcal{M}\big((A,B) \bullet (C,D)\big)$$

$$A \bullet B \bullet C \bullet D \underset{\mathrm{id} \bullet \beta \bullet \mathrm{id}}{\overset{\mathrm{id} \bullet \beta \bullet \mathrm{id}}{\rightleftarrows}} A \bullet C \bullet B \bullet D.$$

Define the morphisms $\varphi_0$ and $\psi_0$ as in (3.10) to be

$$I \xrightarrow{\ \varphi_0\ } \mathcal{M}(I,I) \xrightarrow{\ \psi_0\ } I$$
$$I \xrightarrow{\ \cong\ } I \bullet I \xrightarrow{\ \cong\ } I.$$

The following result describes the monoidal properties of the functor $\mathcal{M}$ with respect to the structure maps $\varphi$ and $\psi$.

**Proposition 3.74 (Joyal and Street).** *We have*

$$\beta \textit{ is a braiding in } (\mathsf{C}, \bullet) \iff (\mathcal{M}, \varphi) \textit{ is strong.}$$
$$\beta \textit{ is a symmetry in } (\mathsf{C}, \bullet) \iff (\mathcal{M}, \varphi) \textit{ is braided strong.}$$

The first equivalence is [184, Proposition 5.2], and the second equivalence is [184, Proposition 5.4].

**Proposition 3.75.** *We have*

$$\begin{aligned} (\mathcal{M}, \varphi) \textit{ is braided strong} &\iff (\mathcal{M}, \varphi, \psi) \textit{ is bistrong} \\ &\iff (\mathcal{M}, \psi) \textit{ is braided costrong.} \end{aligned}$$

Proof. We explain the first equivalence. The backward implication follows from Proposition 3.46. For the forward implication: By Proposition 3.74, $\beta$ is a symmetry and further by Proposition 3.46, $(\mathcal{M}, \varphi, \varphi^{-1})$ is bistrong. Since $\beta$ is a symmetry, we have $\varphi^{-1} = \psi$ which finishes the proof. $\square$

**3.8.2. An alternative description of bilax monoidal functors.** Let $(\mathsf{C}, \bullet)$ and $(\mathsf{D}, \bullet)$ be two monoidal categories and $\mathcal{F}$ be a functor from $\mathsf{C}$ to $\mathsf{D}$. We denote the unit object in both categories by $I$ and write $\mathcal{M}$ for both tensor product functors. Let $\mathcal{F}^0$, $\mathcal{F}_0$, $\mathcal{F}^2$ and $\mathcal{F}_2$ be the functors defined in (3.1) and (3.2).

**Proposition 3.76.** *If $\mathcal{F}$ is (co)lax, then so are $\mathcal{F}_0$ and $\mathcal{F}^0$. If $\mathcal{F}$ is (co)lax and $\mathsf{C}$ and $\mathsf{D}$ are braided, then the functors $\mathcal{F}_2$ and $\mathcal{F}^2$ are also (co)lax.*

Proof. We explain the lax case, the colax case being similar. The assertions about $\mathcal{F}_0$ and $\mathcal{F}^0$ are clear (and may be seen as special cases of the construction of Section 3.4.1). When $\mathsf{C}$ and $\mathsf{D}$ are braided, the tensor product functors $\mathcal{M}$ are lax (in fact, strong) by Proposition 3.74. Since $\mathcal{F}$ is lax the functor $\mathcal{F} \times \mathcal{F}$ is also lax by Proposition 3.6. Note that $\mathcal{F}_2$ and $\mathcal{F}^2$ are defined in terms of $\mathcal{M}$, $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$ via compositions. Hence the result follows from Theorem 3.21. $\square$

For $(\mathcal{F}, \psi)$ a colax monoidal functor, let $(\mathcal{F}_2, \psi_2)$ and $(\mathcal{F}^2, \psi^2)$ be the colax monoidal functors given by the above construction. Explicitly, they are as follows.

$$\begin{tikzcd}
\mathcal{F}(A \bullet C \bullet B \bullet D) \arrow[r, equal] \arrow[d, "\mathcal{F}(\mathrm{id}\bullet\beta\bullet\mathrm{id})"'] & \mathcal{F}_2\big((A,B) \bullet (C,D)\big) \arrow[dd, dashed, "\psi_2"] \\
\mathcal{F}(A \bullet B \bullet C \bullet D) \arrow[d, "\psi_{A\bullet B, C\bullet D}"'] & \\
\mathcal{F}(A \bullet B) \bullet \mathcal{F}(C \bullet D) \arrow[r, equal] & \mathcal{F}_2(A,B) \bullet \mathcal{F}_2(C,D)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(I \bullet I) \arrow[r, equal] \arrow[d, "\cong"'] & \mathcal{F}_2(I,I) \arrow[d, dashed, "(\psi_2)_0"] \\
\mathcal{F}(I) \arrow[r, "\psi_0"'] & I
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(A \bullet C) \bullet \mathcal{F}(B \bullet D) \arrow[r, equal] \arrow[d, "\psi_{A,C}\bullet\psi_{B,D}"'] & \mathcal{F}^2\big((A,B) \bullet (C,D)\big) \arrow[dd, dashed, "\psi^2"] \\
\mathcal{F}(A) \bullet \mathcal{F}(C) \bullet \mathcal{F}(B) \bullet \mathcal{F}(D) \arrow[d, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"'] & \\
\mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \bullet \mathcal{F}(D) \arrow[r, equal] & \mathcal{F}^2(A,B) \bullet \mathcal{F}^2(C,D)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(I) \bullet \mathcal{F}(I) \arrow[r, equal] \arrow[d, "\psi_0\bullet\psi_0"'] & \mathcal{F}^2(I,I) \arrow[d, dashed, "(\psi^2)_0"] \\
I \bullet I \arrow[r, "\cong"'] & I
\end{tikzcd}$$

The lax structures on $\mathcal{F}_2$ and $\mathcal{F}^2$ induced from a lax structure on $\mathcal{F}$ admit similar descriptions.

**Proposition 3.77.** *Let $\mathcal{F}$ be a lax and colax functor with structure maps $\varphi$ and $\psi$ respectively. The following statements are equivalent.*

(i) *$(\mathcal{F}, \varphi, \psi)$ is bilax;*
(ii) *$\varphi\colon \mathcal{F}^2 \Rightarrow \mathcal{F}_2$ and $\varphi_0\colon \mathcal{F}^0 \Rightarrow \mathcal{F}_0$ are morphisms of colax monoidal functors;*
(iii) *$\psi\colon \mathcal{F}_2 \Rightarrow \mathcal{F}^2$ and $\psi_0\colon \mathcal{F}_0 \Rightarrow \mathcal{F}^0$ are morphisms of lax monoidal functors.*

Proof. We indicate how (i) and (ii) are equivalent. The equivalence between (i) and (iii) is similar.

From the explicit definitions of $\psi_2$ and $\psi^2$ given above, one sees that $\varphi\colon \mathcal{F}^2 \Rightarrow \mathcal{F}_2$ being a colax morphism (Definition 3.8) is equivalent to the commutativity of the braiding diagram (3.11) and one of the unitality diagrams (3.12). Similarly, the condition that $\varphi_0\colon \mathcal{F}^0 \Rightarrow \mathcal{F}_0$ is a colax morphism is equivalent to the commutativity of the other two unitality diagrams in Definition 3.3. $\square$

**Remark 3.78.** The above result is the analogue of Proposition 1.11 for bimonoids. It may be used to obtain another proof of Theorem 3.22 as follows. Recall that morphisms of colax monoidal functors are the 2-cells of a 2-category (Section 3.3.3; this uses Theorem 3.21). The structure maps $\varphi\gamma$ of the composite $\mathcal{G}\mathcal{F}$ of two bilax functors (Definition 3.20) are obtained from $\varphi$ and $\gamma$ in terms of this 2-category structure. Now apply Proposition 3.77: $\varphi$ and $\gamma$ are morphisms of colax monoidal functors, hence so is $\varphi\gamma$, and then $\mathcal{GF}$ is bilax. This approach is formalized in Section 6.11.

**3.8.3. Monoidal properties of bilax functors on the category of (co)monoids.** We have seen that a (co)lax functor induces a functor on the category of (co)monoids. If the original functor is bilax, then more can be said about the induced functors, as follows.

**Proposition 3.79.** *If* $(\mathcal{F}, \varphi, \psi)\colon \mathsf{C} \to \mathsf{D}$ *is a bilax monoidal functor, then*

$$(\mathcal{F}, \psi)\colon \mathsf{Mon}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{D})$$

*is a colax monoidal functor and*

$$(\mathcal{F}, \varphi)\colon \mathsf{Comon}(\mathsf{C}) \to \mathsf{Comon}(\mathsf{D})$$

*is a lax monoidal functor.*

Proof. We discuss the first assertion. Since $\mathcal{F}$ is lax and $\mathsf{C}$ and $\mathsf{D}$ are braided, the functors $\mathcal{F}_2$, $\mathcal{F}^2$, $\mathcal{F}_0$ and $\mathcal{F}^0$ are all lax (Proposition 3.76). Further, since $\mathcal{F}$ is bilax, $\psi\colon \mathcal{F}_2 \to \mathcal{F}^2$ and $\psi_0\colon \mathcal{F}_0 \to \mathcal{F}^0$ are morphisms of lax monoidal functors (Proposition 3.77). Now Proposition 3.30 implies that if $A$ and $B$ are monoids, then

$$\psi_{A,B}\colon \mathcal{F}(A \bullet B) \to \mathcal{F}(A) \bullet \mathcal{F}(B) \qquad \text{and} \qquad \psi_0\colon \mathcal{F}(I) \to I$$

are morphisms of monoids. This finishes the proof of the first assertion. □

A similar result for a braided (co)lax functor (whose proof we omit) is given below.

**Proposition 3.80.** *If* $(\mathcal{F}, \varphi)\colon \mathsf{C} \to \mathsf{D}$ *is a braided lax monoidal functor, then*

$$(\mathcal{F}, \varphi)\colon \mathsf{Mon}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{D})$$

*is a lax monoidal functor. Similarly, if* $(\mathcal{F}, \psi)\colon \mathsf{C} \to \mathsf{D}$ *is a braided colax monoidal functor, then*

$$(\mathcal{F}, \psi)\colon \mathsf{Comon}(\mathsf{C}) \to \mathsf{Comon}(\mathsf{D})$$

*is a colax monoidal functor.*

Applying Proposition 3.29 to the lax and colax functors in the above results and using (1.14), one obtains an alternate proof of the facts that bilax functors preserve bimonoids and braided (co)lax functors preserve (co)commutative (co)monoids.

## 3.9. Adjunctions of monoidal functors

We now discuss the notion of adjunction between monoidal categories for various kinds of monoidal functors. We follow the notations of Section A.2.1, where some background information on adjunctions is also given. Throughout this section, $\mathsf{C}$ and $\mathsf{D}$ are monoidal categories and $\bullet$ refers to their tensor products. Work of Kelly on adjunctions between categories with structure includes results on adjunctions between monoidal categories [195, Section 2.1]. We mention that Propositions 3.84 and 3.96 (which we prove directly) are special cases of [195, Theorems 1.2 and 1.4].

The results of this section play an important role in the universal constructions of Chapter 11. Interesting examples of adjunctions between monoidal functors can also be found in Propositions 18.4 and 18.18.

**3.9.1. Colax-lax adjunctions.** Recall from Example 3.12 that

$$\mathrm{Hom}\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$$

is a braided lax monoidal functor. If $\mathcal{F}$ and $\mathcal{G}$ are (braided) colax and (braided) lax monoidal functors respectively, then by Proposition 3.7, Theorem 3.21, and Proposition 3.24, the functors

$$\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \qquad \text{and} \qquad \mathrm{Hom}_{\mathsf{C}}\big(-, \mathcal{G}(-)\big)$$

are (braided) lax monoidal functors from

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}.$$

**Definition 3.81.** Let $(\mathcal{F}, \psi)$ be a (braided) colax monoidal functor and $(\mathcal{G}, \gamma)$ a (braided) lax monoidal functor. We say that they form a pair of (braided) colax-lax adjoint functors if the bijection (A.2) is an isomorphism of (braided) lax functors $\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$.

In the above situation, we also say that the adjunction $(\mathcal{F}, \mathcal{G})$ is (braided) colax-lax. It is clear that $(\mathcal{F}, \mathcal{G})$ is braided colax-lax if $(\mathcal{F}, \mathcal{G})$ is colax-lax and both $\mathcal{F}$ and $\mathcal{G}$ are braided.

**Proposition 3.82.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors between monoidal categories. Assume that $(\mathcal{F}, \psi)$ is a colax monoidal functor and $(\mathcal{G}, \gamma)$ is a lax monoidal functor. Then the following conditions are equivalent.*

(1) *The adjunction $(\mathcal{F}, \mathcal{G})$ is colax-lax.*
(2) *The following two diagrams commute.*

$$\begin{tikzcd}
A \bullet B \arrow[r, "\eta_{A \bullet B}"] \arrow[d, "\eta_A \bullet \eta_B"'] & \mathcal{G}\mathcal{F}(A \bullet B) \arrow[d, "\mathcal{G}(\psi_{A,B})"] \\
\mathcal{G}\mathcal{F}(A) \bullet \mathcal{G}\mathcal{F}(B) \arrow[r, "\gamma_{\mathcal{F}(A),\mathcal{F}(B)}"'] & \mathcal{G}\big(\mathcal{F}(A) \bullet \mathcal{F}(B)\big)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta_I"] \arrow[d, "\mathrm{id}"'] & \mathcal{G}\mathcal{F}(I) \arrow[d, "\mathcal{G}(\psi_0)"] \\
I \arrow[r, "\gamma_0"'] & \mathcal{G}(I)
\end{tikzcd} \tag{3.40}$$

(3) *The following two diagrams commute.*

$$\begin{tikzcd}
\mathcal{F}\big(\mathcal{G}(X) \bullet \mathcal{G}(Y)\big) \arrow[r, "\mathcal{F}(\gamma_{X,Y})"] \arrow[d, "\psi_{\mathcal{G}(X),\mathcal{G}(Y)}"'] & \mathcal{F}\mathcal{G}(X \bullet Y) \arrow[d, "\xi_{X \bullet Y}"] \\
\mathcal{F}\mathcal{G}(X) \bullet \mathcal{F}\mathcal{G}(Y) \arrow[r, "\xi_X \bullet \xi_Y"'] & X \bullet Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(I) \arrow[r, "\mathcal{F}(\gamma_0)"] \arrow[d, "\psi_0"'] & \mathcal{F}\mathcal{G}(I) \arrow[d, "\xi_I"] \\
I \arrow[r, "\mathrm{id}"'] & I
\end{tikzcd} \tag{3.41}$$

Proof. The diagrams (3.40) say that for the bijection in (A.2), the map in one direction is a morphism of lax functors, while the diagrams (3.41) say that the map in the other direction is a morphism of lax functors. $\square$

**Proposition 3.83.** *If $\mathcal{F}$ and $\mathcal{G}$ form a pair of colax-lax adjoint functors between the categories $\mathsf{C}$ and $\mathsf{D}$, then for $C$ a comonoid in $\mathsf{C}$ and $A$ a monoid in $\mathsf{D}$, the bijection (A.2)*

$$\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), A) \cong \mathrm{Hom}_{\mathsf{C}}\big(C, \mathcal{G}(A)\big)$$

*is an isomorphism of convolution monoids.*

Proof. Recall that the convolution monoid is the image of a certain monoid under the lax functor Hom (Section 3.4.5). The result then follows from Definition 3.81 and Proposition 3.30. ☐

**Proposition 3.84.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors. If $\mathcal{F}$ is colax (resp. $\mathcal{G}$ is lax), then there exists a unique lax structure on $\mathcal{G}$ (resp. colax structure on $\mathcal{F}$) such that $(\mathcal{F}, \mathcal{G})$ is a colax-lax adjunction.*

Proof. Let $(\mathcal{F}, \psi)$ be a colax monoidal functor. Use the adjunction (A.2) to define

$$\mathcal{G}(X) \bullet \mathcal{G}(Y) \xrightarrow{\gamma_{X,Y}} \mathcal{G}(X \bullet Y)$$

as the map that corresponds to

$$\mathcal{F}\big(\mathcal{G}(X) \bullet \mathcal{G}(Y)\big) \xrightarrow{\psi_{\mathcal{G}(X),\mathcal{G}(Y)}} \mathcal{F}\mathcal{G}(X) \bullet \mathcal{F}\mathcal{G}(Y) \xrightarrow{\xi_X \bullet \xi_Y} X \bullet Y$$

and $\gamma_0 \colon I \to \mathcal{G}(I)$ as the map that corresponds to $\psi_0 \colon \mathcal{F}(I) \to I$. In view of (A.5), $\gamma$ and $\gamma_0$ are the unique maps for which the diagrams in (3.41) commute. Hence, to complete the proof we only need to show that $(\mathcal{G}, \gamma)$ is indeed a lax monoidal functor.

The associativity (3.5) of $\gamma$ follows from the commutativity of the diagram below.

$$\begin{tikzcd}
 & \mathcal{F}(\mathcal{G}X(\mathcal{G}Y\mathcal{G}Z)) \arrow[rr, "\mathcal{F}(\mathrm{id}\,\gamma_{Y,Z})"] \arrow[dr, "\psi_{\mathcal{G}X,\mathcal{G}Y\mathcal{G}Z}"] & & \mathcal{F}(\mathcal{G}X\mathcal{G}(YZ)) \arrow[rr, "\mathcal{F}(\gamma_{X,YZ})"] \arrow[dr, "\psi_{\mathcal{G}X,\mathcal{G}(YZ)}"] & & \mathcal{F}\mathcal{G}(X(YZ)) \arrow[ddd, "\xi_{X(YZ)}"] \\
\mathcal{F}((\mathcal{G}X\mathcal{G}Y)\mathcal{G}Z) \arrow[ur, "\mathcal{F}(\alpha)"] \arrow[dr, "\psi_{\mathcal{G}X\mathcal{G}Y,\mathcal{G}Z}"] \arrow[dd, "\mathcal{F}(\gamma_{X,Y}\ \mathrm{id})"'] & & \mathcal{F}\mathcal{G}(X)\mathcal{F}(\mathcal{G}Y\mathcal{G}Z) \arrow[rr, "\mathrm{id}\,\mathcal{F}(\gamma_{Y,Z})"] \arrow[dr, "\mathrm{id}\,\psi_{\mathcal{G}Y,\mathcal{G}Z}"] & & \mathcal{F}\mathcal{G}(X)\mathcal{F}\mathcal{G}(YZ) \arrow[ddr, "\xi_X\xi_{YZ}"] & \\
 & \mathcal{F}(\mathcal{G}X\mathcal{G}Y)\mathcal{F}\mathcal{G}(Z) \arrow[dr, "\mathcal{F}(\gamma_{X,Y})\,\mathrm{id}"] \arrow[drr, "\psi_{\mathcal{G}X,\mathcal{G}Y}\,\mathrm{id}"] & & \mathcal{F}\mathcal{G}(X)(\mathcal{F}\mathcal{G}(Y)\mathcal{F}\mathcal{G}(Z)) \arrow[drr, "\xi_X(\xi_Y\xi_Z)"] & & \\
\mathcal{F}(\mathcal{G}(XY)\mathcal{G}Z) \arrow[rr, "\psi_{\mathcal{G}(XY),\mathcal{G}Z}"] \arrow[d, "\mathcal{F}(\gamma_{XY,Z})"'] & & \mathcal{F}\mathcal{G}(XY)\mathcal{F}\mathcal{G}(Z) \arrow[dr, "\xi_{XY}\xi_Z"'] & (\mathcal{F}\mathcal{G}(X)\mathcal{F}\mathcal{G}(Y))\mathcal{F}\mathcal{G}(Z) \arrow[u, "\alpha"] \arrow[d, "(\xi_X\xi_Y)\xi_Z"] & & X(YZ) \\
\mathcal{F}\mathcal{G}((XY)Z) \arrow[rrr, "\xi_{(XY)Z}"'] & & & (XY)Z \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

The hexagon commutes by the associativity of $\psi$. The other squares commute by the definition of $\gamma$ and the naturality of $\psi$ and $\alpha$.

The unitality (3.6) of $\gamma$ follows from the commutativity of the diagram below.

$$\begin{tikzcd}
\mathcal{F}\mathcal{G}(X) \arrow[rrrr, "\mathcal{F}(\mathcal{G}(\lambda))"] \arrow[drrr, "\xi_X"] \arrow[ddrr, "\lambda"] \arrow[dd, "\mathcal{F}(\lambda)"'] & & & & \mathcal{F}\mathcal{G}(IX) \arrow[dddd, "\xi_{IX}"] \\
& & & X \arrow[dddr, "\lambda"] & \\
\mathcal{F}(I\mathcal{G}X) \arrow[r, "\psi_{I,\mathcal{G}X}"] \arrow[dd, "\mathcal{F}(\gamma_0\,\mathrm{id})"'] & \mathcal{F}(I)\mathcal{F}\mathcal{G}(X) \arrow[r, "\psi_0\,\mathrm{id}"] \arrow[d, "\mathcal{F}(\gamma_0)\,\mathrm{id}"] & I\mathcal{F}\mathcal{G}(X) \arrow[ddrr, "\mathrm{id}\,\xi_X"] & & \\
& \mathcal{F}\mathcal{G}(I)\mathcal{F}\mathcal{G}(X) \arrow[drrr, "\xi_I\xi_X"] & & & \\
\mathcal{F}(\mathcal{G}(I)\mathcal{G}X) \arrow[ur, "\psi_{\mathcal{G}(I)\mathcal{G}X}"] \arrow[r, "\mathcal{F}(\gamma_{I,X})"'] & \mathcal{F}\mathcal{G}(IX) \arrow[rrr, "\xi_{IX}"'] & & & IX
\end{tikzcd}$$

The smaller diagrams commute by the naturality of $\xi$, $\lambda$ and $\psi$, by the definition of $\gamma$ and $\gamma_0$, and by the unitality of $\psi$. $\square$

**Proposition 3.85.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors. If $\mathcal{F}$ is braided colax (resp. $\mathcal{G}$ is braided lax), then the unique lax structure on $\mathcal{G}$ (resp. the unique colax structure on $\mathcal{F}$) afforded by Proposition 3.84 is braided.*

Adjunctions can be composed [250, Theorem IV.8.1]; this operation preserves colax-lax adjunctions.

**Proposition 3.86.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors between monoidal categories $\mathsf{C}$ and $\mathsf{D}$. Let $(\mathcal{F}', \mathcal{G}')$ be another pair of adjoint functors between $\mathsf{D}$ and another monoidal category $\mathsf{E}$. If both adjunctions are colax-lax, then so is the adjunction*

$$(\mathcal{F}'\mathcal{F}, \mathcal{G}\mathcal{G}')$$

*between $\mathsf{C}$ and $\mathsf{E}$.*

### 3.9.2. Lax-lax and colax-colax adjunctions.

**Definition 3.87.** Let $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \gamma)$ be (braided) lax monoidal functors. We say that they form a pair of (braided) lax adjoint functors if the unit and counit $\eta$ and $\xi$ are morphisms of (braided) lax monoidal functors, where we view id as a braided lax functor with identity transformations. More explicitly, one requires that the following diagrams commute.

$$\begin{tikzcd}
A \bullet B \arrow[r, "\eta_{A\bullet B}"] \arrow[d, "\eta_A \bullet \eta_B"'] & \mathcal{G}\mathcal{F}(A \bullet B) \\
\mathcal{G}\mathcal{F}(A) \bullet \mathcal{G}\mathcal{F}(B) \arrow[r, "\gamma_{\mathcal{F}(A),\mathcal{F}(B)}"'] & \mathcal{G}\big(\mathcal{F}(A) \bullet \mathcal{F}(B)\big) \arrow[u, "\mathcal{G}(\varphi_{A,B})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta_I"] \arrow[d, "\mathrm{id}"'] & \mathcal{G}\mathcal{F}(I) \\
I \arrow[r, "\gamma_0"'] & \mathcal{G}(I) \arrow[u, "\mathcal{G}(\varphi_0)"']
\end{tikzcd}
\tag{3.42}$$

$$
\begin{tikzcd}
\mathcal{F}\mathcal{G}(X) \bullet \mathcal{F}\mathcal{G}(Y) \arrow[r, "\varphi_{\mathcal{G}(X),\mathcal{G}(Y)}"] \arrow[d, "\xi_X \bullet \xi_Y"'] & \mathcal{F}\big(\mathcal{G}(X) \bullet \mathcal{G}(Y)\big) \arrow[d, "\mathcal{F}(\gamma_{X,Y})"] \\
X \bullet Y & \mathcal{F}\mathcal{G}(X \bullet Y) \arrow[l, "\xi_{X \bullet Y}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\varphi_0"] \arrow[d, "\mathrm{id}"'] & \mathcal{F}(I) \arrow[d, "\mathcal{F}(\gamma_0)"] \\
I & \mathcal{F}\mathcal{G}(I) \arrow[l, "\xi_I"]
\end{tikzcd}
\tag{3.43}
$$

The diagrams in the first (resp. second) row say that $\eta$ (resp. $\xi$) is a morphism of lax monoidal functors.

**Definition 3.88.** Let $(\mathcal{F}, \psi)$ and $(\mathcal{G}, \delta)$ be (braided) colax monoidal functors. We say that they form a pair of (braided) colax adjoint functors if the adjunctions $\eta$ and $\xi$ are morphisms of (braided) colax monoidal functors. Explicitly, the necessary diagrams can be obtained from (3.42) and (3.43) by reversing the arrows labeled $\varphi$ and $\gamma$ and renaming them $\psi$ and $\delta$ respectively.

In the situation of Definitions 3.87 and 3.88, we also say that the adjunction is lax-lax and colax-colax, respectively. It is clear that an adjunction $(\mathcal{F}, \mathcal{G})$ is braided lax-lax (colax-colax) if $(\mathcal{F}, \mathcal{G})$ is lax-lax (colax-colax) and both $\mathcal{F}$ and $\mathcal{G}$ are braided.

**Remark 3.89.** A lax-lax adjunction is the same as an adjunction in the 2-category $\mathsf{lCat}$, in the sense of Section C.1.1. Similarly, a colax-colax adjunction is the same as an adjunction in the 2-category $\mathsf{cCat}$.

The above notions should not be confused with that of *lax adjunctions*, which pertain to the context of *tricategories*, as defined in [347].

**Example 3.90.** An adjunction between categories with finite products is always braided colax-colax, with the canonical braided colax structures of Example 3.19. Dually, an adjunction between categories with finite coproducts is always braided lax-lax.

**Proposition 3.91.** *If $\mathcal{F}$ and $\mathcal{G}$ form a pair of lax adjoint functors between the categories $\mathsf{C}$ and $\mathsf{D}$, then they restrict to a pair of adjoint functors*

$$
\begin{tikzcd}
\mathsf{Mon}(\mathsf{C}) \arrow[r, bend left, "\mathcal{F}"] & \mathsf{Mon}(\mathsf{D}). \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

*A similar result holds in the colax case.*

Proof. We explain the lax case. One needs to check that a morphism of monoids maps to a morphism of monoids under the adjunction. So let $A$ and $X$ be monoids in $\mathsf{C}$ and $\mathsf{D}$ respectively and let $g\colon A \to \mathcal{G}(X)$ be a morphism of monoids in $\mathsf{C}$. Under the adjunction, this corresponds to the map given in (A.5). Since $\mathcal{F}$ is lax, the first map in (A.5), namely $\mathcal{F}(g)$, is a morphism of monoids. Since by assumption $\xi$ is a morphism of lax monoidal functors, the second map in (A.5) is also a morphism of monoids. This completes the check in one direction. For the other direction, which is similar, one uses that $\eta$ is a morphism of lax monoidal functors. $\square$

**Proposition 3.92.** *If $\mathcal{F}$ and $\mathcal{G}$ form a pair of braided lax adjoint functors between the categories $\mathsf{C}$ and $\mathsf{D}$, then they restrict to a pair of adjoint functors*

$$
\begin{tikzcd}
\mathsf{Mon}^{\mathsf{co}}(\mathsf{C}) \arrow[r, bend left, "\mathcal{F}"] & \mathsf{Mon}^{\mathsf{co}}(\mathsf{D}) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathsf{Lie}(\mathsf{C}) \arrow[r, bend left, "\mathcal{F}"] & \mathsf{Lie}(\mathsf{D}). \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

*A similar result holds in the colax case.*

**3.9.3. Colax-lax as a generalization of lax-lax and colax-colax.** We now derive some additional useful properties of the different types of adjunctions between monoidal functors that hold when one of the two functors is strong.

**Proposition 3.93.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors.*

(1) *Suppose $\mathcal{G}$ is lax and $\mathcal{F}$ is strong. View $\mathcal{F}$ as a lax and colax functor as in Proposition 3.44. Then*

$$(\mathcal{F}, \mathcal{G}) \text{ is a colax-lax adjunction} \iff (\mathcal{F}, \mathcal{G}) \text{ is a lax-lax adjunction.}$$

(2) *Suppose $\mathcal{F}$ is colax and $\mathcal{G}$ is strong. View $\mathcal{G}$ as a lax and colax functor as in Proposition 3.44. Then*

$$(\mathcal{F}, \mathcal{G}) \text{ is a colax-lax adjunction} \iff (\mathcal{F}, \mathcal{G}) \text{ is a colax-colax adjunction.}$$

Proof. We prove the first statement. If $\mathcal{F}$ is strong, then by Proposition 3.44 it has a lax structure $\varphi$ and a colax structure $\psi$ such that $\varphi = \psi^{-1}$ and $\varphi_0 = \psi_0^{-1}$. In this situation, the diagrams in (3.42) and (3.43) become equivalent to the diagrams in (3.40) and (3.41), and the result follows. □

Combining Propositions 3.93 and 3.84, we obtain:

**Proposition 3.94.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors and $\mathcal{F}$ (resp. $\mathcal{G}$) be strong. Then there exists a unique lax structure on $\mathcal{G}$ (resp. colax structure on $\mathcal{F}$) such that $(\mathcal{F}, \mathcal{G})$ is a lax-lax (resp. colax-colax) adjunction.*

Combining further with Proposition 3.85, we obtain:

**Proposition 3.95.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors and $\mathcal{F}$ (resp. $\mathcal{G}$) be braided strong. Then there exists a unique braided lax structure on $\mathcal{G}$ (resp. braided colax structure on $\mathcal{F}$) such that $(\mathcal{F}, \mathcal{G})$ is a braided lax-lax (resp. braided colax-colax) adjunction.*

Conversely, the existence of a lax-lax (resp. colax-colax) adjunction implies that the left (resp. right) adjoint is strong.

**Proposition 3.96.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors. If the adjunction is lax-lax (resp. colax-colax), then $\mathcal{F}$ (resp. $\mathcal{G}$) is strong.*

Proof. We prove the first statement only. Assume $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \gamma)$ are lax functors and $(\mathcal{F}, \mathcal{G})$ is a lax-lax adjunction. The idea is to define a colax functor $(\mathcal{F}, \psi)$ using Proposition 3.84 and then show that $\varphi$ and $\psi$ are inverses. Accordingly, define

$$\mathcal{F}(A \bullet B) \xrightarrow{\psi_{A,B}} \mathcal{F}(A) \bullet \mathcal{F}(B)$$

as the map that corresponds to

$$A \bullet B \xrightarrow{\eta_A \bullet \eta_B} \mathcal{G}\mathcal{F}(A) \bullet \mathcal{G}\mathcal{F}(B) \xrightarrow{\gamma_{\mathcal{F}(A),\mathcal{F}(B)}} \mathcal{G}\big(\mathcal{F}(A) \bullet \mathcal{F}(B)\big)$$

under the adjunction (A.2). Similarly, let $\psi_0 \colon \mathcal{F}(I) \to I$ be the map that corresponds to $\gamma_0 \colon I \to \mathcal{G}(I)$. We claim that $\psi$ is the inverse of $\varphi$ and $\psi_0$ is the inverse of $\varphi_0$.

Now consider the following diagram.

$$\begin{tikzcd}
\mathcal{F}(A)\bullet\mathcal{F}(B) \arrow[rr, "\varphi_{A,B}"] \arrow[dr, "\mathcal{F}(\eta_A)\bullet\mathcal{F}(\eta_B)"] \arrow[dd, "\mathrm{id}_{\mathcal{F}(A)\bullet\mathcal{F}(B)}"'] & & \mathcal{F}(A\bullet B) \arrow[dr, "\mathcal{F}(\eta_A\bullet\eta_B)"] \arrow[ddll, dashed, "\psi_{A,B}"] & \\
& \mathcal{F}\mathcal{G}\mathcal{F}(A)\bullet\mathcal{F}\mathcal{G}\mathcal{F}(B) \arrow[rr, "\varphi_{\mathcal{G}\mathcal{F}(A),\mathcal{G}\mathcal{F}(B)}"] \arrow[dl, "\xi_{\mathcal{F}(A)}\bullet\xi_{\mathcal{F}(B)}"] & & \mathcal{F}\left(\mathcal{G}\mathcal{F}(A)\bullet\mathcal{G}\mathcal{F}(B)\right) \arrow[dl, "\mathcal{F}(\gamma_{\mathcal{F}(A),\mathcal{F}(B)})"] \\
\mathcal{F}(A)\bullet\mathcal{F}(B) & & \mathcal{F}\mathcal{G}\left(\mathcal{F}(A)\bullet\mathcal{F}(B)\right) \arrow[ll, "\xi_{\mathcal{F}(A)\bullet\mathcal{F}(B)}"] &
\end{tikzcd}$$

The top front square commutes by naturality of $\varphi$. The bottom front square is a special case of (3.43), it commutes since the adjunction is lax-lax. The front triangle commutes because (A.4) and (A.5) are inverse correspondences. There are two faces on the back, a triangle on the left and a square on the right. The back square commutes by definition of $\psi$. It follows that the back triangle commutes. This says that $\psi_{A,B}\varphi_{A,B} = \mathrm{id}_{\mathcal{F}(A)\bullet\mathcal{F}(B)}$.

Similarly, the diagram

$$\begin{tikzcd}
\mathcal{F}(A\bullet B) & & \mathcal{F}(A\bullet B) \arrow[ll, "\mathrm{id}_{\mathcal{F}(A\bullet B)}"'] \arrow[dl, "\mathcal{F}(\eta_{A\bullet B})"'] \arrow[dr, "\mathcal{F}(\eta_A\bullet\eta_B)"] \arrow[ddll, dashed, "\psi_{A,B}"] & \\
& \mathcal{F}\mathcal{G}\mathcal{F}(A\bullet B) \arrow[ul, "\xi_{\mathcal{F}(A\bullet B)}"] & & \mathcal{F}\left(\mathcal{G}\mathcal{F}(A)\bullet\mathcal{G}\mathcal{F}(B)\right) \arrow[dl, "\mathcal{F}(\gamma_{\mathcal{F}(A),\mathcal{F}(B)})"] \\
\mathcal{F}(A)\bullet\mathcal{F}(B) \arrow[uu, "\varphi_{A,B}"] & & \mathcal{F}\mathcal{G}\left(\mathcal{F}(A)\bullet\mathcal{F}(B)\right) \arrow[ll, "\xi_{\mathcal{F}(A)\bullet\mathcal{F}(B)}"] \arrow[ul, "\mathcal{F}\mathcal{G}(\varphi_{A,B})"'] &
\end{tikzcd}$$

shows that $\varphi_{A,B}\psi_{A,B} = \mathrm{id}_{\mathcal{F}(A\bullet B)}$. Thus, $\varphi$ and $\psi$ are inverses.

A similar argument using the unital counterparts of the above diagrams shows that $\varphi_0$ and $\psi_0$ are inverses. This completes the proof. ☐

**Proposition 3.97.** *Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors between monoidal categories $\mathsf{C}$ and $\mathsf{D}$. Let $(\mathcal{F}', \mathcal{G}')$ be another pair of adjoint functors between $\mathsf{D}$ and another monoidal category $\mathsf{E}$. If both adjunctions are either lax-lax, or colax-colax, then so is the adjunction*

$$(\mathcal{F}'\mathcal{F}, \mathcal{G}\mathcal{G}')$$

*between $\mathsf{C}$ and $\mathsf{E}$.*

Proof. We consider the lax-lax case. Proposition 3.96 implies that $\mathcal{F}$ and $\mathcal{F}'$ are both strong. Hence by Proposition 3.93, one can view $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ as colax-lax adjunctions. Now applying Proposition 3.86, we see that the composite is also colax-lax, and further $\mathcal{F}'\mathcal{F}$ is strong. Applying Proposition 3.93 in the opposite direction, we see that the composite is lax-lax. ☐

## 3.10. The contragredient construction

In this section, we introduce the contragredient construction. Roughly speaking, it allows us to pass from a given situation to its dual situation. The "given situation" could be a monoidal category, or a monoidal functor, or some variation thereof. More general discussions, which build on this one, are given in Sections 6.12 and 7.10.

**3.10.1. Contravariant monoidal functors.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a contravariant functor and let $\mathsf{C}$ and $\mathsf{D}$ be monoidal categories. We say that $\mathcal{F}$ is *contravariant strong* if

$$\mathcal{F}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}, \qquad \text{or equivalently,} \qquad \mathcal{F}\colon \mathsf{C} \to \mathsf{D}^{\mathrm{op}}$$

is strong. Now let $\mathsf{C}$ and $\mathsf{D}$ be braided monoidal categories. We say that $\mathcal{F}$ is *contravariant bilax* (*bistrong*) if

$$\mathcal{F}\colon (\mathsf{C}^{\mathrm{op}}, \bullet, \beta^{\mathrm{op}}) \to (\mathsf{D}, \bullet, \beta), \qquad \text{or equivalently,} \qquad \mathcal{F}\colon (\mathsf{C}, \bullet, \beta) \to (\mathsf{D}^{\mathrm{op}}, \bullet, \beta^{\mathrm{op}})$$

is bilax (bistrong). (The equivalence used in the second definition follows from Proposition 3.7.)

**3.10.2. Contragredient of monoidal categories.** In this discussion, $*$ stands for a contravariant functor, say from $\mathsf{C}$ to $\mathsf{C}'$. We assume that there is another functor from $\mathsf{C}'$ to $\mathsf{C}$, also called $*$, such that

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "*"] & \mathsf{C}' \arrow[l, bend left, "*"] \end{tikzcd} \tag{3.44}$$

is an adjoint equivalence of categories.

If one of the categories, say $\mathsf{C}$ for definiteness, is monoidal with product $\bullet$ and unit $I$, then it induces a monoidal structure on $\mathsf{C}'$ by

$$A \bullet^{\vee} B := (A^* \bullet B^*)^*,$$

with the unit given by $I^{\vee} := I^*$. We say that $\bullet^{\vee}$ is the contragredient of $\bullet$, and that the monoidal category $(\mathsf{C}', \bullet^{\vee})$ is the contragredient of $(\mathsf{C}, \bullet)$. We have that

$$(\bullet^{\vee})^{\vee} \cong \bullet.$$

**Proposition 3.98.** *The functors*

$$\begin{tikzcd} (\mathsf{C}, \bullet) \arrow[r, bend left, "*"] & (\mathsf{C}', \bullet^{\vee}) \arrow[l, bend left, "*"] \end{tikzcd}$$

*are contravariant strong.*

PROOF. It follows from the definition of $\bullet^{\vee}$ that

$$(A \bullet^{\vee} B)^* \cong A^* \bullet B^*, \quad (I^{\vee})^* \cong I \qquad \text{and} \qquad (A \bullet B)^* \cong A^* \bullet^{\vee} B^*, \quad I^* \cong I^{\vee}$$

which implies that the $*$ functors are contravariant strong. $\square$

Now further if $\mathsf{C}$ is braided with braiding $\beta$, then so is $\mathsf{C}'$ with braiding

$$\beta^{\vee}_{B,A} := \beta^*_{A^*,B^*}.$$

We say that $\beta^{\vee}$ is the contragredient of $\beta$ and that the braided monoidal category $(\mathsf{C}', \bullet^{\vee}, \beta^{\vee})$ is the contragredient of $(\mathsf{C}, \bullet, \beta)$. We have that

$$(\beta^{\vee})^{\vee} \cong \beta.$$

**Proposition 3.99.** *The functors*

$$
\begin{tikzcd}
(\mathsf{C}, \bullet, \beta) \arrow[r, bend left, "*"] & (\mathsf{C}', \bullet^\vee, \beta^\vee) \arrow[l, bend left, "*"]
\end{tikzcd}
$$

*are contravariant bistrong.*

Proof. We saw in Proposition 3.98 that the $*$ functors are contravariant strong. Similarly, it follows from the definition of $\beta^\vee$ that the following diagrams commute.

$$
\begin{tikzcd}
(A \bullet^\vee B)^* \arrow[r, "\cong"] \arrow[d, "(\beta^\vee_{B,A})^*"'] & A^* \bullet B^* \arrow[d, "\beta_{A^*,B^*}"] \\
(B \bullet^\vee A)^* \arrow[r, "\cong"'] & B^* \bullet A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \bullet B)^* \arrow[r, "\cong"] \arrow[d, "\beta^*_{B,A}"'] & A^* \bullet^\vee B^* \arrow[d, "\beta^\vee_{A^*,B^*}"] \\
(B \bullet A)^* \arrow[r, "\cong"'] & B^* \bullet^\vee A^*
\end{tikzcd}
$$

This implies that the $*$ functors are contravariant braided strong, which is the same as contravariant bistrong. □

**3.10.3. Contragredient of functors.** Consider the following situation

$$
\mathsf{C} \xrightarrow{*} \mathsf{C}' \xrightarrow{\mathcal{F}} \mathsf{D}' \xrightarrow{*} \mathsf{D} \tag{3.45}
$$

where $\mathcal{F}$ is a covariant functor and the functors $*$ are as per the assumption (3.44). Let $\mathcal{F}^\vee$ denote the above composite, namely

$$
\mathcal{F}^\vee(-) := \mathcal{F}(-^*)^*. \tag{3.46}
$$

We refer to $\mathcal{F}^\vee$ as the *contragredient* of $\mathcal{F}$. Observe that it is a covariant functor.

For a natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$, let $\theta^\vee\colon \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee$ denote the induced natural transformation. Explicitly, it is given by

$$
\mathcal{G}^\vee(A) = \mathcal{G}(A^*)^* \xrightarrow{(\theta_{A^*})^*} \mathcal{F}(A^*)^* = \mathcal{F}^\vee(A). \tag{3.47}
$$

We have that

$$
(\mathcal{F}^\vee)^\vee \cong \mathcal{F},
$$

where it is implicit that the appropriate adjoint $*$ functors are used for defining the contragredient of $\mathcal{F}^\vee$.

**Example 3.100.** In the context of the tensor, symmetric and exterior algebras, the isomorphisms in (2.69) are instances of the contragredient construction. We elaborate a little bit further to make this point clear.

Let $*\colon \mathsf{Vec} \to \mathsf{Vec}$ be the duality functor which sends a vector space to its dual. This is an involutive contravariant bistrong functor on finite-dimensional vector spaces. Therefore, it maps (finite-dimensional) algebras to coalgebras and viceversa.

Let $\mathsf{gAlg}$ and $\mathsf{gCoalg}$ be the categories of (finite-dimensional) graded algebras and graded coalgebras. Consider the functor

$$
\mathcal{T}: \mathsf{Vec} \to \mathsf{gAlg}, \qquad \mathcal{T}(V) := \bigoplus_{k\geq 0} V^{\otimes k}.
$$

The object $\mathcal{T}(V)$ is the tensor algebra. It is graded by the number of tensor factors, and its product is given by concatenation. The contragredient of $\mathcal{T}$ is given by the composite

$$\mathcal{T}^\vee\colon \mathsf{Vec} \xrightarrow{\ *\ } \mathsf{Vec} \xrightarrow{\ \mathcal{T}\ } \mathsf{gAlg} \xrightarrow{\ *\ } \mathsf{gCoalg}.$$

It is clear that the coproduct on $\mathcal{T}^\vee$ is deconcatenation.

A duality functor similar to $*$ can be defined on the category of species (Section 8.6). The analogues of the tensor, symmetric and exterior algebras for species along with their contragredients are treated in Chapter 11.

**Example 3.101.** Another interesting instance of the contragredient construction is provided by the Fock functors which relate species to graded vector spaces. The duality functors on both species and vector spaces play a role here. See Section 15.1.2 for the simplest example of this kind.

**3.10.4. Contragredient of monoidal functors.** The contragredient construction is compatible with monoidal functors. We illustrate this feature with some simple but important results. Given a functor $\mathcal{F}\colon \mathsf{C}' \to \mathsf{D}'$ and a transformation $\varphi$ as in (3.3), consider its contragredient $\mathcal{F}^\vee\colon \mathsf{C} \to \mathsf{D}$ (3.46) and define a transformation $\varphi^\vee$ by

$$(3.48)\quad \begin{aligned}\varphi^\vee_{A,B}\colon \mathcal{F}^\vee(A \bullet^\vee B) = \mathcal{F}(A^* \bullet B^*)^* & \\ \xrightarrow{(\varphi_{A^*,B^*})^*} \big(\mathcal{F}(A^*) \bullet \mathcal{F}(B^*)\big)^* &= \mathcal{F}^\vee(A) \bullet^\vee \mathcal{F}^\vee(B).\end{aligned}$$

Similarly, for $(\mathcal{F},\psi)$ with $\psi$ as in (3.7), one defines $(\mathcal{F}^\vee,\psi^\vee)$.

**Proposition 3.102.** *If $(\mathcal{F},\varphi)\colon \mathsf{C}' \to \mathsf{D}'$ is (braided) lax, then*

$$(\mathcal{F}^\vee,\varphi^\vee)\colon \mathsf{C} \to \mathsf{D}$$

*is (braided) colax. Similarly, if $(\mathcal{F},\psi)$ is (braided) colax, then $(\mathcal{F}^\vee,\psi^\vee)$ is (braided) lax, and if $(\mathcal{F},\varphi,\psi)$ is (braided) bilax, then so is $(\mathcal{F}^\vee,\psi^\vee,\varphi^\vee)$.*

*Further, if $\theta\colon (\mathcal{F},\varphi) \Rightarrow (\mathcal{G},\gamma)$ is a morphism of lax (colax, bilax) functors, then*

$$\theta^\vee\colon (\mathcal{G}^\vee,\gamma^\vee) \Rightarrow (\mathcal{F}^\vee,\varphi^\vee)$$

*is a morphism of colax (lax, bilax) functors.*

Proof. Let $(\mathcal{F},\varphi)$ be lax. The $*$ functors by Proposition 3.98 are contravariant strong. Then applying Theorem 3.21, the following composite of lax functors is lax.

$$\mathsf{C}^{\mathrm{op}} \xrightarrow{\ *\ } \mathsf{C}' \xrightarrow{\ (\mathcal{F},\varphi)\ } \mathsf{D}' \xrightarrow{\ *\ } \mathsf{D}^{\mathrm{op}}$$

Passing to the opposite categories and applying Proposition 3.7, we obtain the functor $\mathcal{F}^\vee$ equipped with a colax structure. It is straightforward to check that the colax structure is given by $\varphi^\vee$.

The remaining claims are proved in a similar manner. $\square$

**Proposition 3.103.** *If $(\mathcal{F},\mathcal{G})$ is a pair of adjoint functors, then so is $(\mathcal{G}^\vee,\mathcal{F}^\vee)$. In addition, if the adjunction $(\mathcal{F},\mathcal{G})$ is lax-lax (resp. colax-colax), then $(\mathcal{G}^\vee,\mathcal{F}^\vee)$ is colax-colax (resp. lax-lax), and if $(\mathcal{F},\mathcal{G})$ is colax-lax, then so is $(\mathcal{G}^\vee,\mathcal{F}^\vee)$.*

PROOF. The proof is summarized in the following diagram.

$$\mathsf{C}' \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{D}' \quad\Longrightarrow\quad \mathsf{C}^{\mathrm{op}} \underset{\mathcal{G}^\vee}{\overset{\mathcal{F}^\vee}{\rightleftarrows}} \mathsf{D}^{\mathrm{op}} \quad\Longrightarrow\quad \mathsf{D} \underset{\mathcal{F}^\vee}{\overset{\mathcal{G}^\vee}{\rightleftarrows}} \mathsf{C}.$$

The content of the first implication is that adjunctions can be composed (Propositions 3.86 and 3.97). The second implication says that passing to the opposite categories switches left and right adjoints. This follows directly from the definition. □

Since the contragredient construction $(-)^\vee$ involves a passage to the opposite categories, it switches left and right adjoints, and lax and colax functors.

**3.10.5. Self-duality.** Now we specialize (3.44) to the situation where $\mathsf{C} = \mathsf{C}'$ and the two $*$ functors coincide. We say that an object $V$ in $\mathsf{C}$ is *self-dual* if $V \cong V^*$.

**Definition 3.104.** A monoidal category $(\mathsf{C}, \bullet)$ is *self-dual* if $\bullet^\vee \cong \bullet$, or more precisely, if

$$\mathrm{id}\colon (\mathsf{C}, \bullet) \to (\mathsf{C}, \bullet^\vee)$$

is a strong functor.

Similarly, a braided monoidal category $(\mathsf{C}, \bullet, \beta)$ is *self-dual* if $\bullet^\vee \cong \bullet$ and $\beta^\vee \cong \beta$, or more precisely, if

$$\mathrm{id}\colon (\mathsf{C}, \bullet, \beta) \to (\mathsf{C}, \bullet^\vee, \beta^\vee)$$

is a bistrong functor.

**Definition 3.105.** Let $\mathsf{C}$ and $\mathsf{D}$ be self-dual braided monoidal categories. A (bilax) functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is *self-dual* if $\mathcal{F}^\vee \cong \mathcal{F}$ as (bilax) functors.

**Proposition 3.106.** *A self-dual bilax functor induces a self-dual functor on the corresponding categories of bimonoids.*

The proof is straightforward.

**Proposition 3.107.** *A self-dual* (*bilax*) *functor preserves self-dual objects* (*bimonoids*).

PROOF. Let $\mathcal{F}$ be a self-dual functor and let $V$ be a self-dual object. Then by assumption,

$$\mathcal{F}(V) \cong \mathcal{F}(V^*) \cong \mathcal{F}(V)^*.$$

Hence $\mathcal{F}(V)$ is self-dual.

This result complemented with Proposition 3.106 yields the claim about self-dual bilax functors. □

**Definition 3.108.** Let $\mathsf{C}$ and $\mathsf{D}$ be self-dual braided monoidal categories, and let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a (bilax) functor. A natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{F}^\vee$ is *self-dual* if $\theta^\vee \cong \theta$.

**Definition 3.109.** A colax-lax adjunction $(\mathcal{F}, \mathcal{G})$ is *self-dual* if $\mathcal{F} \cong \mathcal{G}^\vee$ as colax functors, $\mathcal{G} \cong \mathcal{F}^\vee$ as lax functors, and these isomorphisms are compatible with the unit and counit of the adjunction.

TABLE 3.4. Self-dual functors.

| Self-dual functors | Sections |
|---|---|
| Hadamard functor on species | 8.13 and 9.4 |
| $\mathcal{S}$ and $\mathcal{S}^\vee$ | 11.6.4 |
| $\Lambda$ and $\Lambda^\vee$ | 11.7.6 |
| $\mathcal{T}_0$ | 11.10.3 |
| Bosonic Fock functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ in char 0 | 15.1 |
| Free Fock functor $\mathcal{K}_0$ | 16.1.3 |
| Fermionic Fock functors $\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^\vee_{-1}$ in char 0 | 16.3 |
| Anyonic Fock functors $\Im_q$ | 16.3 |

TABLE 3.5. Self-dual natural transformations.

| Self-dual natural transformations | Section |
|---|---|
| norm map $\kappa\colon \mathcal{T} \to \mathcal{T}^\vee$ | 11.6 |
| $q$-norm map $\kappa_q\colon \mathcal{T}_q \to \mathcal{T}_q^\vee$ | 11.7.5 |
| norm map $\kappa\colon \mathcal{K} \to \mathcal{K}^\vee$ | 15.4 |
| $q$-norm map $\kappa_q\colon \mathcal{K}_q \to \mathcal{K}_q^\vee$ | 16.2 |

**3.10.6. Examples.** The main examples of self-dual functors in this monograph are given below. A more elaborate summary is provided in Table 3.4.

- The functors $\mathcal{S}$, $\Lambda$ and $\mathcal{T}_0$ in Section 2.6.3 are self-dual. In Chapter 11, we construct analogues of these functors with species playing the role of graded vector spaces. It is interesting to note that in contrast to graded vector spaces, the functors $\mathcal{S}$ and $\Lambda$ for species are self-dual, regardless of the characteristic.
- The Hadamard functor on species is a self-dual bilax functor.
- Fock functors provide an important source of self-dual bilax functors. Their decorated and colored versions (not shown in the table) studied in Chapters 19 and 20 provide further examples.

The main examples of self-dual natural transformations are given in Table 3.5. These admit self-dual colored generalizations as well.

An example of a self-dual colax-lax adjunction is given in (8.81).

## 3.11. The image of a morphism of bilax monoidal functors

In an abelian monoidal category, a morphism of bimonoids has an *image* which is itself a bimonoid. Our main goal in this section is to obtain an analogous result for morphisms of bilax monoidal functors (Theorem 3.116). A nice proof of this fact can be given by viewing a morphism between two bilax monoidal functors as a bilax monoidal functor in an appropriate category. This is Proposition 3.111.

**3.11.1. The category of arrows.** Let $\mathsf{D}$ be an arbitrary category. The category $\mathsf{D}^{(2)}$ of *arrows in* $\mathsf{D}$ has for objects the triples $(A, f, B)$ where $A$ and $B$ are objects of $\mathsf{D}$ and $f\colon A \to B$ is a morphism in $\mathsf{D}$. A morphism from $(A, f, B)$ to $(C, g, D)$ is a pair $(h, k)$ of morphisms in $\mathsf{D}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

commutes. Composition and identities in $\mathsf{D}^{(2)}$ are coordinatewise. The category $\mathsf{D}^{(2)}$ is an example of a comma category; see Example A.22.

Suppose $(\mathsf{D}, \bullet)$ is a monoidal category. Then so is $\mathsf{D}^{(2)}$, with tensor product

$$(A, f, B) \bullet (A', f', B') := (A \bullet A', f \bullet f', B \bullet B').$$

The unit object is $(I, \mathrm{id}_I, I)$, where $I$ is the unit object in $\mathsf{D}$. If $\mathsf{D}$ is braided, then so is $\mathsf{D}^{(2)}$, with braiding

$$\begin{tikzcd} (A, f, B) \bullet (A', f', B') \arrow[d, "(\beta_{A,A'}, \beta_{B,B'})"'] \\ (A', f', B') \bullet (A, f, B) \end{tikzcd} \qquad \begin{tikzcd} A \bullet A' \arrow[r, "f \bullet f'"] \arrow[d, "\beta_{A,A'}"'] & B \bullet B' \arrow[d, "\beta_{B,B'}"] \\ A' \bullet A \arrow[r, "f' \bullet f"'] & B' \bullet B \end{tikzcd}$$

where $\beta$ is the braiding in $\mathsf{D}$.

**Proposition 3.110.** *An object $(A, f, B)$ of $\mathsf{D}^{(2)}$ is a monoid if and only if $A$ and $B$ are monoids in $\mathsf{D}$ and $f\colon A \to B$ is a morphism of monoids. The same statement holds replacing monoids by comonoids or bimonoids (the latter if the category $\mathsf{D}$ is braided).*

The proof is straightforward.

**3.11.2. Morphisms of monoidal functors as monoidal functors.** Let $\mathcal{F}$ and $\mathcal{G}$ be functors from a category $\mathsf{C}$ to a category $\mathsf{D}$, and let

$$\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$$

be a natural transformation. Then one can define a functor

$$\mathcal{H}_\theta\colon \mathsf{C} \to \mathsf{D}^{(2)}$$

by

$$A \mapsto \big(\mathcal{F}(A), \theta_A, \mathcal{G}(A)\big), \qquad f \mapsto \big(\mathcal{F}(f), \mathcal{G}(f)\big).$$

Naturality of $\theta$ makes $\mathcal{H}_\theta$ well-defined.

Now suppose that $\mathsf{C}$ and $\mathsf{D}$ are monoidal categories. Recall the notion of morphisms of monoidal functors (Definitions 3.8 and 3.9).

**Proposition 3.111.** *The functor $\mathcal{H}_\theta$ is lax monoidal if and only if the functors $\mathcal{F}$ and $\mathcal{G}$ are lax monoidal and $\theta$ is a morphism of lax monoidal functors. The same statement holds replacing lax for colax or bilax (the latter if the categories $\mathsf{C}$ and $\mathsf{D}$ are braided).*

PROOF. We explain the lax case. Suppose $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \gamma)$ are lax monoidal functors and $\theta$ is a morphism of lax monoidal functors. Then we define $\Phi$ by

$$\begin{tikzcd}
\mathcal{H}_\theta(A) \bullet \mathcal{H}_\theta(A') \arrow[r, equal] \arrow[d, dashed, "\Phi_{A,A'}"'] & \big(\mathcal{F}(A) \bullet \mathcal{F}(A'), \theta_A \bullet \theta_{A'}, \mathcal{G}(A) \bullet \mathcal{G}(A')\big) \arrow[d, "(\varphi_{A,A'}, \gamma_{A,A'})"] \\
\mathcal{H}_\theta(A \bullet A') \arrow[r, equal] & \big(\mathcal{F}(A \bullet A'), \theta_{A \bullet A'}, \mathcal{G}(A \bullet A')\big).
\end{tikzcd}$$

This is a well-defined morphism in $\mathsf{D}^{(2)}$ in view of the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}(A) \bullet \mathcal{F}(A') \arrow[r, "\theta_A \bullet \theta_{A'}"] \arrow[d, "\varphi_{A,A'}"'] & \mathcal{G}(A) \bullet \mathcal{G}(A') \arrow[d, "\gamma_{A,A'}"] \\
\mathcal{F}(A \bullet A') \arrow[r, "\theta_{A \bullet A'}"'] & \mathcal{G}(A \bullet A')
\end{tikzcd}$$

which holds since $\theta$ is a morphism of lax monoidal functors (3.14). We also set

$$\begin{tikzcd}
I_{\mathsf{D}^{(2)}} \arrow[r, equal] \arrow[d, dashed, "\Phi_0"'] & (I, \mathrm{id}_I, I) \arrow[d, "(\varphi_0, \gamma_0)"] \\
\mathcal{H}_\theta(I) \arrow[r, equal] & \big(\mathcal{F}(I), \theta_I, \mathcal{G}(I)\big)
\end{tikzcd}$$

which is well-defined in view of the second diagram in (3.14). The axioms in Definition 3.1 for $\varphi$ and $\gamma$ translate into the corresponding axioms for $\Phi$. Conversely, if $\Phi$ is a lax structure on the functor $\mathcal{H}_\theta$, then its components define lax structures on $\mathcal{F}$ and $\mathcal{G}$ such that $\theta$ is a morphism of lax functors. □

Proposition 3.110 is the special case of Proposition 3.111 in which $\mathsf{C}$ is the one-object monoidal category as in Section 3.4.1.

**3.11.3. The image of a morphism.** Recall that in an *abelian category* [250, Section VIII.3], every morphism $f\colon A \to B$ factors as

(3.49)
$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\
& X \arrow[ur, "m"'] &
\end{tikzcd}$$

with $e$ an epimorphism and $m$ a monomorphism. This is called a *monic-epi factorization* of $f$. The factorization is functorial in the following sense.

**Proposition 3.112.** *Given a commutative diagram in an abelian category*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
A' \arrow[r, "f'"'] & B',
\end{tikzcd}$$

*and monic-epi factorizations of $f$ and $f'$, there is a unique morphism $j\colon X \to X'$ such that the diagrams below commute*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "h"'] & X \arrow[r, "m"] \arrow[d, "j"] & B \arrow[d, "k"] \\ A' \arrow[r, "e'"'] & X' \arrow[r, "m'"'] & B', \end{tikzcd}$$

*where the rows are the given monic-epi factorizations of $f$ and $f'$.*

Proof. This is [250, Proposition VIII.3.1]. □

It follows that if $h$ and $k$ are isomorphisms, then so is $j$. In this sense, monic-epi factorizations are unique up to isomorphism. The maps $m$ and $e$ in (3.49) are called the *image* and *coimage* of $f$ respectively. Sometimes, more loosely, the same terminology is applied to the object $X$ (either term).

**3.11.4. The image functor.** Let $\mathsf{D}$ be an abelian category. We proceed to construct a functor

$$\Im\colon \mathsf{D}^{(2)} \to \mathsf{D}.$$

For each object $(A, f, B)$ of $\mathsf{D}^{(2)}$, we choose a monic-epi factorization as in (3.49) and we let

$$\Im(A, f, B) := X$$

where $X$ is the middle object in the chosen factorization. Given a morphism $(h, k)\colon (A, f, B) \to (A', f', B')$ in $\mathsf{D}^{(2)}$, we let

$$\Im(h, k) := j,$$

where $j$ is the unique morphism relating the chosen monic-epi factorizations of $f$ and $f'$ afforded by Proposition 3.112.

We refer to $\Im\colon \mathsf{D}^{(2)} \to \mathsf{D}$ as the *image functor*. Its functoriality follows from Proposition 3.112.

**Lemma 3.113.** *Let $(\mathsf{D}, \bullet)$ be an abelian monoidal category (Definition 1.8) and let*

$$\begin{tikzcd} A_1 \arrow[rr, "f_1"] \arrow[dr, "e_1"'] & & B_1 \\ & X_1 \arrow[ur, "m_1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_2 \arrow[rr, "f_2"] \arrow[dr, "e_2"'] & & B_2 \\ & X_2 \arrow[ur, "m_2"'] & \end{tikzcd}$$

*be monic-epi factorizations of two morphisms $f_1$ and $f_2$. Then*

$$\begin{tikzcd} A_1 \bullet A_2 \arrow[rr, "f_1 \bullet f_2"] \arrow[dr, "e_1 \bullet e_2"'] & & B_1 \bullet B_2 \\ & X_1 \bullet X_2 \arrow[ur, "m_1 \bullet m_2"'] & \end{tikzcd}$$

*is a monic-epi factorization of $f_1 \bullet f_2$.*

PROOF. We have to check that $e_1 \bullet e_2$ is a monomorphism and $m_1 \bullet m_2$ is an epimorphism. By exactness, the maps

$$e_1 \bullet \mathrm{id}\colon A_1 \bullet A_2 \to X_1 \bullet A_2 \quad \text{and} \quad \mathrm{id} \bullet e_2\colon X_1 \bullet A_2 \to X_1 \bullet X_2$$

are monomorphisms. Hence so is their composite $e_1 \bullet e_2$. For similar reasons, $m_1 \bullet m_2$ is an epimorphism. $\square$

**Proposition 3.114.** *Let* $(\mathsf{D}, \bullet)$ *be an abelian monoidal category. The functor*

$$\Im\colon (\mathsf{D}^{(2)}, \bullet) \to (\mathsf{D}, \bullet)$$

*is strong. If* $(\mathsf{D}, \bullet)$ *is braided, then* $\Im$ *is bistrong.*

PROOF. We define structure maps $\varphi$ and $\varphi_0$ (Definition 3.1). Take two objects in $\mathsf{D}^{(2)}$ and their chosen factorizations as in Lemma 3.113. Let also

$$A_1 \bullet A_2 \xrightarrow{e_{12}} X_{12} \xrightarrow{m_{12}} B_1 \bullet B_2$$

be the chosen factorization of $f_1 \bullet f_2$. Lemma 3.113 and uniqueness of factorizations (Proposition 3.112) allows us to define

$$\varphi_{(A_1,f_1,B_1),(A_2,f_2,B_2)}\colon \Im(A_1, f_1, B_1) \bullet \Im(A_2, f_2, B_2) \to \Im(A_1 \bullet A_2, f_1 \bullet f_2, B_1 \bullet B_2)$$

as the unique isomorphism such that the following diagram commutes

$$\begin{tikzcd}
A_1 \bullet A_2 \arrow[r, "e_1 \bullet e_2"] \arrow[d, equal] & X_1 \bullet X_2 \arrow[r, "m_1 \bullet m_2"] \arrow[d, dashed, "\varphi"'] & B_1 \bullet B_2 \arrow[d, equal] \\
A_1 \bullet A_2 \arrow[r, "e_{12}"'] & X_{12} \arrow[r, "m_{12}"'] & B_1 \bullet B_2.
\end{tikzcd} \tag{3.50}$$

The identity of the unit object of $\mathsf{D}$ can be factored through the unit object as $\mathrm{id}_I = \mathrm{id}_I \mathrm{id}_I$. We let

$$\varphi_0\colon I \to \Im(I, \mathrm{id}_I, I)$$

be the isomorphism afforded by Proposition 3.112.

Now let $(A_3, f_3, B_3)$ be a third object of $\mathsf{D}^{(2)}$. For ease of notation, assume the associativity constraints of $(\mathsf{D}, \bullet)$ are identities. We use similar notations to the above for the chosen factorizations of $f_2 \bullet f_3$ and $f_1 \bullet f_2 \bullet f_3$. By definition of $\varphi$, the following diagram commutes.

$$\begin{tikzcd}
A_1 \bullet A_2 \bullet A_3 \arrow[r, "e_1 \bullet e_2 \bullet e_3"] \arrow[d, equal] & X_1 \bullet X_2 \bullet X_3 \arrow[r, "m_1 \bullet m_2 \bullet m_3"] \arrow[d, "\varphi \bullet \mathrm{id}"'] & B_1 \bullet B_2 \bullet B_3 \arrow[d, equal] \\
A_1 \bullet A_2 \bullet A_3 \arrow[r, "e_{12} \bullet e_3"'] \arrow[d, equal] & X_{12} \bullet X_3 \arrow[r, "m_{12} \bullet m_3"'] \arrow[d, "\varphi"'] & B_1 \bullet B_2 \bullet B_3 \arrow[d, equal] \\
A_1 \bullet A_2 \bullet A_3 \arrow[r, "e_{123}"'] & X_{123} \arrow[r, "m_{123}"'] & B_1 \bullet B_2 \bullet B_3.
\end{tikzcd}$$

For the same reasons, a similar diagram with the middle vertical maps being

$$X_1 \bullet X_2 \bullet X_3 \xrightarrow{\mathrm{id} \bullet \varphi} X_1 \bullet X_{23} \xrightarrow{\varphi} X_{123}$$

commutes as well. Then, by uniqueness of factorizations,

$$\varphi(\mathrm{id} \bullet \varphi) = \varphi(\varphi \bullet \mathrm{id}).$$

Thus, the associativity condition in Definition 3.1 holds. The unital condition can be verified similarly, and hence $(\Im, \varphi)$ is a lax monoidal functor. Since $\varphi$ is an isomorphism, it is strong.

If the category $\mathsf{D}$ is braided, then the strong monoidal functor $(\Im, \varphi)$ is braided. This follows from the commutativity of the diagram

$$\begin{tikzcd}
A_1 \bullet A_2 \arrow[r, "e_1 \bullet e_2"] \arrow[d, "\beta_{A_1,A_2}"'] & X_1 \bullet X_2 \arrow[r, "m_1 \bullet m_2"] \arrow[d, "\beta_{X_1,X_2}"'] & B_1 \bullet B_2 \arrow[d, "\beta_{B_1,B_2}"] \\
A_2 \bullet A_1 \arrow[r, "e_2 \bullet e_1"'] & X_2 \bullet X_1 \arrow[r, "m_2 \bullet m_1"'] & B_2 \bullet B_1,
\end{tikzcd}$$

which holds by naturality of $\beta$. Hence, by Proposition 3.46, the functor $\Im$ is bistrong. $\square$

Let $\mathcal{P}_1$ and $\mathcal{P}_2 \colon \mathsf{D}^{(2)} \to \mathsf{D}$ be the canonical projections, that is,

$$\mathcal{P}_1(A, f, B) = A \qquad \text{and} \qquad \mathcal{P}_2(A, f, B) = B.$$

They are strong monoidal functors $\mathsf{D}^{(2)} \to \mathsf{D}$ (bistrong if $\mathsf{D}$ is braided). Moreover, there are transformations

$$\mathcal{P}_1 \Rightarrow \Im \Rightarrow \mathcal{P}_2 \tag{3.51}$$

defined by

$$\begin{tikzcd}
\mathcal{P}_1(A, f, B) \arrow[r, dashed] \arrow[d, equal] & \Im(A, f, B) \arrow[r, dashed] \arrow[d, equal] & \mathcal{P}_2(A, f, B) \arrow[d, equal] \\
A \arrow[r, "e"'] & X \arrow[r, "m"'] & B,
\end{tikzcd}$$

where the bottom row is the chosen factorization of $f$.

**Proposition 3.115.** *The transformations (3.51) are morphisms of (bi)strong monoidal functors.*

Proof. Naturality follows from the functoriality of factorizations (Proposition 3.112) and conditions (3.14) follow from the definition of $\varphi$ in (3.50). $\square$

**3.11.5. The image of a morphism of monoidal functors.** We are now in position to prove the main result of this section. Let $\mathsf{C}$ be an arbitrary monoidal category and $\mathsf{D}$ an abelian monoidal category. Let

$$\mathcal{F} \colon \mathsf{C} \to \mathsf{D} \quad \text{and} \quad \mathcal{G} \colon \mathsf{C} \to \mathsf{D}$$

be two functors and

$$\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$$

a natural transformation. Let $\Im_\theta$ denote the composite of functors

$$\mathsf{C} \xrightarrow{\mathcal{H}_\theta} \mathsf{D}^{(2)} \xrightarrow{\Im} \mathsf{D},$$

where $\mathcal{H}_\theta$ is the functor of Section 3.11.2 and $\Im$ is the image functor of Section 3.11.4. The functor $\Im_\theta$ sends an object $A$ in $\mathsf{C}$ to the image of the morphism $\theta_A \colon \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathsf{D}$.

**Theorem 3.116.** *In the above situation, if $\mathcal{F}$ and $\mathcal{G}$ are lax monoidal functors and $\theta$ is a morphism of lax monoidal functors, then*

$$\Im_\theta \colon \mathsf{C} \to \mathsf{D}$$

*is a lax monoidal functor. Moreover, $\theta$ factors as a composite of morphisms of lax monoidal functors*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, Rightarrow, "\theta"] \arrow[dr, Rightarrow] & & \mathcal{G} \\ & \Im_\theta . \arrow[ur, Rightarrow] & \end{tikzcd}$$

*The same result holds replacing lax for colax or bilax (the latter if $\mathsf{C}$ and $\mathsf{D}$ are braided).*

Proof. We explain the lax case. By construction $\Im_\theta$ is the composite of the lax monoidal functor $\mathcal{H}_\theta$ (Proposition 3.111) and the strong monoidal functor $\Im$ (Proposition 3.114), so Theorem 3.22 applies. Note that the composite of $\mathcal{H}_\theta$ and $\mathcal{P}_1$ is $\mathcal{F}$, and the composite of $\mathcal{H}_\theta$ and $\mathcal{P}_2$ is $\mathcal{G}$. The factors of $\theta$ are the compositions of the morphisms of Proposition 3.115 with the functor $\mathcal{H}_\theta$. They are morphisms of lax monoidal functors by Theorem 3.21. $\square$

**Remark 3.117.** The construction of the image functor $\Im$ involved a global choice of factorizations. Changing the choice leads to an isomorphic bistrong monoidal functor (again by uniqueness of factorizations). Suppose the category $\mathsf{D}$ is the category of (graded) vector spaces, or more generally, the category of modules over a commutative ring. In such a case there are two canonical choices of monic-epi factorizations (3.49). Namely, one can choose the middle object $X$ as the classical image of $f$ (a subobject of $B$) or as the classical coimage of $f$ (a quotient of $A$). It follows that both choices lead to isomorphic monoidal functors $\Im_\theta^1$ and $\Im_\theta^2$. One thus obtains a diagram of morphisms of monoidal functors

$$\begin{tikzcd} \mathcal{F} \arrow[rr, Rightarrow, "\theta"] \arrow[dr, Rightarrow] & & \mathcal{G} \\ & \Im_\theta^1 \cong \Im_\theta^2 . \arrow[ur, Rightarrow] & \end{tikzcd}$$

This is the situation encountered in Part III of the monograph; see Sections 15.4.3, 16.3.5, 19.2.1, 19.7.2 and 20.2.3. Further, in some of these situations, $\theta$ is given by symmetrization (an instance of the norm map in group theory). In that case, the image $\Im_\theta^2$ can be identified with invariants and the coimage $\Im_\theta^1$ with coinvariants, provided the field characteristic is 0.

**Remark 3.118.** While abelian monoidal categories constitute a natural context in which to formulate Theorem 3.116, this and the other results of this section hold in greater generality. In fact, all that is needed is the existence of functorial monic-epi factorizations in the category $\mathsf{D}$ (as in Proposition 3.112), and the fact that the tensor product of $\mathsf{D}$ preserves monomorphisms and epimorphisms. This holds not only in abelian monoidal categories (as in Definition 1.8), but also in $(\mathsf{Set}, \times)$, and in many other situations: indeed, it holds in any *topos* viewed as a monoidal category under Cartesian product. This follows from [251, Propositions IV.6.1–2]; see also [60, Corollaries 5.9.2 and 5.9.4] and [59, Proposition 2.3.4]. Very general conditions under which monic-epi factorizations exist and are unique are given in [58, Section 4.4] and [59, Section 2.1].

**3.11.6. Self-duality of the image.** In order to be able to discuss issues related to self-duality, we combine the above setup with that of the contragredient construction (Section 3.10). Accordingly, we assume that $\mathsf{C}$ and $\mathsf{D}$ are equipped with contravariant $*$ functors as in (3.44) (taking $\mathsf{C}' = \mathsf{C}$). This induces a $*$ functor on the category of arrows $\mathsf{D}^{(2)}$ by

$$(A, f, B)^* := (B^*, f^*, A^*).$$

Let $\mathcal{H}_\theta$ be as in Section 3.11.2. Then

$$(\mathcal{H}_\theta)^\vee = \mathcal{H}_{\theta^\vee},$$

with definitions as in (3.46) and (3.47). This follows from the following calculation.

$$\begin{aligned}(\mathcal{H}_\theta)^\vee(A) &= \big(\mathcal{F}(A^*), \theta_{A^*}, \mathcal{G}(A^*)\big)^* \\ &= (\mathcal{G}(A^*)^*, \theta^*_{A^*}, \mathcal{F}(A^*)^*) \\ &= \big(\mathcal{G}^\vee(A), \theta^\vee_A, \mathcal{F}^\vee(A)\big) \\ &= \mathcal{H}_{\theta^\vee}(A).\end{aligned}$$

In particular, if $\theta$ is a self-dual transformation $\mathcal{F} \Rightarrow \mathcal{F}^\vee$, then $\mathcal{H}_\theta$ is a self-dual functor (Definitions 3.105 and 3.108).

Now assume further that $\mathsf{D}$ is an abelian category such that monic-epi factorizations in $\mathsf{D}$ are compatible with the $*$ functor. Explicitly, this means that the dual of a monic-epi factorization of $f$ as in (3.49) yields a monic-epi factorization of $f^*$. With this assumption, it follows that the image functor $\Im$ of Section 3.11.4 is self-dual. Since $\Im_\theta$ is the composite of $\mathcal{H}_\theta$ and $\Im$, it follows that

$$(\Im_\theta)^\vee = \Im_{\theta^\vee}.$$

In particular, if $\theta$ is a self-dual transformation $\mathcal{F} \Rightarrow \mathcal{F}^\vee$, then $\Im_\theta$ is a self-dual functor.

If, in addition, we assume that $\mathsf{C}$ and $\mathsf{D}$ are (braided) monoidal categories, then by employing Propositions 3.111 and 3.114, and Theorem 3.116, one sees that the above results generalize to that setting. Among these, we highlight the following result.

**Proposition 3.119.** *Let $\mathsf{C}$ be a self-dual braided monoidal category, $\mathsf{D}$ be a self-dual braided abelian monoidal category, and let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a bilax functor. If $\theta\colon \mathcal{F} \to \mathcal{F}^\vee$ is a self-dual morphism of bilax functors, then the image $\Im_\theta$ is a self-dual bilax functor.*

The image of the norm transformation between full Fock functors and its deformed, decorated and colored versions are examples of this kind. They are discussed in Part III of the monograph; see the sections cited in Remark 3.117.

CHAPTER 4

# Operad Lax Monoidal Functors

The analogies between the notion of monoid and that of lax monoidal functor, and between the notion of commutative monoid and that of braided lax monoidal functor were explained in Chapter 3; see in particular Section 3.4. Just as there are other types of monoids besides associative and commutative, there are other types of monoidal functors. This is the topic of the present chapter. Section 4.1 serves as motivation by introducing a number of such types of monoids and monoidal functors. In Sections 4.2, 4.3, and 4.4, these notions are treated in full generality by making use of the notion of operad. The main result is Theorem 4.28 which describes the structure on a composite of two monoidal functors of such general types. This involves the Hadamard product of operads. Operads and the necessary background are discussed in Appendix B.

We mention in passing and without further comment that many other constructions of Chapter 3 also generalize to the setting of operads.

Subsequent chapters in Part I develop the theory of monoidal functors in a direction different from the present chapter and do not logically depend on this one.

## 4.1. Other types of monoids and monoidal functors

There has been interest in the recent literature in various types of algebras, beyond the classical associative, commutative, and Lie algebras. In particular, dendriform and Zinbiel algebras have been the object of study by Loday and others [235].

In Section 4.1.1 we define these objects in the general context of monoidal categories and review their most basic properties. Then we consider the question of comparing such objects in two different monoidal categories. This leads to various notions of monoidal functor, which accompany each notion of monoid just as associative and commutative monoids go along with lax and braided lax monoidal functors. These are introduced in Section 4.1.2. Section 4.1.3 provides some results on the effect of a monoidal functor of a given type on a monoid of another type.

**4.1.1. Other types of monoids.** In Chapter 1, we considered three *types* of monoids: associative monoids (simply called monoids in Definition 1.9), commutative monoids (Definition 1.17), and Lie monoids (Definition 1.25). The first type can be defined in any monoidal category, the second in any braided monoidal category, and the third in any linear symmetric monoidal category (see Remark 1.27).

We turn now to other types of monoids of interest.

**Definition 4.1.** Let $(\mathsf{M}, \bullet)$ be a linear monoidal category. A *dendriform monoid* in $\mathsf{M}$ is a triple $(A, \prec, \succ)$ where $A$ is an object of $\mathsf{M}$,

$$\prec\colon A \bullet A \to A \qquad\qquad \succ\colon A \bullet A \to A$$

are maps in $\mathsf{M}$, and the following diagrams commute

$$\begin{tikzcd} A \bullet A \bullet A \arrow[r, "\mathrm{id}\bullet *"] \arrow[d, "\prec\bullet\mathrm{id}"'] & A \bullet A \arrow[d, "\prec"] \\ A \bullet A \arrow[r, "\prec"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \bullet A \bullet A \arrow[r, "\mathrm{id}\bullet\prec"] \arrow[d, "\succ\bullet\mathrm{id}"'] & A \bullet A \arrow[d, "\succ"] \\ A \bullet A \arrow[r, "\prec"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \bullet A \bullet A \arrow[r, "\mathrm{id}\bullet\succ"] \arrow[d, "*\bullet\mathrm{id}"'] & A \bullet A \arrow[d, "\succ"] \\ A \bullet A \arrow[r, "\succ"'] & A \end{tikzcd}$$

where $* = \prec + \succ$.

**Definition 4.2.** Let $(\mathsf{M}, \bullet, \beta)$ be a linear symmetric monoidal category. A *Zinbiel monoid* in $\mathsf{M}$ is a pair $(A, \mu)$ where $A$ is an object in $\mathsf{M}$, $\mu\colon A \bullet A \to A$ is a map in $\mathsf{M}$, and the following diagram commutes.

$$\begin{tikzcd} A \bullet A \bullet A \arrow[r, "\mathrm{id}\bullet(\mu+\mu\beta)"] \arrow[d, "\mu\bullet\mathrm{id}"'] & A \bullet A \arrow[d, "\mu"] \\ A \bullet A \arrow[r, "\mu"'] & A \end{tikzcd}$$

We point out that none of these axioms involve the unit object of $\mathsf{M}$; Definitions 4.1 and 4.2 may just as well be stated in a nonunital linear (symmetric) monoidal category $\mathsf{M}$.

**Remark 4.3.** When $\mathsf{M}$ is the category of vector spaces, one recovers the notions of dendriform and Zinbiel algebras introduced by Loday. Zinbiel algebras appeared in [236]; see also [238, Section 7.1]. The same objects had been considered earlier by Schützenberger [325, pp. 18–19, identity (SO)]. For dendriform algebras, see [238, Definition 5.1].

The following results are straightforward. They extend well-known results for dendriform and Zinbiel algebras to the context of linear (symmetric) monoidal categories. They are of the same sort as Proposition 1.26, which constructed a Lie monoid out of an associative monoid.

**Proposition 4.4.** *Let $(\mathsf{M}, \bullet)$ be a linear monoidal category and $(A, \prec, \succ)$ a dendriform monoid. Define*

$$*\colon A \bullet A \to A$$

*by*

$$* = \prec + \succ,$$

*as in Definition 4.1. Then $(A, *)$ is a nonunital associative monoid.*

**Proposition 4.5.** *Let $(\mathsf{M}, \bullet)$ be a linear symmetric monoidal category and $(A, \mu)$ a Zinbiel monoid. Define*

$$\prec := \mu\colon A \bullet A \to A \qquad \text{and} \qquad \succ := \mu\beta\colon A \bullet A \to A.$$

*Then $(A, \prec, \succ)$ is a dendriform monoid. Moreover, the nonunital associative monoid of Proposition 4.4 is commutative.*

**4.1.2. Other types of monoidal functors.** For each type of monoid, there is a corresponding type of monoidal functor. Lax monoidal functors (Definition 3.1) correspond to associative monoids and braided lax monoidal functors (Definition 3.11) correspond to commutative monoids. We now define the types of functor corresponding to Lie, dendriform, and Zinbiel monoids. The definitions below are to be compared with Definitions 1.25, 4.1, and 4.2.

**Notation 4.6.** The notation $(\mathcal{F}, \varphi)$ stands for a functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ between monoidal categories, not necessarily unital, along with a natural transformation

$$\mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\varphi_{A,B}} \mathcal{F}(A \bullet B) \tag{4.1}$$

between the functors $\mathcal{F}^2$ and $\mathcal{F}_2$ as in (3.3). The monoidal categories may be assumed to be linear and symmetric, depending on the context.

We let $\varphi_{(A,B),C}$ be the natural transformation given by the composite

$$\mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \xrightarrow{\varphi_{A,B} \bullet \mathrm{id}} \mathcal{F}(A \bullet B) \bullet \mathcal{F}(C) \xrightarrow{\varphi_{A \bullet B, C}} \mathcal{F}(A \bullet B \bullet C). \tag{4.2}$$

The natural transformation $\varphi_{A,(B,C)}$ is defined similarly.

We say that the transformation $\varphi$ is *associative* if

$$\varphi_{(A,B),C} = \varphi_{A,(B,C)}.$$

In this case, one may drop the brackets and denote this transformation by $\varphi_{A,B,C}$.

**Definition 4.7.** Let $\mathsf{C}$ be a symmetric monoidal category and $\mathsf{D}$ a linear symmetric monoidal category. We say that $(\mathcal{F}, \varphi)$ is *Lie-lax monoidal* if $\varphi$ satisfies the *antisymmetry* relation:

$$\begin{aligned} &\big(\mathcal{F}(A)\mathcal{F}(B) \xrightarrow{\varphi} \mathcal{F}(AB)\big) \\ &\qquad + \big(\mathcal{F}(A)\mathcal{F}(B) \to \mathcal{F}(B)\mathcal{F}(A) \xrightarrow{\varphi} \mathcal{F}(BA) \to \mathcal{F}(AB)\big) = 0, \end{aligned} \tag{4.3}$$

and satisfies the *Jacobi identity*: The sum of the three morphisms below is zero.

$$\begin{gathered} \mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C) \xrightarrow{\varphi_{(A,B),C}} \mathcal{F}(ABC) \\ \mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C) \longrightarrow \mathcal{F}(C)\mathcal{F}(A)\mathcal{F}(B) \xrightarrow{\varphi_{(C,A),B}} \mathcal{F}(CAB) \longrightarrow \mathcal{F}(ABC) \\ \mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C) \longrightarrow \mathcal{F}(B)\mathcal{F}(C)\mathcal{F}(A) \xrightarrow{\varphi_{(B,C),A}} \mathcal{F}(BCA) \longrightarrow \mathcal{F}(ABC) \end{gathered} \tag{4.4}$$

The unlabeled arrows in (4.3) and (4.4) denote the canonical morphisms induced by the symmetries of the categories $\mathsf{C}$ and $\mathsf{D}$. The monoidal operations have been omitted for simplicity. We follow the same convention in subsequent definitions.

**Definition 4.8.** Let $\mathsf{C}$ be a monoidal category and $\mathsf{D}$ a linear monoidal category. We say that $(\mathcal{F}, \varphi^{\prec}, \varphi^{\succ})$ is *dendriform-lax monoidal* if

$$\varphi^{\prec}, \varphi^{\succ} \colon \mathcal{F}(A) \bullet \mathcal{F}(B) \to \mathcal{F}(A \bullet B)$$

and the following diagrams commute

$$\begin{tikzcd}
\mathcal{F}(A)\bullet\mathcal{F}(B)\bullet\mathcal{F}(C) \arrow[r, "\mathrm{id}\bullet\varphi"] \arrow[d, "\varphi^{\prec}\bullet\mathrm{id}"'] & \mathcal{F}(A)\bullet\mathcal{F}(B\bullet C) \arrow[d, "\varphi^{\prec}"] \\
\mathcal{F}(A\bullet B)\bullet\mathcal{F}(C) \arrow[r, "\varphi^{\prec}"'] & \mathcal{F}(A\bullet B\bullet C)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(A)\bullet\mathcal{F}(B)\bullet\mathcal{F}(C) \arrow[r, "\mathrm{id}\bullet\varphi^{\succ}"] \arrow[d, "\varphi\bullet\mathrm{id}"'] & \mathcal{F}(A)\bullet\mathcal{F}(B\bullet C) \arrow[d, "\varphi^{\succ}"] \\
\mathcal{F}(A\bullet B)\bullet\mathcal{F}(C) \arrow[r, "\varphi^{\succ}"'] & \mathcal{F}(A\bullet B\bullet C)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(A)\bullet\mathcal{F}(B)\bullet\mathcal{F}(C) \arrow[r, "\mathrm{id}\bullet\varphi^{\prec}"] \arrow[d, "\varphi^{\succ}\bullet\mathrm{id}"'] & \mathcal{F}(A)\bullet\mathcal{F}(B\bullet C) \arrow[d, "\varphi^{\succ}"] \\
\mathcal{F}(A\bullet B)\bullet\mathcal{F}(C) \arrow[r, "\varphi^{\prec}"'] & \mathcal{F}(A\bullet B\bullet C)
\end{tikzcd}$$

where $\varphi = \varphi^{\prec} + \varphi^{\succ}$.

**Definition 4.9.** Let $\mathsf{C}$ be a symmetric monoidal category and $\mathsf{D}$ a linear symmetric monoidal category. We say that $(\mathcal{F},\varphi)$ is *Zinbiel-lax monoidal* if $\varphi$ satisfies the following identity.

$$\begin{aligned}
(4.5)\quad & \big(\mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C) \xrightarrow{\varphi_{(A,B),C}} \mathcal{F}(ABC)\big) \\
&= \big(\mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C) \xrightarrow{\varphi_{A,(B,C)}} \mathcal{F}(ABC)\big) \\
&\quad + \big(\mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C) \to \mathcal{F}(A)\mathcal{F}(C)\mathcal{F}(B) \xrightarrow{\varphi_{A,(C,B)}} \mathcal{F}(ACB) \to \mathcal{F}(ABC)\big)
\end{aligned}$$

The following results are analogues of Propositions 1.26, 4.4, and 4.5. The proofs offer no difficulty.

**Proposition 4.10.** *Let $\mathsf{C}$ be a monoidal category and $\mathsf{D}$ a linear symmetric monoidal category. Let $(\mathcal{F},\varphi)$ a lax monoidal functor from $\mathsf{C}$ to $\mathsf{D}$ (not necessarily unital). Define*

$$\varphi^{-} := \mathcal{F}(A)\bullet\mathcal{F}(B) \to \mathcal{F}(A\bullet B)$$

*by*

$$\begin{aligned}
&\big(\mathcal{F}(A)\mathcal{F}(B) \xrightarrow{\varphi^{-}_{A,B}} \mathcal{F}(AB)\big) \\
&\qquad = \big(\mathcal{F}(A)\mathcal{F}(B) \xrightarrow{\varphi_{A,B}} \mathcal{F}(AB)\big) \\
&\qquad\quad - \big(\mathcal{F}(A)\mathcal{F}(B) \xrightarrow{\beta} \mathcal{F}(B)\mathcal{F}(A) \xrightarrow{\varphi_{B,A}} \mathcal{F}(BA) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(AB)\big).
\end{aligned}$$

*Then $(\mathcal{F},\varphi^{-})$ is a Lie-lax monoidal functor.*

**Proposition 4.11.** *Let $\mathsf{C}$ be a monoidal category and $\mathsf{D}$ a linear monoidal category. Let $(\mathcal{F}, \varphi^{\prec}, \varphi^{\succ})$ be a dendriform-lax monoidal functor from $\mathsf{C}$ to $\mathsf{D}$. Define*

$$\varphi\colon \mathcal{F}(A) \bullet \mathcal{F}(B) \to \mathcal{F}(A \bullet B)$$

*by*

$$\varphi = \varphi^{\prec} + \varphi^{\succ},$$

*as in Definition 4.8. Then $(\mathcal{F}, \varphi)$ is a nonunital lax monoidal functor.*

**Proposition 4.12.** *Let $\mathsf{C}$ be a symmetric monoidal category and $\mathsf{D}$ a linear symmetric monoidal category. Let $(\mathcal{F}, \varphi)$ be a Zinbiel-lax monoidal functor. Define*

$$\varphi^{\prec} := \varphi \qquad \text{and} \qquad \varphi^{\succ} := \mathcal{F}(\beta)\varphi\beta.$$

*Then $(\mathcal{F}, \varphi^{\prec}, \varphi^{\succ})$ is a dendriform-lax monoidal functor. Moreover, the nonunital lax monoidal functor of Proposition 4.11 is braided.*

**4.1.3. Transformation of monoids under monoidal functors.** Lax monoidal functors preserve monoids (Proposition 3.29) and braided lax monoidal functors preserve commutative monoids (Proposition 3.37). It is natural to ask what type of functors would preserve other types of monoids. By analogy with the above cases, it may seem that Lie lax functors (Definition 4.7) would preserve Lie monoids. However, something else is true; namely that braided lax monoidal functors preserve Lie monoids.

**Proposition 4.13.** *Let $(\mathcal{F}, \varphi)$ be a linear braided lax monoidal functor between linear symmetric monoidal categories $\mathsf{C}$ and $\mathsf{D}$, and $(L, \gamma)$ a Lie monoid in $\mathsf{C}$. Then $\mathcal{F}(L)$ is a Lie monoid in $\mathsf{D}$ with structure map given by the composite*

$$\mathcal{F}(L) \bullet \mathcal{F}(L) \xrightarrow{\varphi_{L,L}} \mathcal{F}(L \bullet L) \xrightarrow{\mathcal{F}(\gamma)} \mathcal{F}(L).$$

Proof. We verify the Jacobi axiom in Definition 1.25 for $\mathcal{F}(L)$. We need to show that the sum of the following composites, for $i = 0, 1, 2$, is zero:

$$\begin{tikzcd}
\mathcal{F}(L) \bullet \mathcal{F}(L) \bullet \mathcal{F}(L) \arrow[r, "\varphi_{L,L} \bullet \mathrm{id}"] & \mathcal{F}(L \bullet L) \bullet \mathcal{F}(L) \arrow[r, "\mathcal{F}(\gamma) \bullet \mathrm{id}"] & \mathcal{F}(L) \bullet \mathcal{F}(L) \arrow[r, "\varphi_{L,L}"] & \mathcal{F}(L \bullet L) \arrow[d, "\mathcal{F}(\gamma)"] \\
\mathcal{F}(L) \bullet \mathcal{F}(L) \bullet \mathcal{F}(L) \arrow[u, "\xi^i"] & & & \mathcal{F}(L),
\end{tikzcd}$$

where $\xi$ is as in Definition 1.25.

By naturality and associativity of $\varphi$ (Definition 3.1), the above equals the composite

$$\mathcal{F}(L) \bullet \mathcal{F}(L) \bullet \mathcal{F}(L) \xrightarrow{\xi^i} \mathcal{F}(L) \bullet \mathcal{F}(L) \bullet \mathcal{F}(L) \xrightarrow{\varphi_{L,L,L}} \mathcal{F}(L \bullet L \bullet L) \xrightarrow{\mathcal{F}(\gamma(\gamma \bullet \mathrm{id}))} \mathcal{F}(L).$$

Moreover, since $(\mathcal{F}, \varphi)$ is braided, $\varphi$ commutes with $\xi$. Therefore, the above equals

$$\mathcal{F}(L) \bullet \mathcal{F}(L) \bullet \mathcal{F}(L) \xrightarrow{\varphi_{L,L,L}} \mathcal{F}(L \bullet L \bullet L) \xrightarrow{\mathcal{F}(\xi^i)} \mathcal{F}(L \bullet L \bullet L) \xrightarrow{\mathcal{F}(\gamma(\gamma \bullet \mathrm{id}))} \mathcal{F}(L).$$

The Jacobi axiom for $L$ and the additivity of $\mathcal{F}$ then imply that the sum of these three maps is zero.

The verification of the antisymmetry axiom is similar. □

Lax monoidal functors preserve dendriform monoids, and braided lax monoidal functors preserve Zinbiel monoids, as we now see.

**Proposition 4.14.** *Let $(\mathcal{F}, \varphi)$ be a linear lax monoidal functor between linear monoidal categories $\mathsf{C}$ and $\mathsf{D}$. Let $(A, \prec, \succ)$ be a dendriform monoid in $\mathsf{C}$. Then $\mathcal{F}(A)$ is a dendriform monoid in $\mathsf{D}$ with structure maps given by*

$$\mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{A,A}} \mathcal{F}(A \bullet A) \xrightarrow{\mathcal{F}(\prec)} \mathcal{F}(A)$$

*and*

$$\mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{A,A}} \mathcal{F}(A \bullet A) \xrightarrow{\mathcal{F}(\succ)} \mathcal{F}(A).$$

*Moreover, if $A$ is a Zinbiel monoid (as in Proposition 4.5) and $\mathcal{F}$ is in addition braided lax monoidal, then $\mathcal{F}(A)$ is a Zinbiel monoid.*

The proof offers no difficulty.

Combining the results of Propositions 3.29, 4.13, and 4.14, we see that braided lax monoidal functors preserve all types of monoids discussed so far. This result is generalized in Corollary 4.37. The following result can also be easily established.

**Proposition 4.15.** *Let $(\mathcal{F}, \varphi)$ be a linear Zinbiel lax monoidal functor between linear symmetric monoidal categories $\mathsf{C}$ and $\mathsf{D}$. Let $(A, \mu)$ be an associative monoid in $\mathsf{C}$. Then $\mathcal{F}(A)$ is a dendriform monoid in $\mathsf{D}$ with structure maps given by*

$$\mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{A,A}} \mathcal{F}(A \bullet A) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(A)$$

*and*

$$\mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\beta} \mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{A,A}} \mathcal{F}(A \bullet A) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(A \bullet A) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(A).$$

*Moreover, if $A$ is a commutative monoid, then $\mathcal{F}(A)$ is a Zinbiel monoid (as in Proposition 4.5).*

The preceding results do not answer all possible questions about transformation of monoids under monoidal functors. What is the result of applying a Lie-lax monoidal functor to an associative monoid? What is the result of composing a Lie-lax monoidal functor with a Zinbiel-lax monoidal functor? There is a simple answer to these questions, and to any question of this type, that we discuss in Section 4.4. It involves the notion of operad. Each operad gives rise to a notion of monoid and of monoidal functor. We discuss these in Sections 4.2 and 4.3.

## 4.2. Types of monoid: the general case

This section assumes some basic familiarity with species and operads. Roughly speaking, a species is a collection of vector spaces, one for each finite set. This is similar to a graded vector space which is a sequence of vector spaces. Species are discussed in detail in Chapter 8. Operads are species with additional structure. They are discussed in full detail in Appendix B.

In this section, we recall the notion of an operad-monoid along with a basic set of examples. Let $\mathbf{p}$ be an operad. Then one may view a $\mathbf{p}$-monoid as a representation of $\mathbf{p}$. It is most common to represent $\mathbf{p}$ in the category of vector spaces; that is, the most basic structure on a $\mathbf{p}$-monoid is that of a vector space (see Remark 4.19). However, it is preferable to work in more generality.

Accordingly, let $\mathsf{M}$ be a symmetric monoidal category. Let $\Bbbk$ be the field over which the operad $\mathbf{p}$ is defined. We assume that $\mathsf{M}$ is linear over $\Bbbk$ (Definition 1.6). This is the category in which we will represent $\mathbf{p}$.

**4.2.1. The endomorphism operad associated to an object.** Let $I$ be a finite set. Given a family $(V_i)_{i\in I}$ of objects of $\mathsf{M}$, we let

$$\bullet_{i\in I} V_i$$

denote their unordered tensor product (Section 1.4). This makes use of the fact that $\mathsf{M}$ is a symmetric monoidal category. If $V_i = V$ for every $i$, we write

$$V^{\bullet I} := \bullet_{i\in I} V_i.$$

For any object $V$ of $\mathsf{M}$, define the endomorphism species

$$\mathbf{End}_V[I] := \mathrm{Hom}_{\mathsf{M}}(V^{\bullet I}, V). \tag{4.6}$$

The linearity assumption on $\mathsf{M}$ turns $\mathbf{End}_V[I]$ into a vector space. A bijection $I \to J$ induces an isomorphism

$$V^{\bullet I} \cong V^{\bullet J}$$

and hence a linear isomorphism $\mathbf{End}_V[J] \cong \mathbf{End}_V[I]$. In this manner, $\mathbf{End}_V$ is a species.

The endomorphism species carries an operad structure as follows. We use the notation of Section B.4.3. Fix a map $f\colon I \to X$. Given morphisms

$$\varrho\colon V^{\bullet X} \to V \qquad \text{and} \qquad \varrho_x\colon V^{\bullet f^{-1}(x)} \to V \quad \text{for each } x \in X,$$

we define a morphism $V^{\bullet I} \to V$ as the composite below.

$$\begin{tikzcd}
V^{\bullet I} \arrow[rr, dashed] \arrow[d, "\cong"'] & & V \arrow[d, equal] \\
\bullet_{x\in X} V^{\bullet f^{-1}(x)} \arrow[r, "\bullet_{x\in X}\varrho_x"'] & V^{\bullet X} \arrow[r, "\varrho"'] & V
\end{tikzcd}$$

This defines the map $\gamma_f$ in (B.13). For any singleton set $\{x\}$, we have

$$V^{\bullet\{x\}} \cong V.$$

Using this identification, one defines the unit map $\eta_x$ in (B.14) to be the map which sends $1 \in \Bbbk$ to the identity morphism $V \to V$. It is straightforward to check that the operad axioms hold.

**4.2.2. Operad-monoids in a linear monoidal category.**

**Definition 4.16.** Let $\mathbf{p}$ be an operad. A $\mathbf{p}$-*monoid* in $\mathsf{M}$ is an object $V$ with a morphism of operads

$$\mathbf{p} \to \mathbf{End}_V,$$

where $\mathbf{End}_V$ is the endomorphism operad.

We now make the notion of $\mathbf{p}$-monoid more explicit. We use the following generic notation for the structure maps of the operad $\mathbf{p}$.

$$\begin{aligned}
\mathbf{p}[X] \otimes \bigotimes_{x\in X} \mathbf{p}[f^{-1}(x)] &\to \mathbf{p}[I], & \qquad \Bbbk &\to \mathbf{p}[\{*\}] \\
a \otimes \bigotimes_{x\in X} a_x &\mapsto c & 1 &\mapsto i.
\end{aligned} \tag{4.7}$$

A $\mathbf{p}$-monoid in $\mathsf{M}$ is an object $V$ with the following structure. For each finite set $I$, there is a morphism

$$Ra\colon V^{\bullet I} \to V \qquad \text{for each } a \in \mathbf{p}[I], \tag{4.8}$$

which is linear in the element $a$, subject to the conditions below.

**Naturality.** For any bijection $\sigma\colon I \to J$, the following diagram commutes.

$$\begin{tikzcd} V^{\bullet J} \arrow[r, "R(\sigma a)"] \arrow[d, "\cong"'] & V \arrow[d, equal] \\ V^{\bullet I} \arrow[r, "Ra"'] & V \end{tikzcd} \tag{4.9}$$

**Substitution compatibility.** Let $a$, $a_x$, $c$ and $i$ be as in (4.7). For each $f\colon I \to X$ the following diagram commutes.

$$\begin{tikzcd} V^{\bullet I} \arrow[rr, "Rc"] \arrow[d, "\cong"'] & & V \arrow[d, equal] \\ \bullet_{x\in X}\Big(\bullet_{i\in f^{-1}(x)} V\Big) \arrow[r, "\bullet_{x\in X} Ra_x"'] & V^{\bullet X} \arrow[r, "Ra"'] & V \end{tikzcd} \tag{4.10}$$

Further, $Ri$ is the identity morphism.

**Definition 4.17.** A morphism $V \to W$ of $\mathbf{p}$-monoids is a map $V \to W$ such that for each $a \in \mathbf{p}[I]$ the following diagram commutes.

$$\begin{tikzcd} V^{\bullet I} \arrow[r, "Ra"] \arrow[d] & V \arrow[d] \\ W^{\bullet I} \arrow[r, "Ra"'] & W \end{tikzcd}$$

A morphism $\mathbf{p} \to \mathbf{q}$ of operads induces a restriction functor from the category of $\mathbf{q}$-monoids to the category of $\mathbf{p}$-monoids.

**Example 4.18.** Consider the associative operad $\mathbf{As}$ (Example B.15). Let $A$ be an $\mathbf{As}$-monoid in a monoidal category $\mathsf{M}$. According to (4.6), the structure on $A$ consists of maps of species

$$A^{\bullet I} \to A,$$

one for each linear order on a finite set $I$, which combine into a morphism of operads $\mathbf{As} \to \mathbf{End}_A$. Naturality on $I$ reduces this to a family of maps of species

$$A^{\bullet n} \to A,$$

one for each $n$, which result in a morphism of operads. Further analysis of the operad structure of $\mathbf{As}$ (its familiar presentation by one generator of degree 2 subject to the associativity relation) shows that the map for $n = 1$ must be the identity and the map for $n = 2$ is associative with respect to the monoidal structure of $\mathsf{M}$. Moreover, this map determines all the higher maps. Thus, $A$ is a monoid in

TABLE 4.1. Operad-monoids and familiar types of monoids.

| Operad-monoid | Familiar monoid | Definition |
|---|---|---|
| **As**-monoid | monoid | 1.9 |
| **Com**-monoid | commutative monoid | 1.17 |
| **Lie**-monoid | Lie monoid | 1.25 |
| **Dend**-monoid | dendriform monoid | 4.1 |
| **Zinb**-monoid | Zinbiel monoid | 4.2 |

the monoidal category $\mathsf{M}$ in the usual sense of Definition 1.9. The map for $n = 2$ is the product and the map for $n = 0$ is the unit.

In summary, an **As**-monoid structure on $A$ is equivalent to a monoid structure on the object $A$ in the monoidal category $\mathsf{M}$.

The analysis in Example 4.18 can also be carried out for the other types of monoid discussed in Section 4.1.1, besides associative monoids. The conclusion is that each type of monoid is a special case of the general notion of operad-monoid, as shown in Table 4.1. The dendriform operad **Dend** is described in [238]. The remaining relevant operads are discussed in Sections B.1.4 and B.4.3.

The content of Table 4.1 is that for the operads $\mathbf{p}$ in question, one has an explicit description of the notion of $\mathbf{p}$-monoids in terms of commutative diagrams, corresponding to their standard presentations as quadratic operads [260, Section 3.2]. In principle, any presentation of an operad $\mathbf{p}$ leads to a similar description of the notion of $\mathbf{p}$-monoids in terms of commutative diagrams. We do not discuss the theory of operad presentations in this work (but see Example B.5).

**Remark 4.19.** When $\mathsf{M}$ is the category of vector spaces, Definition 4.16 recovers the usual notion of $\mathbf{p}$-algebras over an operad $\mathbf{p}$. This extends the notions of algebra mentioned in Remark 4.3.

Other choices of $\mathsf{M}$ lead to the notions of graded and super algebras. A *graded* $\mathbf{p}$-*algebra* is a $\mathbf{p}$-monoid in the category $(\mathsf{gVec}, \cdot, \beta)$ or $(\mathsf{gVec}, \cdot, \beta_{-1})$ (Sections 2.1.3 and 2.3.1). In general, these two versions of graded algebras differ; to distinguish them one sometimes refer to the former as *unsigned* and to the latter as *signed graded algebras*. A *super* $\mathbf{p}$-*algebra* is a $\mathbf{p}$-monoid in the category $(\mathsf{sVec}, \cdot, \beta_{-1})$ of super vector spaces (Section 2.3.8).

We briefly mention the dual notion of operad-comonoids.

**Definition 4.20.** A $\mathbf{p}$-*comonoid* in $\mathsf{M}$ is a $\mathbf{p}$-monoid in the opposite category $\mathsf{M}^{\mathrm{op}}$.

## 4.3. Types of monoidal functor: the general case

We generalize the discussion is the previous section. Instead of representing an operad on an object in a symmetric monoidal category, we represent it on a functor between symmetric monoidal categories. This leads to the notion of an operad-lax monoidal functor. The discussion parallels that in Section 4.2.

**4.3.1. The functors $\mathcal{F}^I$ and $\mathcal{F}_I$.** Let $\mathsf{C}$ be a symmetric monoidal category and $I$ be a finite set. View $I$ as a discrete category: objects are the elements of $I$ and every morphism is an identity. Let $\mathsf{C}^I$ be the category of functors $I \to \mathsf{C}$: objects are functors from $I$ to $\mathsf{C}$ and morphisms are natural transformations. Explicitly, an object is an assignment of an object $A_i$ in $\mathsf{C}$ to every element $i \in I$. We denote this object by $(A_i)_{i\in I}$. A morphism $(A_i)_{i\in I} \to (B_i)_{i\in I}$ is an assignment of a morphism $A_i \to B_i$ for each $i \in I$.

Let $\mathcal{M}$ be the tensor product functor

$$\mathcal{M} \colon \mathsf{C}^I \to \mathsf{C} \qquad\qquad (A_i)_{i\in I} \mapsto \bullet_{i\in I} A_i,$$

where the right-hand side refers to the unordered tensor product over $I$ (Section 1.4).

Now let $(\mathsf{C}, \bullet, \beta)$ and $(\mathsf{D}, \bullet, \beta)$ be two symmetric monoidal categories and let $\mathcal{F}$ be a functor from $\mathsf{C}$ to $\mathsf{D}$. Then, for any finite set $I$, there is an induced functor

$$(\mathcal{F})_{i\in I} \colon \mathsf{C}^I \to \mathsf{D}^I.$$

We denote the unit object in both $\mathsf{C}$ and $\mathsf{D}$ by $K$ and write $\mathcal{M}$ for the tensor product functors in both categories. Define

$$\mathcal{F}^I := \mathcal{M} \circ (\mathcal{F})_{i\in I} \qquad \text{and} \qquad \mathcal{F}_I := \mathcal{F} \circ \mathcal{M}; \tag{4.11}$$

these are functors from $\mathsf{C}^I$ to $\mathsf{D}$.

**Remark 4.21.** The functors $\mathcal{F}^0$, $\mathcal{F}_0$, $\mathcal{F}^2$ and $\mathcal{F}_2$ defined in (3.1) and (3.2) played an important role in Chapter 3. The functors $\mathcal{F}^I$ and $\mathcal{F}_I$ provide a generalization: letting $I$ be the empty set or the set $[2]$ recovers these constructions. Similarly, the tensor product functor $\mathcal{M}$ generalizes the functor by the same name considered in Section 3.8.1.

**4.3.2. The endomorphism operad associated to a functor.** Let $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ be a functor between two symmetric monoidal categories and let $\mathcal{F}^I$ and $\mathcal{F}_I$ be as in (4.11). We further assume that $\mathsf{D}$ is linear over $\Bbbk$. Let $\mathrm{Nat}(\mathcal{F}, \mathcal{G})$ denote the set of natural transformations from $\mathcal{F}$ to $\mathcal{G}$.

Define the *endomorphism species* $\mathbf{End}_{\mathcal{F}}$ by

$$\mathbf{End}_{\mathcal{F}}[I] := \mathrm{Nat}(\mathcal{F}^I, \mathcal{F}_I).$$

This is a vector space because $\mathsf{D}$ is linear. It is clear that a bijection $I \to J$ induces a linear isomorphism

$$\mathrm{Nat}(\mathcal{F}^I, \mathcal{F}_I) \to \mathrm{Nat}(\mathcal{F}^J, \mathcal{F}_J),$$

so $\mathbf{End}_{\mathcal{F}}$ is a species.

More explicitly, an element of $\mathbf{End}_{\mathcal{F}}[I]$ consists of natural morphisms

$$\bullet_{i\in I} \mathcal{F}(A_i) \to \mathcal{F}\Big(\bullet_{i\in I} A_i\Big).$$

Note that

$$\mathbf{End}_{\mathcal{F}}[\emptyset] = \mathrm{Nat}(\mathcal{F}^\emptyset, \mathcal{F}_\emptyset) = \mathrm{Hom}\big(K, \mathcal{F}(K)\big),$$

where the right-hand side is the space of morphisms from the unit object of $\mathsf{D}$ to the image under $\mathcal{F}$ of the unit object of $\mathsf{C}$.

The endomorphism species carries an operad structure as follows. Fix a map $f \colon I \to X$. Given morphisms

$$\bullet_{x\in X} \mathcal{F}(B_x) \to \mathcal{F}\Big(\bullet_{x\in X} B_x\Big) \quad \text{and} \quad \bullet_{i\in f^{-1}(x)} \mathcal{F}(C_i) \to \mathcal{F}\Big(\bullet_{i\in f^{-1}(x)} C_i\Big) \quad \text{for each } x \in X,$$

we define

$$\begin{tikzcd}
\bullet_{i\in I}\mathcal{F}(A_i) \arrow[rr, dashed] \arrow[d, "\cong"'] & & \mathcal{F}\Big(\bullet_{i\in I} A_i\Big) \arrow[d, "\cong"'] \\
\bullet_{x\in X}\Big(\bullet_{i\in f^{-1}(x)}\mathcal{F}(A_i)\Big) \arrow[r] & \bullet_{x\in X}\mathcal{F}\Big(\bullet_{i\in f^{-1}(x)} A_i\Big) \arrow[r] & \mathcal{F}\Big(\bullet_{x\in X}\Big(\bullet_{i\in f^{-1}(x)} A_i\Big)\Big).
\end{tikzcd}$$

This defines the map $\gamma_f$ in (B.13). For a singleton $\{x\}$, the unit map $\eta_x$ in (B.14) sends $1 \in \Bbbk$ to the identity morphism

$$\bullet_{x\in\{x\}}\mathcal{F}(A_x) \cong \mathcal{F}(A_x) \to \mathcal{F}(A_x) \cong \mathcal{F}\Big(\bullet_{x\in\{x\}} A_x\Big).$$

It is straightforward to check that the operad axioms hold.

### 4.3.3. Operad-lax monoidal functors.

**Definition 4.22.** Let $\mathbf{p}$ be an operad. We say that a functor $\mathcal{F}$ is $\mathbf{p}$-*lax monoidal* if there is given a morphism of operads

$$R\colon \mathbf{p} \to \mathbf{End}_{\mathcal{F}},$$

where $\mathbf{End}_{\mathcal{F}}$ is the endomorphism operad.

We say that $R$ is a *representation* of $\mathbf{p}$ on the functor $\mathcal{F}$. The image of $a \in \mathbf{p}[I]$ under this morphism is denoted $Ra\colon \mathcal{F}^I \Rightarrow \mathcal{F}_I$.

We now make the notion of a $\mathbf{p}$-lax functor more explicit. A $\mathbf{p}$-lax functor is a functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ with the following structure. For each finite set $I$, there is a morphism

$$Ra\colon \bullet_{i\in I}\mathcal{F}(A_i) \to \mathcal{F}\Big(\bullet_{i\in I} A_i\Big) \qquad \text{for each } a \in \mathbf{p}[I] \tag{4.12}$$

which is natural in the $A_i$'s and linear in the element $a$, and subject to the conditions below.

**Naturality.** For any bijection $\sigma\colon I \to J$, the following diagram commutes.

$$\begin{tikzcd}
\bullet_{j\in J}\mathcal{F}(A_j) \arrow[r, "R(\sigma a)"] \arrow[d, "\cong"'] & \mathcal{F}\Big(\bullet_{j\in J} A_j\Big) \\
\bullet_{i\in I}\mathcal{F}(A_{\sigma(i)}) \arrow[r, "Ra"'] & \mathcal{F}\Big(\bullet_{i\in I} A_{\sigma(i)}\Big) \arrow[u, "\cong"']
\end{tikzcd} \tag{4.13}$$

Equivalently, we have $\sigma(Ra) = R(\sigma a)$.

**Substitution compatibility.** Let $a$, $a_x$, $c$ and $i$ be as in (4.7). For each $f\colon I \to X$ the following diagram commutes.

$$
\begin{tikzcd}
\bullet_{i\in I} \mathcal{F}(A_i) \arrow[rr, "Rc"] \arrow[d, "\cong"'] & & \mathcal{F}\Big(\bullet_{i\in I} A_i\Big) \arrow[d, "\cong"] \\
\bullet_{x\in X}\Big(\bullet_{i\in f^{-1}(x)} \mathcal{F}(A_i)\Big) \arrow[r, "\bullet_{x\in X} Ra_x"'] & \bullet_{x\in X} \mathcal{F}\Big(\bullet_{i\in f^{-1}(x)} A_i\Big) \arrow[r, "Ra"'] & \mathcal{F}\Big(\bullet_{x\in X}\Big(\bullet_{i\in f^{-1}(x)} A_i\Big)\Big)
\end{tikzcd}
\tag{4.14}
$$

In addition, $Ri$ is the identity morphism.

We briefly explain how these conditions are equivalent to Definition 4.22. Naturality says that $\mathbf{p} \to \mathbf{End}_{\mathcal{F}}$ is a morphism of species and substitution compatibility says further that it is a morphism of operads. More precisely, diagram (4.14) says that the first diagram in (B.4) commutes, while $Ri$ being the identity morphism says that the second diagram in (B.4) commutes.

**Definition 4.23.** A morphism between $\mathbf{p}$-lax functors $\mathcal{F}$ and $\mathcal{G}$ is a natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ such that for each $a \in \mathbf{p}[I]$ the following diagram commutes.

$$
\begin{tikzcd}
\bullet_{i\in I} \mathcal{F}(A_i) \arrow[r, "Ra"] \arrow[d, "\bullet_{i\in I} \theta_{A_i}"'] & \mathcal{F}\Big(\bullet_{i\in I} A_i\Big) \arrow[d, "\theta_{(\bullet_{i\in I} A_i)}"] \\
\bullet_{i\in I} \mathcal{G}(A_i) \arrow[r, "Ra"'] & \mathcal{G}\Big(\bullet_{i\in I} A_i\Big)
\end{tikzcd}
\tag{4.15}
$$

The following is straightforward.

**Proposition 4.24.** *The composite of morphisms of* $\mathbf{p}$*-lax monoidal functors is again a morphism of* $\mathbf{p}$*-lax monoidal functors.*

Composition is clearly associative; thus for two fixed symmetric monoidal categories and an operad $\mathbf{p}$, we have the category of $\mathbf{p}$-lax functors between them. Further, a morphism $\mathbf{p} \to \mathbf{q}$ of operads induces a restriction functor from the category of $\mathbf{q}$-lax functors to the category of $\mathbf{p}$-lax functors.

The types of monoidal functors discussed in Section 4.1.2 are special cases of the general notion of operad-lax functor, as shown in Table 4.2. (Compare with Table 4.1.)

We briefly mention the dual notion to operad-lax functors.

Table 4.2. Operad-lax functors and familiar types of functors.

| Operad-lax functor | Familiar lax functor | Definition |
|---|---|---|
| **As**-lax functor | lax functor | 3.1 |
| **Com**-lax functor | braided lax functor | 3.11 |
| **Lie**-lax functor | Lie-lax functor | 4.7 |
| **Dend**-lax functor | dendriform-lax functor | 4.8 |
| **Zinb**-lax functor | Zinbiel-lax functor | 4.9 |

**Definition 4.25.** A $\mathbf{p}$-*colax monoidal functor* $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is a $\mathbf{p}$-lax monoidal functor $\mathcal{F}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$.

**Remark 4.26.** In [312, Section 3] and [313, Definition 2.3], Richter considers a special case of Definition 4.22: she defines $E_\infty$-lax monoidal functors, where $E_\infty$ stands for a homotopy version of the commutative operad **Com** (Example B.15). It is clear that Richter's definition contains all the ingredients of the general notion.

Her work provides interesting examples of $E_\infty$-lax monoidal functors, such as the cubical construction of Eilenberg and Mac Lane [312, Section 5] and the inverse functor in the Dold-Kan correspondence [313, Section 5]. Richter also explains that the chain complex functors of Section 5.4 are $E_\infty$-colax monoidal; see [312, Section 7] and [313, Section 5]. We say a bit more about this in Section 5.5.5.

**4.3.4. Operad-monoids as operad-lax monoidal functors.** We have seen that (co)monoids can be viewed as (co)lax monoidal functors in Section 3.4.1. We now extend this result to operad-(co)monoids.

Let $\mathsf{I}$ be the one-arrow category. A functor $\mathcal{F}\colon \mathsf{I} \to \mathsf{M}$ is determined by the choice of an object, say $V$, in $\mathsf{M}$. In this situation, the endomorphism operad $\mathbf{End}_{\mathcal{F}}$ associated to the functor $\mathcal{F}$ specializes to the endomorphism operad $\mathbf{End}_V$ associated to the object $V$ (Section 4.2.1).

Suppose $\mathbf{p}$ is an operad. It follows from Definitions 4.16 and 4.22 that $\mathcal{F}$ is $\mathbf{p}$-(co)lax monoidal if and only if $V$ is a $\mathbf{p}$-(co)monoid. This leads to the following result.

**Proposition 4.27.** *The category of* $\mathbf{p}$-*(co)monoids in* $\mathsf{M}$ *is equivalent to the category of* $\mathbf{p}$-*(co)lax monoidal functors from* $\mathsf{I}$ *to* $\mathsf{M}$.

Specializing this result to $\mathbf{p} = \mathbf{As}$ and $\mathbf{p} = \mathbf{Com}$ recovers Propositions 3.25 and 3.27.

## 4.4. Composites of monoidal functors and transformation of monoids

It is natural to wonder about the result of composing operad-lax monoidal functors. An elegant answer to this question is given in Section 4.4.1, with specializations in Sections 4.4.2 and 4.4.3. This is then used in Section 4.4.4 to describe the effect of applying a monoidal functor of a given type to a monoid of another type. This addresses the questions raised at the end of Section 4.1.3 in the proper generality. The statement of the main result involves the Hadamard product on operads. The latter is discussed in detail in Section B.6; we recall below the ideas required for the present discussion.

Suppose that $\mathbf{p}$ and $\mathbf{q}$ are species. Their Hadamard product $\mathbf{p}\times\mathbf{q}$ is defined by $(\mathbf{p}\times\mathbf{q})[I] := \mathbf{p}[I]\otimes\mathbf{q}[I]$. The unit for this product is the exponential species $\mathbf{E}$ given by $\mathbf{E}[I] := \Bbbk$. This product extends to operads. That is, if $\mathbf{p}$ and $\mathbf{q}$ are operads, then there is a canonical structure of an operad on $\mathbf{p} \times \mathbf{q}$. This operation turns the category of operads into a monoidal category; the unit object is the operad **Com** (whose underlying species is $\mathbf{E}$). Comonoids in this category are of particular significance and are called Hopf operads.

**4.4.1. Composites of monoidal functors.** Throughout this discussion, $\mathsf{C}$, $\mathsf{D}$, and $\mathsf{E}$ are symmetric monoidal categories, and $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{E}$ are functors. We also assume that the categories $\mathsf{D}$ and $\mathsf{E}$, and the functor $\mathcal{G}$ are linear over $\Bbbk$.

Let $\mathbf{p}$ and $\mathbf{q}$ be species, and $R\colon \mathbf{p} \to \mathbf{End}_{\mathcal{F}}$ and $R\colon \mathbf{q} \to \mathbf{End}_{\mathcal{G}}$ be maps of species. We proceed to construct a map from the Hadamard product $\mathbf{q} \times \mathbf{p}$ to the endomorphism species of the composite functor

$$\mathcal{G}\mathcal{F}\colon \mathsf{C} \to \mathsf{E}.$$

Define a map of species

$$R\colon \mathbf{q} \times \mathbf{p} \to \mathbf{End}_{\mathcal{G}\mathcal{F}} \tag{4.16}$$

as follows. For $a \in \mathbf{p}[I]$ and $b \in \mathbf{q}[I]$, let $R(b \otimes a)$ be defined by:

$$\begin{tikzcd} \bullet_{i\in I} \mathcal{G}\mathcal{F}(A_i) \arrow[rr, dashed, "R(b\otimes a)"] \arrow[dr, "Rb"'] & & \mathcal{G}\mathcal{F}\Big(\bullet_{i\in I} A_i\Big) \\ & \mathcal{G}\Big(\bullet_{i\in I} \mathcal{F}(A_i)\Big) \arrow[ur, "\mathcal{G}(Ra)"'] & \end{tikzcd} \qquad \begin{tikzcd} I \arrow[rr, dashed, "R(b\otimes a)"] \arrow[dr, "Rb"'] & & \mathcal{G}\mathcal{F}(I) \\ & \mathcal{G}(I). \arrow[ur, "\mathcal{G}(Ra)"'] & \end{tikzcd}$$

Note that for $I$ fixed, $R(b \otimes a)$ is natural in the $A_i$'s and linear in $b \otimes a$ as required. The linearity of $\mathcal{G}$ is important for this conclusion. The diagram on the right explicitly shows the case when $I$ is empty.

One may also work in a setting where $\mathcal{G}$ is not necessarily linear but where the operad $\mathbf{p}$ is linearized (Section B.1.3). In this case, the above definition is only applied to those elements $a$ which lie in the canonical basis of $\mathbf{p}[I]$.

**Theorem 4.28.** *Let $\mathbf{p}$ and $\mathbf{q}$ be operads. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be $\mathbf{p}$-(co)lax and $\mathcal{G}\colon \mathsf{D} \to \mathsf{E}$ be $\mathbf{q}$-(co)lax monoidal functors. Assume that either $\mathbf{p}$ is a linearized operad, or $\mathcal{G}$ is linear. Then $\mathcal{G}\mathcal{F}\colon \mathsf{C} \to \mathsf{E}$ is $(\mathbf{q} \times \mathbf{p})$-(co)lax monoidal.*

Whether one writes $(\mathbf{q} \times \mathbf{p})$-(co)lax or $(\mathbf{p} \times \mathbf{q})$-(co)lax for $\mathcal{G}\mathcal{F}$ is irrelevant, in view of the symmetry $\mathbf{p} \times \mathbf{q} \cong \mathbf{q} \times \mathbf{p}$.

Proof. We need to check that if $R\colon \mathbf{p} \to \mathbf{End}_{\mathcal{F}}$ and $R\colon \mathbf{q} \to \mathbf{End}_{\mathcal{G}}$ are morphisms of operads, then so is

$$R\colon \mathbf{q} \times \mathbf{p} \to \mathbf{End}_{\mathcal{G}\mathcal{F}}$$

as defined in (4.16). For this, it is convenient to work with the explicit definition of operad-lax functors given after Definition 4.22. Thus, we need to check for naturality and substitution compatibility for $R(b \otimes a)$.

The naturality of $R(b \otimes a)$, or equivalently, the commutativity of diagram (4.13) for $R(b \otimes a)$ follows directly from the same property for $Ra$ and $Rb$. The check for substitution compatibility is more interesting. We follow the notation of Section B.6.2 for the Hadamard product of operads.

By definition $R(j \otimes i)$ is the composite map

$$\mathcal{G}\mathcal{F}(A) \xrightarrow{Rj} \mathcal{G}\mathcal{F}(A) \xrightarrow{\mathcal{G}(Ri)} \mathcal{G}\mathcal{F}(A).$$

Since $Rj$ and $Ri$ are both the identity morphisms, it follows that $R(j \otimes i)$ is also the identity morphism.

We fix $f\colon I \to X$ and for simplicity we identify the tensors

$$\bullet_{i\in I} \quad \text{and} \quad \bullet_{x\in X} \bullet_{i\in f^{-1}(x)}.$$

The diagram (4.14) for $R(b \otimes a)$ commutes by the commutativity of the following diagram.

$$
\begin{tikzcd}
\bullet_{x\in X}\bullet_{i\in f^{-1}(x)} \mathcal{G}\mathcal{F}(A_i) \arrow[rr, "Rd"] \arrow[d, "\bullet_{x\in X} Rb_x"] & & \mathcal{G}\Big(\bullet_{x\in X}\bullet_{i\in f^{-1}(x)} \mathcal{F}(A_i)\Big) \arrow[ddl, "\mathcal{G}(\bullet_{x\in X} Ra_x)"'] \arrow[dd, "\mathcal{G}(Rc)"] \\
\bullet_{x\in X} \mathcal{G}\Big(\bullet_{i\in f^{-1}(x)} \mathcal{F}(A_i)\Big) \arrow[urr, "Rb"'] \arrow[d, "\bullet_{x\in X} \mathcal{G}(Ra_x)"] & & \\
\bullet_{x\in X} \mathcal{G}\mathcal{F}(\bullet_{i\in f^{-1}(x)} A_i) \arrow[r, "Rb"'] & \mathcal{G}\Big(\bullet_{x\in X} \mathcal{F}\Big(\bullet_{i\in f^{-1}(x)} A_i\Big)\Big) \arrow[r, "\mathcal{G}(Ra)"'] & \mathcal{G}\mathcal{F}\Big(\bullet_{x\in X}\bullet_{i\in f^{-1}(x)} A_i\Big)
\end{tikzcd}
$$

The triangle on the left commutes since diagram (4.14) commutes for $Rb$, which we remind is a consequence of $R\colon \mathbf{q} \to \mathbf{End}_{\mathcal{G}}$ being a morphism of operads. The commutativity of the triangle on the right is $\mathcal{G}$ applied to the fact that diagram (4.14) commutes for $Ra$, which follows from $R\colon \mathbf{p} \to \mathbf{End}_{\mathcal{F}}$ being a morphism of operads. The quadrilateral commutes by the naturality of $Rb$. $\square$

**Remark 4.29.** Note that in the proof above we only used the naturality of $Rb$ and not that of $Ra$. This proof may be compared with that of Theorem 3.21, where only the naturality of $\gamma$ is used.

The following result complements Theorem 4.28. The proof is omitted.

**Proposition 4.30.** *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be operads. Let* $\mathcal{F}, \mathcal{F}'\colon \mathsf{C} \to \mathsf{D}$ *be* $\mathbf{p}$-*(co)lax and* $\mathcal{G}, \mathcal{G}'\colon \mathsf{D} \to \mathsf{E}$ *be* $\mathbf{q}$-*(co)lax monoidal functors. Assume that either* $\mathbf{p}$ *is a linearized operad, or both* $\mathcal{G}$ *and* $\mathcal{G}'$ *are linear. Then*

(i) *If* $\mathcal{F} \Rightarrow \mathcal{F}'$ *is a morphism of* $\mathbf{p}$-*(co)lax monoidal functors, then the induced natural transformation* $\mathcal{G}\mathcal{F} \Rightarrow \mathcal{G}\mathcal{F}'$ *is a morphism of* $(\mathbf{q} \times \mathbf{p})$-*(co)lax monoidal functors.*

(ii) *If* $\mathcal{G} \Rightarrow \mathcal{G}'$ *is a morphism of* $\mathbf{q}$-*(co)lax monoidal functors, then the induced natural transformation* $\mathcal{G}\mathcal{F} \Rightarrow \mathcal{G}'\mathcal{F}$ *is a morphism of* $(\mathbf{q} \times \mathbf{p})$-*(co)lax monoidal functors.*

**4.4.2. Specialization: either $\mathcal{F}$ or $\mathcal{G}$ is braided lax.** The operad **Com** is the unit object for the Hadamard product. For this reason, the situation in Theorem 4.28 is particularly nice when one of the functors is braided lax. We briefly discuss this case.

**Corollary 4.31.** *Let* $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ *and* $\mathcal{G}\colon \mathsf{D} \to \mathsf{E}$ *be functors.*

(i) *Let* $\mathbf{q}$ *be an operad. Suppose* $\mathcal{F}$ *is braided (co)lax monoidal and* $\mathcal{G}$ *is* $\mathbf{q}$-*(co)lax monoidal. Then* $\mathcal{G}\mathcal{F}$ *is* $\mathbf{q}$-*(co)lax monoidal.*

(ii) *Let* $\mathbf{p}$ *be an operad. Suppose* $\mathcal{F}$ *is* $\mathbf{p}$-*(co)lax monoidal and* $\mathcal{G}$ *is braided (co)lax monoidal. Assume that either* $\mathbf{p}$ *is linearized or* $\mathcal{F}$ *is linear. Then* $\mathcal{G}\mathcal{F}$ *is* $\mathbf{p}$-*(co)lax monoidal.*

*In addition, post (or pre) composing by a* $\mathbf{p}$-*(co)lax monoidal functor turns a morphism of braided (co)lax monoidal functors into a morphism of* $\mathbf{p}$-*(co)lax monoidal* *functors. Similarly, pre* (*or post*) *composing by a braided* (*co*)*lax monoidal functor preserves morphisms of* **p**-(*co*)*lax monoidal functors.*

Proof. We explain (i); the proof of (ii) is similar. A braided (co)lax monoidal functor is the same as a **Com**-(co)lax monoidal functor (Table 4.2). In addition, **Com** is a linearized operad. Hence, the composite $\mathcal{GF}$ is $(\mathbf{Com}\times\mathbf{q})$-(co)lax monoidal, by Theorem 4.28. Since $\mathbf{Com}\times\mathbf{p}\cong\mathbf{p}$ as operads, the result follows. $\square$

More explicitly, let $(\mathcal{F},\varphi,\varphi_0)$ be a braided lax functor and let $\varphi_I\colon \mathcal{F}^I\to\mathcal{F}_I$ be the transformation

$$\varphi_{(\mathop{\bullet}\limits_{i\in I}A_i)}\colon \mathop{\bullet}\limits_{i\in I}\mathcal{F}(A_i)\to\mathcal{F}\Big(\mathop{\bullet}\limits_{i\in I}A_i\Big)$$

obtained by iterating $\varphi$. If $I$ is empty, then $\varphi_I$ is the same as $\varphi_0$ and if $I=[2]$, then $\varphi_I$ is the same as $\varphi$. Now let $\mathcal{G}$ be a **p**-lax functor. Then the composite $\mathcal{GF}$ is a **p**-lax functor as follows. For $a\in\mathbf{p}[I]$, we define:

$$\begin{tikzcd}
\mathop{\bullet}\limits_{i\in I}\mathcal{GF}(A_i) \arrow[rr, dashed, "Ra"] \arrow[dr, "Ra"'] & & \mathcal{GF}\Big(\mathop{\bullet}\limits_{i\in I}A_i\Big) \\
& \mathcal{G}\Big(\mathop{\bullet}\limits_{i\in I}\mathcal{F}(A_i)\Big). \arrow[ur, "\mathcal{G}(\varphi_{(\mathop{\bullet}\limits_{i\in I}A_i)})"'] &
\end{tikzcd}$$

The **p**-lax structure for the functor $\mathcal{FG}$ may be similarly described.

**4.4.3. Specialization: $\mathbf{p}=\mathbf{q}$.** Let **p** be a linearized operad. This means that **p** is obtained by linearizing a set operad; thus each $\mathbf{p}[I]$ comes equipped with a canonical basis. This yields morphisms of operads:

$$\mathbf{p}\to\mathbf{p}\times\mathbf{p} \quad\text{and}\quad \mathbf{p}\to\mathbf{Com}. \tag{4.17}$$

It follows that any $(\mathbf{p}\times\mathbf{p})$-(co)lax or braided (co)lax functor is canonically a **p**-(co)lax functor. In particular, the identity functor is **p**-lax.

Theorem 4.28 now implies the following:

**Theorem 4.32.** *Let* **p** *be a linearized operad. The composite of two* **p**-*(co)lax functors is again* **p**-*(co)lax. In addition, pre or post composing by a* **p**-*(co)lax functor preserves morphisms between* **p**-*(co)lax functors.*

Explicitly, for an element $a$ in the canonical basis of $\mathbf{p}[I]$, the structure map $Ra$ for $\mathcal{GF}$ is given by the composite:

$$\mathop{\bullet}\limits_{i\in I}\mathcal{GF}(A_i)\xrightarrow{\ Ra\ }\mathcal{G}\left(\mathop{\bullet}\limits_{i\in I}\mathcal{F}(A_i)\right)\xrightarrow{\mathcal{G}(Ra)}\mathcal{GF}\left(\mathop{\bullet}\limits_{i\in I}A_i\right).$$

We extend by linearity to all elements of $\mathbf{p}[I]$.

Theorem 4.32 leads to the following result. For a linearized operad **p**, there is a 2-category whose 0-cells are symmetric monoidal categories, 1-cells are **p**-(co)lax functors, and 2-cells are morphisms between **p**-(co)lax functors.

Recall that a Hopf operad is a comonoid in the category of operads with respect to the Hadamard product. The morphisms (4.17) turn any linearized operad into a Hopf operad. The result of Theorem 4.32 continues to hold for arbitrary Hopf operads, provided the functors are linear:

**Theorem 4.33.** *Let* $\mathbf{p}$ *be a Hopf operad. The composite of two linear* $\mathbf{p}$*-(co)lax functors is again linear* $\mathbf{p}$*-(co)lax. In addition, pre or post composing by a linear* $\mathbf{p}$*-(co)lax functor preserves morphisms between linear* $\mathbf{p}$*-(co)lax functors.*

Note that the structure map $Ra$ for $\mathcal{GF}$ written for the linearized case above is *not* linear in the element $a$. This was the reason for writing it only for elements in the canonical basis. In the Hopf operad case, the structure map $Ra$ for $\mathcal{GF}$, for any element $a$, is given by the composite:

$$\underset{i\in I}{\bullet}\mathcal{GF}(A_i) \xrightarrow{Ra_{(2)}} \mathcal{G}\left(\underset{i\in I}{\bullet}\mathcal{F}(A_i)\right) \xrightarrow{\mathcal{G}(Ra_{(1)})} \mathcal{GF}\left(\underset{i\in I}{\bullet}A_i\right)$$

where Sweedler's notation has been employed to express the coproduct of the element $a$: $\Delta(a) = a_{(1)} \otimes a_{(2)}$. One may equally well use $a_{(1)}$ on the first arrow and $a_{(2)}$ on the second; this leads to a second $\mathbf{p}$-monoidal structure on $\mathcal{GF}$. If $\mathbf{p}$ is cocommutative, the two structures coincide.

Theorem 4.33 implies the following. For a Hopf operad $\mathbf{p}$, there is a 2-category whose 0-cells are linear symmetric monoidal categories, 1-cells are linear $\mathbf{p}$-(co)lax monoidal functors and 2-cells are morphisms of $\mathbf{p}$-(co)lax monoidal functors.

**4.4.4. Transformation of monoids under monoidal functors: the general case.** Proposition 4.27 shows that operad-(co)monoids can be viewed as operad-(co)lax monoidal functors. This relation enables us to derive results about operad-(co)monoids from results about operad-(co)lax monoidal functors.

Theorems 4.28, 4.32 and 4.33 yield the following important results.

**Theorem 4.34.** *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be operads and* $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ *a* $\mathbf{q}$*-lax monoidal functor. Assume that either* $\mathbf{p}$ *is linearized or* $\mathcal{F}$ *is linear. Then* $\mathcal{F}$ *sends a* $\mathbf{p}$*-monoid in* $\mathsf{C}$ *to a* $(\mathbf{p}\times\mathbf{q})$*-monoid in* $\mathsf{D}$*.*

**Theorem 4.35.** *Let* $\mathbf{p}$ *be a linearized operad and* $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ *a* $\mathbf{p}$*-lax monoidal functor. Then* $\mathcal{F}$ *sends a* $\mathbf{p}$*-monoid in* $\mathsf{C}$ *to a* $\mathbf{p}$*-monoid in* $\mathsf{D}$*.*

*Let* $\mathbf{p}$ *be a Hopf operad and* $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ *a linear* $\mathbf{p}$*-lax monoidal functor. Then* $\mathcal{F}$ *sends a* $\mathbf{p}$*-monoid in* $\mathsf{C}$ *to a* $\mathbf{p}$*-monoid in* $\mathsf{D}$*.*

The operad **Com** is linearized and also the unit object for the Hadamard product. Specializing either $\mathbf{p}$ or $\mathbf{q}$ to **Com** yields the following corollaries.

**Corollary 4.36.** *A* $\mathbf{p}$*-lax monoidal functor sends a commutative monoid to a* $\mathbf{p}$*-monoid.*

**Corollary 4.37.** *A braided lax monoidal functor preserves* $\mathbf{p}$*-monoids for any linearized operad* $\mathbf{p}$*. A linear braided lax monoidal functor preserves* $\mathbf{p}$*-monoids for any operad* $\mathbf{p}$*.*

The special cases $\mathbf{p} = \mathbf{Com}$ and **Lie** recover the results in Propositions 3.37 and 4.13. The other results about transformations of monoids in Section 4.1.3 can also be derived from Theorem 4.34. For instance, to obtain the result on preservation of dendriform monoids in Proposition 4.14, one argues as follows. If $\mathcal{F}$ is **As**-lax monoidal and $A$ is a **Dend**-monoid, then $\mathcal{F}(A)$ is a $(\mathbf{Dend}\times\mathbf{As})$-monoid. Now, there is a morphism of operads $\mathbf{Dend} \to \mathbf{Dend}\times\mathbf{As}$. (This morphism is responsible for the well-known fact that the tensor product of a dendriform monoid with an associative monoid is a dendriform monoid.) Restricting along this morphism one deduces a **Dend**-monoid structure on $\mathcal{F}(A)$.

CHAPTER 5

# Bilax Monoidal Functors in Homological Algebra

The notion of bilax monoidal functor between braided monoidal categories is of central importance to this work. This chapter discusses what may be the most classical example of a bilax monoidal functor in mathematical nature. The familiar construction of a chain complex out of a simplicial module defines a functor between symmetric monoidal categories, and the classical maps of Eilenberg–Zilber and Alexander–Whitney turn it into a bilax monoidal functor. This holds both for the unnormalized and normalized versions of the construction. The latter is an example of a normal bilax monoidal functor in the sense of Definition 3.39 (and the reason for such terminology).

This chapter is meant to serve as motivation for the study of bilax monoidal functors. The results given here are not needed in the rest of the monograph.

Detailed discussions on simplicial modules and chain complexes can be found in the books by Goerss and Jardine [147], Loday [237, Appendix B], Mac Lane [249], May [262], or Weibel [372, Chapter 8]. We provide the necessary background in order to keep our exposition self-contained.

**Organization.** We start by setting up the notation and reviewing basic notions about simplicial modules in Section 5.1. Motivations from topological spaces are given in Section 5.2. The Alexander–Whitney and Eilenberg–Zilber maps are discussed in Section 5.3, where we also give the key compatibility between these maps that gives rise to a bilax monoidal functor (Lemma 5.5). The chain complex functors appear in Section 5.4 along with the main result that they are bilax, which is given in Theorem 5.6. While not formulated in these exact terms in the literature, this result pertains to the folklore of simplicial algebra. It was brought to our attention by Clemens Berger.

It is important to remark that we work with ordinary morphisms of chain complexes, not chain homotopy classes. If we pass to the homotopy category of chain complexes, then the chain complex functors become bistrong. As explained in Section 3.6.1, in this situation the bilax axiom simplifies, and one does not need to confront it explicitly. In addition, this suffices for the applications to the construction of products in (co)homology. This may perhaps explain the lack of treatment in the literature of the general notion of bilax monoidal functors.

In Section 5.5 we state a number of well-known results which may be seen as consequences of the theorem, mainly regarding the existence of products in (co)homology. These include the cup product and the Pontrjagin product. Finally, in Section 5.6 we discuss the possibility of obtaining a one-parameter deformation of the chain complex functor. This can be done successfully provided that the boundary maps are set aside (Theorem 5.17). If boundaries are to be kept, then a partial

TABLE 5.1. Monoidal categories related to $\Bbbk$-modules.

| Notation | Description |
|---|---|
| $\mathsf{gMod}$ | Graded $\Bbbk$-modules |
| $\mathsf{dgMod_a}$ | Chain complexes of $\Bbbk$-modules (Differential graded modules) |
| $\overline{\mathsf{dgMod_a}}$ | Chain complexes of $\Bbbk$-modules up to homotopy |
| $\mathsf{dgMod^c}$ | Cochain complexes of $\Bbbk$-modules |
| $\overline{\mathsf{dgMod^c}}$ | Cochain complexes of $\Bbbk$-modules up to homotopy |
| $\mathsf{gMod_a}$ | Graded $\Bbbk$-modules with annihilation operators |
| $\mathsf{gMod^c}$ | Graded $\Bbbk$-modules with creation operators |
| $\mathsf{dgMod}_N$ | $N$-complexes of $\Bbbk$-modules |

result is still true (Proposition 5.22). We make here use of some constructions of Kapranov [187] and of Dubois-Violette [105].

**Commutative rings versus fields.** For the most part of this monograph, we work with vector spaces over a field $\Bbbk$. This chapter deals with notions related to homology for which it is desirable to allow more general scalars. The basic notions about graded vector spaces discussed in Chapter 2 carry over to graded modules *mutatis mutandis*. Accordingly, throughout this chapter, $\Bbbk$ denotes a commutative ring, $\mathsf{Mod}$ denotes the category of $\Bbbk$-modules, and so on. The notations are summarized in Table 5.1. A comparison with Table 2.3 shows that we essentially use all the terminology of Chapter 2 with $\mathsf{Mod}$ replacing $\mathsf{Vec}$.

**A historical note.** Simplicial sets were introduced by Eilenberg and Zilber, who called them complete semi-simplicial complexes [122, Section 8]. The Alexander–Whitney map has its origins in the work of several authors on the cup product in algebraic topology. The cup product appears explicitly in works of Alexander [19, Equation (9:1)], Čech [78, Section 7] and Whitney [375, Section 6]. The idea was introduced independently by Alexander and Kolmogorov at a 1935 conference in Moscow. The relevant papers are [17, 18, 205, 206]. The Eilenberg–Zilber map is implicit in a paper of Eilenberg and Zilber [123, Section 1]; it appears explicitly in work of Eilenberg and Mac Lane [119, Section 5]. The compatibility between the two maps is treated in a paper of Eilenberg and Moore [121, Section 17]. The Pontrjagin product first appeared in the work of Pontrjagin [298]. More historical information can be found in [99, 176, 376].

We thank Ryan Budney, Allen Hatcher, Joseph Neisendorfer and Paul Selick for help with these references.

## 5.1. The simplicial category and simplicial modules

In this section, we review basic notions about simplicial modules.

**5.1.1. Simplicial objects.** The *simplicial category* $\Delta$ has for objects the non-negative integers $0, 1, 2, \ldots$. The morphisms from $n$ to $m$ are the order-preserving functions

$$\mu\colon \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}. \tag{5.1}$$

TABLE 5.2. Simplicial objects in various categories.

| Category | Simplicial object |
|---|---|
| sets | simplicial set |
| $\Bbbk$-modules | simplicial $\Bbbk$-module |
| groups | simplicial group |
| $\Bbbk$-(co, bi)algebras | simplicial (co, bi)algebra |

Morphisms are composed as ordinary functions.

Let $\mathsf{C}$ be an arbitrary category. A *simplicial object* in $\mathsf{C}$ is a contravariant functor

$$\Delta \to \mathsf{C}.$$

If $X$ is a simplicial object, we let $X_n$ denote the value of the functor $X$ on the object $n$, and if $\mu$ is as above, we let

$$X_\mu \colon X_m \to X_n \tag{5.2}$$

denote the value of the functor $X$ on the morphism $\mu$.

A morphism of simplicial objects $X \to Y$ is a natural transformation of functors, that is, a family of maps $X_n \to Y_n$ in $\mathsf{C}$ commuting with the action of order-preserving functions. This defines the category of simplicial objects in $\mathsf{C}$.

Some familiar categories along with their simplicial objects are summarized in Table 5.2. For example, a *simplicial set* is a simplicial object in the category of sets; for $\Bbbk$ a commutative ring, a *simplicial* $\Bbbk$*-module* is a simplicial object in the category of $\Bbbk$-modules, and so on. For the most part, we deal with the category of simplicial $\Bbbk$-modules. We denote it by $\mathsf{sMod}$ (omitting $\Bbbk$ from the notation). The category of simplicial sets will be denoted by $\mathsf{sSet}$.

**Remark 5.1.** There is a related notion called *semi-simplicial object* in which $\Delta$ is replaced by the subcategory whose morphisms are *injective* order-preserving maps [372, Definition 8.1.9]. Semi-simplicial sets were introduced by Eilenberg and Zilber [122, Section 1]. These are called $\Delta$-complexes by Hatcher [160] and *triangulated spaces* by Gelfand and Manin [143, Section I.1]. Kozlov uses the short form *trisp* [207, Section 2.3].

### 5.1.2. Special morphisms in the simplicial category.

We set up the notation for some special morphisms of the category $\Delta$. For each $i = 0, \dots, n$, define maps

$$\delta_i \colon \{0, 1, \dots, n-1\} \to \{0, \dots, n\} \qquad \text{and} \qquad \sigma_i \colon \{0, 1, \dots, n+1\} \to \{0, \dots, n\}$$

as follows. The map $\delta_i$ is the unique injective order-preserving map whose image misses $i$. The map $\sigma_i$ is the unique surjective order-preserving map which identifies $i$ and $i+1$. Explicitly,

$$\delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \qquad \text{and} \qquad \sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

These morphisms of $\Delta$ are called the *face* and *degeneracy maps*, respectively. The dependence of these maps on $n$ is implicit. They may be visualized as shown in Figure 5.1.

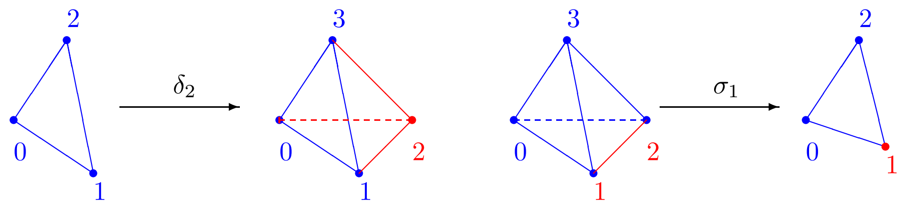



FIGURE 5.1. Faces and degeneracies.

When a simplicial object $X$ is given, it is customary to write, for each $i = 0, \dots, n$,

$$d_i := X_{\delta_i} \colon X_n \to X_{n-1} \quad (n \geq 1) \quad \text{and} \quad s_i := X_{\sigma_i} \colon X_n \to X_{n+1} \quad (n \geq 0). \tag{5.3}$$

Any morphism in $\Delta$ is a composite of face and degeneracy maps [249, Lemma VIII.5.1]. Moreover, the category $\Delta$ admits a presentation in which these maps are the generators, and the relations are certain simple commutation relations among faces and degeneracies. As a consequence, one has the following result.

**Lemma 5.2** ([249, Theorem VIII.5.2])**.** *To define a simplicial object $X$ in a category $\mathsf{C}$, it suffices to specify a sequence of objects $X_n$ of $\mathsf{C}$, $n \geq 0$, and maps*

$$d_i \colon X_n \to X_{n-1} \quad \textit{for } n \geq 1,\ 0 \leq i \leq n$$

*and*

$$s_i \colon X_n \to X_{n+1} \quad \textit{for } n \geq 0,\ 0 \leq i \leq n$$

*in $\mathsf{C}$, such that the following relations hold:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \textit{for } i < j, \\ s_i s_j &= s_{j+1} s_i \quad \textit{for } i \leq j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \textit{for } i < j, \\ \mathrm{id}_{X_n} & \textit{for } i = j \textit{ or } i = j+1, \\ s_j d_{i-1} & \textit{for } i > j+1. \end{cases} \end{aligned} \tag{5.4}$$

**5.1.3. The tensor product of simplicial modules.** The tensor product of two simplicial $\Bbbk$-modules $X$ and $Y$ is $X \times Y$ defined by

$$(X \times Y)_n := X_n \otimes Y_n \quad \text{and} \quad (X \times Y)_\mu := X_\mu \otimes Y_\mu$$

for every $n \geq 0$ and every morphism $\mu$ in $\Delta$. The unit object $E$ is defined by

$$E_n := \Bbbk \qquad \text{and} \qquad E_\mu := \mathrm{id}_\Bbbk$$

for every $n \geq 0$ and every morphism $\mu$ in $\Delta$. The symmetry $\beta$ is the trivial switch

$$X_n \otimes Y_n \xrightarrow{\cong} Y_n \otimes X_n, \qquad x \otimes y \mapsto y \otimes x.$$

This turns $\mathsf{sMod}$ into a symmetric monoidal category.

The same construction can be done for $\mathsf{sSet}$ by replacing the tensor product by the Cartesian product. In this case, the monoidal structure is the categorical product. Note that to every simplicial set, one can associate a simplicial module by linearization. This yields a functor

$$\mathsf{sSet} \to \mathsf{sMod} \tag{5.5}$$

which is evidently bistrong.

## 5.2. Topological spaces and simplicial sets

Historically, homology was first defined for topological spaces and then abstracted to simplicial sets. In this section, we discuss the monoidal properties of two classical functors which relate topological spaces and simplicial sets. One is the singular complex functor of Eilenberg [116] (a precursor is in the paper by Lefschetz [224]) and the other is the geometric realization functor of Milnor [273].

This section is for motivation purposes only; subsequent sections do not logically depend on this one.

**5.2.1. The singular complex functor.** Let $(\mathsf{Top}, \times, \beta)$ be the symmetric monoidal category of topological spaces with tensor product being the Cartesian product. Consider the singular complex functor

$$\mathcal{S}\colon (\mathsf{Top}, \times, \beta) \to (\mathsf{sSet}, \times, \beta) \tag{5.6}$$

which sends a topological space $T$ to the simplicial set $X$ where $X_n$ consists of the singular $n$-simplices in $X$, that is, all continous maps from the standard $n$-simplex to $X$. Note that a morphism $\mu$ as in (5.1) can be viewed as a continous map from the standard $n$-simplex to the standard $m$-simplex. The map $X_\mu$ of (5.2) is then defined as the pull-back along $\mu$ viewed in this manner.

Since the monoidal structure on both categories is the categorical product, $\mathcal{S}$ carries a canonical braided colax structure (Example 3.19). In particular, for topological spaces $T$ and $U$, there is a natural transformation

$$\mathcal{S}(T \times U) \to \mathcal{S}(T) \times \mathcal{S}(U)$$

constructed from the universal property of products. Since a continous map into $T \times U$ is the same as a pair of continous maps, one into $T$ and another into $U$, it follows that $\mathcal{S}$ is in fact bistrong.

**5.2.2. The geometric realization functor.** The singular complex functor admits a left adjoint. This is the functor which sends a simplicial set $X$ to its geometric realization $|X|$. We denote this functor by

$$|\cdot|\colon (\mathsf{sSet}, \times, \beta) \to (\mathsf{Top}, \times, \beta). \tag{5.7}$$

For the same reason as the singular complex functor, $|\cdot|$ carries a canonical braided colax structure. In particular, for simplicial sets $X$ and $Y$, there is a natural transformation

$$|X \times Y| \to |X| \times |Y|.$$

It follows from general considerations (Example 3.90) that the adjunction $(|\cdot|, \mathcal{S})$ is braided colax-colax.

**5.2.3. Homotopy categories.** Recall that there is a notion of homotopy for continous maps. This gives rise to the homotopy category of topological spaces which we denote by $\overline{\mathsf{Top}}$: Objects are topological spaces and morphisms are continous maps up to homotopy. Note that there is a bistrong functor $\mathsf{Top} \to \overline{\mathsf{Top}}$ which sends a topological space to itself and a continous map to its homotopy class.

There is an analogous notion of homotopy for simplicial maps [147, Section I.6], [262, Definition 5.1] or [372, Section 8.3.11]. This gives rise to the homotopy category of simplicial modules which we denote by $\overline{\mathsf{sMod}}$. As for topological spaces, the canonical functor $\mathsf{sMod} \to \overline{\mathsf{sMod}}$ is bistrong.

*Warning.* The situation for simplicial sets is more delicate. The homotopy relation for simplicial maps between simplicial sets is *not* an equivalence relation. To overcome this problem one needs to restrict to the subcategory of *fibrant simplicial sets*, also called *Kan complexes*. These are simplicial sets which satisfy the *Kan condition* [147, Section I.3], [262, Definition 1.3] or [372, Section 8.2.7]. The original source is [185]. The singular complex of a topological space is indeed a fibrant simplicial set. Further, these simplicial sets are closed under taking Cartesian product. We denote the subcategory of fibrant simplicial sets by $\mathsf{fsSet}$ and its homotopy category by $\overline{\mathsf{fsSet}}$.

The following commutative diagrams relate the various homotopy categories.

$$\begin{tikzcd} \mathsf{fsSet} \arrow[r] \arrow[d] & \mathsf{sMod} \arrow[d] \\ \overline{\mathsf{fsSet}} \arrow[r] & \overline{\mathsf{sMod}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Top} \arrow[r, "\mathcal{S}"] \arrow[d] & \mathsf{fsSet} \arrow[d] \\ \overline{\mathsf{Top}} \arrow[r] & \overline{\mathsf{fsSet}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{fsSet} \arrow[r, "|\cdot|"] \arrow[d] & \mathsf{Top} \arrow[d] \\ \overline{\mathsf{fsSet}} \arrow[r] & \overline{\mathsf{Top}} \end{tikzcd} \tag{5.8}$$

The functors in the top horizontal rows are as in (5.5), (5.6) and (5.7). The first two diagrams are diagrams of bistrong functors, while the third is a diagram of braided colax functors. Composing the first two yields the commutative diagram

$$\begin{tikzcd} \mathsf{Top} \arrow[r] \arrow[d] & \mathsf{sMod} \arrow[d] \\ \overline{\mathsf{Top}} \arrow[r] & \overline{\mathsf{sMod}} \end{tikzcd} \tag{5.9}$$

of bistrong functors.

## 5.3. Alexander–Whitney and Eilenberg–Zilber

In this section, we review the classical maps of Alexander–Whitney and Eilenberg–Zilber. They provide the structure maps for the colax and lax structures of the chain complex functors. We also establish a compatibility lemma which allows us to deduce (in Section 5.4) that the lax and colax structures are compatible, in the sense that they make the chain complex functors bilax monoidal.

**5.3.1. Alexander–Whitney.** Fix $n \geq 0$. For each pair of indices $h, k$ with $0 \leq h \leq k \leq n$, define

$$\delta_{h,k} \colon \{0, 1, \ldots, k-h\} \to \{0, 1, \ldots, n\}$$

by

$$\delta_{h,k}(i) := i + h. \tag{5.10}$$

The dependence of $\delta_{h,k}$ on $n$ is not reflected in the notation, but $n$ will always be clear from the context.

The map $\delta_{h,k}$ is order-preserving, injective, and its image is $\{h, h+1, \ldots, k\}$. It is the following composite of face maps:

$$\begin{aligned} \{0,1,\ldots,k-h\} &\xrightarrow{\delta_0} \{0,1,\ldots,k-h+1\} \to \cdots \xrightarrow{\delta_{h-1}} \{0,1,\ldots,k-h+h\} \\ &= \{0,1,\ldots,k\} \xrightarrow{\delta_{k+1}} \{0,1,\ldots,k+1\} \to \cdots \xrightarrow{\delta_n} \{0,1,\ldots,n\}. \end{aligned} \tag{5.11}$$

Let $X$ and $Y$ be simplicial modules and $n \geq 0$. The *Alexander–Whitney map*

$$\psi_{X,Y} \colon X_n \otimes Y_n \to \bigoplus_{i=0}^{n} X_i \otimes Y_{n-i} \tag{5.12}$$

is given by

$$\psi_{X,Y}(x \otimes y) := \sum_{i=0}^{n} X_{\delta_{0,i}}(x) \otimes Y_{\delta_{i,n}}(y) = \sum_{i=0}^{n} (d_{i+1} \cdots d_n)(x) \otimes (d_0 \cdots d_{i-1})(y).$$

The second expression follows from (5.11).

**Remark 5.3.** There is a companion to the Alexander–Whitney map obtained by interchanging the roles of $X$ and $Y$. We denote it by ${}^b\psi$. It is given by

$$({}^b\psi)_{X,Y}(x \otimes y) := \sum_{i=0}^{n} X_{\delta_{i,n}}(x) \otimes Y_{\delta_{0,i}}(y) = \sum_{i=0}^{n} (d_0 \cdots d_{i-1})(x) \otimes (d_{i+1} \cdots d_n)(y).$$

There is no reason to view either $\psi$ or ${}^b\psi$ as more fundamental than the other.

**5.3.2. Eilenberg–Zilber.** Let $p, q \geq 0$. Given a subset $S \subseteq \{0, 1, \ldots, p+q-1\}$ with $|S| = p$, define

$$\sigma_S \colon \{0, 1, \ldots, p+q\} \to \{0, 1, \ldots, q\}$$

by

$$\sigma_S(i) := i - |\{j \in S : j < i\}|. \tag{5.13}$$

The dependence of $\sigma_S$ on $q$ is not reflected in the notation, but $q$ will always be clear from the context.

The map $\sigma_S$ is order-preserving and surjective. Indeed, it is the unique order-preserving surjective map which identifies $i$ and $i+1$ for each $i \in S$. Write $S = \{i_1, \ldots, i_p\}$ with $0 \leq i_1 < \cdots < i_p \leq p+q-1$. The map $\sigma_S$ is the following composite of degeneracy maps:

$$\{0, 1, \ldots, p+q\} \xrightarrow{\sigma_{i_p}} \{0, 1, \ldots, p+q-1\} \to \cdots \xrightarrow{\sigma_{i_1}} \{0, 1, \ldots, p+q-p\}. \tag{5.14}$$

In the same situation, we set

$$\epsilon(S) := \sum_{j=1}^{p} i_j - (j-1).$$

**Remark 5.4.** The function $\epsilon$ is called the *signature* by Mac Lane [249, Chapter VIII.8]. It is essentially the Schubert statistic (2.13). The precise connection is

$$\epsilon(S) = \mathrm{sch}_{p+q}(S+1),$$

where

$$S + 1 = \{i_1 + 1, \ldots, i_p + 1\} \subseteq [p+q].$$

We will need some elementary properties of the Schubert statistic (Section 2.2.1) later in Sections 5.4 and 5.6.

Let $X$ and $Y$ be simplicial modules and $p, q \geq 0$. The *Eilenberg–Zilber map*

$$\varphi_{X,Y} \colon X_p \otimes Y_q \to X_{p+q} \otimes Y_{p+q} \tag{5.15}$$

is given by

$$\begin{aligned}\varphi_{X,Y}(x\otimes y) &:= \sum_{\substack{S\sqcup T=\{0,1,\dots,p+q-1\}\\ |S|=p,|T|=q}} (-1)^{\epsilon(S)} X_{\sigma_T}(x)\otimes Y_{\sigma_S}(y)\\ &= \sum_{\substack{S\sqcup T=\{0,1,\dots,p+q-1\}\\ S=\{i_1<\cdots<i_p\}\\ T=\{j_1<\cdots<j_q\}}} (-1)^{\epsilon(S)} (s_{j_q}\cdots s_{j_1})(x)\otimes (s_{i_p}\cdots s_{i_1})(y).\end{aligned}$$

The second expression follows from (5.14).

The map $\varphi_{X,Y}$ was introduced by Eilenberg and Mac Lane [119, Section 5] in relation to the Eilenberg–Zilber theorem [123, Section 1]. It is also referred to as the *simplicial cross product* [67, Section IV.16] or [160, Section 3.B, p. 277].

**5.3.3. A compatibility lemma.** The lemma below will allow us to relate the Alexander–Whitney and Eilenberg–Zilber maps. We need some notation. Given $0\le h\le k\le n$ and a subset $S\subseteq\{0,1,\dots,n-1\}$, let

$$S(h,k) := S\cap\{h,h+1,\dots,k-1\}-h \tag{5.16}$$

be the subset of $\{0,1,\dots,k-h-1\}$ obtained by subtracting $h$ from each element in the intersection of $S$ and $\{h,h+1,\dots,k-1\}$.

**Lemma 5.5.** *Let $p,q\ge 0$, $n=p+q$, and $0\le h\le k\le n$. Consider a disjoint decomposition $S\sqcup T=\{0,1,\dots,n-1\}$ with $|S|=p$ and $|T|=q$. Let*

$$i := |T(0,h)| \qquad \textit{and} \qquad j := |T(0,k)|.$$

*Then the following diagram commutes.*

$$\begin{tikzcd} \{0,1,\dots,k-h\} \arrow[r, "\delta_{h,k}"] \arrow[d, "\sigma_{S(h,k)}"'] & \{0,1,\dots,n\} \arrow[d, "\sigma_S"] \\ \{0,1,\dots,j-i\} \arrow[r, "\delta_{i,j}"'] & \{0,1,\dots,q\} \end{tikzcd} \tag{5.17}$$

Proof. First note that, since $|T|=q$, $0\le i\le j\le q$, so $\delta_{i,j}$ maps as stated. Also, since $S$ and $T$ are complementary,

$$|S\cap\{0,1,\dots,h-1\}| = h-i \qquad\text{and}\qquad |S\cap\{0,1,\dots,k-1\}| = k-j;$$

therefore,

$$|S\cap\{h,h+1,\dots,k-1\}| = k-h-(j-i)$$

and $\sigma_{S(h,k)}$ maps as stated.

Let $x\in\{0,1,\dots,k-h\}$. We compute using (5.10) and (5.13):

$$\sigma_S\delta_{h,k}(x) = \sigma_S(x+h) = x+h-|\{y\in S : y<x+h\}|.$$

We have

$$\begin{aligned}|\{y \in S : y < x+h\}| &= |\{y \in S \cap \{0,1,\dots,h-1\} : y < x+h\}| \\ &\qquad + |\{y \in S \cap \{h,h+1,\dots,k-1\} : y < x+h\}| \\ &= |S \cap \{0,1,\dots,h-1\}| \\ &\qquad + |\{y-h \in S \cap \{h,h+1,\dots,k-1\} - h : y-h < x\}| \\ &= h - i + |\{z \in S(h,k) : z < x\}|.\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma_S \delta_{h,k}(x) &= x + h - \big(h - i + |\{z \in S(h,k) : z < x\}|\big) \\ &= x + i - |\{z \in S(h,k) : z < x\}| = \delta_{i,j}\sigma_{S(h,k)}(x).\end{aligned}$$ □

The above lemma can be rephrased in geometric terms. Interpret the set $\{0,1,\dots,n\}$ as the standard $n$-simplex, and denote it by $\Delta_n$. Further, interpret an element $i$ of a subset $S \subseteq \{0,1,\dots,n-1\}$ as the edge joining $i$ and $i+1$ in $\Delta_n$. With this understanding, the map $\delta_{h,k}$ embeds $\Delta_{k-h}$ into $\Delta_n$ using the vertices from $h$ to $k$, while the map $\sigma_S$ contracts the edges corresponding to the elements of $S$. Diagram (5.17) can be rewritten as follows.

$$\begin{tikzcd} \Delta_{k-h} \arrow[r, "\delta_{h,k}"] \arrow[d, "\sigma_{S(h,k)}"'] & \Delta_n \arrow[d, "\sigma_S"] \\ \Delta_{j-i} \arrow[r, "\delta_{i,j}"'] & \Delta_q \end{tikzcd}$$

The horizontal maps are embeddings, while the vertical maps contract edges in the manner explained above.

We will apply this lemma in the next section in the special cases when the embedded simplices are either the initial segment or the final segment of the bigger simplex.

## 5.4. The chain complex functors

The goal of this section is to construct bilax functors from the category of simplicial modules to the category of chain complexes. Chain complexes were reviewed in Section 2.7; we follow the notation of Table 5.1.

**5.4.1. From simplicial modules to chain complexes.** From a simplicial module $X$ one constructs a chain complex $\mathcal{C}(X)$ as follows. The $n$-th component of $\mathcal{C}(X)$ is simply $X_n$ and the $n$-th component of the boundary map ($n \geq 1$) is

$$\partial_n \colon X_n \to X_{n-1}, \qquad \partial_n := \sum_{i=0}^{n} (-1)^i d_i, \tag{5.18}$$

where $d_i \colon X_n \to X_{n-1}$ is as in (5.3). Then $\mathcal{C}(X)$ is a chain complex called the *unnormalized chain complex* of the simplicial module $X$.

Let $\mathcal{N}_n(X)$ be the submodule of $X_n$ defined by

$$\mathcal{N}_0(X) := X_0 \qquad \text{and} \qquad \mathcal{N}_n(X) := \bigcap_{i=0}^{n-1} \ker(d_i \colon X_n \to X_{n-1}).$$

Note the index $i$ runs up to $n-1$ only. The boundary map sends $\mathcal{N}_n(X)$ to $\mathcal{N}_{n-1}(X)$. Therefore, $\mathcal{N}(X)$ is a chain subcomplex of $\mathcal{C}(X)$. It is called the *normalized chain complex* of the simplicial module $X$.

A morphism of simplicial modules $X \to Y$ induces morphisms of complexes

$$\mathcal{C}(X) \to \mathcal{C}(Y) \qquad \text{and} \qquad \mathcal{N}(X) \to \mathcal{N}(Y).$$

We obtain two functors

$$\mathcal{C}\colon \mathsf{sMod} \to \mathsf{dgMod_a} \qquad \text{and} \qquad \mathcal{N}\colon \mathsf{sMod} \to \mathsf{dgMod_a}$$

called the *unnormalized* and *normalized chain complex functors*, respectively.

The inclusion defines a natural transformation

$$\mathcal{N} \Rightarrow \mathcal{C}.$$

The normalized chain complex functor $\mathcal{N}$ has a canonical complement in $\mathcal{C}$. This is the functor $\mathcal{D}$ consisting of "degenerate elements":

$$\mathcal{D}_n(X) := \sum_{i=0}^{n-1} s_i(\mathcal{C}_{n-1}(X)).$$

One can check that [372, Lemma 8.3.7]

$$X_n = \mathcal{N}_n(X) \oplus \mathcal{D}_n(X).$$

This yields a canonical projection $X_n \twoheadrightarrow \mathcal{N}_n(X)$ and hence a natural transformation

$$\mathcal{C} \Rightarrow \mathcal{N}.$$

**5.4.2. Monoidal properties of the chain complex functors.** We proceed to discuss the monoidal structure of the chain complex functors. For that, recall that

$$(\mathsf{sMod}, \times, \beta, E) \qquad \text{and} \qquad (\mathsf{dgMod_a}, \cdot, \beta_{-1}, 1)$$

are both symmetric monoidal categories. The former is discussed in Section 5.1 and the latter in Section 2.7.

The Alexander–Whitney map (5.12) and the Eilenberg–Zilber map (5.15) commute with the boundary maps and define morphisms of complexes

$$\mathcal{C}(X) \cdot \mathcal{C}(Y) \underset{\psi_{X,Y}}{\overset{\varphi_{X,Y}}{\rightleftarrows}} \mathcal{C}(X \times Y). \tag{5.19}$$

Moreover, since they are given in terms of actions of morphisms of $\Delta$, these maps are natural in $X$ and $Y$. For Alexander–Whitney, we will verify more general properties in Proposition 5.20.

Consider the maps

$$1 \underset{\psi_0}{\overset{\varphi_0}{\rightleftarrows}} \mathcal{C}(E) \tag{5.20}$$

whose components

$$1_n \underset{\psi_0}{\overset{\varphi_0}{\rightleftarrows}} E_n$$

are the identity if $n = 0$ and $0$ if $n > 0$.

**Theorem 5.6.** *The chain complex functor*

$$(\mathcal{C}, \varphi, \psi) \colon (\mathsf{sMod}, \times, \beta) \to (\mathsf{dgMod_a}, \cdot, \beta_{-1})$$

*is bilax monoidal.*

Proof. The associativity axioms (3.5) for Alexander–Whitney and Eilenberg–Zilber can be understood as follows. If $X, Y$, and $Z$ are simplicial modules, then both

$$(\psi_{X,Y} \cdot \mathrm{id}_Z)\psi_{X\times Y,Z} \quad \text{and} \quad (\mathrm{id}_X \cdot \psi_{Y,Z})\psi_{X,Y\times Z}$$

are given by

$$\begin{aligned} X_n \otimes Y_n \otimes Z_n &\to \bigoplus_{0\le h\le k\le n} X_h \otimes Y_{k-h} \otimes Z_{n-k} \\ x \otimes y \otimes z &\mapsto \sum_{0\le h\le k\le n} X_{\delta_{0,h}}(x) \otimes Y_{\delta_{h,k}}(y) \otimes Z_{\delta_{k,n}}(z). \end{aligned}$$

Also, both

$$\varphi_{X\times Y,Z}(\varphi_{X,Y} \cdot \mathrm{id}_Z) \quad \text{and} \quad \varphi_{X,Y\times Z}(\mathrm{id}_X \cdot \varphi_{Y,Z})$$

are given by

$$\begin{aligned} X_p \otimes Y_q \otimes Z_r &\to X_{p+q+r} \otimes Y_{p+q+r} \otimes Z_{p+q+r} \\ x \otimes y \otimes z &\mapsto \sum_{\substack{S\sqcup T\sqcup U=\{0,1,\ldots,p+q+r-1\} \\ |S|=p,|T|=q,|U|=r}} (-1)^{\epsilon(S,T)} X_{\sigma_{T\sqcup U}}(x) \otimes Y_{\sigma_{S\sqcup U}}(y) \otimes Z_{\sigma_{S\sqcup T}}(z), \end{aligned}$$

where $\epsilon(S,T)$ is given by any of the two following expressions.

$$\epsilon(S,T) := \epsilon(S \sqcup T) + \epsilon(\bar{S}) = \epsilon(S) + \epsilon(\bar{T}) \tag{5.21}$$

Here $\bar{S}$ and $\bar{T}$ are the images of $S$ and $T$ under the unique order-preserving bijections

$$S \sqcup T \to \{0,1,\ldots,p+q-1\} \quad \text{and} \quad T \sqcup U \to \{0,1,\ldots,q+r-1\},$$

respectively. The equality in (5.21) holds by (2.17).

The unit axioms (3.6), (3.12), and (3.13) are all straightforward.

The main point is the verification of the braiding axiom (3.11). It takes the following form

$$\begin{tikzcd}[column sep=small] & \mathcal{C}(W\times X)\cdot\mathcal{C}(Y\times Z) \arrow[dl, "\varphi_{W\times X,Y\times Z}"'] \arrow[dr, "\psi_{W,X}\cdot\psi_{Y,Z}"] & \\ \mathcal{C}(W\times X\times Y\times Z) \arrow[d, "\mathcal{C}(\mathrm{id}\times\beta\times\mathrm{id})"'] & & \mathcal{C}(W)\cdot\mathcal{C}(X)\cdot\mathcal{C}(Y)\cdot\mathcal{C}(Z) \arrow[d, "\mathrm{id}\cdot\beta_{-1}\cdot\mathrm{id}"] \\ \mathcal{C}(W\times Y\times X\times Z) \arrow[dr, "\psi_{W\times Y,X\times Z}"'] & & \mathcal{C}(W)\cdot\mathcal{C}(Y)\cdot\mathcal{C}(X)\cdot\mathcal{C}(Z) \arrow[dl, "\varphi_{W,Y}\cdot\varphi_{X,Z}"] \\ & \mathcal{C}(W\times Y)\cdot\mathcal{C}(X\times Z) & \end{tikzcd} \tag{5.22}$$

where $W$ is a fourth simplicial module. Take

$$w \in W_p,\ x \in X_p,\ y \in Y_q,\ z \in Z_q,$$

so that $(w \otimes x) \otimes (y \otimes z)$ is an element of degree $p+q$ in $\mathcal{C}(W \times X) \cdot \mathcal{C}(Y \times Z)$. Starting from this element and going clockwise around the diagram one obtains

$$\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{I,J}\sum_{S,T}(-1)^{(p-i)j}(-1)^{\epsilon(I)}(-1)^{\epsilon(S)} \\ \times \big(W_{\delta_{0,i}\sigma_J}(w) \otimes Y_{\delta_{0,j}\sigma_I}(y)\big) \otimes \big(X_{\delta_{i,p}\sigma_T}(x) \otimes Z_{\delta_{j,q}\sigma_S}(z)\big).$$

The inner sums are over subsets $I, J, S, T$ such that

$$I \sqcup J = \{0, 1, \ldots, i+j-1\},\ |I| = i,\ |J| = j,$$

and

$$S \sqcup T = \{0, 1, \ldots, p+q-i-j-1\},\ |S| = p-i,\ |T| = q-j.$$

The sign $(-1)^{(p-i)j}$ is introduced by the braiding of $\mathsf{dgMod_a}$.

On the other hand, starting from the same element but going counterclockwise around the diagram, one obtains

$$\sum_{h=0}^{p+q}\sum_{A,B}(-1)^{\epsilon(A)}\big(W_{\sigma_B\delta_{0,h}}(w) \otimes Y_{\sigma_A\delta_{0,h}}(y)\big) \otimes \big(X_{\sigma_B\delta_{h,p+q}}(x) \otimes Z_{\sigma_A\delta_{h,p+q}}(z)\big).$$

The inner sum is over subsets $A, B$ such that

$$A \sqcup B = \{0, 1, \ldots, p+q-1\},\ |A| = p,\ |B| = q.$$

Given $A$ and $B$ as above, define

$$I := A(0,h),\ J := B(0,h),\ S := A(h, p+q),\ T := B(h, p+q),$$

with notation as in (5.16). In particular, $h = i+j$. This sets up a bijection between pairs $(A, B)$ and tuples $(I, J, S, T)$ as above. Moreover, the corresponding terms in each of the two summations obtained above agree, by virtue of Lemma 5.5. Further, by (2.18), we have

$$(p-i)j + \epsilon(I) + \epsilon(S) = \epsilon(A) \tag{5.23}$$

which shows that the signs agree. $\square$

Alexander–Whitney and Eilenberg–Zilber project onto (but do not restrict to) the normalized complex yielding natural transformations

$$\mathcal{N}(X) \cdot \mathcal{N}(Y) \underset{\psi_{X,Y}}{\overset{\varphi_{X,Y}}{\rightleftarrows}} \mathcal{N}(X \times Y). \tag{5.24}$$

The same is true of the unit maps. This yields

$$1 \underset{\psi_0}{\overset{\varphi_0}{\rightleftarrows}} \mathcal{N}(E). \tag{5.25}$$

In fact, in contrast to (5.20), these maps are inverse isomorphisms. We deduce:

**Corollary 5.7.** *The normalized chain complex functor*

$$(\mathcal{N}, \varphi, \psi)\colon (\mathsf{sMod}, \times, \beta) \to (\mathsf{dgMod_a}, \cdot, \beta_{-1})$$

*is bilax monoidal and the transformation $\mathcal{C} \Rightarrow \mathcal{N}$ is a morphism of bilax monoidal functors. Moreover, $\mathcal{N}$ is normal in the sense of Definition 3.39.*

We now turn to the commutativity properties of the chain complex functors. The Eilenberg–Zilber map commutes with the braidings. This is explained below.

**Proposition 5.8.** *The chain complex functors*

$$(\mathcal{C}, \varphi),\ (\mathcal{N}, \varphi)\colon (\mathsf{sMod}, \times, \beta) \to (\mathsf{dgMod}_{\mathsf{a}}, \cdot, \beta_{-1})$$

*are braided lax.*

Proof. We have to verify the commutativity of the right rectangle in (3.16) involving the braidings and the Eilenberg–Zilber map. It suffices to do it for the unnormalized chain complex functor. Starting from $x \otimes y \in X_p \otimes Y_q$ and going clockwise we obtain

$$\sum_{S,T} (-1)^{\epsilon(S)} Y_{\sigma_S}(y) \otimes X_{\sigma_T}(x),$$

while going counterclockwise we obtain

$$(-1)^{pq} \sum_{S,T} (-1)^{\epsilon(T)} Y_{\sigma_S}(y) \otimes X_{\sigma_T}(x).$$

In both cases the sum is over subsets $S, T$ such that

$$S \sqcup T = \{0, 1, \ldots, p+q-1\},\ |S| = p,\ |T| = q.$$

The obtained elements agree since

$$(-1)^{pq+\epsilon(T)} = (-1)^{pq-\epsilon(T)} = (-1)^{\epsilon(S)}$$

by (2.15). □

In contrast, the Alexander–Whitney map does not commute with the braidings. The conjugate of $\psi$ by the braidings is precisely the map ${}^b\psi$ given in Remark 5.3. So neither $\mathcal{C}$ nor $\mathcal{N}$ is braided colax (but see Section 5.5.5). It is natural to wonder whether ${}^b\psi$ is compatible with $\varphi$. The answer is positive in view of Proposition 3.16: $(\mathcal{C}, \varphi, {}^b\psi)$ and $(\mathcal{N}, \varphi, {}^b\psi)$ are bilax monoidal.

A key property of bilax functors is that they preserve bimonoids and that of braided lax functors is that they preserve commutative monoids. Recall from Section 2.7 that a monoid in the category of chain complexes is a *differential graded algebra* (with differential of degree $-1$). Similarly, (co, bi)monoids are differential graded (co, bi)algebras. Further, recall that a monoid in the category of simplicial modules is the same thing as a *simplicial algebra*, that is, a simplicial object in the category of $\Bbbk$-algebras. Similarly, (co, bi)monoids are simplicial (co, bi)algebras. These observations imply the following well-known fact.

**Corollary 5.9.** *If $X$ is a simplicial bialgebra, then $\mathcal{C}(X)$ and $\mathcal{N}(X)$ are differential graded bialgebras. Further, if $X$ is commutative, then so are $\mathcal{C}(X)$ and $\mathcal{N}(X)$.*

**Remark 5.10.** The fact that the chain complex functors should be examples of bilax monoidal functors was pointed out to us by Clemens Berger. Closely related properties are noted in the literature.

The fact that $\varphi_{X,Y}$ and $\psi_{X,Y}$ are natural morphisms of chain complexes (5.19) is given in [249, Theorems VIII.8.5 and VIII.8.8]. The fact that they project onto the normalized complexes (5.24) is given in [249, Corollaries VIII.8.6 and VIII.8.9] or [262, Propositions 29.8 and 29.9]. The fact that the chain complex functors are colax and lax is given in [249, Proposition VIII.8.7 and Exercise VIII.8.2] and in [262, Propositions 29.8 and 29.9], as well as many other places. The fact that they are braided lax is given in [119, Theorem 5.2]; this is the starting point for the work of Richter [313]. The braiding axiom (5.22) appears in several textbooks [125, Section 4.b], [328, Section 5.5], and [361, Section 9.7, Problem 5]. It can be traced back to the work of Eilenberg and Moore [121, p. 232].

Certain consequences of the fact that $\mathcal{N}$ is a normal bilax monoidal functor have been remarked in the literature. For instance, Mac Lane [249, Corollary VIII.8.9] notes that

$$\psi_{X,Y}\varphi_{X,Y} = \mathrm{id}_{\mathcal{N}(X)\cdot\mathcal{N}(Y)},$$

which we know is a general fact for normal bilax functors (Proposition 3.41, item (ii)). Working in the more general setting of *crossed complexes*, Tonks notes the commutativity of the following diagrams [362, Proposition 2.2.13].

$$\begin{tikzcd}
\mathcal{N}(X)\cdot\mathcal{N}(Y\times Z) \arrow[r, "\mathrm{id}\cdot\psi_{Y,Z}"] \arrow[d, "\varphi_{X,Y\times Z}"'] & \mathcal{N}(X)\cdot\mathcal{N}(Y)\cdot\mathcal{N}(Z) \arrow[d, "\varphi_{X,Y}\cdot\mathrm{id}"] \\
\mathcal{N}(X\times Y\times Z) \arrow[r, "\psi_{X\times Y,Z}"'] & \mathcal{N}(X\times Y)\cdot\mathcal{N}(Z)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{N}(X\times Y)\cdot\mathcal{N}(Z) \arrow[r, "\psi_{X,Y}\cdot\mathrm{id}"] \arrow[d, "\varphi_{X\times Y,Z}"'] & \mathcal{N}(X)\cdot\mathcal{N}(Y)\cdot\mathcal{N}(Z) \arrow[d, "\mathrm{id}\cdot\varphi_{Y,Z}"] \\
\mathcal{N}(X\times Y\times Z) \arrow[r, "\psi_{X,Y\times Z}"'] & \mathcal{N}(X)\cdot\mathcal{N}(Y\times Z)
\end{tikzcd}$$

This is also a general fact for normal bilax monoidal functors (Proposition 3.41, items (iv) and (v)). On the other hand, such conditions do not suffice to imply that a given functor is bilax (see Example 3.42).

**Question 5.11.** It would be interesting to know if Theorem 5.6 extends to the context of crossed complexes. In other words, is Tonks' fundamental crossed complex another example of a normal bilax monoidal functor? Tonks' diagrams suggest that the answer is probably positive.

## 5.5. The (co)homology functors. Cup and Pontrjagin products

In this section, we compose the chain complex functors of the previous section with the (co)homology functors of Section 2.7.5. We study their monoidal properties and state some well-known results which follow. We follow the notation of Table 5.1.

**5.5.1. The chain complex functors up to homotopy.** The chain complex functors are compatible with homotopies. In other words, a homotopy between simplicial maps induces a chain homotopy between the corresponding chain maps. Let $\overline{\mathcal{C}}$ and $\overline{\mathcal{N}}$ be defined by the commutativity of the following diagrams.

$$\begin{tikzcd}
\mathsf{sMod} \arrow[r, "\mathcal{C}"] \arrow[d] & \mathsf{dgMod_a} \arrow[d] \\
\overline{\mathsf{sMod}} \arrow[r, "\overline{\mathcal{C}}"'] & \overline{\mathsf{dgMod_a}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{sMod} \arrow[r, "\mathcal{N}"] \arrow[d] & \mathsf{dgMod_a} \arrow[d] \\
\overline{\mathsf{sMod}} \arrow[r, "\overline{\mathcal{N}}"'] & \overline{\mathsf{dgMod_a}}
\end{tikzcd}$$

The left vertical functor appears in (5.9) while the right vertical functor was defined in (2.73). They are both bistrong: the main observation is that the objects in the source and target categories are the same. By construction, $\overline{\mathcal{C}}$ and $\overline{\mathcal{N}}$ are bilax monoidal and $\overline{\mathcal{C}} \Rightarrow \overline{\mathcal{N}}$ is a morphism of bilax monoidal functors. In fact:

**Proposition 5.12.** *The functors $\overline{\mathcal{C}}$ and $\overline{\mathcal{N}}$ are bistrong, and $\overline{\mathcal{C}} \Rightarrow \overline{\mathcal{N}}$ is an isomorphism of bistrong monoidal functors.*

PROOF. The fact that $\overline{\mathcal{C}}$ is bistrong follows from the Eilenberg–Zilber theorem: $\varphi_{X,Y}$ and $\psi_{X,Y}$ are both chain equivalences. Further they are inverse to each other. (The latter may also be deduced from the former using Proposition 3.46.) A good account of the Eilenberg–Zilber theorem is given in Mac Lane's book [249, Section VIII.8, Theorems 8.1, 8.5 and 8.8]. Some other sources are [67, Corollary VI.1.4], [262, Section 29, Corollary 29.10], [282, Section 59], [335, Chapter 5, Section 3, Theorems 6, 7 and 8], [361, Theorem 9.7.1] and [372, Section 8.5].

Recall the normalization theorem [249, Theorem VIII.6.1] or [262, Corollary 22.3] which says that the natural transformation $\mathcal{C} \Rightarrow \mathcal{N}$ induces an isomorphism between the functors $\overline{\mathcal{C}}$ and $\overline{\mathcal{N}}$. This implies that $\overline{\mathcal{N}}$ is bistrong and isomorphic to $\overline{\mathcal{C}}$ as a bistrong functor. □

The Dold–Kan theorem [101, 186] states that the normalized chain complex functor defines an equivalence of categories

$$\mathcal{N} \colon \mathsf{sMod} \to \mathsf{dgMod_a}.$$

For later references, see [262, Theorem 22.4] or [372, Theorem 8.4.1]. The inverse functor is described for instance in [372, Section 8.4.4]. This equivalence is compatible with homotopies (so $\overline{\mathcal{N}}$ is also an equivalence of categories). Further, it is an adjoint equivalence [372, Exercise 8.4.2]. The monoidal properties of the adjunction (Section 3.9) are studied in [77, 313, 326].

**5.5.2. Monoidal properties of duality.** Recall that duality interchanges chain and cochain complexes (Section 2.7). Further, if the complexes are assumed to be finite-dimensional in each component, then the duality functor is bistrong. However, one frequently meets complexes which are in fact infinite-dimensional in all components; take, for example, the singular chain complex of a topological space. To include such situations, we work in full generality. Note that for any complexes $K$ and $L$, there is always a map

$$K^* \cdot L^* \to (K \cdot L)^*.$$

While this map is not an isomorphism in general, we have:

**Proposition 5.13.** *The duality functor*

$$(\overline{\mathsf{dgMod_a}})^{\mathrm{op}} \to \overline{\mathsf{dgMod^c}}$$

*is braided lax, where $(-)^{\mathrm{op}}$ refers to the opposite category.*

**5.5.3. Monoidal properties of (co)homology.** Recall that the (co)homology functors are defined on the homotopy category of (co)chain complexes. We now study their monoidal properties.

**Proposition 5.14.** *The homology and cohomology functors*

$$\mathcal{H}_\bullet \colon \overline{\mathsf{dgMod_a}} \to \mathsf{gMod} \qquad \textit{and} \qquad \mathcal{H}^\bullet \colon \overline{\mathsf{dgMod^c}} \to \mathsf{gMod}$$

*of (2.74) are braided lax. Further, if $\Bbbk$ is a field, then they are bistrong.*

Proof. We first explain the homology case. The tensor product of cycles is a cycle, and the tensor product of a cycle and a boundary, in either order, is a boundary. For any chain complexes $K$ and $L$, this yields maps

$$\mathcal{H}_\bullet(K) \cdot \mathcal{H}_\bullet(L) \to \mathcal{H}_\bullet(K \cdot L). \tag{5.26}$$

This along with the obvious isomorphism $1 \to \mathcal{H}_\bullet(1)$ turn $\mathcal{H}_\bullet$ into a braided lax functor. This is straightforward to verify.

The Künneth formula says that (5.26) fits into a short exact sequence (the third term involves the Tor functor). The statement in varying levels of generality can be found in a number of textbooks. We mention [69, Proposition (0.8)], [160, Theorem 3B.5], [249, Theorems V.10.1 and V.10.2], [282, Section 58], [335, Chapter 5, Section 3, Theorem 3] or [372, Theorem 3.6.3]. The origin of this formula is in the papers of Künneth [214, 215]. We want to apply it in the case when $\Bbbk$ is a field. In this situation, all $\Bbbk$-modules are free, so the Tor term vanishes implying that (5.26) is an isomorphism. Hence, in this case, the functor $\mathcal{H}_\bullet$ is bistrong.

The preceding is true even if one works with chain complexes which are $\mathbb{Z}$-graded. In this situation, the distinction between chain and cochain complexes disappears since one can pass from one situation to the other by changing the degree of each component to its negative. The assertions for the cohomology functor follow from this observation. An explicit formulation of the Künneth formula for cohomology of cochain complexes is given in [335, Chapter 5, Section 4, Theorem 2]. $\square$

**Remark 5.15.** A Künneth formula for cohomology of *chain* complexes is given in [282, Section 60, Theorem 60.3].

**Proposition 5.16.** *If $\Bbbk$ is a field, then the homology and cohomology functors are contragredients of each other.*

Proof. Let us first work over any commutative ring $\Bbbk$. For any chain complex $K$, there is a surjective map

$$\mathcal{H}^\bullet(K^*) \to \mathcal{H}_\bullet(K)^*.$$

This map fits into a short exact sequence (the third term involves the Ext functor). This is known as the universal coefficient theorem for cohomology, see [160, Theorem 3.2], [249, Theorem III.4.1 and Corollary III.4.2] or [335, Chapter 5, Section 5, Theorem 3]. If $\Bbbk$ is a field, then the Ext term vanishes and the above map is an isomorphism. $\square$

To summarize, if $\Bbbk$ is a field, then homology and cohomology are dual notions, and both are bistrong as functors.

**5.5.4. Cup and Pontrjagin products.** The (co)homology of a simplicial module $X$ is defined to be

$$\mathcal{H}_\bullet(X) := \mathcal{H}_\bullet\big(\bar{\mathcal{C}}(X)\big) \qquad \text{and} \qquad \mathcal{H}^\bullet(X) := \mathcal{H}^\bullet\big(\bar{\mathcal{C}}(X)^*\big),$$

where $\bar{\mathcal{C}}(X)^*$ denotes the cochain complex which is the dual of $\bar{\mathcal{C}}(X)$.

We now discuss the cup and Pontrjagin products in the context of simplicial modules. By composing with the bistrong functors (5.5) and (5.6), they can be carried over to the context of topological spaces. Both products can be defined over any commutative ring $\Bbbk$.

We begin with the cup product. Consider the composition of functors

$$\mathcal{H}^\bullet\colon (\overline{\mathsf{sMod}})^{\mathrm{op}} \xrightarrow{\overline{\mathcal{C}}} (\overline{\mathsf{dgMod}_{\mathsf{a}}})^{\mathrm{op}} \xrightarrow{(-)^*} \overline{\mathsf{dgMod}^{\mathsf{c}}} \xrightarrow{\mathcal{H}^\bullet} \mathsf{gMod}$$

where $(-)^{\mathrm{op}}$ refers to the opposite category. We continue to denote the composite by $\mathcal{H}^\bullet$ since it yields the cohomology of a simplicial module. The first functor in the composite is bistrong, while the remaining two are braided lax; so the composite is braided lax.

Suppose $X$ is a simplicial set. Then its linearization $\Bbbk X$ is a simplicial coalgebra (via the diagonal). This is a cocommutative comonoid in $\overline{\mathsf{sMod}}$ or equivalently a commutative monoid in $(\overline{\mathsf{sMod}})^{\mathrm{op}}$. Hence $\mathcal{H}^\bullet(\Bbbk X)$, which is its image under a braided lax functor, is a graded commutative algebra. This is the *cup product* in cohomology.

Note that the Alexander–Whitney map does not turn $\mathcal{C}$ into a braided colax functor; however it does turn $\overline{\mathcal{C}}$ into a braided colax functor. In other words, the left rectangle in (3.16) involving the braidings and the Alexander–Whitney map commutes up to homotopy. For this reason, one says that the differential graded coalgebra $\mathcal{C}(\Bbbk X)$ is cocommutative *up to homotopy*. It is this fact which ensures that the cup product is commutative. A direct proof of commutativity of the cup product is given in Hatcher's book [160, Theorem 3.14]. The main step in the proof is indeed the construction of a chain homotopy.

We now discuss the Pontrjagin product. Consider the composition of functors

$$\mathcal{H}_\bullet\colon \overline{\mathsf{sMod}} \xrightarrow{\overline{\mathcal{C}}} \overline{\mathsf{dgMod}_{\mathsf{a}}} \xrightarrow{\mathcal{H}_\bullet} \mathsf{gMod}$$

We continue to denote it by $\mathcal{H}_\bullet$ since it yields the homology of a simplicial module. The first functor in the composite is bistrong, while the second is braided lax; so the composite is braided lax.

Suppose that $G$ is a simplicial group, that is, a simplicial object in the category of groups. Then its linearization $\Bbbk G$ is a simplicial bialgebra, and in particular, a simplicial algebra. It follows that $\mathcal{H}_\bullet(\Bbbk G)$ is a graded algebra. This is the *Pontrjagin product* in homology. If $G$ is abelian, then this product is commutative. There is no subtle chain homotopy issue here since the Eilenberg–Zilber map does turn $\mathcal{C}$ into a braided lax functor. Since $\Bbbk G$ is a simplicial bialgebra, Corollary 5.9 shows that $\mathcal{C}(\Bbbk G)$ and $\mathcal{N}(\Bbbk G)$ are differential graded bialgebras. Further, if we assume that $\Bbbk$ is a field, then $\mathcal{H}_\bullet(\Bbbk G)$ is a graded bialgebra.

**5.5.5. The chain complex functors and $E_\infty$-coalgebras.** Recall that the chain complex functors $\mathcal{C}$ and $\mathcal{N}$ are not braided colax. For this reason, for a general simplicial set $X$, the differential graded coalgebras $\mathcal{C}(\Bbbk X)$ and $\mathcal{N}(\Bbbk X)$ may fail to be cocommutative. However, it is known from classical work of Dold [102] that these coalgebras satisfy a *higher homotopy version* of cocommutativity. This statement is a refinement of the fact that $\overline{\mathcal{C}}(\Bbbk X)$ is a cocommutative comonoid in the homotopy category $\overline{\mathsf{dgMod}_{\mathsf{a}}}$ of chain complexes. More precise versions of this statement are given by Smirnov [331, Corollary to Proposition 5, and Proposition 8] and by Berger and Fresse [39].

The cocommutativity up to higher homotopies of these coalgebras may be deduced from corresponding properties of the chain complex functors. This has been explained by Richter. In [312, Section 7] and [313, Section 5], Richter shows that the chain complex functors are $E_\infty$-colax, where $E_\infty$ stands for a homotopy version of the commutative operad **Com** (Example B.15).

In Section 4.3.3, for any operad $\mathbf{p}$, we defined a notion of $\mathbf{p}$-lax monoidal functors and the dual notion of $\mathbf{p}$-colax monoidal functors. A braided colax monoidal functor is the same thing as a **Com**-colax monoidal functor (Table 4.2). Thus, an $E_\infty$-colax monoidal functor is a homotopy version of a braided colax monoidal functor. Moreover, it follows from Corollary 4.36 that applying an $E_\infty$-colax monoidal functor to a cocommutative comonoid results in a $E_\infty$-comonoid. Combined with Richter's result, this recovers the fact that $\mathcal{C}(\Bbbk X)$ and $\mathcal{N}(\Bbbk X)$ are $E_\infty$-coalgebras.

## 5.6. A $q$-analogue of the chain complex functor

Throughout this section, $\Bbbk$ is a commutative ring and $q \in \Bbbk$ is a fixed scalar.

**5.6.1. In search of $q$-analogues.** We discuss the possibility of deforming the chain complex functor

$$(\mathcal{C}, \varphi, \psi)\colon (\mathsf{sMod}, \times, \beta) \to (\mathsf{dgMod}_{\mathsf{a}}, \cdot, \beta_{-1})$$

of Section 5.4. More precisely, we would like to replace the target category by the category of $N$-complexes $(\mathsf{dgMod}_N, \cdot_q, \beta_q)$ of Section 2.9. However, we saw in Proposition 2.27 that the only choices, other than the one already considered, are

$$(\mathsf{gMod}, \cdot, \beta_q), \qquad (\mathsf{gMod}_{\mathsf{a}}, \cdot, \beta), \qquad \text{and} \qquad (\mathsf{gMod}_{\mathsf{a}}, \cdot_{-1}, \beta_{-1}).$$

These are the categories of graded $\Bbbk$-modules, and of graded $\Bbbk$-modules with annihilation operators (the latter has two choices for tensor products and braidings).

In Section 5.6.2 we construct a one-parameter deformation $\varphi_q$ of the Eilenberg–Zilber map which together with the Alexander–Whitney map $\psi$ yields a bilax monoidal functor

$$(\mathcal{C}, \varphi_q, \psi)\colon (\mathsf{sMod}, \times, \beta) \to (\mathsf{gMod}, \cdot, \beta_q). \tag{5.27}$$

This is the content of Theorem 5.17 and gives a positive answer for the first example. For the third example, there is an obvious bistrong functor

$$(\mathsf{dgMod}_{\mathsf{a}}, \cdot, \beta_{-1}) \to (\mathsf{gMod}_{\mathsf{a}}, \cdot_{-1}, \beta_{-1}),$$

so we can precompose this with the chain complex functor to again get a positive answer. For the second example, we do not have any result of this kind.

We know from Proposition 2.24 and Remark 2.25 that $(\mathsf{dgMod}_N, \cdot_q)$ is a monoidal category if $N = \infty$ or $N = 1$ and $q$ is arbitrary, or if $2 \leq N < \infty$ and

$$\Bbbk \text{ is an integral domain, and } q \in \Bbbk \text{ is a primitive } N\text{-th root of unity.} \tag{5.28}$$

We ask whether in this situation, the functor

$$(\mathsf{sMod}, \times) \to (\mathsf{dgMod}_N, \cdot_q)$$

defined by appropriately deforming (5.18) is either lax or colax. We will show in Proposition 5.22 that there is a positive answer in the colax case.

**5.6.2. Alexander–Whitney and $q$-Eilenberg–Zilber.** Consider the functor

$$\mathcal{C}\colon \mathsf{sMod} \to \mathsf{gMod}.$$

that simply forgets the face and degeneracy maps. Thus, if $X$ is a simplicial module, the $n$-th component of $\mathcal{C}(X)$ is $X_n$. We proceed to turn $\mathcal{C}$ into a bilax monoidal functor as mentioned in (5.27).

Let $X$ and $Y$ be simplicial modules. The colax structure map

$$\psi_{X,Y}\colon \mathcal{C}(X \times Y) \to \mathcal{C}(X) \cdot \mathcal{C}(Y)$$

will be the Alexander–Whitney map (5.12).

The lax structure map

$$(\varphi_q)_{X,Y}\colon \mathcal{C}(X) \cdot \mathcal{C}(Y) \to \mathcal{C}(X \times Y)$$

will be a $q$-deformation of the Eilenberg–Zilber map, with $q = -1$ recovering the classical map (5.15). It is defined as follows. Let $i, j \geq 0$. We define

$$(\varphi_q)_{X,Y}\colon X_i \otimes Y_j \to X_{i+j} \otimes Y_{i+j}$$

by

$$(\varphi_q)_{X,Y}(x \otimes y) := \sum_{\substack{S \sqcup T = \{0,1,\ldots,i+j-1\} \\ |S|=i, |T|=j}} q^{\epsilon(S)} X_{\sigma_T}(x) \otimes Y_{\sigma_S}(y),$$

where $\sigma_S$ and $\sigma_T$ are as in (5.13).

The unit structure maps $\psi_0$ and $\varphi_0$ are as before (5.20).

**Theorem 5.17.** *The functor*

$$(\mathcal{C}, \varphi_q, \psi)\colon (\mathsf{sMod}, \times, \beta) \to (\mathsf{gMod}, \cdot, \beta_q)$$

*is bilax monoidal.*

Proof. The assertions that only involve $\psi$ follow from Theorem 5.6, since the above functor is obtained by composing the chain complex functor with the forgetful functor $\mathsf{dgMod_a} \to \mathsf{gMod}$ which is strong.

Consider the associativity and unitality axioms for $(\mathcal{C}, \varphi_q)$. As in the proof of Theorem 5.6, we find that both maps

$$(\varphi_q)_{X \times Y, Z}\big((\varphi_q)_{X,Y} \cdot \mathrm{id}_Z\big) \quad \text{and} \quad (\varphi_q)_{X, Y \times Z}(\mathrm{id}_X \cdot (\varphi_q)_{Y,Z})$$

from

$$X_i \otimes Y_j \otimes Z_k \to X_{i+j+k} \otimes Y_{i+j+k} \otimes Z_{i+j+k}$$

are given by

$$x \otimes y \otimes z \mapsto \sum_{\substack{S \sqcup T \sqcup U = \{0,1,\ldots,i+j+k-1\} \\ |S|=i, |T|=j, |U|=k}} q^{\epsilon(S,T)} X_{\sigma_{T \sqcup U}}(x) \otimes Y_{\sigma_{S \sqcup U}}(y) \otimes Z_{\sigma_{S \sqcup T}}(z),$$

where $\epsilon(S,T)$ is as in (5.21). Thus, diagram (3.5) commutes and $\varphi_q$ is associative. The commutativity of diagrams (3.6) follows from (2.14); thus $\varphi_q$ is unital.

Finally, we verify the braiding axiom (3.11). This is (5.22) with $\varphi_q$ replacing $\varphi$. The extra ingredient is the verification of

$$q^{(p-i)j + \epsilon(I) + \epsilon(S)} = q^{\epsilon(A)}$$

which holds by (5.23). ∎

Recall that the chain complex functor $(\mathcal{C}, \varphi)$ is braided lax (Proposition 5.8). For the deformed functor $(\mathcal{C}, \varphi_q)$ we have the following property.

**Proposition 5.18.** *The Eilenberg–Zilber maps $\varphi_q$ and $\varphi_{q^{-1}}$ are related by conjugation by $\beta_q$. More precisely, for any simplicial modules $X$ and $Y$, the following diagram commutes.*

$$\begin{tikzcd} \mathcal{C}(X)\cdot\mathcal{C}(Y) \arrow[r, "\varphi_q"] \arrow[d, "\beta_q"'] & \mathcal{C}(X\times Y) \arrow[d, "\mathcal{C}(\beta)"] \\ \mathcal{C}(Y)\cdot\mathcal{C}(X) \arrow[r, "\varphi_{q^{-1}}"'] & \mathcal{C}(Y\times X) \end{tikzcd}$$

Proof. As in the proof of Proposition 5.8, starting from $x\otimes y\in X_i\otimes Y_j$ and going clockwise we obtain

$$\sum_{S,T} q^{\epsilon(S)} Y_{\sigma_S}(y)\otimes X_{\sigma_T}(x),$$

while going counterclockwise we obtain

$$q^{ij}\sum_{S,T} q^{-\epsilon(T)} Y_{\sigma_S}(y)\otimes X_{\sigma_T}(x).$$

In both cases the sum is over subsets $S, T$ such that

$$S\sqcup T=\{0,1,\dots,i+j-1\},\ |S|=i,\ |T|=j.$$

The obtained elements agree since

$$q^{ij-\epsilon(T)}=q^{\epsilon(S)}$$

by (2.15). $\square$

**5.6.3. The boundary maps of Kapranov and Dubois-Violette.** To a simplicial module $X$, we associate an $\infty$-complex $\mathcal{C}_q(X)$, or equivalently, a graded module with annihilation operators as follows. The underlying graded module is $\mathcal{C}(X)$. Thus, the $n$-th component of $\mathcal{C}_q(X)$ is $X_n$. The boundary map of $\mathcal{C}_q(X)$ has components

$$X_n\to X_{n-1},\qquad \partial_n:=\sum_{i=0}^{n-1} q^i d_i - q^{n-1}d_n, \tag{5.29}$$

for each $n\geq 1$.

A morphism of simplicial modules commutes with the boundary maps (5.29). We thus obtain a functor

$$\mathcal{C}_q\colon \mathsf{sMod}\to \mathsf{gMod_a}.$$

In addition, if $q\in \Bbbk$ is an $N$-th root of unity and $q\neq 1$, then the boundary map satisfies $\partial^N=0$ [104, Lemma 3]. Thus, we obtain in this case a functor

$$\mathcal{C}_q\colon \mathsf{sMod}\to \mathsf{dgMod}_N.$$

**Remark 5.19.** The boundary map (5.29) was introduced by Dubois-Violette [104, Example 2] (in the dual context of cochain complexes). Kapranov had considered a different boundary map with $n$-th component

$$\sum_{i=0}^{n} q^i d_i. \tag{5.30}$$

Note that both boundary maps reduce to (5.18) when $q = -1$. In addition, if $q$ is an $N$-th root of unity and $q \neq 1$, then (5.30) also satisfies $\partial^N = 0$ [187, Proposition 0.2].

We show below that the boundary map (5.29) commutes with the Alexander–Whitney map, while (5.30) does not (unless $q = -1$). On the other hand, simple calculations show that neither boundary commutes with the $q$-Eilenberg–Zilber map.

**Proposition 5.20.** *The Alexander–Whitney map commutes with the boundary maps of $\mathcal{C}_q(X \times Y)$ and $\mathcal{C}_q(X) \cdot_q \mathcal{C}_q(Y)$.*

Proof. We have to verify the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{C}_q(X \times Y) \arrow[r, "\partial"] \arrow[d, "\psi"'] & \mathcal{C}_q(X \times Y) \arrow[d, "\psi"] \\
\mathcal{C}_q(X) \cdot_q \mathcal{C}_q(Y) \arrow[r, "\partial\cdot\mathrm{id}+\tau\cdot\partial"'] & \mathcal{C}_q(X) \cdot_q \mathcal{C}_q(Y)
\end{tikzcd}$$

in which all boundary maps are instances of (5.29) and $\tau$ is as in (2.76).

Take $x \in X_n$, $y \in Y_n$. Applying $\partial$ to $x \otimes y$ we obtain

$$\sum_{h=0}^{n-1} q^h d_h(x) \otimes d_h(y) - q^{n-1} d_n(x) \otimes d_n(y).$$

Applying $\psi$ to this element we obtain

$$\sum_{h=0}^{n-1} q^h \sum_{j=0}^{n-1} (d_{j+1} \cdots d_{n-1} d_h)(x) \otimes (d_0 \cdots d_{j-1} d_h)(y) \\ - q^{n-1} \sum_{j=0}^{n-1} (d_{j+1} \cdots d_{n-1} d_n)(x) \otimes (d_0 \cdots d_{j-1} d_n)(y)$$

Using the simplicial relations (5.4), we see that

$$d_{j+1} \cdots d_{n-1} d_h = \begin{cases} d_h d_{j+2} \cdots d_n & \text{if } h \leq j, \\ d_{j+1} \cdots d_n & \text{if } h \geq j+1, \end{cases}$$

and

$$d_0 \cdots d_{j-1} d_h = \begin{cases} d_0 \cdots d_j & \text{if } h \leq j, \\ d_{h-j} d_0 \cdots d_{j-1} & \text{if } h \geq j+1. \end{cases}$$

Therefore,

$$\psi\partial(x \otimes y) = A + B - C$$

where the elements $A$, $B$, and $C$ are defined as follows.

$$\begin{aligned}
A :=& \sum_{j=0}^{n-1}\sum_{h=0}^{j} q^h (d_h d_{j+2}\cdots d_n)(x)\otimes (d_0\cdots d_j)(y),\\
B :=& \sum_{j=0}^{n-1}\sum_{h=j+1}^{n-1} q^h (d_{j+1}\cdots d_n)(x)\otimes (d_{h-j}d_0\cdots d_{j-1})(y),\\
C :=& q^{n-1}\sum_{j=0}^{n-1}(d_{j+1}\cdots d_n)(x)\otimes (d_{n-j}d_0\cdots d_{j-1})(y).
\end{aligned}$$

Consider now the other side of the diagram. Applying $\psi$ to $x\otimes y$ we obtain

$$\sum_{i=0}^{n}(d_{i+1}\cdots d_n)(x)\otimes (d_0\cdots d_{i-1})(y).$$

Applying $\partial\otimes \mathrm{id}$ and $\tau\otimes\partial$ to this element we obtain respectively

$$\sum_{i=1}^{n}\sum_{r=0}^{i-1} q^r (d_r d_{i+1}\cdots d_n)(x)\otimes (d_0\cdots d_{i-1})(y) \\ - \sum_{i=1}^{n} q^{i-1}(d_i d_{i+1}\cdots d_n)(x)\otimes (d_0\cdots d_{i-1})(y),$$

and

$$\sum_{i=0}^{n-1}\sum_{s=0}^{n-i-1} q^i (d_{i+1}\cdots d_n)(x)\otimes q^s (d_s d_0\cdots d_{i-1})(y) \\ - \sum_{i=0}^{n-1} q^i (d_{i+1}\cdots d_n)(x)\otimes q^{n-i-1}(d_{n-i}d_0\cdots d_{i-1})(y).$$

Let $A'$, $B'$, $B''$, and $C'$ be the four summations in the two preceding formulas, so that

$$(\partial\otimes\mathrm{id}+\tau\otimes\partial)(x\otimes y) = A' - B' + B'' - C'.$$

Simple changes of variables show that

$$A = A' \qquad \text{and} \qquad C = C'.$$

On the other hand, another application of the simplicial relations shows that

$$B' = \sum_{i=1}^{n} q^{i-1}(d_i d_{i+1}\cdots d_n)(x)\otimes (d_0 d_0\cdots d_{i-2})(y).$$

In addition, letting $j := i$ and $h := i+s$ we see that

$$\begin{aligned}
B'' &= \sum_{j=0}^{n-1}\sum_{h=j}^{n-1} q^h (d_{j+1}\cdots d_n)(x)\otimes (d_{h-j}d_0\cdots d_{j-1})(y)\\
&= \sum_{j=0}^{n-1} q^j (d_{j+1}\cdots d_n)(x)\otimes (d_0 d_0\cdots d_{j-1})(y)\\
&\qquad + \sum_{j=0}^{n-1}\sum_{h=j+1}^{n-1} q^h (d_{j+1}\cdots d_n)(x)\otimes (d_{h-j}d_0\cdots d_{j-1})(y)\\
&= B' + B.
\end{aligned}$$

Thus
$$B = B'' - B'$$
and
$$\psi\partial(x \otimes y) = (\partial \otimes \mathrm{id} + \tau \otimes \partial)(x \otimes y)$$
as needed. □

**Remark 5.21.** If we use Kapranov's boundary map (5.30) instead of Dubois-Violette's (5.29), the somewhat delicate cancellation $B = B'' - B'$ that took place in the above proof does not hold anymore, and the commutativity between the Alexander–Whitney maps and the boundary maps is lost.

Proposition 5.20 allows us to view $\psi_{X,Y}$ as a morphism in $\mathsf{dgMod}_N$. Suppose that $N = \infty$, or else that $2 \leq N < \infty$ and (5.28) is satisfied. In particular, $\mathsf{dgMod}_N$ is a monoidal category. Under these hypotheses:

**Proposition 5.22.** *The functor*
$$(\mathcal{C}_q, \psi) \colon (\mathsf{sMod}, \times) \to (\mathsf{dgMod}_N, \cdot_q)$$
*is colax monoidal.*

Proof. This follows from the fact that the functor
$$(\mathcal{C}, \psi) \colon (\mathsf{sMod}, \times) \to (\mathsf{gMod}, \cdot)$$
is colax monoidal (Theorem 5.17). □

CHAPTER 6

# 2-Monoidal Categories

In Chapter 3 we studied braided monoidal categories and bilax monoidal functors in detail. We now consider 2-monoidal categories. The main motivation for considering these objects is that they provide a more general and natural context for bilax monoidal functors, as well as for braided (co)lax monoidal functors.

A category is 2-monoidal when it possesses two monoidal structures related by an interchange law. Notions of this sort have been considered in the literature. We provide a precise definition in Section 6.1 which differs from those notions in various respects, as we explain in Remark 6.2. Coherence for 2-monoidal categories is discussed in Section 6.2.

A braided monoidal category provides an example of a 2-monoidal category in which the two monoidal structures coincide. The interchange law is built from the braiding. This is explained in Section 6.3. Strong 2-monoidal categories (those for which the interchange law and other structure maps are invertible) are necessarily of this form. For arbitrary 2-monoidal categories the two monoidal structures may be different, and the notion of 2-monoidal category is more general than that of a braided monoidal category. Several examples of this kind are discussed in Section 6.4.

In a 2-monoidal category one may define bimonoids, double monoids, and double comonoids. When a braided monoidal category is viewed as a 2-monoidal category, a double (co)monoid is simply a (co)commutative monoid (Proposition 6.29). These and related notions along with many examples are studied in Sections 6.5, 6.6 and 6.7. Similarly, monoidal functors between 2-monoidal categories may be bilax, double lax, or double colax, as discussed in Section 6.8. These notions generalize those of bilax, braided lax, and braided colax monoidal functor between braided monoidal categories (Proposition 6.59). Sections 6.9 and 6.10 provide examples. For instance: In a monoidal category with coproducts, the free monoid on an object admits a description which is similar to that of the tensor algebra on a vector space. This construction gives rise to a bilax monoidal functor between 2-monoidal categories.

An important feature of the theory of 2-monoidal categories presented here is that it emerges from considerations of higher category theory. For instance, a 2-monoidal category is precisely a pseudomonoid in an appropriate monoidal 2-category, and a bilax functor is a morphism of pseudomonoids. This viewpoint is explained in Section 6.11. The necessary background on 2-categories is given in Appendix C.

In Section 6.12 we extend the contragredient construction of Section 3.10 to the context of 2-monoidal categories.

## 6.1. The basic theory of 2-monoidal categories

This section deals with basic notions related to 2-monoidal categories. These categories are equipped with two monoidal structures related by an interchange law. They are such that each monoidal structure defines a monoidal functor with respect to the other. We also define strong 2-monoidal categories and braided 2-monoidal categories; both notions are closely related to the familiar notion of braided monoidal categories.

### 6.1.1. Definition.

**Definition 6.1.** A 2-*monoidal category* is a five tuple $(\mathsf{C}, \diamond, I, \star, J)$ where $(\mathsf{C}, \diamond, I)$ and $(\mathsf{C}, \star, J)$ are monoidal categories with units $I$ and $J$ respectively, along with a transformation (called the *interchange law*)

$$\zeta_{A,B,C,D} \colon (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D) \tag{6.1}$$

which is natural in $A, B, C$ and $D$, and three morphisms

$$\Delta_I \colon I \to I \star I, \qquad \mu_J \colon J \diamond J \to J, \qquad \iota_J = \epsilon_I \colon I \to J, \tag{6.2}$$

such that the axioms below are satisfied.

**Associativity.** The following diagrams commute.

$$\begin{tikzcd}
\big((A \star B) \diamond (C \star D)\big) \diamond (E \star F) \arrow[r, "\alpha"] \arrow[d, "\zeta \diamond \mathrm{id}"'] & (A \star B) \diamond \big((C \star D) \diamond (E \star F)\big) \arrow[d, "\mathrm{id} \diamond \zeta"] \\
\big((A \diamond C) \star (B \diamond D)\big) \diamond (E \star F) \arrow[d, "\zeta"'] & (A \star B) \diamond \big((C \diamond E) \star (D \diamond F)\big) \arrow[d, "\zeta"] \\
((A \diamond C) \diamond E) \star ((B \diamond D) \diamond F) \arrow[r, "\alpha \star \alpha"'] & \big(A \diamond (C \diamond E)\big) \star \big(B \diamond (D \diamond F)\big)
\end{tikzcd} \tag{6.3}$$

$$\begin{tikzcd}
((A \star B) \star C) \diamond ((D \star E) \star F) \arrow[r, "\alpha \diamond \alpha"] \arrow[d, "\zeta"'] & \big(A \star (B \star C)\big) \diamond \big(D \star (E \star F)\big) \arrow[d, "\zeta"] \\
\big((A \star B) \diamond (D \star E)\big) \star (C \diamond F) \arrow[d, "\zeta \star \mathrm{id}"'] & (A \diamond D) \star \big((B \star C) \diamond (E \star F)\big) \arrow[d, "\mathrm{id} \star \zeta"] \\
\big((A \diamond D) \star (B \diamond E)\big) \star (C \diamond F) \arrow[r, "\alpha"'] & (A \diamond D) \star \big((B \diamond E) \star (C \diamond F)\big)
\end{tikzcd} \tag{6.4}$$

**Unitality.** The following diagrams commute.

$$\begin{tikzcd}
I \diamond (A \star B) \arrow[r, "\Delta_I \diamond \mathrm{id}"] & (I \star I) \diamond (A \star B) \arrow[d, "\zeta"] \\
A \star B \arrow[u, "\lambda_{A \star B}"] \arrow[r, "\lambda_A \star \lambda_B"'] & (I \diamond A) \star (I \diamond B)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \star B) \diamond I \arrow[r, "\mathrm{id} \diamond \Delta_I"] & (A \star B) \diamond (I \star I) \arrow[d, "\zeta"] \\
A \star B \arrow[u, "\rho_{A \star B}"] \arrow[r, "\rho_A \star \rho_B"'] & (A \diamond I) \star (B \diamond I)
\end{tikzcd} \tag{6.5}$$

$$
\begin{tikzcd}
J \star (A \diamond B) & (J \diamond J) \star (A \diamond B) \arrow[l, "\mu_J \star \mathrm{id}"'] \\
A \diamond B \arrow[u, "\lambda_{A \diamond B}"] \arrow[r, "\lambda_A \diamond \lambda_B"'] & (J \star A) \diamond (J \star B) \arrow[u, "\zeta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \diamond B) \star J & (A \diamond B) \star (J \diamond J) \arrow[l, "\mathrm{id} \star \mu_J"'] \\
A \diamond B \arrow[u, "\rho_{A \diamond B}"] \arrow[r, "\rho_A \diamond \rho_B"'] & (A \star J) \diamond (B \star J) \arrow[u, "\zeta"']
\end{tikzcd}
\tag{6.6}
$$

**Compatibility of units.** The units $I$ and $J$ are compatible in the following sense.

$$(J, \mu_J, \iota_J) \text{ is a monoid in } (\mathsf{C}, \diamond, I). \tag{6.7}$$

$$(I, \Delta_I, \epsilon_I) \text{ is a comonoid in } (\mathsf{C}, \star, J). \tag{6.8}$$

The arrows labeled $\alpha$, $\lambda$ and $\rho$ in the diagrams above refer to the associativity and unit constraints in either monoidal category.

Sometimes we write $(\mathsf{C}, \diamond, \star)$ for a 2-monoidal category with the units, interchange law (6.1), and structure maps (6.2) being understood. The interchange law is sometimes called the *middle-four interchange* [90]. The structure map $\iota_J = \epsilon_I$ is determined by the rest of the structure; see Proposition 6.9.

**Remark 6.2.** Let us argue in favor of our definition of 2-monoidal categories. Related notions appear in the literature but with important differences.

The *two-fold monoidal categories* of Balteanu and Fiedorowicz [31, 204] involve two monoidal structures which are required to be strict and to share the unit object ($I = J$). While the former assumption may not be crucial, the latter fails in many of the examples we are interested in (see Section 6.4). Two-fold monoidal categories thus appear as a somewhat unnatural special case of 2-monoidal categories. Forcey, Siehler, and Sowers [132] improve on this notion by removing the strictness assumption and allowing the unit objects to be distinct, but the structure maps $\Delta_I$ and $\mu_J$ (6.2) are assumed to be isomorphisms. This again fails in most of our examples.

Another related notion appears in recent work of Vallette [363, Section 1.2], under the name of *lax* 2*-monoidal category*; this notion involves fewer structure morphisms and fewer axioms than our notion of 2-monoidal category. We do not know of any examples in which only these axioms are satisfied. In [363, Section 1.3], Vallette defines a colax 2-monoidal category and then combines the two to define a 2-monoidal category. This is different from what we do (compare his definition with the alternative definition we give in Proposition 6.4) and again leaves out most of our examples.

In addition to the examples of Section 6.4, support for Definition 6.1 is provided by various results of later sections, such as Proposition 6.4 and most notably Proposition 6.73, which shows that our notion of 2-monoidal category is an instance of a general notion in higher category theory (that of a pseudomonoid in a monoidal 2-category).

**Definition 6.3.** We say that a 2-monoidal category is *strong* if the structure morphisms (6.1) and (6.2) are isomorphisms.

The notion of a strong 2-monoidal category is not truly a new one: a result of Joyal and Street implies that this notion is equivalent to that of a braided monoidal category. We recall this fact with more explanation in Section 6.3.

**6.1.2. Opposite and transposes.** Let $(\mathsf{C}, \diamond, I, \star, J)$ be a 2-monoidal category with interchange law $\zeta$. The symmetry between the two monoidal structures which is evident in Definition 6.1 implies that $(\mathsf{C}^{\mathrm{op}}, \star, J, \diamond, I)$ is also a 2-monoidal category. We denote it simply by $\mathsf{C}^{\mathrm{op}}$ and call it the *opposite* 2-monoidal category of $\mathsf{C}$. The interchange law $\zeta^{\mathrm{op}}$ is given by the maps

$$\zeta^{\mathrm{op}}_{A,B,C,D} \colon (A \diamond B) \star (C \diamond D) \to (A \star C) \diamond (B \star D) \tag{6.9}$$

of the category $\mathsf{C}^{\mathrm{op}}$ which correspond to the maps

$$\zeta_{A,C,B,D} \colon (A \star C) \diamond (B \star D) \to (A \diamond B) \star (C \diamond D)$$

of the category $\mathsf{C}$.

Define a new monoidal structure $\tilde{\diamond}$ on $\mathsf{C}$ by

$$A \mathbin{\tilde{\diamond}} B := B \diamond A.$$

The category $(\mathsf{C}, \tilde{\diamond}, I, \star, J)$ is 2-monoidal, with interchange law $\zeta^{t_\diamond}$ given by the maps

$$\zeta^{t_\diamond}_{A,B,C,D} \colon (A \star B) \mathbin{\tilde{\diamond}} (C \star D) \to (A \mathbin{\tilde{\diamond}} C) \star (B \mathbin{\tilde{\diamond}} D) \tag{6.10}$$

which are equal to the maps

$$\zeta_{C,D,A,B} \colon (C \star D) \diamond (A \star B) \to (C \diamond A) \star (D \diamond B).$$

We denote this 2-monoidal category by $\mathsf{C}^{t_\diamond}$ and call it the $\diamond$-*transpose* of $\mathsf{C}$.

The $\star$-*transpose* is the 2-monoidal category $(\mathsf{C}, \diamond, I, \tilde{\star}, J)$ with $A \mathbin{\tilde{\star}} B := B \star A$ and interchange law $\zeta^{t_\star}_{A,B,C,D} := \zeta_{B,A,D,C}$. We denote it by $\mathsf{C}^{t_\star}$.

Combining the two constructions we obtain the *transpose* $(\mathsf{C}, \tilde{\diamond}, I, \tilde{\star}, J)$, denoted $\mathsf{C}^t$. The interchange law is $\zeta^t_{A,B,C,D} := \zeta_{D,C,B,A}$.

Finally, we mention that if $(\mathsf{C}, \diamond, \star)$ is a strong 2-monoidal category, then so is $(\mathsf{C}, \star, \diamond)$. The interchange law for the latter is $\zeta^{-1}$.

**6.1.3. The interchange law: a pictorial representation.** The transformation $\zeta$ is called an *interchange law* between the two monoidal structures on $\mathsf{C}$. We represent the two terms in (6.1) by means of pictures as below.

$$\frac{A \quad B}{C \quad D} \longrightarrow \begin{array}{c|c} A & B \\ C & D \end{array} \tag{6.11}$$

The lines indicate which operation takes precedence: the horizontal line in the picture on the left indicates that the operation $\star$ is first applied to each pair of objects on the same side of the line, then the operation $\diamond$ is applied to the results; the vertical line gives precedence to the operation $\diamond$.

Other pictures involving more objects can be similarly interpreted. For instance, diagrams (6.3) and (6.4) can be represented as follows.

$$
\begin{tikzcd}
\begin{array}{cc} A & B \\ \hline C & D \\ \hline E & F \end{array} \arrow[r] \arrow[d] & \begin{array}{c|c} A & B \\ C & D \\ \hline \multicolumn{2}{c}{E \quad F} \end{array} \arrow[d] \\
\begin{array}{c|c} \multicolumn{2}{c}{A \quad B} \\ \hline C & D \\ E & F \end{array} \arrow[r] & \begin{array}{c|c} A & B \\ C & D \\ E & F \end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{array}{ccc} A & B & C \\ \hline D & E & F \end{array} \arrow[r] \arrow[d] & \begin{array}{cc|c} A & B & C \\ \cline{1-2} D & E & F \end{array} \arrow[d] \\
\begin{array}{c|cc} A & B & C \\ \cline{2-3} D & E & F \end{array} \arrow[r] & \begin{array}{c|c|c} A & B & C \\ D & E & F \end{array}
\end{tikzcd}
\tag{6.12}
$$

(The associativity constraints have been suppressed for convenience.)

Similarly, diagrams (6.5) and (6.6) can be expressed as follows.

$$
\begin{tikzcd}
\dfrac{I \quad I}{A \quad B} \arrow[d] & A \quad B \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & \dfrac{A \quad B}{I \quad I} \arrow[d] \\
\begin{array}{c|c} I & I \\ A & B \end{array} \arrow[r] & A \quad B & \begin{array}{c|c} A & B \\ I & I \end{array} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\dfrac{J \quad A}{J \quad B} \arrow[d] & \dfrac{A}{B} \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & \dfrac{A \quad J}{B \quad J} \arrow[d] \\
\begin{array}{c|c} J & A \\ J & B \end{array} \arrow[r] & \dfrac{A}{B} & \begin{array}{c|c} A & J \\ B & J \end{array} \arrow[l]
\end{tikzcd}
\tag{6.13}
$$

The unit and counit conditions in (6.7) and (6.8) can be expressed as follows.

$$
\begin{tikzcd}
I \mid I \arrow[d] & I \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & I \mid I \arrow[d] \\
J \mid I \arrow[r] & I & I \mid J \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\dfrac{J}{I} \arrow[d] & J \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & \dfrac{I}{J} \arrow[d] \\
\dfrac{J}{J} \arrow[r] & J & \dfrac{J}{J} \arrow[l]
\end{tikzcd}
\tag{6.14}
$$

**6.1.4. An equivalent description of 2-monoidal categories.** There are alternative descriptions for the notion of 2-monoidal category. One is given in the result below. Two other equivalent descriptions are given in Proposition 6.73; also see Remark 6.74.

Let $(\mathsf{I}, \bullet)$ be the one-arrow category and let $*$ denote its unique object.

**Proposition 6.4.** *Let $(\mathsf{C}, \diamond, I)$ and $(\mathsf{C}, \star, J)$ be monoidal categories equipped with structure morphisms (6.1) and (6.2). Then $(\mathsf{C}, \diamond, I, \star, J)$ is a (strong) 2-monoidal category if and only if both conditions* (i) *and* (ii) *below hold.*

(i) *The functors*

$$\star \colon (\mathsf{C} \times \mathsf{C}, \diamond) \to (\mathsf{C}, \diamond) \quad (A, B) \mapsto A \star B \qquad \mathcal{J} \colon \mathsf{I} \to (\mathsf{C}, \diamond) \quad * \mapsto J$$

*are lax (strong).*

(ii) *The functors*

$$\diamond \colon (\mathsf{C} \times \mathsf{C}, \star) \to (\mathsf{C}, \star) \quad (A, B) \mapsto A \diamond B \qquad \mathcal{I} \colon \mathsf{I} \to (\mathsf{C}, \star) \quad * \mapsto I$$

*are colax (costrong).*

Proof. One checks that condition (i) is equivalent to axioms (6.3), (6.5) and (6.7). Similarly, condition (ii) is equivalent to axioms (6.4), (6.6) and (6.8). $\square$

Other types of compatibilities between the functors $\diamond$ and $\star$ lead to higher monoidal categories, which are the object of Chapter 7. For instance, the case when $(\mathsf{C}, \star)$ is braided and the functor $\diamond\colon (\mathsf{C} \times \mathsf{C}, \star) \to (\mathsf{C}, \star)$ is bilax occurs when $(\mathsf{C}, \star, \diamond, \star)$ is a 3-monoidal category. The case when both monoidal categories are braided and each functor is braided bilax with respect to the other occurs when $(\mathsf{C}, \star, \diamond, \star, \diamond)$ and $(\mathsf{C}, \diamond, \star, \diamond, \star)$ are 4-monoidal categories. These facts are discussed in more detail in Proposition 7.4 and 7.27.

**6.1.5. Braided 2-monoidal categories.** Braided monoidal categories formalize the concept of commutativity for monoidal categories. We briefly discuss the analogous concept for 2-monoidal categories.

**Definition 6.5.** A 2-monoidal category $(\mathsf{C}, \diamond, I, \star, J)$ is $\diamond$-*braided* if the monoidal category $(\mathsf{C}, \diamond, I)$ is braided and the following diagrams commute.

$$
\begin{tikzcd}
(A \star B) \diamond (C \star D) \arrow[r, "\zeta"] \arrow[d, "\beta"'] & (A \diamond C) \star (B \diamond D) \arrow[d, "\beta \star \beta"] \\
(C \star D) \diamond (A \star B) \arrow[r, "\zeta"'] & (C \diamond A) \star (D \diamond B)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & J \diamond J \arrow[dl, "\mu_J"'] \arrow[dd, "\beta"] \\
J & \\
 & J \diamond J \arrow[ul, "\mu_J"]
\end{tikzcd}
\tag{6.15}
$$

Similarly, a 2-monoidal category $(\mathsf{C}, \diamond, I, \star, J)$ is $\star$-*braided* if the monoidal category $(\mathsf{C}, \star, J)$ is braided and the following diagrams commute.

$$
\begin{tikzcd}
(A \star B) \diamond (C \star D) \arrow[r, "\zeta"] \arrow[d, "\beta \diamond \beta"'] & (A \diamond C) \star (B \diamond D) \arrow[d, "\beta"] \\
(B \star A) \diamond (D \star C) \arrow[r, "\zeta"'] & (B \diamond D) \star (A \diamond C)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I \star I \arrow[dd, "\beta"] \\
I \arrow[ur, "\Delta_I"] \arrow[dr, "\Delta_I"'] & \\
 & I \star I
\end{tikzcd}
\tag{6.16}
$$

Further, a 2-monoidal category is *braided* if it is braided with respect to both monoidal structures.

The first diagrams in (6.15) and (6.16) can be shown pictorially as follows.

$$
\begin{tikzcd}
\dfrac{A \quad B}{C \quad D} \arrow[r, "\zeta"] \arrow[d, "\beta"'] & \begin{array}{c|c} A & B \\ C & D \end{array} \arrow[d, "\beta \star \beta"] \\
\dfrac{C \quad D}{A \quad B} \arrow[r, "\zeta"'] & \begin{array}{c|c} C & D \\ A & B \end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
\dfrac{A \quad B}{C \quad D} \arrow[r, "\zeta"] \arrow[d, "\beta \diamond \beta"'] & \begin{array}{c|c} A & B \\ C & D \end{array} \arrow[d, "\beta"] \\
\dfrac{B \quad A}{D \quad C} \arrow[r, "\zeta"'] & \begin{array}{c|c} B & A \\ D & C \end{array}
\end{tikzcd}
\tag{6.17}
$$

In the first diagram, we switch objects vertically, while in the second diagram, we switch them horizontally.

Definition 6.5 can be understood in terms of the functors of Proposition 6.4:

**Proposition 6.6.** *A 2-monoidal category is $\diamond$-braided if and only if the functors $\star$ and $\mathcal{J}$ are braided lax. Similarly, a 2-monoidal category is $\star$-braided if and only if the functors $\diamond$ and $\mathcal{I}$ are braided colax.*

## 6.2. Coherence

There is a coherence result for 2-monoidal categories, which we briefly indicate. It can be deduced from a coherence result on lax functors given in [230].

First, let $A$ and $B$ be objects of a monoidal 2-category $\mathsf{C}$. If two morphisms $A \to B$ are constructed out of the structure maps in $\mathsf{C}$ (including the structure constraints of the monoidal categories $(\mathsf{C}, \diamond, I)$ and $(\mathsf{C}, \star, J)$), then they coincide. For instance, the following diagrams, in which each arrow involves a unit constraint of one of the monoidal categories (and hence is invertible), commute.

(6.18)
$$\begin{tikzcd}
 & & J \arrow[dll] \arrow[d] \arrow[drr] & & \\
I \diamond J \arrow[d] & & J \star J \arrow[dll] \arrow[drr] & & J \diamond I \arrow[d] \\
(I \diamond J) \star J \arrow[drr] & & & & J \star (J \diamond I) \arrow[dll] \\
 & & (I \diamond J) \star (J \diamond I) & &
\end{tikzcd}$$

Second, let $A_{ij}$ be objects in a 2-monoidal category $\mathsf{C}$, with $1 \leq i \leq m$, $1 \leq j \leq n$. In this situation, there is a morphism

(6.19)
$$\mathop{\diamond}_{j \in [n]} \Big( \mathop{\star}_{i \in [m]} A_{ij} \Big) \longrightarrow \mathop{\star}_{i \in [m]} \Big( \mathop{\diamond}_{j \in [n]} A_{ij} \Big)$$

constructed out of the structure maps in $\mathsf{C}$, and it is the unique such. The notation stands for unordered tensor products (Section 1.4). Since these are taken over the canonically ordered sets $[m]$ and $[n]$, they may also be interpreted as unbracketed tensor products.

The first array in Table 6.1 shows instances of these canonical morphisms, along with their $m$ and $n$ values. Each entry in the next two arrays refers to an axiom and the corresponding $m$ and $n$ value. In each axiom, one equalizes certain morphisms to guarantee uniqueness in (6.19). If either $m$ or $n$ is 1, then the morphism in (6.19) is the identity.

Table 6.1. Coherence for 2-monoidal categories.

| $m$ | $n$ | Morphism |
|---|---|---|
| 2 | 2 | $\zeta$ |
| 2 | 0 | $\Delta_I$ |
| 0 | 2 | $\mu_J$ |
| 0 | 0 | $\iota_J = \epsilon_I$ |

| $m$ | $n$ | Mor. | Eqn. |
|---|---|---|---|
| 2 | 3 | | (6.3) |
| 3 | 2 | | (6.4) |
| 2 | 1 | id | (6.5) |
| 1 | 2 | id | (6.6) |

| $m$ | $n$ | Mor. | Eqn. |
|---|---|---|---|
| 3 | 0 | | (6.8) |
| 0 | 3 | | (6.7) |
| 1 | 0 | id | (6.8) |
| 0 | 1 | id | (6.7) |

We provide an example of the uniqueness in (6.19). The composite of the morphisms below

$$\begin{array}{cc} A & B \\ \hline C & D \end{array} \longrightarrow \begin{array}{ccc} A & J & B \\ \hline I & I & I \\ \hline C & J & D \end{array} \longrightarrow \begin{array}{c|c|c} A & J & B \\ I & I & I \\ C & J & D \end{array} \longrightarrow \begin{array}{c|c|c} A & J & B \\ I & J & I \\ C & J & D \end{array} \longrightarrow \begin{array}{c|c} A & B \\ C & D \end{array}$$

is equal to the interchange law (6.11).

We now discuss some diagrams that commute in any 2-monoidal category. Their commutativity follows from coherence, but we provide direct proofs. When considering the diagrams below, recall that $\iota_J = \epsilon_I$ according to Definition 6.1. Among other things, we show below that this map is determined by the interchange law and the unit constraints of the two monoidal categories. The latter occur in the arrows labeled $\cong$ in the diagrams below.

**Proposition 6.7.** *In any 2-monoidal category, the following diagrams commute.*

$$\begin{tikzcd} I \arrow[r, "\iota_J"] \arrow[d, "\Delta_I"'] & J \arrow[d, "\cong"] \\ I \star I \arrow[r, "\iota_J \star \iota_J"'] & J \star J \end{tikzcd} \qquad \begin{tikzcd} J \diamond J \arrow[r, "\epsilon_I \diamond \epsilon_I"] \arrow[d, "\mu_J"'] & I \diamond I \arrow[d, "\cong"] \\ J \arrow[r, "\epsilon_I"'] & I \end{tikzcd}$$

$$\begin{tikzcd} I \diamond I \arrow[r, "\cong"] \arrow[d, "\Delta_I \diamond \Delta_I"'] & I \arrow[r, "\Delta_I"] & I \star I \arrow[d, "\cong"] \\ (I \star I) \diamond (I \star I) \arrow[rr, "\zeta"'] & & (I \diamond I) \star (I \diamond I) \end{tikzcd} \qquad \begin{tikzcd} J \diamond J \arrow[r, "\mu_J"] \arrow[d, "\cong"'] & J \arrow[r, "\cong"] & J \star J \\ (J \star J) \diamond (J \star J) \arrow[rr, "\zeta"'] & & (J \diamond J) \star (J \diamond J) \arrow[u, "\mu_J \star \mu_J"'] \end{tikzcd}$$

Proof. The first diagram is a consequence of counitality for the comonoid $(I, \Delta_I, \epsilon_I)$, since $\epsilon_I = \iota_J$. The third diagram follows from either diagram in (6.5). The other two follow by passing to the opposite category. $\square$

**Proposition 6.8.** *For any objects $A$ and $B$ of a 2-monoidal category, the following diagrams commute.*

$$\begin{tikzcd} A \star B \arrow[r, "\cong"] \arrow[d, "\cong"'] & I \diamond (A \star B) \arrow[d, "\cong"] \\ A \star (I \diamond B) \arrow[d, "\mathrm{id} \star (\epsilon_I \diamond \mathrm{id})"'] & (I \star J) \diamond (A \star B) \arrow[d, "\zeta"] \\ A \star (J \diamond B) \arrow[r, "\cong"'] & (I \diamond A) \star (J \diamond B) \end{tikzcd} \qquad \begin{tikzcd} A \star B \arrow[r, "\cong"] \arrow[d, "\cong"'] & (A \star B) \diamond I \arrow[d, "\cong"] \\ A \star (B \diamond I) \arrow[d, "\mathrm{id} \star (\mathrm{id} \diamond \epsilon_I)"'] & (A \star B) \diamond (I \star J) \arrow[d, "\zeta"] \\ A \star (B \diamond J) \arrow[r, "\cong"'] & (A \diamond I) \star (B \diamond J) \end{tikzcd}$$

$$
\begin{tikzcd}
A \star B \arrow[r, "\cong"] \arrow[d, "\cong"'] & (A \star B) \diamond I \arrow[d, "\cong"] \\
(A \diamond I) \star B \arrow[d, "(\mathrm{id} \diamond \epsilon_I) \star \mathrm{id}"'] & (A \star B) \diamond (J \star I) \arrow[d, "\zeta"] \\
(A \diamond J) \star B \arrow[r, "\cong"'] & (A \diamond J) \star (B \diamond I)
\end{tikzcd}
\qquad
\begin{tikzcd}
A \star B \arrow[r, "\cong"] \arrow[d, "\cong"'] & I \diamond (A \star B) \arrow[d, "\cong"] \\
(I \diamond A) \star B \arrow[d, "(\epsilon_I \diamond \mathrm{id}) \star \mathrm{id}"'] & (J \star I) \diamond (A \star B) \arrow[d, "\zeta"] \\
(J \diamond A) \star B \arrow[r, "\cong"'] & (J \diamond A) \star (I \diamond B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \diamond B & J \star (A \diamond B) \arrow[l, "\cong"'] \\
A \diamond (J \star B) \arrow[u, "\cong"] & (J \diamond I) \star (A \diamond B) \arrow[u, "\cong"'] \\
A \diamond (I \star B) \arrow[u, "\mathrm{id} \diamond (\iota_J \star \mathrm{id})"] & (J \star A) \diamond (I \star B) \arrow[u, "\zeta"'] \arrow[l, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \diamond B & (A \diamond B) \star J \arrow[l, "\cong"'] \\
A \diamond (B \star J) \arrow[u, "\cong"] & (A \diamond B) \star (J \diamond I) \arrow[u, "\cong"'] \\
A \diamond (B \star I) \arrow[u, "\mathrm{id} \diamond (\mathrm{id} \star \iota_J)"] & (A \star J) \diamond (B \star I) \arrow[u, "\zeta"'] \arrow[l, "\cong"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \diamond B & (A \diamond B) \star J \arrow[l, "\cong"'] \\
(A \star J) \diamond B \arrow[u, "\cong"] & (A \diamond B) \star (I \diamond J) \arrow[u, "\cong"'] \\
(A \star I) \diamond B \arrow[u, "(\mathrm{id} \star \iota_J) \diamond \mathrm{id}"] & (A \star I) \diamond (B \star J) \arrow[u, "\zeta"'] \arrow[l, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \diamond B & J \star (A \diamond B) \arrow[l, "\cong"'] \\
(J \star A) \diamond B \arrow[u, "\cong"] & (I \diamond J) \star (A \diamond B) \arrow[u, "\cong"'] \\
(I \star A) \diamond B \arrow[u, "(\epsilon_J \star \mathrm{id}) \diamond \mathrm{id}"] & (I \star A) \diamond (J \star B) \arrow[u, "\zeta"'] \arrow[l, "\cong"]
\end{tikzcd}
$$

Proof. We fill in the first diagram with commutative pieces as follows.

$$
\begin{tikzcd}
A \star B \arrow[r, "\cong"] \arrow[dd, "\cong"'] & I \diamond (A \star B) \arrow[r, "\cong"] \arrow[d, "\Delta_I \star \mathrm{id}"'] & (I \star J) \diamond (A \star B) \arrow[ddd, "\zeta"] \\
& (I \star I) \diamond (A \star B) \arrow[ur, "(\mathrm{id} \star \epsilon_I) \diamond \mathrm{id}"'] \arrow[d, "\zeta"'] & \\
A \star (I \diamond B) \arrow[r, "\cong"] \arrow[d, "\mathrm{id} \star (\epsilon_I \diamond \mathrm{id})"'] & (I \diamond A) \star (I \diamond B) \arrow[dr, "\mathrm{id} \star (\epsilon_I \diamond \mathrm{id})"'] & \\
A \star (J \diamond B) \arrow[rr, "\cong"'] & & (I \diamond A) \star (J \diamond B)
\end{tikzcd}
$$

In this diagram, the top left corner commutes by (6.5), and the triangle next to it by counitality (6.8). The remaining pieces commute by naturality.

The commutativity of the remaining diagrams follows by passing to the transpose or opposite categories. □

The following result shows that $\iota_J = \epsilon_I$ is determined by the interchange law and the unit constraints of the two monoidal categories. The horizontal arrows in the diagrams below are uniquely determined in view of (6.18) (and similar diagrams).

**Proposition 6.9.** *In any 2-monoidal category, the following diagrams commute.*

$$\begin{tikzcd}
(I \star J) \diamond (J \star I) \arrow[d, "\zeta_{I,J,J,I}"'] & I \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\iota_J = \epsilon_I"'] & (J \star I) \diamond (I \star J) \arrow[d, "\zeta_{J,I,I,J}"] \\
(I \diamond J) \star (J \diamond I) & J \arrow[l, "\cong"] \arrow[r, "\cong"'] & (J \diamond I) \star (I \diamond J)
\end{tikzcd}$$

Proof. These can be deduced from the first two diagrams in Proposition 6.8 by setting $A = J$, $B = I$. ☐

In pictorial notation, this can be shown as follows.

$$\begin{tikzcd}
\begin{array}{cc} I & J \\ \hline J & I \end{array} \arrow[d] & I \arrow[l] \arrow[r] \arrow[d] & \begin{array}{cc} J & I \\ \hline I & J \end{array} \arrow[d] \\
\begin{array}{c|c} I & J \\ J & I \end{array} & J \arrow[l] \arrow[r] & \begin{array}{c|c} J & I \\ I & J \end{array}
\end{tikzcd}$$

### 6.3. Braided monoidal categories as 2-monoidal categories

In this section, we relate 2-monoidal categories to braided monoidal categories. More precisely, we show that a strong 2-monoidal category (Definition 6.3) is the same as a braided monoidal category. Further, a strong braided 2-monoidal category is the same as a symmetric monoidal category.

**Proposition 6.10.** *A braided monoidal category gives rise to a strong 2-monoidal category both of whose monoidal structures are identical.*

Proof. Let $(\mathsf{C}, \bullet, I, \beta)$ be a braided monoidal category. Then define a strong 2-monoidal category $(\mathsf{C}, \bullet, I, \bullet, I)$ both of whose monoidal structures are identical, with the interchange law given by the following composite

$$(A \bullet B) \bullet (C \bullet D) \xrightarrow{\cong} A \bullet (B \bullet C) \bullet D \xrightarrow{\mathrm{id} \bullet \beta \bullet \mathrm{id}} A \bullet (C \bullet B) \bullet D \xrightarrow{\cong} (A \bullet C) \bullet (B \bullet D)$$

and the remaining structure morphisms given by

$$\lambda_I \colon I \to I \bullet I, \qquad \lambda_I^{-1} \colon I \bullet I \to I, \qquad \mathrm{id} \colon I \to I.$$

The arrows labeled $\cong$ are the unique isomorphisms obtained from the associativity constraint in $\mathsf{C}$, while $\lambda$ refers to the left unit constraint in $\mathsf{C}$. It does not matter whether one uses $\lambda$ or $\rho$ since $\lambda_I = \rho_I$. It is straightforward to check that the 2-monoidal category axioms hold. ☐

Proposition 6.10 admits the following converse.

**Proposition 6.11.** *Let $(\mathsf{C}, \diamond, I, \star, J)$ be a strong 2-monoidal category. Then the monoidal categories $(\mathsf{C}, \diamond, I)$ and $(\mathsf{C}, \star, J)$ are braided and isomorphic as braided monoidal categories.*

*Moreover, the interchange law arises from the braiding as in Proposition 6.10 (and the isomorphisms between $\diamond$ and $\star$).*

PROOF. We outline the main steps. The units $I$ and $J$ are isomorphic, since the 2-monoidal category is strong. Choosing $B$ and $C$ equal to either unit object in (6.1), we deduce that the two monoidal structures are isomorphic. Finally, choosing $A$ and $D$ equal to either unit object in (6.1), we obtain the braiding. Coherence guarantees that axioms (1.5) hold.

We now show that the braiding determines the interchange law. For simplicity of notation, we identify the two monoidal structures. Consider the following composites.

$$\frac{A}{D} \longrightarrow \frac{A \quad I}{I \quad D} \xrightarrow{\varsigma} \begin{array}{c|c} A & I \\ I & D \end{array} \longrightarrow A \mid D$$

$$\frac{B}{C} \longrightarrow \frac{I \quad B}{C \quad I} \xrightarrow{\varsigma} \begin{array}{c|c} I & B \\ C & I \end{array} \longrightarrow C \mid B$$

By construction, the first composite is the identity, while the second composite is the braiding. One can then deduce that the following composites are both the identity.

$$\frac{A}{C \quad D} \longrightarrow \frac{A \quad I}{C \quad D} \xrightarrow{\varsigma} \begin{array}{c|c} A & I \\ C & D \end{array} \longrightarrow \begin{array}{c|c} A & \\ C & D \end{array}$$

$$\frac{A \quad B}{D} \longrightarrow \frac{A \quad B}{I \quad D} \xrightarrow{\varsigma} \begin{array}{c|c} A & B \\ I & D \end{array} \longrightarrow \begin{array}{c|c} & B \\ A & D \end{array}$$

From these observations, we derive the required result: The composite

$$\frac{A \quad B}{C \quad D} \longrightarrow \begin{array}{cc} A & I \\ \hline I & B \\ \hline C & I \\ \hline I & D \end{array} \longrightarrow \begin{array}{c|c} A & I \\ I & B \\ C & I \\ I & D \end{array} \longrightarrow \begin{array}{c|c} A & B \\ C & D \end{array},$$

with the middle map defined by (6.19), is the interchange law and it is induced from the braiding. $\square$

**Remark 6.12.** Propositions 6.10 and 6.11 are reformulations of results of Joyal and Street [184, Propositions 5.2 and 5.3]. It follows that there is an equivalence between the notions of braided monoidal category and strong 2-monoidal category. In particular, any strong 2-monoidal category arises from a braided monoidal category as in Proposition 6.10, up to a 2-strong monoidal functor. (Strong monoidal functors are discussed in Section 6.8.4 below.)

Proposition 6.11 is a categorical version of the Eckmann–Hilton argument (see Proposition 6.29 and Remark 6.30). Another result in this direction is given by Kock [204, Proposition 3.2], who shows that a 2-monoidal category with strict associativity and strict interchange must come from a symmetric monoidal category (by means of the construction of Proposition 6.10) and moreover, the symmetry must satisfy $\beta_{X,X} = \mathrm{id}_{X \bullet X}$ for every object $X$.

**Proposition 6.13.** *Let* $\mathsf{C}$ *be a braided monoidal category viewed as a 2-monoidal category. Then the following are equivalent.*

(i) $\mathsf{C}$ *is* $\diamond$*-braided.*
(ii) $\mathsf{C}$ *is* $\star$*-braided.*
(iii) $\beta$ *is a symmetry.*

Proof. Let $(\mathcal{M}, \varphi, \psi)$ be as in Section 3.8.1. From the definitions or from Proposition 6.6, one has the following.

$$\text{(i)} \iff (\mathcal{M}, \varphi) \text{ is braided strong.}$$
$$\text{(ii)} \iff (\mathcal{M}, \psi) \text{ is braided costrong.}$$

Propositions 3.74 and 3.75 say that the statements on the right are both equivalent to $(\mathcal{M}, \varphi, \psi)$ being bistrong and further equivalent to $\beta$ being a symmetry. □

Thus, braided monoidal categories are same as strong 2-monoidal categories and symmetric monoidal categories are same as of strong braided 2-monoidal categories.

Recall the notion of lax braided monoidal categories from Definition 1.5. The following result, which generalizes Proposition 6.10, is of interest to us.

**Proposition 6.14.** *A lax braided monoidal category gives rise to a 2-monoidal category whose monoidal structures are identical.*

We use the notation $(\mathsf{C}, \bullet, \bullet)$ to denote a 2-monoidal category which arises from this construction.

**Example 6.15.** The braiding $\beta$ on graded vector spaces admit a one-parameter deformation denoted by $\beta_q$, see (2.50). Of special interest is the value $q = 0$. It is distinguished by the property that $\beta_0$ is not invertible, hence $\beta_0$ is not a braiding, however it is a lax braiding. The above result then implies that $(\mathsf{gVec}, \cdot, \beta_q)$ is a 2-monoidal category for all values of $q$.

**Remark 6.16.** One may wonder about the converse of Proposition 6.14: If the two monoidal structures of a 2-monoidal category coincide, is the interchange law necessarily of the form $\mathrm{id} \bullet \beta \bullet \mathrm{id}$ for a lax braiding $\beta$?

Proposition 6.11 says that the answer is positive in the strong case. Moreover, its proof shows that if $(\mathsf{C}, \bullet, I, \bullet, I, \zeta)$ is 2-monoidal, then defining $\beta_{B,C}$ by

$$\mathrm{id}_I \bullet \beta_{B,C} \bullet \mathrm{id}_I = \zeta_{I,B,C,I},$$

one obtains a lax braiding $\beta$ on $(\mathsf{C}, \bullet, I)$.

However, it does not follow (and is not true in general) that $\zeta = \mathrm{id} \bullet \beta \bullet \mathrm{id}$; see Example 6.24. So, the answer to the above question is in general negative.

## 6.4. Examples of 2-monoidal categories

There is an abundance of 2-monoidal categories in nature. In this section, we provide a variety of examples. The categories are constructed from diverse objects such as graphs, posets, vector spaces, and bimodules over a commutative algebra. We learned Examples 6.17 and 6.18 below from Steve Chase.

**Example 6.17.** Let $X$ be a set. A *directed graph with vertex set* $X$ is a triple $(A, s, t)$ where $A$ is a set and

$$A \underset{t}{\overset{s}{\rightrightarrows}} X$$

is a pair of maps, called the *source* and *target maps*, respectively. The elements $a \in A$ are called *arrows* and may be represented as follows.

$$s(a) \xrightarrow{a} t(a)$$

A morphism $(A, s, t) \to (B, s, t)$ of directed graphs with vertex set $X$ is a map $f\colon A \to B$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "s"'] & & B \arrow[dl, "s"] \\ & X & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "t"'] & & B \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

commute.

We often write $A$ instead of $(A, s, t)$ and understand that $s$ and $t$ are given.

Let $\mathsf{C}$ be the category of directed graphs with vertex set $X$. Given two such graphs $A$ and $B$, define

$$A \diamond B := \{(a, b) \in A \times B : s(a) = t(b)\}$$
$$A \star B := \{(a, b) \in A \times B : s(a) = s(b) \text{ and } t(a) = t(b)\}.$$

Thus $A \diamond B$ consists of pairs of arrows *in series* and $A \star B$ consists of pairs of arrows *in parallel*; their elements may be respectively represented as follows.

$$\begin{tikzcd} * \arrow[r, bend left, "b"] & * \arrow[r, bend left, "a"] & * \end{tikzcd} \qquad\qquad \begin{tikzcd} * \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & * \end{tikzcd}$$

We turn $A \diamond B$ and $A \star B$ into directed graphs with vertex set $X$ by defining, for $(a, b) \in A \diamond B$,

$$s(a, b) := s(b) \qquad \text{and} \qquad t(a, b) := t(a),$$

and for $(a, b) \in A \star B$,

$$s(a, b) := s(a) = s(b) \qquad \text{and} \qquad t(a, b) := t(a) = t(b).$$

These operations turn the category of directed graphs with vertex set $X$ into a 2-monoidal category $(\mathsf{C}, \diamond, \star)$. Let $A$, $B$, $C$, $D$ be directed graphs with vertex set $X$. The interchange law (6.1)

$$\zeta_{A,B,C,D}\colon (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

simply sends $(a, b, c, d)$ to $(a, c, b, d)$. Note that the elements of these two sets may be represented as follows: for $(a, b, c, d) \in (A \star B) \diamond (C \star D)$,

$$\begin{tikzcd} * \arrow[r, bend left, "c"] \arrow[r, bend right, "d"'] & * \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & * \end{tikzcd}$$

and for $(a, c, b, d) \in (A \diamond C) \star (B \diamond D)$,

$$\begin{tikzcd} & * \arrow[dr, bend left, "a"] & \\ * \arrow[ur, bend left, "c"] \arrow[dr, bend right, "d"'] & & * \\ & * \arrow[ur, bend right, "b"'] & \end{tikzcd}$$

so $\zeta$ is well-defined, but in general is not bijective. The unit object $I$ of $(\mathsf{C},\diamond)$ is the *discrete graph* $(X, \mathrm{id}, \mathrm{id})$ and the unit object $J$ of $(\mathsf{C},\star)$ is the *complete graph* $(X \times X, p_1, p_2)$ with $p_1(x,y) = x$ and $p_2(x,y) = y$. The structure map $\Delta_I$ is the identity. The other maps in (6.2) are uniquely determined since the object $J$ is terminal in the category $\mathsf{C}$.

**Example 6.18.** Let $\Bbbk$ be a commutative ring. Let $K$ be a $\Bbbk$-algebra and $\mathsf{C}$ the category of $K$-bimodules. An object of $\mathsf{C}$ is a $\Bbbk$-module $M$ with left and right actions of $K$ such that

$$(a \cdot m) \cdot b = a \cdot (m \cdot b)$$

for all $a, b \in K$, $m \in M$.

Let $M \star N$ denote the tensor product over $K$ of two $K$-bimodules $M$ and $N$:

$$M \star N := M \otimes_K N = M \otimes N/(m \cdot a \otimes n \equiv m \otimes a \cdot n),$$

where $M \otimes N$ denotes tensor product over $\Bbbk$. The category $\mathsf{C}$ is monoidal under tensor product over $K$. The unit object is $K$ viewed as a $K$-bimodule by left and right multiplication.

Suppose from now on that $K$ is commutative. Then a $K$-bimodule $M$ may also be viewed as a $(K \otimes K)$-bimodule via

$$(a \otimes b) \cdot m := a \cdot m \cdot b =: m \cdot (a \otimes b).$$

Let $M \diamond N$ denote the tensor product over $K \otimes K$ of two $K$-bimodules $M$ and $N$:

$$M \diamond N := M \otimes_{K \otimes K} N = M \otimes N/(a \cdot m \cdot b \otimes n \equiv m \otimes a \cdot n \cdot b).$$

Then $M \diamond N$ is again a $K$-bimodule under

$$a \cdot (m \otimes n) := a \cdot m \otimes n = m \otimes a \cdot n \quad \text{and} \quad (m \otimes n) \cdot b := m \otimes n \cdot b = m \cdot b \otimes n.$$

Therefore, tensor product over $K \otimes K$ defines a second monoidal structure on $\mathsf{C}$. The unit object is $K \otimes K$ viewed as a $K$-bimodule by

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya.$$

Let

$$\gamma_{M,N} \colon M \otimes N \to M \star N \quad \text{and} \quad \varphi_{M,N} \colon M \otimes N \to M \diamond N \tag{6.20}$$

denote the canonical projections.

The above operations turn the category of $K$-bimodules into a 2-monoidal category

$$(\mathsf{C}, \diamond, K \otimes K, \star, K).$$

Let $M$, $N$, $P$ and $Q$ be $K$-bimodules. The interchange law (6.1)

$$\zeta_{A,B,C,D} \colon (M \star N) \diamond (P \star Q) \to (M \diamond P) \star (N \diamond Q)$$

is the unique map making the following diagram commutative.

$$\begin{tikzcd}
M \otimes N \otimes P \otimes Q \arrow[rr, "\mathrm{id}_M \otimes \beta_{N,P} \otimes \mathrm{id}_Q"] \arrow[d, "\gamma_{M,N} \otimes \gamma_{P,Q}"'] & & M \otimes P \otimes N \otimes Q \arrow[d, "\varphi_{M,P} \otimes \varphi_{N,Q}"] \\
(M \star N) \otimes (P \star Q) \arrow[d, "\varphi_{M \star N, P \star Q}"'] & & (M \diamond P) \otimes (N \diamond Q) \arrow[d, "\gamma_{M \diamond P, N \diamond Q}"] \\
(M \star N) \diamond (P \star Q) \arrow[rr, dashed, "\zeta"'] & & (M \diamond P) \star (N \diamond Q)
\end{tikzcd} \tag{6.21}$$

It is easy to see that $\zeta$ is well-defined and that the axioms in Definition 6.1 are satisfied.

**Example 6.19.** Throughout this example, we employ the terminology and notations of Section A.1. Let $(\mathsf{C}, \diamond, I)$ be an arbitrary monoidal category. Suppose that in the category $\mathsf{C}$ all finite products exist and consider the corresponding cartesian monoidal category $(\mathsf{C}, \times, J)$, as in Example 1.4. Then $(\mathsf{C}, \diamond, I, \times, J)$ is a 2-monoidal category: The interchange law (6.1)

$$\zeta_{A,B,C,D} \colon (A \times B) \diamond (C \times D) \to (A \diamond C) \times (B \diamond D)$$

is defined by

$$\zeta_{A,B,C,D} := (\pi_A^{A \times B} \diamond \pi_C^{C \times D},\, \pi_B^{A \times B} \diamond \pi_D^{C \times D}).$$

The structure maps (6.2) are as follows: $\Delta_I \colon I \to I \times I$ is the diagonal (Example 1.19), and $\mu_J \colon J \diamond J \to J$ and $\iota_J = \epsilon_I \colon I \to J$ are the unique maps to the terminal object $J$.

Let $(\mathsf{C}, \star, J)$ be an arbitrary monoidal category, and dually suppose that in the category $\mathsf{C}$ all finite coproducts exist. Let $(\mathsf{C}, \amalg, I)$ be the corresponding cocartesian monoidal category, as in Example 1.4. Then $(\mathsf{C}, \amalg, I, \star, J)$ is a 2-monoidal category. The interchange law

$$\zeta_{A,B,C,D} \colon (A \star B) \amalg (C \star D) \to (A \amalg C) \star (B \amalg D)$$

is defined by

$$\zeta_{A,B,C,D} := \begin{pmatrix} \iota_A^{A \amalg C} \star \iota_B^{B \amalg D} \\ \iota_C^{A \amalg C} \star \iota_D^{B \amalg D} \end{pmatrix}.$$

Note that Example 6.17 is of the first form above: the categorical product of two directed graphs $A$ and $B$ is the graph $A \star B$ defined in that example. On the other hand, in Example 6.18, neither operation is the categorical product or coproduct in the category of bimodules.

Finally, suppose that $\mathsf{C}$ is a category with both finite products and finite coproducts. Then $(\mathsf{C}, \amalg, I, \times, J)$ is a 2-monoidal category. This may be seen as a special case of either of the two constructions above. It is important to note that both constructions yield the same structure. For instance, in both cases the interchange law is given by

$$\begin{pmatrix} \iota_A^{A \amalg C} \pi_A^{A \times B} & \iota_B^{B \amalg D} \pi_B^{A \times B} \\ \iota_C^{A \amalg C} \pi_C^{C \times D} & \iota_D^{B \amalg D} \pi_D^{C \times D} \end{pmatrix} \colon (A \times B) \amalg (C \times D) \to (A \amalg C) \times (B \amalg D).$$

**Example 6.20.** Let $P$ be a partially ordered set. It gives rise to a category $\mathsf{C}$ in which the objects are the elements of $P$ and there is exactly one morphism from $x$ to $y$ if $x \leq y$; otherwise there are no morphisms from $x$ to $y$. If every pair of elements $a, b \in P$ has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$, then $P$ is said to be a *lattice*. Suppose in addition that $P$ has a minimum element 0 and a maximum element 1. In this case, $\mathsf{C}$ carries a structure of 2-monoidal category with

$$\diamond = \vee, \; I = 0, \; \star = \wedge, \; J = 1.$$

The interchange law is the unique map

$$\zeta \colon (a \wedge b) \vee (c \wedge d) \to (a \vee c) \wedge (b \vee d)$$

which exists since $(a \wedge b) \vee (c \wedge d) \leq (a \vee c) \wedge (b \vee d)$. The structure maps $\Delta_I$ and $\mu_J$ are identities, while $\iota_J = \epsilon_I$ is the unique map $0 \to 1$. The axioms in Definition 6.1 are satisfied since any diagram in $\mathsf{C}$ commutes, given that there is at most one morphism between any two objects.

Note that $\wedge$ is the categorical product and $\vee$ is the categorical coproduct in $\mathsf{C}$, so this 2-monoidal category is a special case of those in Example 6.19.

**Remark 6.21.** The 2-monoidal category $(\mathsf{C}, \diamond, \star)$ of directed graphs of Example 6.17 is $\star$-braided. More generally, the 2-monoidal categories $(\mathsf{C}, \diamond, \times)$ and $(\mathsf{C}, \amalg, \star)$ of Example 6.19 are $\times$-braided and $\amalg$-braided respectively. It follows that the 2-monoidal category $(\mathsf{C}, \vee, \wedge)$ of Example 6.20 is braided.

**Example 6.22.** Recall the Hadamard and Cauchy products on graded vector spaces (Example 6.78). Let $A$, $B$, $C$ and $D$ be graded vector spaces. Define the interchange law

$$\zeta \colon (A \times B) \cdot (C \times D) \to (A \cdot C) \times (B \cdot D)$$

$$\bigoplus_{n=s+t} (A_s \otimes B_s) \otimes (C_t \otimes D_t) \to \Big( \bigoplus_{n=s_1+t_1} A_{s_1} \otimes C_{t_1} \Big) \otimes \Big( \bigoplus_{n=s_2+t_2} B_{s_2} \otimes D_{t_2} \Big)$$

to be the natural embedding given by switching the middle factors. This turns $(\mathsf{gVec}, \cdot, \times)$ into a braided 2-monoidal category with the unit morphisms

$$\Delta_1 \colon 1 \to 1 \times 1, \qquad \mu_E \colon E \cdot E \to E, \qquad \iota_E = \epsilon_1 \colon 1 \to E,$$

defined as follows. The first and third map are identities on the $\emptyset$-component. Under the identification of $E$ with $\Bbbk[x]$, the map $\mu_E$ is the usual product on polynomials.

Interestingly, one can also define an interchange law in the other direction. We let it be the surjection which is identity if $s_1 = s_2$ and $t_1 = t_2$ and zero otherwise. This turns $(\mathsf{gVec}, \times, \cdot)$ also into a braided 2-monoidal category with the unit morphisms

$$\Delta_E \colon E \to E \cdot E, \qquad \mu_1 \colon 1 \times 1 \to 1, \qquad \iota_1 = \epsilon_E \colon E \to 1,$$

defined as follows. The last two maps are identities on the $\emptyset$-component. Under the identication of $E$ with $\Bbbk[x]$, the map $\Delta_E$ is

$$x^n \mapsto \sum_{i=0}^{n} x^i \otimes x^{n-i}.$$

Note that the two examples are distinguished by the order in which the monoidal structures are listed. We will see later that the two constructions are in fact the contragredients of each other (Section 6.12).

In Section 8.1, we define analogues of the Hadamard and Cauchy products for species. By proceeding along similar lines, one obtains that $(\mathsf{Sp}, \cdot, \mathbf{1}, \times, \mathbf{E})$ and $(\mathsf{Sp}, \times, \mathbf{E}, \cdot, \mathbf{1})$ are braided 2-monoidal categories (Proposition 8.68).

**Example 6.23.** We now discuss an interchange law between the Hadamard and substitution product of graded vector spaces. Let $A$, $B$, $C$ and $D$ be graded vector spaces. Define

$$\zeta\colon (A \times B) \circ (C \times D) \to (A \circ C) \times (B \circ D)$$

to be the map induced by the isomorphism

$$(A_k \otimes B_k) \otimes \Big( \bigotimes_{h=1}^{k} C_{i_h} \otimes D_{i_h} \Big) \to \Big( A_k \otimes \bigotimes_{h=1}^{k} C_{i_h} \Big) \otimes \Big( B_k \otimes \bigotimes_{h=1}^{k} D_{i_h} \Big)$$

which reorders the tensor factors. This turns $(\mathsf{gVec}, \circ, \times)$ into a 2-monoidal category (with the unit morphisms defined appropriately).

The analogue of this construction for species yields a 2-monoidal category $(\mathsf{Sp}, \circ, \times)$, which is discussed in Section B.6.

**Example 6.24.** We know from Section 6.3 that one source of 2-monoidal categories is provided by braided monoidal categories (or even just lax braided monoidal categories). One uses the same operation twice and the braiding gives rise to the interchange law. The example about to be discussed exhibits a fairly particular feature: the two monoidal operations coincide, but the interchange law is not given by a braiding, not even a lax braiding.

Let $\mathsf{Vec}$ be the category of vector spaces. We consider the following monoidal structure on this category:

$$V \odot W := V \oplus W \oplus (V \otimes W).$$

On morphisms, we correspondingly set

$$f \odot g := f \oplus g \oplus (f \otimes g).$$

The operation $\odot$ is associative and turns $\mathsf{Vec}$ into a monoidal category in which the unit object is the zero vector space. One way to see this is by appealing to the functor

$$(\mathsf{Vec}, \odot, 0) \to (\mathsf{Vec}, \otimes, \Bbbk), \quad V \mapsto \Bbbk \oplus V,$$

which is strong monoidal. (The existence of this functor does not trivialize the example under discussion in any way.) More explicitly, note that both $(A \odot B) \odot C$ and $A \odot (B \odot C)$ are canonically isomorphic to

$$A \odot B \odot C := A \oplus B \oplus C \oplus (A \otimes B) \oplus (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes B \otimes C).$$

More generally,

$$A_1 \odot \cdots \odot A_n = \bigoplus_{\substack{1 \le k \le n \\ i_1 < \cdots < i_k}} A_{i_1} \otimes \cdots \otimes A_{i_k}.$$

Note that in order to define a linear map between two spaces $A_1 \odot \cdots \odot A_n$ and $B_1 \odot \cdots \odot B_m$ it suffices to specify a linear map between each pair of components $A_{i_1} \otimes \cdots \otimes A_{i_k}$ and $B_{j_1} \otimes \cdots \otimes B_{j_h}$.

We define an interchange law between $\odot$ and itself as follows. We let

$$\zeta : A \odot B \odot C \odot D \to A \odot C \odot B \odot D$$

be given by the identity map on the components

$$A,\ B,\ C,\ D,\ A\otimes B,\ A\otimes C,\ B\otimes D,\ C\otimes D,\ A\otimes B\otimes D,\ \text{and}\ A\otimes C\otimes D.$$

On all other components, $\zeta$ is 0.

Note that $\zeta$ vanishes on all components containing $B\otimes C$ as a subfactor and for this reason no switch of factors is ever involved. But note also that $\zeta$ vanishes on the component $A\otimes D$. This is an important point: any map of the form

$$\mathrm{id}\odot\beta\odot\mathrm{id}\colon A\odot B\odot C\odot D\to A\odot C\odot B\odot D$$

is necessarily the identity on $A\otimes D$. For this reason, $\zeta$ does not arises from a braiding (or a lax braiding).

The unit maps (6.2) are all defined to be 0. The unit axioms (6.5)–(6.8) are thus trivially satisfied.

It remains to verify the associativity axioms (6.3)–(6.4). Since all components involved are either the identity or 0, it suffices to verify that for each diagram the two sides vanish on the same components. For diagram (6.3), the two sides vanish precisely on the components

$$A\otimes D,\ A\otimes D\otimes E,\ A\otimes D\otimes F,\ A\otimes D\otimes E\otimes F,$$
$$A\otimes F,\ C\otimes F,\ \text{and}\ A\otimes C\otimes F$$

as well as on any components containing

$$B\otimes C,\ B\otimes E,\ \text{and}\ D\otimes E$$

as subfactors. For diagram (6.4) the situation is similar.

We conclude that $(\mathsf{Vec},\odot,\odot)$ is a 2-monoidal category with interchange law $\zeta$. As explained above, it does not arise from a lax braided monoidal category via the construction of Proposition 6.14.

## 6.5. Bimonoids and double (co)monoids

Recall that one can define bimonoids and (co)commutative (co)monoids in any braided monoidal category. In this section we discuss the analogues of these in a 2-monoidal category.

Bimonoids generalize in a straightforward manner. The idea is as follows. For a bimonoid (Definition 1.10), the braiding $\beta$ occurs in diagram (1.9) where it relates two tensors both with 4 factors; so it can be replaced without difficulty by the interchange law. In contrast, for a (co)commutative (co)monoid (Definition 1.17), the braiding relates two tensors both with 2 factors. As a result, it is not immediately clear how to proceed. We use the term double (co)monoid for the generalization of a (co)commutative (co)monoid.

A summary of different types of "monoids" is given in Table 6.2.

### 6.5.1. Bimonoids.

**Definition 6.25.** Let $(\mathsf{C},\diamond,I,\star,J)$ be a 2-monoidal category. A *bimonoid* in $\mathsf{C}$ is a quintuple $(H,\mu,\iota,\Delta,\epsilon)$ where $(H,\mu,\iota)$ is a monoid in $(\mathsf{C},\diamond,I)$, $(H,\Delta,\epsilon)$ is a comonoid in $(\mathsf{C},\star,J)$, and the two structures are compatible in the sense that the following four diagrams commute.

$$
\begin{tikzcd}
(H \star H) \diamond (H \star H) \arrow[rr, "\zeta"] & & (H \diamond H) \star (H \diamond H) \arrow[d, "\mu\star\mu"] \\
H \diamond H \arrow[u, "\Delta\diamond\Delta"] \arrow[r, "\mu"'] & H \arrow[r, "\Delta"'] & H \star H
\end{tikzcd}
\tag{6.22}
$$

$$
\begin{tikzcd}
H \diamond H \arrow[r, "\epsilon\diamond\epsilon"] \arrow[d, "\mu"'] & J \diamond J \arrow[d, "\mu_J"] \\
H \arrow[r, "\epsilon"'] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\iota"] \arrow[d, "\Delta_I"'] & H \arrow[d, "\Delta"] \\
I \star I \arrow[r, "\iota\star\iota"'] & H \star H
\end{tikzcd}
\tag{6.23}
$$

$$
\begin{tikzcd}
& H \arrow[dr, "\epsilon"] & \\
I \arrow[ur, "\iota"] \arrow[rr, "\iota_J = \epsilon_I"'] & & J
\end{tikzcd}
\tag{6.24}
$$

A morphism of bimonoids is a morphism of the underlying monoids and comonoids.

**Remark 6.26.** The notion of bimonoid is self-dual in the following sense. Let $(H, \mu, \iota, \Delta, \epsilon)$ be a bimonoid in $(\mathsf{C}, \diamond, I, \star, J)$. Switching $\mu$ with $\Delta$ and $\iota$ with $\epsilon$ we obtain a bimonoid in the opposite category $(\mathsf{C}^{\mathrm{op}}, \star, J, \diamond, I)$ of Section 6.1.2. In more detail, let $\overline{H}$ be $H$ viewed as an object of $\mathsf{C}^{\mathrm{op}}$ and let

$$
\begin{aligned}
\overline{\mu} &:= \Delta \in \mathrm{Hom}_{\mathsf{C}}(H \star H, H) = \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(\overline{H}, \overline{H} \star \overline{H}),\\
\overline{\iota} &:= \epsilon \in \mathrm{Hom}_{\mathsf{C}}(H, J) = \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(J, \overline{H}),\\
\overline{\Delta} &:= \mu \in \mathrm{Hom}_{\mathsf{C}}(H, H \diamond H) = \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(\overline{H} \diamond \overline{H}, \overline{H}),\\
\overline{\epsilon} &:= \iota \in \mathrm{Hom}_{\mathsf{C}}(I, H) = \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(\overline{H}, I).
\end{aligned}
$$

Then $(\overline{H}, \overline{\mu}, \overline{\iota}, \overline{\Delta}, \overline{\epsilon})$ is a bimonoid in $(\mathsf{C}^{\mathrm{op}}, \star, J, \diamond, I)$.

**Proposition 6.27.** *The unit objects $I$ and $J$ carry a unique bimonoid structure. In addition, $\iota_J = \epsilon_I \colon I \to J$ is a morphism of bimonoids.*

Proof. Consider the object $I$. Since it is the unit for $\diamond$, it carries a unique monoid structure given by the identity maps. By (6.8), $(I, \Delta_I, \epsilon_I)$ is a comonoid. The structures are compatible in view of the last two diagrams in Proposition 6.7. This is the unique comonoid structure compatible with the monoid structure, in view of (6.23) and (6.24). The statement for $J$ follow by duality. Finally, the first two diagrams in Proposition 6.7 imply that $\iota_J = \epsilon_I$ is a morphism of bimonoids. ☐

**6.5.2. Double (co)monoids.** We now turn to the analogue of a commutative monoid for 2-monoidal categories; we call it a double monoid. Informally, it is an object with two monoid structures which commute with each other; hence the name. This is not to be confused with commutative bimonoids.

**Definition 6.28.** A *double monoid* in a 2-monoidal category $(\mathsf{C}, \diamond, \star)$ is an object $A$ equipped with morphisms

$$
A \diamond A \to A, \quad I \to A, \qquad A \star A \to A, \quad J \to A,
$$

which turn $A$ into a monoid in both $(\mathsf{C}, \diamond, I)$ and $(\mathsf{C}, \star, J)$ and the following diagrams commute.

(6.25)
$$\begin{tikzcd}
(A \star A) \diamond (A \star A) \arrow[rr] \arrow[d] & & (A \diamond A) \star (A \diamond A) \arrow[d] \\
A \diamond A \arrow[r] & A & A \star A \arrow[l]
\end{tikzcd}$$

(6.26)
$$\begin{tikzcd}
J \diamond J \arrow[r] \arrow[d] & A \diamond A \arrow[d] & & I \star I \arrow[r] & A \star A \arrow[d] \\
J \arrow[r] & A & & I \arrow[r] \arrow[u] & A
\end{tikzcd}$$

(6.27)
$$\begin{tikzcd}
& A & \\
I \arrow[ur] \arrow[rr] & & J \arrow[ul]
\end{tikzcd}$$

Dually, one can define a *double comonoid*.

A morphism between two double (co)monoids is a morphism of the two underlying (co)monoids.

The second diagram in (6.26) may be omitted from the definition since its commutativity follows from the remaining assumptions: Recall that $J$ is the unit object for $\star$, $I$ is a comonoid with respect to $\star$ with $I \to J$ being the counit map, and $A$ is a monoid with respect to $\star$ with $J \to A$ being the unit map. Thus we have a commutative diagram

$$\begin{tikzcd}
I \star I \arrow[r] & J \star J \arrow[r] \arrow[d, equal] & A \star A \arrow[d] \\
I \arrow[r] \arrow[u] & J \arrow[r] & A
\end{tikzcd}$$

whose composite is the second diagram in (6.26).

Let $(\mathsf{C}, \bullet, \bullet)$ be a 2-monoidal category arising from a braided monoidal category as in Section 6.3.

**Proposition 6.29 (Eckmann–Hilton argument).** *A* (*co*)*commutative* (*co*)*monoid in* $\mathsf{C}$ *gives rise to a double* (*co*)*monoid, for which both* (*co*)*monoid structures are identical.*

*Conversely, let $A$ be a double* (*co*)*monoid in* $(\mathsf{C}, \bullet, \bullet)$. *Then the two* (*co*)*products on $A$ coincide and are* (*co*)*commutative.*

Proof. The first statement is clear. For the converse, we explain the monoid case. Diagram (6.27) shows that the two unit maps of $A$ coincide. Let us denote them by $\iota \colon I \to A$. Let $\mu_1, \mu_2 \colon A \bullet A \to A$ be the two products of $A$. The following diagram commutes.

$$\begin{tikzcd}
 & A \bullet I \bullet I \bullet A \arrow[r, "\mathrm{id}\bullet\iota\bullet\iota\bullet\mathrm{id}"] \arrow[dd, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"] & A \bullet A \bullet A \bullet A \arrow[r, "\mu_1\bullet\mu_1"] \arrow[dd, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"'] & A \bullet A \arrow[dr, "\mu_2"] & \\
A \bullet A \arrow[ur, "\rho\bullet\lambda"] \arrow[dr, "\rho\bullet\lambda"'] & & & & A \\
 & A \bullet I \bullet I \bullet A \arrow[r, "\mathrm{id}\bullet\iota\bullet\iota\bullet\mathrm{id}"'] & A \bullet A \bullet A \bullet A \arrow[r, "\mu_2\bullet\mu_2"'] & A \bullet A \arrow[ur, "\mu_1"'] &
\end{tikzcd}$$

Indeed, the triangle commutes by (1.8), the square by naturality, and the pentagon by (6.25). The unitality of $\mu_1$ and $\mu_2$ now tells us that the map along the top is $\mu_2$ and that along the bottom is $\mu_1$. Thus, $\mu_1 = \mu_2$. Let us denote this common product by $\mu$. A similar argument proves the commutativity of the following diagram

$$\begin{tikzcd}
A \bullet A \arrow[r, "\lambda\bullet\rho"] \arrow[dd, "\beta"'] & I \bullet A \bullet A \bullet I \arrow[r, "\iota\bullet\mathrm{id}\bullet\mathrm{id}\bullet\iota"] \arrow[dd, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"] & A \bullet A \bullet A \bullet A \arrow[r, "\mu\bullet\mu"] \arrow[dd, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"'] & A \bullet A \arrow[dr, "\mu"] & \\
 & & & & A \\
A \bullet A \arrow[r, "\lambda\bullet\rho"'] & I \bullet A \bullet A \bullet I \arrow[r, "\iota\bullet\mathrm{id}\bullet\mathrm{id}\bullet\iota"'] & A \bullet A \bullet A \bullet A \arrow[r, "\mu\bullet\mu"'] & A \bullet A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

from which we deduce that $\mu\beta = \mu$. Thus, $(A, \mu, \iota)$ is a commutative monoid. □

To summarize, in the context of braided monoidal categories, commutative and double monoids are equivalent notions.

**Remark 6.30.** The classical Eckmann–Hilton argument [109] or [226, Lemma 1.2.4] goes as follows.

Consider a set with two binary operations $+$ and $\times$ and two elements 0 and 1. Consider the axioms

$$\begin{aligned}
(x + y) \times (z + t) &= (x \times z) + (y \times t), \\
x + 0 &= x = 0 + x, \\
x \times 1 &= x = 1 \times x.
\end{aligned}$$

This is not quite a double monoid (since we are not assuming associativity), but is very close. By setting $x = t = 1$ and $y = z = 0$, we deduce $1 = 0$. Then, by setting $y = z = 0$, we get that the operations $*$ and $+$ coincide. Next, by setting $x = t = 0$, we get that $+$ is commutative. Further manipulations give that $+$ is associative.

Note that the proof of Proposition 6.29 made no use of associativity. This may be deduced from the other assumptions, as in the classical case.

Several results related to the Eckmann–Hilton argument appear in this monograph. Apart from Proposition 6.29, we have Propositions 3.46, 6.11, and 6.59.

**6.5.3. Commutative bimonoids and commutative double monoids.** We now briefly discuss the notion of commutativity for bimonoids and double monoids. The context for this is more special: that of braided 2-monoidal categories (Section 6.1.5).

**Definition 6.31.** A *commutative bimonoid* in a $\diamond$-braided 2-monoidal category is a bimonoid $(H, \mu, \iota, \Delta, \epsilon)$ such that $(H, \mu, \iota)$ is commutative.

A *cocommutative bimonoid* in a $\star$-braided 2-monoidal category is a bimonoid $(H, \mu, \iota, \Delta, \epsilon)$ such that $(H, \Delta, \epsilon)$ is cocommutative.

**Proposition 6.32.** *Let $(H, \mu, \Delta)$ be a bimonoid in a 2-monoidal category $\mathsf{C}$.*

(i) *If $\mathsf{C}$ is $\diamond$-braided, then $H^{\mathrm{op}} := (H, \mu\beta, \Delta)$ is again a bimonoid.*
(ii) *If $\mathsf{C}$ is $\star$-braided, then ${}^{\mathrm{cop}}H := (H, \mu, \beta\Delta)$ is again a bimonoid.*

The proof is straightforward. We also note that the three statements in each set below are equivalent.

$$H \text{ is a commutative bimonoid;} \quad \mathrm{id}\colon H \to H^{\mathrm{op}} \text{ is a morphism of bimonoids;} \quad \mu = \mu\beta. \tag{6.28}$$

$$H \text{ is a cocommutative bimonoid;} \quad \mathrm{id}\colon H \to {}^{\mathrm{cop}}H \text{ is a morphism of bimonoids;} \quad \Delta = \beta\Delta. \tag{6.29}$$

**Remark 6.33.** The results of Proposition 6.32 are consistent with those of Proposition 1.21 for bimonoids in a braided monoidal category. Indeed, when the 2-monoidal category arises from a braided monoidal category, the hypotheses in (i) and (ii) above are both equivalent to $\beta = \beta^{-1}$, by Proposition 6.13. Therefore, in this situation $H^{\mathrm{op}} = {}^{\mathrm{op}}H$ and ${}^{\mathrm{cop}}H = H^{\mathrm{cop}}$.

**Definition 6.34.** A *(co)commutative double (co)monoid* in a $\diamond$-braided 2-monoidal category is a double (co)monoid in the 2-monoidal category such that the coproduct with respect to $\diamond$ is (co)commutative.

A *(co)commutative double (co)monoid* in a $\star$-braided 2-monoidal category is a double (co)monoid in the 2-monoidal category such that the coproduct with respect to $\star$ is (co)commutative.

**6.5.4. Alternative descriptions of bimonoids and double (co)monoids.** Recall that a bimonoid in a braided monoidal category can be interpreted as a monoid in the category of comonoids, or viceversa. We show that the same is true for a bimonoid in a 2-monoidal category. We also give analogous results for double (co)monoids.

**Proposition 6.35.** *Let $(\mathsf{C}, \diamond, I, \star, J)$ be a 2-monoidal category. If $A$ and $B$ are monoids in $(\mathsf{C}, \diamond, I)$, then so is $A \star B$. Moreover, $(\mathsf{Mon}(\mathsf{C}, \diamond, I), \star, J)$ is a monoidal category. Similarly, if $C$ and $D$ are comonoids in $(\mathsf{C}, \star, J)$, then so is $C \diamond D$, and $(\mathsf{Comon}(\mathsf{C}, \star, J), \diamond, I)$ is a monoidal category.*

Proof. We explain the first statement, the second being similar. If $A$ and $B$ are monoids in $(\mathsf{C}, \diamond, I)$, then by applying the first lax functor in Proposition 6.4, part (i), it follows that $A \star B$ is also a monoid. Explicitly, the structure is as follows.

$$(A \star B) \diamond (A \star B) \xrightarrow{\ \zeta\ } (A \diamond A) \star (B \diamond B) \xrightarrow{\ \mu\star\mu\ } A \star B.$$
$$I \xrightarrow{\ \Delta_I\ } I \star I \xrightarrow{\ \iota\star\iota\ } A \star B.$$

The second lax functor in Proposition 6.4, part (i), says that $J$ is also a monoid.

The only thing left to check is that the associativity and unit constraints of $(\mathsf{C}, \star, J)$ induce morphisms of monoids. This is omitted. $\square$

Let $\mathsf{Bimon}(\mathsf{C}, \diamond, \star)$, $\mathsf{dMon}(\mathsf{C}, \diamond, \star)$ and $\mathsf{dComon}(\mathsf{C}, \diamond, \star)$ denote the categories of bimonoids, double monoids and double comonoids respectively in $(\mathsf{C}, \diamond, \star)$. These and other similar notations are given in Table 6.2.

**Proposition 6.36.** *There are canonical equivalences of categories*

$$\mathsf{Bimon}(\mathsf{C},\diamond,\star) \cong \mathsf{Comon}(\mathsf{Mon}(\mathsf{C},\diamond),\star) \cong \mathsf{Mon}(\mathsf{Comon}(\mathsf{C},\star),\diamond),$$
$$\mathsf{dMon}(\mathsf{C},\diamond,\star) \cong \mathsf{Mon}(\mathsf{Mon}(\mathsf{C},\diamond),\star),$$
$$\mathsf{dComon}(\mathsf{C},\diamond,\star) \cong \mathsf{Comon}(\mathsf{Comon}(\mathsf{C},\star),\diamond).$$

Proposition 6.36 provides alternative descriptions for bimonoids and double (co)monoids. They generalize the corresponding statements for braided monoidal categories (1.14). For example, the middle statement generalizes the fact that a commutative monoid is a monoid in the category of monoids. The verification of the above propositions is straightforward. The proof brings into play all the 2-monoidal category axioms.

**Proposition 6.37.** *If* $(\mathsf{C},\diamond,\star)$ *is* $\diamond$*-braided, then* $(\mathsf{Mon}^{\mathsf{co}}(\mathsf{C},\diamond),\star)$ *is a monoidal category and* $(\mathsf{Comon}(\mathsf{C},\star),\diamond)$ *is a braided monoidal category. Further, there are canonical equivalences of categories*

$$\begin{aligned}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C},\diamond,\star) &\cong \mathsf{Comon}(\mathsf{Mon}^{\mathsf{co}}(\mathsf{C},\diamond),\star) \cong \mathsf{Mon}^{\mathsf{co}}(\mathsf{Comon}(\mathsf{C},\star),\diamond)\\ &\cong \mathsf{Mon}(\mathsf{Comon}(\mathsf{Mon}(\mathsf{C},\diamond),\star),\diamond).\end{aligned}$$

A similar interpretation may be given for ${}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C},\diamond,\star)$. Observe that commutative bimonoids or $(2,1)$-monoids are obtained by a three step process suggesting that they belong to the world of 3-monoidal categories. This point is addressed in more detail below.

**6.5.5. A summary of different types of monoids.** We have discussed various notions of "monoids" in "monoidal categories", starting with monoids and comonoids in a monoidal category; bimonoids, commutative monoids and cocommutative comonoids in a braided monoidal category; bimonoids, double monoids and double comonoids in a 2-monoidal category; and (co)commutative bimonoids and (co)commutative double (co)monoids in a braided 2-monoidal category.

A summary of all these objects, along with the notations that we use for the corresponding categories is given in Table 6.2. The first column suggests a systematic nomenclature for organizing these objects: a monoid is a $(1,0)$-monoid, a

TABLE 6.2. Categories of different types of monoids.

| $(0,0)$ | $\mathsf{C}$ | Object |
|---|---|---|
| $(1,0)$ | $\mathsf{Mon}(\mathsf{C},\diamond)$ | Monoid |
| $(0,1)$ | $\mathsf{Comon}(\mathsf{C},\star)$ | Comonoid |
| $(2,0)$ | $\mathsf{dMon}(\mathsf{C},\diamond,\star)$ | Double monoid |
| $(1,1)$ | $\mathsf{Bimon}(\mathsf{C},\diamond,\star)$ | Bimonoid |
| $(0,2)$ | $\mathsf{dComon}(\mathsf{C},\diamond,\star)$ | Double comonoid |
| $(3,0)$ | $\mathsf{dMon}^{\mathsf{co}}(\mathsf{C},\diamond,\star)$ | Comm. double monoid |
| $(2,1)$ | $\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C},\diamond,\star)$ | Comm. bimonoid |
| $(1,2)$ | ${}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C},\diamond,\star)$ | Cocomm. bimonoid |
| $(0,3)$ | ${}^{\mathsf{co}}\mathsf{dComon}(\mathsf{C},\diamond,\star)$ | Cocomm. double comonoid |

comonoid is a $(0,1)$-monoid, a bimonoid is a $(1,1)$-monoid, and so forth. An object in the category is to be viewed as a $(0,0)$-monoid. This notation suggests that one can define the notion of a $(i,j)$-monoid in a higher monoidal category with $i+j$ monoidal structures. This is the subject of Chapter 7; see in particular Sections 7.4 and 7.7.

We emphasize that for $(2,1)$-monoids we assume $(\mathsf{C},\diamond)$ is braided, for $(1,2)$-monoids we assume $(\mathsf{C},\star)$ is braided, and for $(3,0)$- and $(0,3)$-monoids we assume either $(\mathsf{C},\diamond)$ or $(\mathsf{C},\star)$ is braided. We will see later that braided 2-monoidal categories are examples of 3-monoidal categories. The latter provide a natural context for these four types of monoids. This situation is analogous to that for bimonoids: Bimonoids can be defined in a braided monoidal category but the natural context for them is a 2-monoidal category.

## 6.6. Modules and comodules over a bimonoid

Recall the notion of a (co)module over a (co)monoid from Definition 1.12. The category of (co)modules over a bimonoid is a monoidal category. We discussed this result in the context of braided monoidal categories in Section 1.2.3. We now generalize it to the context of 2-monoidal categories.

Let $(\mathsf{C},\diamond,I,\star,J)$ be a 2-monoidal category. Let $A$ and $B$ be monoids in $(\mathsf{C},\diamond,I)$ and let $(M,\chi)$ and $(N,\rho)$ be left modules over $A$ and $B$ respectively. Consider the monoid $A\star B$ resulting from Proposition 6.35.

**Proposition 6.38.** *In this situation, $M\star N$ is a left $A\star B$-module with structure map*

$$(A\star B)\diamond(M\star N)\xrightarrow{\zeta}(A\diamond M)\star(B\diamond N)\xrightarrow{\chi\star\rho}M\star N.$$

Now let $A$ be a bimonoid in $\mathsf{C}$. The maps

$$\Delta\colon A\to A\star A \qquad\text{and}\qquad \epsilon\colon A\to J.$$

are morphisms of monoids in $(\mathsf{C},\diamond,I)$. It follows that if $M$ and $N$ are left $A$-modules, then $M\star N$ is again a left $A$-module with structure map:

$$A\diamond(M\star N)\xrightarrow{\Delta\diamond\mathrm{id}}(A\star A)\diamond(M\star N)\xrightarrow{\zeta}(A\diamond M)\star(A\diamond N)\xrightarrow{\chi\star\rho}M\star N.$$

It is also clear that if $f\colon M\to M'$ and $g\colon N\to N'$ are morphisms of left $A$-modules, then so is $f\star g\colon M\star N\to M'\star N'$.

Further, $J$ is an $A$-module with structure map:

$$A\diamond J\xrightarrow{\epsilon\diamond\mathrm{id}}J\diamond J\xrightarrow{\mu_J}J.$$

Let $\mathsf{Mod}_A$ denote the category of left modules over the bimonoid $A$. The above constructions yield functors

$$\star\colon \mathsf{Mod}_A\times\mathsf{Mod}_A\to\mathsf{Mod}_A \qquad\text{and}\qquad \mathsf{I}\to\mathsf{Mod}_A, \tag{6.30}$$

where $\mathsf{I}$ is the one-arrow category.

**Proposition 6.39.** *For a bimonoid $A$ in $(\mathsf{C},\diamond,\star)$, the category $\mathsf{Mod}_A$ equipped with the functors (6.30) and the associative and unit constraints induced from $(\mathsf{C},\star,J)$ is a monoidal category.*

PROOF. The only thing left to check is that the associative and unit constraints of $(\mathsf{C}, \star)$ are morphisms of $A$-modules. The first check follows from the coassociativity of $\Delta$ and (6.4), while the second check follows from the counitality of $\epsilon$ and (6.6). $\square$

**Proposition 6.40.** *Let* $\mathsf{C}$ *be a* $\star$*-braided* 2*-monoidal category. For a cocommutative bimonoid* $A$, *the monoidal category* $\mathsf{Mod}_A$ *is braided, with the braiding induced from* $(\mathsf{C}, \star, \beta)$. *Further, if* $\beta$ *is a symmetry, then so is the induced braiding.*

PROOF. The proof is straightforward and follows from (6.16). $\square$

Given a morphism $A \to B$ of (cocommutative) bimonoids, the restriction functor $\mathsf{Mod}_B \to \mathsf{Mod}_A$ on the module categories is a (braided) strong monoidal functor.

Let $\mathsf{Comod}^A$ denote the category of left comodules over a comonoid $A$. The above discussion can also be carried out for the category $\mathsf{Comod}^A$. If $A$ is a bimonoid and $M$ and $N$ are left $A$-comodules, then $M \diamond N$ is again a left $A$-comodule:

$$M \diamond N \longrightarrow (A \star M) \diamond (A \star N) \xrightarrow{\zeta} (A \diamond A) \star (M \diamond N) \xrightarrow{\mu \star \mathrm{id}} A \star (M \diamond N).$$

**Proposition 6.41.** *For a bimonoid* $A$ *in* $(\mathsf{C}, \diamond, \star)$, *the category* $\mathsf{Comod}^A$ *is a monoidal category, whose structure is induced from* $(\mathsf{C}, \diamond, I)$. *Further, if* $\mathsf{C}$ *is* $\diamond$*-braided and* $A$ *is commutative, then* $\mathsf{Comod}^A$ *is braided.*

## 6.7. Examples of bimonoids and double monoids

We now describe bimonoids and double monoids in explicit terms in some of the 2-monoidal categories of Section 6.4.

**Example 6.42.** Let $(\mathsf{C}, \diamond, \times)$ be a 2-monoidal category as in Example 6.19, in which the second monoidal structure is cartesian. Recall from Example 1.19 that

$$\mathsf{Comon}(\mathsf{C}, \times) \cong \mathsf{C}.$$

It follows that

$$\mathsf{Bimon}(\mathsf{C}, \diamond, \times) \cong \mathsf{Mon}(\mathsf{C}, \diamond).$$

Dually, let $(\mathsf{C}, \amalg, \star)$ be a 2-monoidal category in which the first structure is cocartesian. Then

$$\mathsf{Bimon}(\mathsf{C}, \amalg, \star) \cong \mathsf{Comon}(\mathsf{C}, \star).$$

Finally, if $(\mathsf{C}, \amalg, \times)$ is the 2-monoidal category that combines both situations, then

$$\mathsf{Bimon}(\mathsf{C}, \amalg, \times) \cong \mathsf{C}.$$

**Example 6.43.** Consider the category $(\mathsf{C}, \diamond, \star)$ of directed graphs with vertex set $X$ as in Example 6.17. A monoid in $(\mathsf{C}, \diamond)$ is precisely a small category with object set $X$. According to Example 6.42, so is a bimonoid in $(\mathsf{C}, \diamond, \star)$, since the operation $\star$ is the categorical product.

A monoid in $(\mathsf{C}, \star)$ is a graph in which for each $x, y \in X$, the set of arrows $\{a \in A \mid s(a) = x,\ t(a) = y\}$ is a monoid. A double monoid in $(\mathsf{C}, \diamond, \star)$ is a category with object set $X$ enriched over the category of monoids in $\mathsf{Set}$ (Section C.3.1).

**Example 6.44.** Let $K$ be a commutative algebra and $(\mathsf{C}, \diamond, \star)$ be the category of $K$-bimodules as in Example 6.18. Let $(M, \mu, \iota)$ be a monoid in $(\mathsf{C}, \diamond)$.

Composing $\mu$ with the projection $\varphi\colon M \otimes M \to M \diamond M$ (6.20) we obtain an associative product on $M$

$$M \otimes M \xrightarrow{\varphi} M \diamond M \xrightarrow{\mu} M.$$

We denote the product of $m, n \in M$ simply by $mn$.

We use $\iota\colon K \otimes K \to M$ to define two maps as follows.

$$\begin{aligned} s\colon K \to M \qquad & s(a) := \iota(a \otimes 1), \\ t\colon K \to M \qquad & t(a) := \iota(1 \otimes a). \end{aligned}$$

The following properties can be deduced from the unit axioms in Definition 1.9:

$$\begin{aligned} a \cdot m &= s(a)m = ms(a) \\ m \cdot b &= t(b)m = mt(b). \end{aligned}$$

Choosing $a = b = 1$ we deduce (from the unit axiom for the actions of $K$ on $M$) that the element $\iota(1 \otimes 1) = s(1) = t(1)$ is a unit for the product on $M$ defined above. It also follows that the images of $s$ and $t$ are contained in the center of the algebra $M$. Moreover, from the associativity of the actions of $K$ on $M$ we deduce that $s$ and $t$ are morphisms of algebras.

In summary, $M$ is a unital associative algebra and $s, t\colon K \to M$ are morphisms of algebras whose images are contained in the center of $M$. Thus, $M$ is precisely a $K$-*algebroid* in the sense of Ravenel [301, Section A1.1] (this reference deals directly with Hopf algebroids). The same objects ($K$-algebroids) are simply called *graphs* by Maltsiniotis [254]. The construction of a $K$-algebroid from a monoid in $(\mathsf{C}, \diamond)$ is reversible, so that $K$-algebroids and monoids in $(\mathsf{C}, \diamond)$ are equivalent notions.

A comonoid in $(\mathsf{C}, \star)$ is sometimes called a $K$-*coalgebroid* [95, 254] and more often a $K$-*coring* [74, 349].

A $K$-*bialgebroid* [254, 301] can be defined as a comonoid in $(\mathsf{Mon}(\mathsf{C}, \diamond), \star)$. Thus, $K$-bialgebroids and bimonoids in $(\mathsf{C}, \diamond, \star)$ are equivalent notions.

**Remark 6.45.** There is a more general notion of bialgebroid that allows for the base algebra $K$ to be noncommutative and requires only that the images of $s$ and $t$ in $H$ commute elementwise [73, 321]; see [190] for a survey of related notions.

A related (and older) notion is that of $\times_K$-*bialgebras*. These were introduced by Sweedler [351, Section 5] in the commutative case and Takeuchi [355, Section 4] in the noncommutative case.

It is unclear to us whether these objects can be seen as bimonoids in a certain 2-monoidal category. It seems that to capture this notion in full generality one needs a more general setting than that of 2-monoidal categories in which the associativity constraints are allowed to be lax (along the lines of Section D.3). We do not consider such objects in this monograph.

**Example 6.46.** Let $P$ be a lattice and $(\mathsf{C}, \vee, \wedge)$ the 2-monoidal category of Example 6.20. Any object $x$ of $\mathsf{C}$ (element of $P$) carries a unique bimonoid structure. The maps $\mu\colon x \vee x \to x$ and $\Delta\colon x \to x \wedge x$ are identities, while $\iota\colon 0 \to x$ and $\epsilon\colon x \to 1$ are the unique maps. The bimonoid is commutative and cocommutative. This is a special case of Example 6.42.

**Example 6.47.** Consider now the 2-monoidal category $(\mathsf{C}, \odot, \odot)$ of Example 6.24. As discussed in that example, maps between iterated $\odot$-products are determined by their components. We display such components in matrix notation. For instance, a map $A \odot B \to C$ is determined by the matrix

$$\begin{pmatrix} x & y & z \end{pmatrix}$$

where $x : A \to C$, $y : B \to C$ and $z : A \otimes B \to C$ are its components, so that the map is $x + y + z$. Similarly, the product $f \odot g$ of two maps $f : A \to C$ and $g : B \to D$ is determined by the matrix

$$\begin{pmatrix} f & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & f \otimes g \end{pmatrix}.$$

Let $(A, \mu, \iota)$ be a monoid in $(\mathsf{C}, \odot, 0)$. Since the unit object is 0, we have $\iota = 0$. Write $\mu = \begin{pmatrix} x & y & m \end{pmatrix}$. The unit axioms in Definition 1.9 force $x = y = \mathrm{id}_A$, while the associativity axiom simply boils down to that of the map $m : A \otimes A \to A$. In conclusion, a monoid in $(\mathsf{C}, \odot, 0)$ is the same as a not necessarily unital algebra. Let us write $\mu_+ := m$ for the nonunital multiplication.

Similarly, a comonoid in $(\mathsf{C}, \odot, 0)$ is the same as a not necessarily counital coalgebra.

Now suppose $(A, \mu, \Delta)$ is a bimonoid in $(\mathsf{C}, \odot, 0)$. The unit and counit maps are 0, and hence the only relevant axiom in Definition 6.25 is (6.22). According to the preceding discussion, we may display the components of $\mu : A \odot A \to A$ and $\Delta : A \to A \odot A$ as follows:

$$\mu = \begin{pmatrix} \mathrm{id} & \mathrm{id} & \mu_+ \end{pmatrix} \qquad \Delta = \begin{pmatrix} \mathrm{id} \\ \mathrm{id} \\ \Delta_+ \end{pmatrix}.$$

Their composite $A \odot A \to A \odot A$ is therefore given by

$$\Delta\mu = \begin{pmatrix} \mathrm{id} & \mathrm{id} & \mu_+ \\ \mathrm{id} & \mathrm{id} & \mu_+ \\ \Delta_+ & \Delta_+ & \Delta_+\mu_+ \end{pmatrix}.$$

On the other hand, making use of the definition of the interchange law $\zeta$ in Example 6.24 to go around the top of diagram (6.22), we find that

$$(\mu \odot \mu)\zeta(\Delta \odot \Delta) = \begin{pmatrix} \mathrm{id} & \mathrm{id} & \mu_+ \\ \mathrm{id} & \mathrm{id} & \mu_+ \\ \Delta_+ & \Delta_+ & (\mu_+ \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_+) + (\mathrm{id} \otimes \mu_+)(\Delta_+ \otimes \mathrm{id}) \end{pmatrix}.$$

Therefore,

$$\Delta_+\mu_+ = (\mu_+ \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_+) + (\mathrm{id} \otimes \mu_+)(\Delta_+ \otimes \mathrm{id}).$$

Equivalently, employing the notations $\mu_+(a \otimes b) = ab$ and $\Delta_+(a) = a_1 \otimes a_2$, the condition is

$$\Delta_+(ab) = ab_1 \otimes b_2 + a_1 \otimes a_2 b. \tag{6.31}$$

Conversely, given a nonunital multiplication $\mu_+$ and a noncounital comultiplication $\Delta_+$ on $A$ that satisfy (6.31), we obtain a bimonoid in $(\mathsf{C}, \odot, 0)$ by reversing the above construction. Such objects $(A, \mu_+, \Delta_+)$ are called *infinitesimal bialgebras*. In conclusion, a bimonoid in $(\mathsf{C}, \odot, 0)$ is the same as an infinitesimal bialgebra. It follows that this notion is self-dual (Remark 6.26).

Infinitesimal bialgebras were introduced by Joni and Rota [179, Section XII] and further studied by Ehrenborg and Readdy [113], Aguiar [5, 7, 8, 9] and Voiculescu [369, 370]. These objects should not be confused with the 0-bialgebras of Section 2.3.6, for which a similar terminology is often employed.

**Example 6.48.** We briefly discuss double monoids in $(\mathsf{C}, \odot, \odot)$. Note that the Eckmann–Hilton argument (Proposition 6.29) does not apply because we are not dealing with a braided monoidal category. In fact, its conclusion does not hold, as we see next.

Let $(A, \mu, \nu)$ be a double monoid in $(\mathsf{C}, \odot, \odot)$. Axiom (6.25) forces $\mu_+ = 0$ and imposes no conditions on $\nu_+$. Thus, in this situation, monoids and double monoids are equivalent notions.

**Example 6.49.** We expand on Example 6.47 by discussing modules over a bimonoid in the 2-monoidal category $(\mathsf{C}, \odot, \odot)$.

First, we consider modules over monoids. Let $A$ be a monoid in $(\mathsf{C}, \odot, \odot)$ with product $\mu : A \odot A \to A$ and let $(M, \chi)$ be a left module over it. Write

$$\chi = \begin{pmatrix} \chi_0 & \chi_0' & \chi_+ \end{pmatrix}$$

for the components of $\chi : A \odot M \to M$. The unit axiom for $\chi$ makes $\chi_0' = \mathrm{id}_M$. Associativity boils down to the following conditions on the maps $\chi_0 : A \to M$ and $\chi_+ : A \otimes M \to M$:

$$\chi_+(\mu_+ \otimes \mathrm{id}_M) = \chi_+(\mathrm{id}_A \otimes \chi_+),$$
$$\chi_+(\mathrm{id}_A \otimes \chi_0) = \chi_0 \mu_+.$$

Equivalently, $(M, \chi_+)$ is a nonunital left module over the nonunital algebra $A$, and $\chi_0 : A \to M$ is a morphism of modules.

Second, we discuss tensor products of monoids and modules. Let $B$ be another monoid in $(\mathsf{C}, \odot, \odot)$. By Proposition 6.35, there is a monoid structure on $A \odot B$. The corresponding nonunital product on $(A \odot B) \otimes (A \odot B)$ is given by

$$(a, b, u \otimes v) \otimes (a', b', u' \otimes v') \mapsto (aa',\, bb',\, au' \otimes v' + u \otimes vb'),$$

where $a, a', u \in A$ and $b, b', v \in B$.

Now let $(N, \rho)$ be a left module over $B$. By Proposition 6.38, there is a left $A \odot B$-module structure on $M \odot N$. A calculation shows that the structure maps are

$$A \odot B \to M \odot N$$
$$(a, b, u \otimes v) \mapsto \big(\chi_0(a),\, \rho_0(b),\, \chi_0(u) \otimes \rho_0(v)\big),$$

and

$$(A \odot B) \otimes (M \odot N) \to M \odot N$$
$$(a, b, u \otimes v) \otimes (m, n, x \otimes y) \mapsto \big(a \cdot m,\, b \cdot n,\, a \cdot x \otimes y + \chi_0(u) \otimes v \cdot n\big),$$

where $a \in A$, $b \in B$, $u \otimes v \in A \otimes B$, $m \in M$, $n \in N$, $x \otimes y \in M \otimes N$,

$$\chi_+(a \otimes m) = a \cdot m \quad \text{and} \quad \rho_+(b \otimes n) = b \cdot n.$$

Finally, suppose that $(A, \mu, \Delta)$ is a bimonoid in $(\mathsf{C}, \odot, \odot)$, or equivalently that $(A, \mu_+, \Delta_+)$ is an infinitesimal bialgebra, and let $(M, \chi)$ and $(N, \rho)$ be two left $A$-modules. According to Proposition 6.39, there is an $A$-module structure on $M \odot N$.

Combining the preceding calculations we obtain that the structure maps of $M \odot N$ are

$$\begin{aligned} A &\to M \odot N \\ a &\mapsto \big(\chi_0(a),\ \rho_0(a),\ \chi_0(a_1) \otimes \rho_0(a_2)\big), \end{aligned}$$

and

$$\begin{aligned} A \otimes (M \odot N) &\to M \odot N \\ a \otimes (m, n, x \otimes y) &\mapsto \big(a \cdot m,\ a \cdot n,\ a \cdot x \otimes y + \chi_0(a_1) \otimes a_2 \cdot n\big), \end{aligned}$$

where the notation is as above and in addition $\Delta_+(a) = a_1 \otimes a_2$.

The preceding constructions for modules over nonunital algebras and infinitesimal bialgebras improve upon the results in [9, Appendix A].

## 6.8. Bilax and double (co)lax monoidal functors

There are three different types of monoidal functors between 2-monoidal categories: bilax, double lax and double colax. They correspond to the three types of monoids in a 2-monoidal category: bimonoid, double monoid and double comonoid. The latter were discussed in Section 6.5.

**6.8.1. Bilax monoidal functors.** The theory of bilax monoidal functors for braided monoidal categories can be extended to the more general context of 2-monoidal categories. We now explain this briefly.

**Definition 6.50.** Let $(\mathsf{C}, \diamond, I, \star, J)$ and $(\mathsf{D}, \diamond, I, \star, J)$ be 2-monoidal categories. A *bilax functor* is a 3-tuple $(\mathcal{F}, \varphi, \psi)$ where

- $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is a functor,
- $(\mathcal{F}, \varphi)\colon (\mathsf{C}, \diamond, I) \to (\mathsf{D}, \diamond, I)$ is lax monoidal,
- $(\mathcal{F}, \psi)\colon (\mathsf{C}, \star, J) \to (\mathsf{D}, \star, J)$ is colax monoidal,

and the conditions below are satisfied.

**Interchanging.** The following diagram commutes

$$\begin{tikzcd}[column sep=tiny] & \mathcal{F}(A \star B) \diamond \mathcal{F}(C \star D) \arrow[dl, "\varphi_{A\star B, C\star D}"'] \arrow[dr, "\psi_{A,B} \diamond \psi_{C,D}"] & \\ \mathcal{F}\big((A \star B) \diamond (C \star D)\big) \arrow[d, "\mathcal{F}(\zeta_{A,B,C,D})"'] & & \big(\mathcal{F}(A) \star \mathcal{F}(B)\big) \diamond \big(\mathcal{F}(C) \star \mathcal{F}(D)\big) \arrow[d, "\zeta_{\mathcal{F}(A),\mathcal{F}(B),\mathcal{F}(C),\mathcal{F}(D)}"] \\ \mathcal{F}\big((A \diamond C) \star (B \diamond D)\big) \arrow[dr, "\psi_{A\diamond C, B\diamond D}"'] & & \big(\mathcal{F}(A) \diamond \mathcal{F}(C)\big) \star \big(\mathcal{F}(B) \diamond \mathcal{F}(D)\big) \arrow[dl, "\varphi_{A,C} \star \varphi_{B,D}"] \\ & \mathcal{F}(A \diamond C) \star \mathcal{F}(B \diamond D) & \end{tikzcd} \tag{6.32}$$

where $\zeta$ denotes the interchange law in either 2-monoidal category.

**Unitality.** The following diagrams commute.

$$\begin{tikzcd} I \arrow[r, "\varphi_0"] \arrow[d, "\Delta_I"'] & \mathcal{F}(I) \arrow[r, "\mathcal{F}(\Delta_I)"] & \mathcal{F}(I \star I) \arrow[d, "\psi_{I,I}"] \\ I \star I \arrow[rr, "\varphi_0 \star \varphi_0"'] & & \mathcal{F}(I) \star \mathcal{F}(I) \end{tikzcd} \qquad \begin{tikzcd} J & \mathcal{F}(J) \arrow[l, "\psi_0"'] & \mathcal{F}(J \diamond J) \arrow[l, "\mathcal{F}(\mu_J)"'] \\ J \diamond J \arrow[u, "\mu_J"] & & \mathcal{F}(J) \diamond \mathcal{F}(J) \arrow[ll, "\psi_0 \diamond \psi_0"] \arrow[u, "\varphi_{J,J}"'] \end{tikzcd} \tag{6.33}$$

$$\begin{tikzcd} \mathcal{F}(I) \arrow[r, "\mathcal{F}(\epsilon_I = \iota_J)"] & \mathcal{F}(J) \arrow[d, "\psi_0"] \\ I \arrow[u, "\varphi_0"] \arrow[r, "\epsilon_I = \iota_J"'] & J \end{tikzcd} \tag{6.34}$$

**Definition 6.51.** Let $(\mathcal{F}, \varphi, \psi)$ and $(\mathcal{G}, \gamma, \delta)$ be bilax functors between 2-monoidal categories $\mathsf{C}$ and $\mathsf{D}$. A morphism from $\mathcal{F}$ to $\mathcal{G}$ of bilax functors is a natural transformation $\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$ such that $(\mathcal{F}, \varphi) \Rightarrow (\mathcal{G}, \gamma)$ is a morphism of lax functors and $(\mathcal{F}, \psi) \Rightarrow (\mathcal{G}, \delta)$ is a morphism of colax functors.

**Proposition 6.52.** *There is a* 2*-category whose* 0*-cells,* 1*-cells, and* 2*-cells are respectively* 2*-monoidal categories, bilax monoidal functors, and their morphisms.*

In particular, the composite of bilax functors is again bilax. Further, this composition is compatible with morphisms of bilax functors. These checks proceed along the same lines as in the context of braided monoidal categories (Section 3.3.3) and are omitted. For an alternative approach, see Remark 6.76.

**Corollary 6.53.** *A bilax functor preserves bimonoids and morphisms between bimonoids.*

Proof. This follows from Proposition 6.52 and the observation that the category of bimonoids is equivalent to the category of bilax monoidal functors from $\mathsf{I}$ to $(\mathsf{C}, \diamond, \star)$, where $\mathsf{I}$ is the one-arrow category. $\square$

### 6.8.2. Double (co)lax monoidal functors.

Recall the notion of braided (co)lax monoidal functors between braided monoidal categories. We now extend this notion to 2-monoidal categories. This parallels the passage from (co)commutative (co)monoids to double (co)monoids.

**Definition 6.54.** Let $(\mathsf{C}, \diamond, I, \star, J)$ and $(\mathsf{D}, \diamond, I, \star, J)$ be 2-monoidal categories. A *double lax monoidal functor* is a 3-tuple $(\mathcal{F}, \varphi, \gamma)$ where

- $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ is a functor,
- $(\mathcal{F}, \varphi) \colon (\mathsf{C}, \diamond, I) \to (\mathsf{D}, \diamond, I)$ is lax monoidal,
- $(\mathcal{F}, \gamma) \colon (\mathsf{C}, \star, J) \to (\mathsf{D}, \star, J)$ is lax monoidal,

and the conditions below are satisfied.

**Interchanging.** The following diagram commutes

$$\begin{tikzcd}
\big(\mathcal{F}(A) \star \mathcal{F}(B)\big) \diamond \big(\mathcal{F}(C) \star \mathcal{F}(D)\big) \arrow[r, "\zeta"] \arrow[d, "\gamma_{A,B} \diamond \gamma_{C,D}"'] & \big(\mathcal{F}(A) \diamond \mathcal{F}(C)\big) \star \big(\mathcal{F}(B) \diamond \mathcal{F}(D)\big) \arrow[d, "\varphi_{A,C} \star \varphi_{B,D}"] \\
\mathcal{F}(A \star B) \diamond \mathcal{F}(C \star D) \arrow[d, "\varphi_{A\star B, C\star D}"'] & \mathcal{F}(A \diamond C) \star \mathcal{F}(B \diamond D) \arrow[d, "\gamma_{A\diamond C, B\diamond D}"] \\
\mathcal{F}\big((A \star B) \diamond (C \star D)\big) \arrow[r, "\mathcal{F}(\zeta)"'] & \mathcal{F}\big((A \diamond C) \star (B \diamond D)\big)
\end{tikzcd} \tag{6.35}$$

where $\zeta$ denotes the interchange law in either 2-monoidal category.

**Unitality.** The following diagrams commute.

$$\begin{tikzcd}
I \arrow[r, "\varphi_0"] \arrow[d, "\Delta_I"'] & \mathcal{F}(I) \arrow[r, "\mathcal{F}(\Delta_I)"] & \mathcal{F}(I \star I) \\
I \star I \arrow[rr, "\varphi_0 \star \varphi_0"'] & & \mathcal{F}(I) \star \mathcal{F}(I) \arrow[u, "\gamma_{I,I}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[r, "\gamma_0"] & \mathcal{F}(J) & \mathcal{F}(J \diamond J) \arrow[l, "\mathcal{F}(\mu_J)"'] \\
J \diamond J \arrow[u, "\mu_J"] \arrow[rr, "\gamma_0 \diamond \gamma_0"'] & & \mathcal{F}(J) \diamond \mathcal{F}(J) \arrow[u, "\varphi_{J,J}"']
\end{tikzcd} \tag{6.36}$$

$$\begin{tikzcd}
\mathcal{F}(I) \arrow[r, "\mathcal{F}(\epsilon_I = \iota_J)"] & \mathcal{F}(J) \\
I \arrow[u, "\varphi_0"] \arrow[r, "\epsilon_I = \iota_J"'] & J \arrow[u, "\gamma_0"']
\end{tikzcd} \tag{6.37}$$

**Definition 6.55.** A *double colax monoidal functor* is a 3-tuple $(\mathcal{F}, \psi, \delta)$ where $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ is a functor, $(\mathcal{F}, \psi) \colon (\mathsf{C}, \diamond, I) \to (\mathsf{D}, \diamond, I)$ and $(\mathcal{F}, \delta) \colon (\mathsf{C}, \star, J) \to (\mathsf{D}, \star, J)$ are both colax monoidal and satisfy axioms dual to those in Definition 6.54. Namely, one replaces $\varphi$ by $\psi$, $\gamma$ by $\delta$ and reverses the arrows with those labels.

**Definition 6.56.** Let $(\mathcal{F}, \varphi, \gamma)$ and $(\mathcal{G}, \varphi', \gamma')$ be double lax functors between 2-monoidal categories $\mathsf{C}$ and $\mathsf{D}$. A morphism from $\mathcal{F}$ to $\mathcal{G}$ of double lax functors is a natural transformation $\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$ such that $(\mathcal{F}, \varphi) \Rightarrow (\mathcal{G}, \varphi')$ and $(\mathcal{F}, \gamma) \Rightarrow (\mathcal{G}, \gamma')$ are morphisms of lax functors.

Morphisms between double colax functors are defined similarly.

The category of double (co)monoids is equivalent to the category of double (co)lax monoidal functors from $\mathsf{I}$ to $(\mathsf{C}, \diamond, \star)$, where $\mathsf{I}$ is the one-arrow category.

**Proposition 6.57.** *There is a* 2*-category whose* 0*-cells,* 1*-cells, and* 2*-cells are respectively* 2*-monoidal categories, double* (*co*)*lax monoidal functors and their morphisms.*

It follows that:

**Corollary 6.58.** *A double* (*co*)*lax functor preserves double* (*co*)*monoids and morphisms between double* (*co*)*monoids.*

We present the functor version of Proposition 6.29. Let $(\mathsf{C}, \bullet, \bullet)$ and $(\mathsf{D}, \bullet, \bullet)$ be strong 2-monoidal categories arising from braided monoidal categories as in Section 6.3.

**Proposition 6.59.** *A braided (co)lax functor from* $\mathsf{C}$ *to* $\mathsf{D}$ *gives rise to a double (co)lax functor for which both (co)lax structures are identical.*

*Conversely, let* $\mathcal{F}\colon (\mathsf{C}, \bullet, \bullet) \to (\mathsf{D}, \bullet, \bullet)$ *be a double (co)lax functor. Then the two (co)lax structures coincide and are braided (co)lax.*

Proof. For the first statement, the main diagram to check is (6.35). This diagram can be broken up using the associativity of $\varphi = \gamma$ so that one of the pieces is the square for the braided lax axiom (3.16). Details are omitted.

For the converse, let $\varphi$ and $\gamma$ be two lax structures on $\mathcal{F}$. Axiom (6.37) gives that $\varphi_0 = \gamma_0$. Next by setting $B = C = I$ in (6.35), it follows that $\varphi = \gamma$. Finally by setting $A = D = I$ in (6.35), it follows that $\varphi$ is braided, which finishes the proof. $\square$

This is a variant of the Eckmann–Hilton argument; see Proposition 6.29 and Remark 6.30. To summarize, in the context of braided monoidal categories, braided lax and double lax functors are equivalent notions.

**Remark 6.60.** Suppose there is given a binary operation on a set $X$. The set of axioms

$$(x + y) + (z + t) = (x + z) + (y + t)$$
$$x + 0 = x = 0 + x$$

is clearly equivalent to the set of axioms

$$(x + y) + z = x + (y + z)$$
$$x + y = y + x$$
$$x + 0 = x = 0 + x.$$

Corresponding to each set of axioms there is a different complex for calculating the homology theory of abelian groups of Eilenberg and Mac Lane [118, 120]. The complex corresponding to the first set is Eilenberg and Mac Lane's *cubical construction*. In this connection, see in particular [247, Section 1].

The first set of axioms is analogous to the axioms for a double lax monoidal functor (Definition 6.54) and the second set is analogous to the axioms for a braided lax monoidal functor (Definitions 3.1 and 3.11). Proposition 6.59 states that the equivalence remains true in the categorical setting.

**6.8.3. Braided bilax and double (co)lax monoidal functors.** We now discuss functors that go along with commutative bimonoids and double monoids. We recall that the context for this is that of braided 2-monoidal categories.

**Definition 6.61.** A $\diamond$-braided bilax functor between $\diamond$-braided 2-monoidal categories is a bilax functor between them which is braided lax.

A morphism between $\diamond$-braided bilax functors is a morphism between the underlying bilax functors.

**Definition 6.62.** A $\diamond$-braided double (co)lax functor between $\diamond$-braided 2-monoidal categories is a double (co)lax functor between them which is braided (co)lax.

A morphism between $\diamond$-braided double (co)lax functors is a morphism between the underlying double (co)lax functors.

$$\begin{tikzcd} 4 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 2 \arrow[r] & 1 \end{tikzcd}$$

double lax
lax-lax
$(2,0)$

$$\begin{tikzcd} 2 \arrow[r] \arrow[d] & 4 \arrow[d] \\ 1 \arrow[r] & 2 \end{tikzcd}$$

bilax
lax-colax
$(1,1)$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 2 \arrow[r] & 4 \end{tikzcd}$$

double colax
colax-colax
$(0,2)$

FIGURE 6.1. The interchanging axiom for various monoidal functors.

Similarly, one can define $\star$-braided bilax and $\star$-braided double (co)lax functors and morphisms between them.

**6.8.4. 2-strong monoidal functors.** There is a striking similarity between the axioms for a double lax functor, a bilax functor and a double colax functor. Let us concentrate on the interchanging axiom.

Figure 6.1 shows a condensed version of the interchanging axiom for various monoidal functors. The numbers 1, 2 and 4 indicate the number of occurrences of $\mathcal{F}$ in that entry. A comparison of the three pictures shows that one can pass from one to the other by reversing some of the arrows. To bring out this similarity, we also employ the terminology *lax-colax* for bilax, *lax-lax* for double lax and *colax-colax* for double colax. We also suggest a two tuple notation, with the two coordinates indicating the number of monoidal structures for which the functor is lax and colax respectively.

**Proposition 6.63.** *Suppose $(\mathcal{F}, \varphi)$ is strong and $(\mathcal{F}, \psi)$ is colax. Then $(\mathcal{F}, \varphi, \psi)$ is bilax if and only if $(\mathcal{F}, \varphi^{-1}, \psi)$ is double colax. Similarly, suppose $(\mathcal{F}, \varphi)$ is lax and $(\mathcal{F}, \psi)$ is costrong. Then $(\mathcal{F}, \varphi, \psi)$ is bilax if and only if $(\mathcal{F}, \varphi, \psi^{-1})$ is double lax.*

In the former situation, we say that $\mathcal{F}$ is strong-colax, or equivalently, costrong-colax, while in the latter, we say that $\mathcal{F}$ is lax-costrong, or equivalently, lax-strong. It follows that

strong-strong, strong-costrong, and costrong-costrong

are equivalent notions. We call a functor of this kind a 2-*strong functor*. It can be simultaneously viewed as a $(2,0)$-, a $(1,1)$- and a $(0,2)$-functor.

**Example 6.64.** A 2-strong functor between strong 2-monoidal categories is a familiar notion. Recall that a strong 2-monoidal category is equivalent to a braided monoidal category. In this context, strong-costrong is the same as bistrong by definition and (co)strong-(co)strong is the same as braided (co)strong by Proposition 6.59. As a consequence, we obtain Proposition 3.46 which says that braided (co)strong and bistrong functors are equivalent notions. Though the proof is essentially the same as before, the perspective provided here is new. To summarize, a 2-strong functor between strong 2-monoidal categories is the same as a braided strong (bistrong) functor between braided monoidal categories.

The similarity between bilax functors and double (co)lax functors was not manifest earlier between bilax functors and braided (co)lax functors. This is because the lax-lax axiom (6.35) simplifies to the braided lax axiom (3.16) in the context of braided monoidal categories, and so one directly works with that.

**6.8.5. Alternative descriptions of the various monoidal functors.** Let $(\mathsf{C}, \diamond, \star)$ and $(\mathsf{D}, \diamond, \star)$ be 2-monoidal categories and suppose $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is $\diamond$-lax and $\star$-colax functor with structure maps $\varphi$ and $\psi$ respectively. Let $\mathsf{I}$ be the one-arrow category. Recall the composite functors

$$\mathcal{F}^2, \mathcal{F}_2\colon \mathsf{C} \times \mathsf{C} \to \mathsf{D} \quad \text{and} \quad \mathcal{F}^0, \mathcal{F}_0\colon \mathsf{I} \to \mathsf{D}$$

defined in (3.1) and (3.2). Consider the following diagrams.

$$\text{(6.38)} \qquad \begin{tikzcd}
(\mathsf{C} \times \mathsf{C}, \diamond) \arrow[r, "\star"] \arrow[d, "\mathcal{F} \times \mathcal{F}"'] & (\mathsf{C}, \diamond) \arrow[d, "\mathcal{F}"] \arrow[dl, phantom, "\Swarrow \psi"] \\
(\mathsf{D} \times \mathsf{D}, \diamond) \arrow[r, "\star"'] & (\mathsf{D}, \diamond)
\end{tikzcd} \qquad \begin{tikzcd}
& (\mathsf{C}, \diamond) \arrow[dr, "\mathcal{F}"] \arrow[d, phantom, "\Downarrow \psi_0"] & \\
\mathsf{I} \arrow[ur] \arrow[rr] & {} & (\mathsf{D}, \diamond)
\end{tikzcd}$$

The composites in the first diagram are $\mathcal{F}_2$ and $\mathcal{F}^2$. Since $\star$ is lax monoidal (Proposition 6.4), both $\mathcal{F}_2$ and $\mathcal{F}^2$ can be turned into lax monoidal functors

$$(\mathsf{C} \times \mathsf{C}, \diamond) \to (\mathsf{C}, \diamond).$$

For the same reason, $\mathcal{F}_0$ and $\mathcal{F}^0$ can be turned into lax monoidal functors

$$\mathsf{I} \to (\mathsf{C}, \diamond).$$

Explicitly, the lax structures are as follows.

$$\begin{tikzcd}
\mathcal{F}(A \star B) \diamond \mathcal{F}(C \star D) \arrow[r, equal] \arrow[d, "\varphi_{A \star B, C \star D}"'] & \mathcal{F}_2(A, B) \diamond \mathcal{F}_2(C, D) \arrow[dd, dashed, "\varphi_2"] \\
\mathcal{F}\big((A \star B) \diamond (C \star D)\big) \arrow[d, "\mathcal{F}(\zeta)"'] & \\
\mathcal{F}\big((A \diamond C) \star (B \diamond D)\big) \arrow[r, equal] & \mathcal{F}_2\big((A, B) \diamond (C, D)\big)
\end{tikzcd} \qquad \begin{tikzcd}
\mathcal{F}(I) \arrow[d, "\mathcal{F}(\Delta)"'] & I \arrow[l, "\varphi_0"'] \arrow[d, dashed, "(\varphi_2)_0"] \\
\mathcal{F}(I \star I) \arrow[r, equal] & \mathcal{F}_2(I, I)
\end{tikzcd}$$

$$\begin{tikzcd}
\big(\mathcal{F}(A) \star \mathcal{F}(B)\big) \diamond \big(\mathcal{F}(C) \star \mathcal{F}(D)\big) \arrow[r, equal] \arrow[d, "\zeta"'] & \mathcal{F}^2(A, B) \diamond \mathcal{F}^2(C, D) \arrow[dd, dashed, "\varphi^2"] \\
\big(\mathcal{F}(A) \diamond \mathcal{F}(C)\big) \star \big(\mathcal{F}(B) \diamond \mathcal{F}(D)\big) \arrow[d, "\varphi_{A,C} \star \varphi_{B,D}"'] & \\
\mathcal{F}(A \diamond C) \star \mathcal{F}(B \diamond D) \arrow[r, equal] & \mathcal{F}^2\big((A, B) \diamond (C, D)\big)
\end{tikzcd} \qquad \begin{tikzcd}
I \star I \arrow[d, "\varphi_0 \star \varphi_0"'] & I \arrow[l, "\Delta"'] \arrow[d, dashed, "(\varphi^2)_0"] \\
\mathcal{F}(I) \star \mathcal{F}(I) \arrow[r, equal] & \mathcal{F}^2(I, I)
\end{tikzcd}$$

Up to this point, we have made no use of the $\star$-colax structure $\psi$. It is now natural to ask for the conditions under which $\psi\colon \mathcal{F}_2 \Rightarrow \mathcal{F}^2$ and $\psi_0\colon \mathcal{F}_0 \Rightarrow \mathcal{F}^0$ are morphisms of lax functors.

Dually, the $\star$-colax structure $\psi$ of $\mathcal{F}$ may be used to turn $\mathcal{F}_2, \mathcal{F}^2, \mathcal{F}_0$ and $\mathcal{F}^0$ into colax monoidal functors

$$(\mathsf{C} \times \mathsf{C}, \star) \to (\mathsf{C}, \star) \quad \text{and} \quad \mathsf{I} \to (\mathsf{C}, \star).$$

In this case one may ask whether $\varphi$ is a morphism of colax functors.

Similar questions can be posed in the situation when $\mathcal{F}$ is $\diamond$-lax and $\star$-lax or when $\mathcal{F}$ is $\diamond$-colax and $\star$-colax. The answers are summarized below. In each case, the functors $\mathcal{F}_2, \mathcal{F}^2, \mathcal{F}_0$ and $\mathcal{F}^0$ are endowed with the appropriate lax or colax structure.

**Proposition 6.65.** *Let* $\mathsf{C}$ *and* $\mathsf{D}$ *be* 2*-monoidal categories and* $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ *a functor.*

(a) *Suppose* $\varphi$ *is a* $\diamond$*-lax structure on* $\mathcal{F}$ *and* $\psi$ *is a* $\star$*-colax structure on* $\mathcal{F}$*. The following statements are equivalent.*
  (i) $(\mathcal{F}, \varphi, \psi)$ *is bilax;*
  (ii) $\varphi\colon \mathcal{F}^2 \Rightarrow \mathcal{F}_2$ *and* $\varphi_0\colon \mathcal{F}^0 \Rightarrow \mathcal{F}_0$ *are morphisms of colax functors;*
  (iii) $\psi\colon \mathcal{F}_2 \Rightarrow \mathcal{F}^2$ *and* $\psi_0\colon \mathcal{F}_0 \Rightarrow \mathcal{F}^0$ *are morphisms of lax functors.*

(b) *Suppose* $\varphi$ *is a* $\diamond$*-lax structure on* $\mathcal{F}$ *and* $\gamma$ *is a* $\star$*-lax structure on* $\mathcal{F}$*. The following statements are equivalent.*
  (i) $(\mathcal{F}, \varphi, \gamma)$ *is double lax;*
  (ii) $\gamma\colon \mathcal{F}^2 \Rightarrow \mathcal{F}_2$ *and* $\gamma_0\colon \mathcal{F}^0 \Rightarrow \mathcal{F}_0$ *are morphisms of lax functors.*

(c) *Suppose* $\psi$ *is a* $\diamond$*-colax structure on* $\mathcal{F}$ *and* $\delta$ *is a* $\star$*-colax structure on* $\mathcal{F}$*. The following statements are equivalent.*
  (i) $(\mathcal{F}, \psi, \delta)$ *is double colax;*
  (ii) $\psi\colon \mathcal{F}_2 \Rightarrow \mathcal{F}^2$ *and* $\psi_0\colon \mathcal{F}_0 \Rightarrow \mathcal{F}^0$ *are morphisms of colax functors.*

The proof is straightforward. The first set of statements generalizes the result of Proposition 3.77 from braided monoidal categories to 2-monoidal categories.

## 6.9. Examples of bilax and double (co)lax monoidal functors

We provide examples of monoidal functors between some of the 2-monoidal categories of Section 6.4.

**Example 6.66.** Let $X$ be a set and $(\mathsf{C}_X, \diamond, \star)$ the 2-monoidal category of graphs of Example 6.17. Consider also the symmetric monoidal category of sets under Cartesian product as a 2-monoidal category $(\mathsf{Set}, \times, \times)$ (Section 6.3). The forgetful functor

$$\mathcal{F}\colon \mathsf{C}_X \to \mathsf{Set}, \qquad \mathcal{F}(A, s, t) := A$$

which sends a graph to its underlying set of arrows, is double colax. The structure transformations

$$\psi_{A,B}\colon A \diamond B \to A \times B \qquad \text{and} \qquad \delta_{A,B}\colon A \star B \to A \times B$$

are simply the inclusion maps.

Now let $K$ be a commutative algebra and $(\mathsf{C}_K, \diamond, \star)$ the 2-monoidal category of $K$-bimodules of Example 6.18. Consider $\Bbbk$-modules under tensor product as a 2-monoidal category $(\mathsf{Mod}, \otimes, \otimes)$. The forgetful functor

$$\mathcal{F}\colon \mathsf{C}_K \to \mathsf{Mod}, \qquad \mathcal{F}(M) := M$$

is double lax. The structure transformations

$$\gamma_{M,N}\colon M \otimes N \to M \star N \qquad \text{and} \qquad \varphi_{M,N}\colon M \otimes N \to M \diamond N$$

are the projections defined in (6.20). The interchanging axiom in Definition 6.55 holds precisely by construction of the interchange law in $\mathsf{C}_K$ (diagram (6.21)).

The two preceding examples are related as follows. Given a set $A$, let $\Bbbk^A$ denote the space of all functions $A \to \Bbbk$. This defines the *dual linearization functor*

$$\Bbbk^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Vec}.$$

The inclusion

$$\Bbbk^A \otimes \Bbbk^B \hookrightarrow \Bbbk^{A \times B}$$

turns $\Bbbk^{(-)}$ into a braided lax monoidal functor $(\mathsf{Set}^{\mathrm{op}}, \times) \to (\mathsf{Vec}, \otimes)$, or equivalently (Proposition 6.59), a double lax monoidal functor

$$(\mathsf{Set}^{\mathrm{op}}, \times, \times) \to (\mathsf{Vec}, \otimes, \otimes).$$

Now let $X$ be a fixed set and view $\Bbbk^X$ as a commutative algebra under pointwise product. Given a graph $(A, s, t)$ with vertex set $X$, the following actions turn $\Bbbk^A$ into a $\Bbbk^X$-bimodule:

$$(f \cdot h)(a) := f(a)h\big(s(a)\big) \qquad \text{and} \qquad (h \cdot f)(a) := h\big(t(a)\big)f(a)$$

for $f \in \Bbbk^A$, $h \in \Bbbk^X$, and $a \in A$. Thus the dual linearization functor restricts to a functor

$$\Bbbk^{(-)} \colon (\mathsf{C}_X)^{\mathrm{op}} \to \mathsf{C}_{\Bbbk^X}.$$

Let $A$ and $B$ be graphs with vertex set $X$. It is easy to see that there are unique maps as indicated below.

(6.39)
$$\begin{tikzcd}
\Bbbk^A \otimes \Bbbk^B \arrow[r, hook] \arrow[d, two heads, "\varphi"'] & \Bbbk^{A\times B} \arrow[d, two heads, "\Bbbk^{\delta}"] \\
\Bbbk^A \diamond \Bbbk^B \arrow[r, dashed] & \Bbbk^{A\star B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk^A \otimes \Bbbk^B \arrow[r, hook] \arrow[d, two heads, "\gamma"'] & \Bbbk^{A\times B} \arrow[d, two heads, "\Bbbk^{\psi}"] \\
\Bbbk^A \star \Bbbk^B \arrow[r, dashed] & \Bbbk^{A\diamond B}
\end{tikzcd}$$

For instance, the existence of the map on the right is due to the fact that for $f \in \Bbbk^A$, $g \in \Bbbk^B$, $h \in \Bbbk^X$, and $(a, b) \in A \diamond B$, we have

$$\begin{aligned}(f \cdot h \otimes g)(a, b) &= (f \cdot h)(a)g(b) = f(a)h\big(s(a)\big)g(b) \\ &= f(a)h\big(t(b)\big)g(b) = f(a)(h \cdot g)(b) = (f \otimes h \cdot g)(a, b)\end{aligned}$$

(we used that $s(a) = t(b)$ for $(a, b) \in A \diamond B$).

The maps in (6.66) turn $\Bbbk^{(-)}$ into a double lax monoidal functor

$$(\mathsf{C}_X^{\mathrm{op}}, \star, \diamond) \to (\mathsf{C}_{\Bbbk^X}, \diamond, \star)$$

(note the reversal of the order of the operations in the opposite of $\mathsf{C}_X$). Moreover, diagrams (6.66) imply that

$$\begin{tikzcd}
(\mathsf{C}_X)^{\mathrm{op}} \arrow[r, "\mathcal{F}^{\mathrm{op}}"] \arrow[d, "\Bbbk^{(-)}"'] & \mathsf{Set}^{\mathrm{op}} \arrow[d, "\Bbbk^{(-)}"] \\
\mathsf{C}_{\Bbbk^X} \arrow[r, "\mathcal{F}"'] & \mathsf{Vec}
\end{tikzcd}$$

is a commutative diagram of double lax monoidal functors.

**Example 6.67.** Let $P$ and $Q$ be two lattices and let $\mathsf{C}_P$ and $\mathsf{C}_Q$ be the associated 2-monoidal categories, as in Example 6.20. Let $f \colon P \to Q$ be an order-preserving map:

$$x \leq y \in P \implies f(x) \leq f(y) \in Q.$$

We then have a bilax monoidal functor

$$(\mathcal{F}, \varphi, \gamma) \colon (\mathsf{C}_P, \vee, \wedge) \to (\mathsf{C}_Q, \vee, \wedge)$$

as follows. The functor $\mathcal{F}$ is the map $f$ on objects, and is well-defined on morphisms since $f$ is order-preserving. For the same reason, $f(a) \vee f(b) \leq f(a \vee b)$ and $f(a \wedge b) \leq f(a) \wedge f(b)$, and this gives rise to the transformations

$$\varphi\colon f(a) \vee f(b) \to f(a \vee b) \qquad \text{and} \qquad \psi\colon f(a \wedge b) \to f(a) \wedge f(b).$$

The maps $\varphi_0\colon 0 \to f(0)$ and $\psi_0\colon f(1) \to 1$ are also well-defined. If $f$ is a morphism of lattices, then $\mathcal{F}$ is bistrong.

**Example 6.68.** Let $(\mathsf{C}, \diamond, \times)$ be a 2-monoidal category as in Example 6.19, in which the second monoidal structure is cartesian. Let $\mathsf{D}$ be another such category and let $(\mathcal{F}, \varphi) : (\mathsf{C}, \diamond) \to (\mathsf{D}, \diamond)$ be a lax monoidal functor. According to Example 3.19, $\mathcal{F}$ carries a canonical colax monoidal structure

$$\psi_{A,B}\colon \mathcal{F}(A \times B) \to \mathcal{F}(A) \times \mathcal{F}(B).$$

It turns out that

$$(\mathcal{F}, \varphi, \psi)\colon (\mathsf{C}, \diamond, \times) \to (\mathsf{D}, \diamond, \times)$$

is in fact bilax monoidal. Let us verify the commutativity of diagram (6.32). We claim that both sides yield the same map defined by the commutativity of the diagram below (the map is shown dotted).

$$\begin{tikzcd}
\mathcal{F}(A) \diamond \mathcal{F}(C) \arrow[r, "\varphi_{A,C}"] & \mathcal{F}(A \diamond C) \\
\mathcal{F}(A \times B) \diamond \mathcal{F}(C \times D) \arrow[u, "\mathcal{F}(\pi_A)\diamond\mathcal{F}(\pi_C)"] \arrow[d, "\mathcal{F}(\pi_B)\diamond\mathcal{F}(\pi_D)"'] \arrow[r, dashed] & \mathcal{F}(A \diamond C) \times \mathcal{F}(B \diamond D) \arrow[u, "\pi_{\mathcal{F}(A\diamond C)}"'] \arrow[d, "\pi_{\mathcal{F}(B\diamond D)}"] \\
\mathcal{F}(B) \diamond \mathcal{F}(D) \arrow[r, "\varphi_{B,D}"'] & \mathcal{F}(B \diamond D)
\end{tikzcd}$$

To prove this, one may work separately with each square. The proof involving the top square is the commutativity of the diagram below. The other is similar.

$$\begin{tikzcd}
 & \mathcal{F}(A \times B) \diamond \mathcal{F}(C \times D) \arrow[dl, "\varphi_{A\times B, C\times D}"'] \arrow[dd, "\mathcal{F}(\pi_A)\diamond\mathcal{F}(\pi_C)"] \arrow[dr, "\psi_{A,B}\diamond\psi_{C,D}"] & \\
\mathcal{F}\big((A \times B) \diamond (C \times D)\big) \arrow[d, "\mathcal{F}(\zeta_{A,B,C,D})"'] \arrow[ddr, "\mathcal{F}(\pi_A\diamond\pi_C)"] & & \big(\mathcal{F}(A) \times \mathcal{F}(B)\big) \diamond \big(\mathcal{F}(C) \times \mathcal{F}(D)\big) \arrow[dl, "\pi_{\mathcal{F}(A)}\diamond\pi_{\mathcal{F}(C)}"'] \arrow[d, "\zeta_{\mathcal{F}(A),\mathcal{F}(B),\mathcal{F}(C),\mathcal{F}(D)}"] \\
\mathcal{F}\big((A \diamond C) \times (B \diamond D)\big) \arrow[d, "\psi_{A\diamond C, B\diamond D}"'] \arrow[dr, "\mathcal{F}(\pi_{A\diamond C})"] & \mathcal{F}(A) \diamond \mathcal{F}(C) \arrow[d, "\varphi_{A,C}"] & \big(\mathcal{F}(A) \diamond \mathcal{F}(C)\big) \times \big(\mathcal{F}(B) \diamond \mathcal{F}(D)\big) \arrow[l, "\pi_{\mathcal{F}(A)\diamond\mathcal{F}(C)}"] \arrow[d, "\varphi_{A,C}\times\varphi_{B,D}"] \\
\mathcal{F}(A \diamond C) \times \mathcal{F}(B \diamond D) \arrow[r, "\pi_{\mathcal{F}(A\diamond C)}"'] & \mathcal{F}(A \diamond C) & \mathcal{F}(A \diamond C) \times \mathcal{F}(B \diamond D) \arrow[l, "\pi_{\mathcal{F}(A\diamond C)}"]
\end{tikzcd}$$

Above, the triangles involving $\zeta$ commute by the definition of the interchange law in Example 6.19, and the triangles involving $\psi$ by the definition of the colax structure in Example 3.19. This completes the check.

Thus, in this situation every lax monoidal functor carries a canonical bilax monoidal structure. Similarly, every colax monoidal functor carries a canonical double colax monoidal structure.

There are dual statements for functors between 2-monoidal categories of the form $(\mathsf{C}, \amalg, \star)$, where the first structure is cocartesian. For categories of the form $(\mathsf{C}, \amalg, \times)$, any functor carries a canonical bilax monoidal structure. This generalizes the construction for bicartesian monoidal categories in Example 3.19.

## 6.10. The free monoid functor as a bilax monoidal functor

In this section, we construct the free bimonoid on a comonoid in a monoidal category (satisfying some additional hypotheses). Related ideas in the context of species are discussed in detail in Chapter 11; see Section 11.2 in particular. The relevance of this construction to the present chapter is that it provides an example of a bilax functor.

**6.10.1. The free monoid.** Let $(\mathsf{C}, \star, J)$ be a monoidal category with countable coproducts such that for any object $A$, the functors

$$A \star (-) \colon \mathsf{C} \to \mathsf{C} \qquad \text{and} \qquad (-) \star A \colon \mathsf{C} \to \mathsf{C}$$

preserve these coproducts. Namely, for objects $B_1, B_2, \dots, B_n, \dots$ in $\mathsf{C}$, the maps

$$\coprod_n (A \star B_n) \xrightarrow{\cong} A \star \Big(\coprod_n B_n\Big) \qquad \text{and} \qquad \coprod_n (B_n \star A) \xrightarrow{\cong} \Big(\coprod_n B_n\Big) \star A$$

defined by the universal property of coproducts are isomorphisms (as indicated).

Let $I$ be an initial object in $\mathsf{C}$. We further assume that the canonical map $I \to A \star I$ and $I \to I \star A$ are isomorphisms for every object $A$. In particular, tensoring with $A$ on either side is a strong monoidal functor on $(\mathsf{C}, \amalg, I)$.

Now define the free monoid functor

$$\mathcal{T} \colon (\mathsf{C}, \amalg, I) \to (\mathsf{C}, \star, J) \qquad\qquad \mathcal{T}(A) := \coprod_{n \geq 0} A^{\star n}$$

where $A^{\star n}$ is the unbracketed tensor product of $A$ with itself $n$ times. In particular, $A^{\star 0} = J$.

We now proceed to turn $\mathcal{T}$ into a lax functor. Define morphisms

$$\mathcal{T}(A) \star \mathcal{T}(B) \to \mathcal{T}(A \amalg B) \qquad \text{and} \qquad J \xrightarrow{\cong} \mathcal{T}(I) \tag{6.40}$$

as follows. The second map is defined to be the obvious isomorphism. For the first map, we note that

$$\mathcal{T}(A) \star \mathcal{T}(B) = \coprod_{n,m} A^{\star n} \star B^{\star m} \quad \text{and} \quad \mathcal{T}(A \amalg B) = \coprod_k \coprod_{(C_1, \dots, C_k)} C_1 \star C_2 \star \cdots \star C_k$$

where the sum is over all $(C_1, \dots, C_k)$ such that $C_i$ is equal to either $A$ or $B$. The map between the two is defined in the obvious manner with the $(n, m)$-summand mapping into the $(k = n + m)$-summand.

The lax functor $\mathcal{T}$ induces a functor on monoids, which in view of Example 1.19 maps as follows:

$$\mathcal{T} \colon \mathsf{C} \to \mathsf{Mon}(\mathsf{C}, \star, J).$$

Explicitly, the product of $\mathcal{T}(A)$ is given by the canonical identification of $A^{\star n} \star A^{\star m}$ with $A^{\star(m+n)}$, and the unit is the canonical map from $J = A^{\star 0}$ into the coproduct.

**Proposition 6.69.** [250, Theorem VII.3.2] *We have that $\mathcal{T}(A)$ is the free monoid on $A$, or equivalently that, $\mathcal{T}$ is the left adjoint of the forgetful functor.*

**Remark 6.70.** There are situations in which one is interested in the free monoid on a monoidal category and the above hypotheses are not satisfied. The free operad on a species is an example of this situation [260, Section 1.9]. For recent work in this direction, see [364].

**6.10.2. The free bimonoid on a comonoid.** Recall from Example 6.19 that $(\mathsf{C}, \amalg, \star)$ is a 2-monoidal category. Now assume that $(\mathsf{C}, \star, J)$ is braided. We proceed to turn $\mathcal{T}$ into a bilax functor

$$\mathcal{T} \colon (\mathsf{C}, \amalg, \star) \to (\mathsf{C}, \star, \star).$$

The lax structure is as defined in (6.40). For the colax structure, define the morphisms

$$\mathcal{T}(A \star B) \to \mathcal{T}(A) \star \mathcal{T}(B) \qquad \text{and} \qquad \mathcal{T}(J) \to J$$

or equivalently,

$$\coprod_k (A \star B)^{\star k} \to \coprod_{n,m} A^{\star n} \star B^{\star m} \qquad \text{and} \qquad \coprod_n J^{\star n} \to J$$

as follows. The first map is defined by rearranging the factors using the braiding and the second map is defined by summing the isomorphisms $J^{\star n} \to J$. It is straightforward to check that $\mathcal{T}$ is bilax.

Recall from Example 6.42 that a bimonoid in $(\mathsf{C}, \amalg, \star)$ is the same as a comonoid in $(\mathsf{C}, \star)$. The bilax functor then induces a functor on bimonoids:

$$\mathcal{T} \colon \mathsf{Comon}(\mathsf{C}, \star) \to \mathsf{Bimon}(\mathsf{C}, \star, \star).$$

**Proposition 6.71.** *We have that $\mathcal{T}(A)$ is the free bimonoid on a comonoid $A$, or equivalently that, $\mathcal{T}$ is the left adjoint of the forgetful functor from bimonoids to comonoids.*

Proof. Consider the adjunction

$$\begin{tikzcd} (\mathsf{C}, \star) \arrow[r, bend left, "\mathcal{T}"] & (\mathsf{Mon}(\mathsf{C}, \star), \star) \arrow[l, bend left, "f\ell"] \end{tikzcd}$$

of Proposition 6.69. It is clear that the forgetful functor is strong. Proposition 3.94 implies that there is a unique colax structure on $\mathcal{T}$ such that the above adjunction is colax-colax. One checks that this colax structure on $\mathcal{T}$ matches the one that was defined above. The result now follows from Proposition 3.91. □

## 6.11. 2-monoidal categories viewed as pseudomonoids

The contents of this section rely on several notions pertaining to 2-categories discussed in Appendix C such as monoidal 2-categories and pseudomonoids.

The 2-categories of relevance to the present chapter are summarized in Table 6.3. The first on the list is $\mathsf{Cat}$, whose objects are categories and whose 1-cells are functors. The other categories similarly involve monoidal or 2-monoidal categories and functors; the 2-categories $\mathsf{Cat}$, $\mathsf{lCat}$ and $\mathsf{cCat}$ were introduced in Section 3.3.3. The 2-cells for $\mathsf{Cat}$ are natural transformations; the 2-cells in the other

TABLE 6.3. 2-categories related to $\mathsf{Cat}$.

| 2-category | 0-cell | 1-cell | 2-cell |
|---|---|---|---|
| $\mathsf{Cat}$ | category | functor | nat. transf. |
| $\mathsf{lCat}$ | monoidal category | lax monoidal functor | Def. 3.8 |
| $\mathsf{cCat}$ | monoidal category | colax monoidal functor | Def. 3.8 |
| $\mathsf{llCat}$ | 2-monoidal category | double lax monoidal functor | Def. 6.56 |
| $\mathsf{lcCat}$ | 2-monoidal category | bilax monoidal functor | Def. 6.51 |
| $\mathsf{ccCat}$ | 2-monoidal category | double colax monoidal functor | Def. 6.56 |

$$\begin{tikzcd}[column sep=small]
 & & \mathsf{Cat} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \mathsf{lCat} \arrow[dl, dash] \arrow[dr, dash] & & \mathsf{cCat} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathsf{llCat} & & \mathsf{lcCat} & & \mathsf{ccCat}
\end{tikzcd}$$

FIGURE 6.2. The lax and colax constructions on $\mathsf{Cat}$.

2-categories do not have a special name. For example, a 2-cell in $\mathsf{lCat}$ is a morphism between lax functors in the sense of Definition 3.8.

Recall that for a monoidal category, one has the monoid and comonoid constructions, namely $\mathsf{Mon}(-)$ and $\mathsf{Comon}(-)$ (Section 1.2.7). Similarly, for a monoidal 2-category, one has the lax and colax constructions, denoted by $\mathsf{l}(-)$ and $\mathsf{c}(-)$. A monoidal 2-category is a 2-category with a compatible monoidal structure. The lax and colax constructions refer to the passage from a monoidal 2-category to the category of pseudomonoids therein.

The goal of this section is to show that the 2-categories of Table 6.3 arise by means of the lax and colax constructions. This is detailed in Figure 6.2. Applying the lax construction takes us one step down to the left, while applying the colax construction takes us one step down to the right. The starting point is the monoidal 2-category $\mathsf{Cat}$.

This claim is proved in Propositions 6.72 and 6.75 below. There is an inductive aspect to the argument, with the proof of the latter result relying on the proof of the former.

**Proposition 6.72.** *A pseudomonoid in* $\mathsf{Cat}$ *is precisely a monoidal category. Moreover,*

$$\mathsf{lCat} = \mathsf{l}(\mathsf{Cat}) \qquad \textit{and} \qquad \mathsf{cCat} = \mathsf{c}(\mathsf{Cat}).$$

This result is given in [264, Example 2.4].

PROOF. Let $\mathsf{I}$ be the one-arrow category. According to Definition C.3, a pseudomonoid in the monoidal 2-category $\mathsf{Cat}$ is a category $\mathsf{C}$ equipped with functors

$$\mathsf{C} \times \mathsf{C} \to \mathsf{C} \qquad \text{and} \qquad \mathsf{I} \to \mathsf{C},$$

and natural transformations as in (C.3)–(C.4) which satisfy axioms (C.5)–(C.6). We explain how this corresponds to a monoidal category structure on $\mathsf{C}$ (Definition 1.1).

The former functor is the tensor product and the transformation $\alpha$ is the associativity constraint. The unit object is the image of the unique object of $\mathsf{I}$ under the latter functor.

The three pentagons in the front of (C.5) correspond to the three arrows along the bottom of pentagon (1.1). The two pentagons in the back (other than the base) of (C.5) correspond to the top of pentagon (1.1). Thus, axiom (C.5) becomes axiom (1.1). Similarly, axiom (C.6) becomes axiom (1.2).

In summary, a pseudomonoid in $\mathsf{Cat}$ is precisely a monoidal category.

To complete the proof of $\mathsf{lCat} = \mathsf{l}(\mathsf{Cat})$ we need to consider morphisms of pseudomonoids, and morphisms between them. (The proof of $\mathsf{cCat} = \mathsf{c}(\mathsf{Cat})$ is similar.) A lax morphism between pseudomonoids (Definition C.4) is the same as a lax monoidal functor between monoidal categories (Definition 3.1). Indeed, the natural transformations (C.7) correspond to the structure morphisms (3.3) and (3.4) and the 2-cell diagrams (C.8) and (C.9) correspond to the associativity and unitality axioms, namely (3.5) and (3.6).

Similarly, a morphism between lax morphisms (Definition C.5) is the same as a morphism between lax monoidal functors (Definition 3.8) with (C.10) reducing to (3.14). $\square$

**Proposition 6.73.** *A pseudomonoid in* $\mathsf{lCat}$ *or* $\mathsf{cCat}$ *is precisely a 2-monoidal category.*

Proof. We use Definition C.3 to understand a pseudomonoid in $\mathsf{lCat}$ explicitly. To start with, we require an object in $\mathsf{lCat}$. This is a monoidal category, which for definiteness, we call $(\mathsf{C}, \diamond, I)$. Next, we require lax monoidal functors

$$(\star, \varphi, \varphi_0) \colon (\mathsf{C} \times \mathsf{C}, \diamond) \to (\mathsf{C}, \diamond) \qquad \text{and} \qquad (\mathcal{J}, \varphi', \varphi_0') \colon \mathsf{I} \to (\mathsf{C}, \diamond), \tag{6.41}$$

where $\mathsf{I}$ is the one-arrow category. More explicitly, we require morphisms

$$\zeta := \varphi \colon (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$
$$\Delta_I := \varphi_0 \colon I \to I \star I, \qquad \mu_J := \varphi' \colon J \diamond J \to J, \qquad \iota_J = \epsilon_I := \varphi_0' \colon I \to J$$

where $J$ is the image under $\mathcal{J}$ of the unique object in $\mathsf{I}$. The notations on the left are chosen to make the connection with Definition 6.1 clearer. One readily checks that the associativity and unitality of the lax monoidal functor $(\star, \varphi, \varphi_0)$ is the same as (6.3) and (6.5) while that of $(\mathcal{J}, \varphi', \varphi_0')$ is the same as (6.7). The discussion so far matches with the first part of Proposition 6.4.

Before proceeding further, we set up some notation. Let $\mathcal{F}$ and $\mathcal{G}$ be two functors from $\mathsf{C} \times \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ defined by

$$\mathcal{F}(A, B, C) := A \star (B \star C) \qquad \text{and} \qquad \mathcal{G}(A, B, C) := (A \star B) \star C.$$

Let $\mathcal{L}$ and $\mathcal{R}$ be two functors from $\mathsf{C}$ to itself defined by

$$\mathcal{L}(A) := J \star A \qquad \text{and} \qquad \mathcal{R}(A) := A \star J.$$

As specified in (C.3)–(C.4), we require morphisms of lax monoidal functors $\alpha \colon \mathcal{F} \Rightarrow \mathcal{G}$, $\lambda \colon \mathrm{id} \Rightarrow \mathcal{L}$ and $\rho \colon \mathrm{id} \Rightarrow \mathcal{R}$. This provides natural morphisms

$$A \star (B \star C) \to (A \star B) \star C, \quad A \to J \star A, \quad A \to A \star J.$$

The fact that $\alpha$ is a morphism of lax monoidal functors is the same as (6.4) and the coassociativity of $\Delta_I$ in (6.8). The fact that $\lambda$ and $\rho$ are morphisms of lax functors is the same as (6.6) and the counitality of $\Delta_I$ in (6.8). This accounts for all the 2-monoidal category axioms, each one appearing exactly once.

The final requirement, namely (C.5)–(C.6), says that $\alpha$, $\lambda$ and $\rho$ satisfy the pentagon and triangle axioms (as in Proposition 6.72). This requires $(\mathsf{C}, \star, J)$ to be a monoidal category, which is the last piece of the puzzle.

To summarize, we have shown that a pseudomonoid in $\mathsf{lCat}$ is the same as a 2-monoidal category.

The argument for $\mathsf{cCat}$ proceeds along the same lines. For the notation to match up, we have to start with the monoidal category $(\mathsf{C}, \star, J)$ and define colax monoidal functors using $\diamond$ and $\mathcal{I}$ as in the second part of Proposition 6.4. As expected, the axioms emerge in a different order; however, we do get all of them, each one appearing exactly once, as before. $\square$

**Remark 6.74.** The above result provides two alternative definitions for a 2-monoidal category. Further, the proof shows that these definitions are obtained by expanding conditions (i) and (ii) respectively of Proposition 6.4.

**Proposition 6.75.** *We have*

$$\mathsf{llCat} = \mathsf{l}(\mathsf{lCat}), \qquad \mathsf{lcCat} = \mathsf{c}(\mathsf{lCat}) = \mathsf{l}(\mathsf{cCat}), \qquad \mathsf{ccCat} = \mathsf{c}(\mathsf{cCat}). \tag{6.42}$$

Proof. The lax and colax constructions applied to $\mathsf{lCat}$ or $\mathsf{cCat}$ yield pseudomonoids therein, and according to Proposition 6.73 these are 2-monoidal categories. Hence, the 0-cells of all 2-categories in (6.42) are 2-monoidal categories.

We now turn our attention to the 1-cells. The four arguments to be made are essentially the four parts of Proposition 6.65. For concreteness, we explain one of them. Let $\mathcal{F}$ be a 1-cell in $\mathsf{c}(\mathsf{lCat})$ connecting the 0-cells $\mathsf{C}$ and $\mathsf{D}$. Recall that $\mathsf{C}$ and $\mathsf{D}$ are equipped with (6.41). We now understand $\mathcal{F}$ explicitly using Definition C.4. Since we are doing the colax construction, the arrows on the 2-cells in the diagrams should be reversed. To start with, there exists a suitable $\varphi$ so that

$$(\mathcal{F}, \varphi) \colon (\mathsf{C}, \diamond) \to (\mathsf{D}, \diamond)$$

is lax. The next requirement is given by (C.7). For this particular example, the same diagrams are shown in (6.38). As in the proof of Proposition 6.72, the conditions (C.8) and (C.9) imply that there exists a suitable $\psi$ so that

$$(\mathcal{F}, \psi) \colon (\mathsf{C}, \star) \to (\mathsf{D}, \star)$$

is colax. However, observe that (C.7) requires something more; namely that $\psi$ be a morphism of lax functors. By one part of Proposition 6.65, this is the same as $(\mathcal{F}, \varphi, \psi)$ being bilax or lax-colax.

The situation for 2-cells is simpler. Continuing with the previous example, let $\mathcal{F} \Rightarrow \mathcal{G}$ be a 2-cell in $\mathsf{c}(\mathsf{lCat})$. To start with, it is a 2-cell in $\mathsf{lCat}$ which means that $(\mathcal{F}, \varphi) \Rightarrow (\mathcal{G}, \gamma)$ is a morphism of lax functors. The condition (C.10) further says that $(\mathcal{F}, \psi) \Rightarrow (\mathcal{G}, \delta)$ is a morphism of colax functors. This completes the proof. $\square$

**Remark 6.76.** The middle claim in (6.42) shows that bilax monoidal functors are 1-cells of a 2-category. In particular, this shows that the composite of bilax functors is again bilax. A direct proof of this was given in Theorem 3.22. The above proof is more abstract; the main idea involved in it was also explained earlier in Remark 3.78. Similar remarks apply to double (co)lax monoidal functors.

TABLE 6.4. More 2-categories related to $\mathsf{Cat}$.

| 2-category | 0-cell | 1-cell |
|---|---|---|
| $\mathsf{bCat}$ | braided monoidal category | bilax monoidal functor |
| $\mathsf{blCat}$ | braided monoidal category | braided lax monoidal functor |
| $\mathsf{bcCat}$ | braided monoidal category | braided colax monoidal functor |
| $\mathsf{cblCat}$ | $\diamond$-braided 2-monoidal category | $\diamond$-braided bilax monoidal functor |
| $\mathsf{lbcCat}$ | $\star$-braided 2-monoidal category | $\star$-braided bilax monoidal functor |

We now consider 2-categories involving braided monoidal categories, which have been considered explicitly in this monograph. They are summarized in Table 6.4. The 2-cells in $\mathsf{bCat}$ are the morphisms of bilax monoidal functors in the sense of Definition 3.9. Note that the braiding does not play a role in this definition. Similarly, the 2-cells in the other 2-categories are the morphisms of the underlying (co, bi)lax monoidal functors.

The following result describes the behavior of the lax and colax constructions in this context. For the definitions related to braided 2-monoidal categories, see Section 6.1.5. The proof is omitted.

**Proposition 6.77.** *A pseudomonoid in* $\mathsf{bcCat}$ *is precisely a* $\star$*-braided* 2*-monoidal category. Similarly, a pseudomonoid in* $\mathsf{blCat}$ *is precisely a* $\diamond$*-braided* 2*-monoidal category. Moreover, we have*

$$\mathsf{cblCat} = \mathsf{c}(\mathsf{blCat}) \qquad \textit{and} \qquad \mathsf{lbcCat} = \mathsf{l}(\mathsf{bcCat}).$$

## 6.12. Contragredience for 2-monoidal categories

In this section, we return to the contragredient construction of Section 3.10 and generalize it to the context of 2-monoidal categories. The present discussion assumes familiarity with the contents of Section 3.10.

**6.12.1. Contravariant monoidal functors.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a contravariant functor. Now let $\mathsf{C}$ and $\mathsf{D}$ be 2-monoidal categories. We say that $\mathcal{F}$ is *contravariant* 2*-strong* if

$$\mathcal{F}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}, \qquad \text{or equivalently,} \qquad \mathcal{F}\colon \mathsf{C} \to \mathsf{D}^{\mathrm{op}}$$

is 2-strong. We say that $\mathcal{F}$ is a *contravariant bilax functor* if

$$\mathcal{F}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}, \qquad \text{or equivalently,} \qquad \mathcal{F}\colon \mathsf{C} \to \mathsf{D}^{\mathrm{op}}$$

is bilax.

**6.12.2. Contragredient of 2-monoidal categories.** Now let

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "*"] & \mathsf{C}' \arrow[l, bend left, "*"] \end{tikzcd}$$

be a contravariant adjoint equivalence of categories, as in (3.44).

Let, say $(\mathsf{C}, \diamond, \star)$, be a 2-monoidal category with structure morphisms $\zeta$, $\Delta_I$, $\mu_J$ and $\iota_J = \epsilon_I$. Using this data, one can define a 2-monoidal structure on $\mathsf{C}'$. We use the notation

$$(\mathsf{C}', \star^{\vee}, \diamond^{\vee})$$

for the 2-monoidal category,

$$\zeta^\vee : (A \diamond^\vee B) \star^\vee (C \diamond^\vee D) \to (A \star^\vee C) \diamond^\vee (B \star^\vee D)$$

for the interchange law, and

$$\Delta_I^\vee : I^* \star^\vee I^* \to I^*, \qquad \mu_J^\vee : J^* \to J^* \diamond^\vee J^*, \qquad \iota_J^\vee = \epsilon_I^\vee : J^* \to I^*.$$

for the structure morphisms. These are defined as follows.

The monoidal structures $\star^\vee$ and $\diamond^\vee$ are given by

$$A \star^\vee B := (A^* \star B^*)^* \qquad \text{and} \qquad A \diamond^\vee B := (A^* \diamond B^*)^*.$$

The correponding unit objects are $J^*$ and $I^*$. For the interchange law $\zeta^\vee$: Let $A$, $B$, $C$ and $D$ be objects of $\mathsf{C}'$. Then $A^*$, $C^*$, $B^*$ and $D^*$ are objects of $\mathsf{C}$. Evaluating the interchange law $\zeta$ of $\mathsf{C}$ on these objects (in that order), one obtains a morphism

$$(A^* \star C^*) \diamond (B^* \star D^*) \to (A^* \diamond B^*) \star (C^* \diamond D^*).$$

Applying the $*$ functor to this morphism yields

$$\big((A^* \diamond B^*) \star (C^* \diamond D^*)\big)^* \to \big((A^* \star C^*) \diamond (B^* \star D^*)\big)^*$$

which is a morphism in $\mathsf{C}'$. This is defined to be $\zeta^\vee$ evaluated on $A$, $B$, $C$ and $D$. The morphisms $\Delta_I^\vee$, $\mu_J^\vee$ and $\iota_J^\vee = \epsilon_I^\vee$ are defined similarly.

It is straightforward to check that $(\mathsf{C}', \star^\vee, \diamond^\vee)$ is a 2-monoidal category. We call it the *contragredient* of $(\mathsf{C}, \diamond, \star)$.

**Example 6.78.** The 2-monoidal categories of graded vector spaces with finite-dimensional components

$$(\mathsf{gVec}, \cdot, \times) \qquad \text{and} \qquad (\mathsf{gVec}, \times, \cdot)$$

of Example 6.22 are contragredients of each other. The essential check is that $\mathsf{gVec}$ is self-dual with respect to the Cauchy and Hadamard products, and the interchange laws in the two cases are contragredients of each other.

Additional examples of this kind, in which $\mathsf{C} = \mathsf{C}'$ and this category is self-dual with respect to both $\diamond$ and $\star$ (so the contragredient has the effect of switching the order of the tensor products), are discussed in Section 8.13.5. They involve species instead of graded vector spaces.

**Proposition 6.79.** *The functors*

$$\begin{tikzcd}(\mathsf{C}, \diamond, \star) \arrow[r, bend left, "*"] & (\mathsf{C}', \star^\vee, \diamond^\vee) \arrow[l, bend left, "*"]\end{tikzcd}$$

*are contravariant* 2*-strong.*

The proof is straightforward.

**6.12.3. Contragredient of monoidal functors.** Consider the situation

$$\mathcal{F}^\vee : \mathsf{C} \xrightarrow{\;*\;} \mathsf{C}' \xrightarrow{\;\mathcal{F}\;} \mathsf{D}' \xrightarrow{\;*\;} \mathsf{D}$$

as in (3.45). For a natural transformation $\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$, let $\theta^\vee : \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee$ denote the induced natural transformation.

**Proposition 6.80.** *Let* $\mathsf{C}'$ *and* $\mathsf{D}'$ *be* 2*-monoidal categories. If* $\mathcal{F}\colon \mathsf{C}' \to \mathsf{D}'$ *is a double lax* (*bilax, double colax*) *monoidal functor, then* $\mathcal{F}^\vee\colon \mathsf{C} \to \mathsf{D}$ *is a double colax* (*bilax, double lax*) *monoidal functor.*

*Further, if* $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ *is a morphism of double lax* (*bilax, double colax*) *functors, then* $\theta^\vee\colon \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee$ *is a morphism of double colax* (*bilax, double lax*) *functors.*

We elaborate further on the above construction. The categories $\mathsf{C}$ and $\mathsf{D}$ are given the 2-monoidal structures contragredient to those of $\mathsf{C}'$ and $\mathsf{D}'$. The colax structure(s) of $\mathcal{F}^\vee$ are constructed from the lax structure(s) of $\mathcal{F}$ and viceversa, as in Proposition 3.102. The proof makes use of Proposition 6.79 and follows along the lines of the proof given for Proposition 3.102.

**6.12.4. Self-duality.** For completeness, we record some basic definitions and results related to self-duality. Here we work in the situation where $\mathsf{C} = \mathsf{C}'$ and where this category is equipped with a self-adjoint $*$ functor.

**Definition 6.81.** A 2-monoidal category $(\mathsf{C}, \diamond, \star)$ is *self-dual* if

$$\mathrm{id}\colon (\mathsf{C}, \diamond, \star) \to (\mathsf{C}, \star^\vee, \diamond^\vee)$$

is a 2-strong equivalence.

**Definition 6.82.** Let $\mathsf{C}$ and $\mathsf{D}$ be self-dual 2-monoidal categories. A bilax functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is *self-dual* if $\mathcal{F}^\vee \cong \mathcal{F}$ as bilax functors.

**Proposition 6.83.** *A self-dual bilax functor induces a self-dual functor on the corresponding categories of bimonoids. In particular, it preserves self-dual bimonoids.*

The proof of the first claim is straightforward. The second claim follows from the first by Proposition 3.107.

**Definition 6.84.** Let $\mathsf{C}$ and $\mathsf{D}$ be self-dual 2-monoidal categories, and let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a bilax functor. A natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{F}^\vee$ of bilax functors is *self-dual* if $\theta^\vee \cong \theta$.

CHAPTER 7

# Higher Monoidal Categories

In previous chapters, we studied monoidal and 2-monoidal categories in detail. These are the $n = 1$ and $n = 2$ cases of the general notion which occupies us in this chapter. To start off, it is natural to ask whether there is a reasonable notion of a category with three compatible monoidal structures. The answer is in the affirmative, and in fact, using this notion, one can then define without difficulty categories with any number of monoidal structures. Roughly speaking, $n$-monoidal categories are categories with $n$ ordered monoidal structures related by $\binom{n}{2}$ interchange laws. They are not to be confused with monoidal $n$-categories (which would be $n$-categories with a monoidal structure).

The $n = 3$ case is very interesting and along with the previous two cases plays an important role in the general theory. We start by discussing 3-monoidal categories in Section 7.1. A symmetric monoidal category provides an example of a 3-monoidal category in which all three monoidal structures coincide. The interchange laws are built from the symmetry. This is explained in Section 7.2. Strong 3-monoidal categories (those for which the interchange laws and other structure maps are invertible) are necessarily of this form. More examples of 3-monoidal categories are provided in Section 7.3. Monoids in a 3-monoidal category and functors between 3-monoidal categories are discussed in Sections 7.4 and 7.5.

Higher monoidal categories and their functors are discussed in Sections 7.6 and 7.8. Section 7.7 deals with the $n + 1$ notions of monoid one may define in an $n$-monoidal category. In Section 7.9 we explain how $n$-monoidal categories can be interpreted as pseudomonoids in appropriate monoidal 2-categories. We also explain how the lax and colax constructions give rise to all notions of monoidal functors between higher monoidal categories. This is a continuation of the discussion in Section 6.11. In Section 7.10 we extend the contragredient construction of Sections 3.10 and 6.12 to the context of higher monoidal categories.

## 7.1. 3-monoidal categories

In this section we develop the basic theory of 3-monoidal categories. The definition consists of a category equipped with three ordered monoidal structures, three interchange laws, one for each pair of monoidal structures, and a bunch of axioms relating them. These interchange laws can be drawn pictorially using the three coordinate planes in three-dimensional space.

We show that a 3-monoidal category can be viewed as consisting of three 2-monoidal categories linked by bilax and double (co)lax monoidal functors. This provides alternative ways of thinking about these objects. Basic constructions on 2-monoidal categories such as taking opposite and transposes also extend to 3-monoidal categories.

### 7.1.1. Definition.

**Definition 7.1.** A 3-*monoidal category* is a seven tuple $(\mathsf{C}, \diamond, I, \star, J, \cdot, K)$ where

$$(\mathsf{C}, \diamond, I, \star, J), \qquad (\mathsf{C}, \star, J, \cdot, K), \qquad \text{and} \qquad (\mathsf{C}, \diamond, I, \cdot, K)$$

are 2-monoidal categories, such that the axioms below are satisfied.

The first assumption entails natural transformations

$$\begin{gathered}(A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)\\ (A \cdot B) \star (C \cdot D) \to (A \star C) \cdot (B \star D)\\ (A \cdot B) \diamond (C \cdot D) \to (A \diamond C) \cdot (B \diamond D)\end{gathered} \tag{7.1}$$

and morphisms

$$\begin{gathered}I \to I \star I,\ I \to I \cdot I, \quad J \to J \cdot J,\ J \diamond J \to J, \quad K \diamond K \to K,\ K \star K \to K\\ I \to J, \qquad J \to K, \qquad I \to K.\end{gathered} \tag{7.2}$$

The axioms to be satisfied are given below. All three natural transformations will be denoted by the same letter $\zeta$, while the other structure maps will go unnamed.

**Interchange axiom.** The following diagram commutes.

$$\begin{tikzcd}
((A_1 \cdot B_1) \star (A_2 \cdot B_2)) \diamond ((C_1 \cdot D_1) \star (C_2 \cdot D_2)) \arrow[r, "\zeta \diamond \zeta"] \arrow[d, "\zeta"'] & ((A_1 \star A_2) \cdot (B_1 \star B_2)) \diamond ((C_1 \star C_2) \cdot (D_1 \star D_2)) \arrow[d, "\zeta"] \\
((A_1 \cdot B_1) \diamond (C_1 \cdot D_1)) \star ((A_2 \cdot B_2) \diamond (C_2 \cdot D_2)) \arrow[d, "\zeta \star \zeta"'] & ((A_1 \star A_2) \diamond (C_1 \star C_2)) \cdot ((B_1 \star B_2) \diamond (D_1 \star D_2)) \arrow[d, "\zeta \cdot \zeta"] \\
((A_1 \diamond C_1) \cdot (B_1 \diamond D_1)) \star ((A_2 \diamond C_2) \cdot (B_2 \diamond D_2)) \arrow[r, "\zeta"'] & ((A_1 \diamond C_1) \star (A_2 \diamond C_2)) \cdot ((B_1 \diamond D_1) \star (B_2 \diamond D_2))
\end{tikzcd} \tag{7.3}$$

**Unitality.** The following diagrams commute.

$$\begin{tikzcd}
(I \star I) \cdot (I \star I) & & I \cdot I \arrow[ll] \\
(I \cdot I) \star (I \cdot I) \arrow[u] & I \star I \arrow[l] & I \arrow[l] \arrow[u]
\end{tikzcd} \tag{7.4}$$

$$\begin{tikzcd}
(J \cdot J) \diamond (J \cdot J) \arrow[d] & & J \diamond J \arrow[ll] \arrow[d] \\
(J \diamond J) \cdot (J \diamond J) \arrow[r] & J \cdot J & J \arrow[l]
\end{tikzcd} \tag{7.5}$$

$$\begin{tikzcd}
(K \star K) \diamond (K \star K) \arrow[rr] \arrow[d] & & K \diamond K \arrow[d] \\
(K \diamond K) \star (K \diamond K) \arrow[r] & K \star K \arrow[r] & K
\end{tikzcd} \tag{7.6}$$

$$\begin{tikzcd} I \arrow[r] \arrow[d] & J \arrow[d] \\ I \cdot I \arrow[r] & J \cdot J \end{tikzcd} \qquad \begin{tikzcd} J \diamond J \arrow[r] \arrow[d] & K \diamond K \arrow[d] \\ J \arrow[r] & K \end{tikzcd} \qquad \begin{tikzcd} I \star I \arrow[r] & K \star K \arrow[d] \\ I \arrow[u] \arrow[r] & K \end{tikzcd} \tag{7.7}$$

$$\begin{tikzcd} & J \arrow[dr] & \\ I \arrow[ur] \arrow[rr] & & K \end{tikzcd} \tag{7.8}$$

We emphasize that in a 3-monoidal category, the order in which the monoidal structures are written is important, as was also the case in a 2-monoidal category.

The last diagram in (7.7) may be omitted from the definition since its commutativity follows from the remaining assumptions: Recall that $J$ is the unit object for $\star$, $I$ is a comonoid with respect to $\star$ with $I \to J$ being the counit map, and $K$ is a monoid with respect to $\star$ with $J \to K$ being the unit map. Thus we have a commutative diagram

$$\begin{tikzcd} I \star I \arrow[r] & J \star J \arrow[r] \arrow[d, equal] & K \star K \arrow[d] \\ I \arrow[u] \arrow[r] & J \arrow[r] & K \end{tikzcd}$$

whose composite is the last diagram in (7.7).

**Definition 7.2.** We say that a 3-monoidal category is *strong* if the structure morphisms (7.1) and (7.2) are isomorphisms.

We will see later in Proposition 7.6 that the notion of a strong 3-monoidal category is equivalent to that of a symmetric monoidal category.

**7.1.2. Opposite and transposes.** We extend the considerations of Section 6.1.2. Let $(\mathsf{C}, \diamond, I, \star, J, \cdot, K)$ be a 3-monoidal category. The monoidal structures in Definition 7.1 have a symmetry which implies that $(\mathsf{C}^{\mathrm{op}}, \cdot, K, \star, J, \diamond, I)$ is also a 3-monoidal category. The interchange laws are as defined in (6.9). We denote the resulting object simply by $\mathsf{C}^{\mathrm{op}}$ and call it the *opposite* 3-monoidal category of $\mathsf{C}$.

Define a new monoidal structure $\tilde{\diamond}$ on $\mathsf{C}$ by

$$A \mathbin{\tilde{\diamond}} B := B \diamond A.$$

The category $(\mathsf{C}, \tilde{\diamond}, I, \star, J, \cdot, K)$ is 3-monoidal. The interchange laws are as defined in (6.10). We denote this 3-monoidal category by $\mathsf{C}^{t_\diamond}$ and call it the $\diamond$-*transpose* of $\mathsf{C}$. The $\star$-*transpose* and $\cdot$-*transpose* of $\mathsf{C}$ are defined similarly.

Combining these constructions we obtain 3-monoidal categories

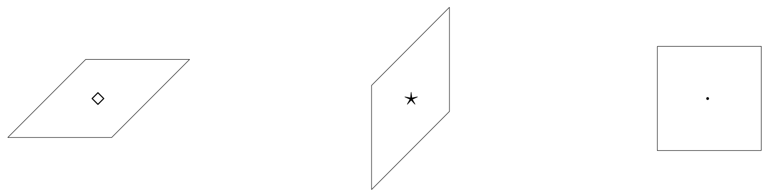
$$(\mathsf{C}, \tilde{\diamond}, I, \tilde{\star}, J, \cdot, K), \quad (\mathsf{C}, \tilde{\diamond}, I, \star, J, \tilde{\cdot}, K), \quad \text{and} \quad (\mathsf{C}, \diamond, I, \tilde{\star}, J, \tilde{\cdot}, K),$$

and finally

$$(\mathsf{C}, \tilde{\diamond}, I, \tilde{\star}, J, \tilde{\cdot}, K)$$

which we call the *transpose* of $\mathsf{C}$ and denote by $\mathsf{C}^t$.

**7.1.3. The interchange axiom: a pictorial representation.** We now give a pictorial way to represent the interchange axiom in a 3-monoidal category $(\mathsf{C}, \diamond, \star, \cdot)$. For that, first recall from (6.11) that the interchange law in a 2-monoidal category $(\mathsf{C}, \diamond, \star)$ can be visualized as a $\star$-line cutting across a $\diamond$-line. In the present situation, we have three monoidal structures, so we use planes in three space rather than lines in two space. For definiteness, we name the three coordinate planes as follows.

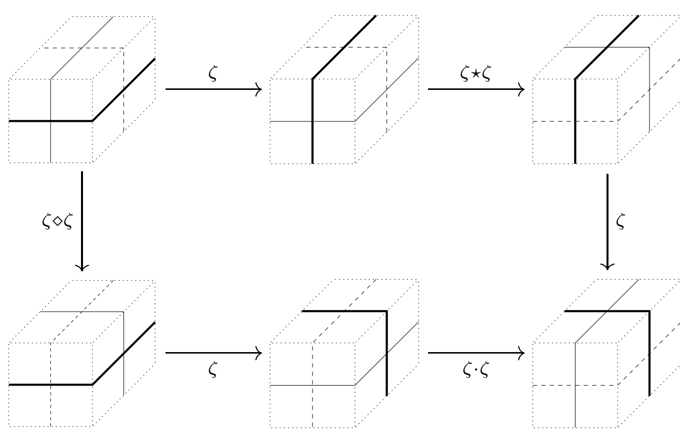



The interchange law between $\star$ and $\diamond$, for example, can be visualized as follows. To start with, the $\diamond$-plane is in one piece while the $\star$-plane is in two pieces, which lie on either side of the $\diamond$-plane. After applying the interchange law, the situation is reversed. We say that the interchange law specifies a procedure for the $\star$-plane to cut across the $\diamond$-plane. Note however that the cutting operation works in only one direction. Following this convention, observe that the three interchange laws (7.1) say that the $\cdot$-plane can cut across both the $\star$- and $\diamond$-planes, and the $\star$-plane can cut across the $\diamond$-plane, but not the other way round.

Building on the above discussion, now imagine the three coordinate planes all together such that one of the planes is in one piece, another is in two pieces and the third is in four pieces. This is a convenient way to picture a linear order on the three planes. For convenience of drawing, we show the plane in one piece with fat lines, the plane in two pieces as usual, and the plane in four pieces with dotted lines.

The following is a pictorial representation of the interchange axiom (7.3).

(7.9)




The cube in the top left corner shows the configuration where the $\diamond$-plane is in one piece, the $\star$-plane is in two pieces and the $\cdot$-plane is in four pieces. The situation is reversed in the cube in the bottom right corner. The six cubes show the six possible configurations or linear orders on the three planes. The unique way of linking these cubes as shown above is the interchange axiom. It may be viewed as the analogue of the braiding hexagon (1.6).

**7.1.4. Equivalent descriptions of 3-monoidal categories.** Recall that there are three types of functors that relate 2-monoidal categories: bilax, double lax and double colax. It is clear from the definition that a 3-monoidal category contains within itself three 2-monoidal categories. Interestingly enough, it also contains a bilax, a double lax and a double colax functor. This is explained below. Furthermore, each one of these can be used to provide an alternative description of a 3-monoidal category. Let $\mathsf{I}$ be the one-arrow category and let $*$ denote its unique object.

**Proposition 7.3.** *Let* $(\mathsf{C}, \diamond, I, \star, J)$, $(\mathsf{C}, \star, J, \cdot, K)$ *and* $(\mathsf{C}, \diamond, I, \cdot, K)$ *be three 2-monoidal categories. Then* $(\mathsf{C}, \diamond, I, \star, J, \cdot, K)$ *is a 3-monoidal category if and only if either of the following three equivalent conditions are satisfied.*

(i) *The functors*

$$\diamond\colon (\mathsf{C}\times\mathsf{C}, \star, \cdot) \to (\mathsf{C}, \star, \cdot) \quad (A, B) \mapsto A \diamond B \qquad\qquad \mathcal{I}\colon \mathsf{I} \to (\mathsf{C}, \star, \cdot) \quad * \mapsto I$$

*are double colax.*

(ii) *The functors*

$$\star\colon (\mathsf{C}\times\mathsf{C}, \diamond, \cdot) \to (\mathsf{C}, \diamond, \cdot) \quad (A, B) \mapsto A \star B \qquad\qquad \mathcal{J}\colon \mathsf{I} \to (\mathsf{C}, \diamond, \cdot) \quad * \mapsto J$$

*are bilax.*

(iii) *The functors*

$$\cdot\colon (\mathsf{C}\times\mathsf{C}, \diamond, \star) \to (\mathsf{C}, \diamond, \star) \quad (A, B) \mapsto A \cdot B \qquad\qquad \mathcal{K}\colon \mathsf{I} \to (\mathsf{C}, \diamond, \star) \quad * \mapsto K$$

*are double lax.*

The conditions on the functors $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ can be rephrased by saying that $I$ is a double comonoid in $(\mathsf{C}, \star, \cdot)$, $J$ is a bimonoid in $(\mathsf{C}, \diamond, \cdot)$ and $K$ is a double monoid in $(\mathsf{C}, \diamond, \star)$.

Proof. The structure morphisms (7.1) and (7.2) are provided to us as part of the hypotheses. Hence in view of Proposition 6.4, we know that the above functors have the appropriate lax and colax properties. The content of the present result then is that the interchange and unitality axioms in a 3-monoidal category are equivalent to the "bi" as well as equivalent to the "double" properties of these functors.

This is a straightforward check. For example, for item (ii), we note that the bilax axioms (6.32), (6.33) and (6.34) for the functor $\star$ reduce to (7.3), (7.4), (7.6) and the last diagram in (7.7), while the bilax axioms for $\mathcal{J}$ account for (7.5), (7.8) and the first two diagrams in (7.7). □

**7.1.5. 3-monoidal categories arising from two tensor products.** It is natural to consider 3-monoidal categories in which there are only two tensor products, with one of them being used twice. A specific context of this kind is considered below. Variations of this idea can be found in Section 7.3.

Let $(\mathsf{C}, \diamond, I, \star, J)$ and $(\mathsf{C}, \star, J, \diamond, I)$ be 2-monoidal categories. Thus, there are two sets of structure morphisms (6.1) and (6.2), with the arrows reversed in one set. In particular, there are two interchange laws:

$$(A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$
$$(A \diamond B) \star (C \diamond D) \to (A \star C) \diamond (B \star D).$$

Assume further that $(\mathsf{C}, \diamond, I)$ is braided. This yields another 2-monoidal category $(\mathsf{C}, \diamond, I, \diamond, I)$ whose structure morphisms are constructed from the braiding (Section 6.3). In this situation:

**Proposition 7.4.** *We have that*

$$(\mathsf{C}, \diamond, \diamond, \star), \ (\mathsf{C}, \diamond, \star, \diamond) \quad \textit{and} \quad (\mathsf{C}, \star, \diamond, \diamond)$$

*are* 3*-monoidal categories if and only if the functors*

$$\star \colon (\mathsf{C} \times \mathsf{C}, \diamond) \to (\mathsf{C}, \diamond) \quad \textit{and} \quad \mathcal{J} \colon \mathsf{I} \to (\mathsf{C}, \diamond)$$

*are braided bilax.*

It is implicit that the structure maps of the above 3-monoidal categories are defined using the structure maps of the 2-monoidal categories that they contain. The condition on the functor $\mathcal{J}$ can be rephrased by saying that $J$ is a commutative and cocommutative bimonoid in $(\mathsf{C}, \diamond)$.

Proof. Applying the three parts of Proposition 7.3 once each, we see that

$$(\mathsf{C}, \diamond, \diamond, \star), \ (\mathsf{C}, \diamond, \star, \diamond) \quad \text{and} \quad (\mathsf{C}, \star, \diamond, \diamond)$$

are 3-monoidal categories if and only if the functors

$$\star \colon (\mathsf{C} \times \mathsf{C}, \diamond, \diamond) \to (\mathsf{C}, \diamond, \diamond) \quad \text{and} \quad \mathcal{J} \colon \mathsf{I} \to (\mathsf{C}, \diamond, \diamond)$$

are double lax, bilax and double colax respectively. In the present situation, by Proposition 6.59, double (co)lax is the same as braided (co)lax. The result follows. □

## 7.2. Symmetric monoidal categories as 3-monoidal categories

We show that a strong 3-monoidal category (Definition 7.2) is the same as a symmetric monoidal category. This is a continuation of the discussion in Section 6.3.

**Proposition 7.5.** *A symmetric monoidal category gives rise to a strong* 3*-monoidal category all of whose monoidal structures are identical.*

Proof. Let $(\mathsf{C}, \bullet, I, \beta)$ be a symmetric monoidal category. Then define a strong 3-monoidal category $(\mathsf{C}, \bullet, I, \bullet, I, \bullet, I)$ all of whose monoidal structures are identical, with the structure maps as defined in the proof of Proposition 6.10. It is straightforward to check that the 3-monoidal category axioms hold. □

Proposition 7.5 admits the following converse.

**Proposition 7.6.** *Let $(\mathsf{C}, \diamond, I, \star, J, \cdot, K)$ be a strong* 3*-monoidal category. Then the monoidal categories $(\mathsf{C}, \diamond, I)$, $(\mathsf{C}, \star, J)$ and $(\mathsf{C}, \cdot, K)$ are symmetric and isomorphic as symmetric monoidal categories.*

*Moreover, the interchange laws arise from the symmetry as in Proposition 7.5 (and the isomorphisms between $\diamond$, $\star$ and $\cdot$).*

Proof. Consider the monoidal categories $(\mathsf{C}, \diamond)$, $(\mathsf{C}, \star)$ and $(\mathsf{C}, \cdot)$. From Proposition 6.11 we know that any two of them are isomorphic. Further these pairwise isomorphisms are compatible: for units, this can be seen from (7.8) and for the monoidal structures, this can be seen by setting all objects except $A_1$ and $D_2$ equal to the unit.

So, for the rest of the proof, we assume that the three monoidal structures are identical. Further, Proposition 6.11 also shows that for any pair of monoidal structures, the interchange law between them is given by a braiding. Let us call these braidings $\beta_{\diamond,\star}$, $\beta_{\star,\cdot}$ and $\beta_{\diamond,\cdot}$. We claim that these braidings are isomorphic and symmetries. This can be deduced by employing the interchange axiom (7.3) thrice as follows.

- Set all objects except $B_1$ and $C_2$ equal to the unit, and deduce that $\beta_{\star,\cdot}$ and $\beta_{\diamond,\cdot}$ are isomorphic.
- Set all objects except $B_2$ and $C_1$ equal to the unit, and deduce that $\beta_{\diamond,\star}$ and $\beta_{\diamond,\cdot}$ are isomorphic.
- Finally, set all objects except $A_2$ and $D_1$ equal to the unit, and deduce that these isomorphic braidings are symmetries.

The result follows. □

We illustrate the above proof on a simple example.

**Example 7.7.** Recall that the category $\mathsf{gVec}$ of graded vector spaces has a monoidal structure given by the Cauchy product and a braiding $\beta_q$ for any nonzero scalar $q$ (2.50).

Fix three nonzero scalars $q_{\diamond,\star}$, $q_{\star,\cdot}$ and $q_{\diamond,\cdot}$. Now define the 2-monoidal category

$$(\mathsf{gVec}, \diamond, \star)$$

with $\diamond$ and $\cdot$ both equal to the Cauchy product and the interchange law given by the braiding $\beta_{q_{\diamond,\star}}$. The 2-monoidal categories $(\mathsf{gVec}, \star, \cdot)$ and $(\mathsf{gVec}, \diamond, \cdot)$ are defined similarly using the scalars $q_{\star,\cdot}$ and $q_{\diamond,\cdot}$.

Proposition 7.6 implies that

$$(\mathsf{gVec}, \diamond, \star, \cdot) \text{ is a 3-monoidal category } \iff q_{\diamond,\star} = q_{\star,\cdot} = q_{\diamond,\cdot} = \pm 1.$$

Let us prove this directly. Observe that the interchange axiom (7.3) holds if and only if

$$q_{\diamond,\star}^{(a_2+b_2)(c_1+d_1)}\, q_{\diamond,\cdot}^{b_1c_1+b_2c_2}\, q_{\star,\cdot}^{(b_1+d_1)(a_2+c_2)} = q_{\diamond,\star}^{a_2c_1+b_2d_1}\, q_{\diamond,\cdot}^{(b_1+b_2)(c_1+c_2)}\, q_{\star,\cdot}^{b_1a_2+d_1c_2}$$

for any nonnegative integers $a_2, b_1, b_2, c_1, c_2$, and $d_1$. By cancelling common terms, this is further equivalent to the identity

$$q_{\diamond,\star}^{b_2c_1+a_2d_1}\, q_{\star,\cdot}^{d_1a_2+b_1c_2} = q_{\diamond,\cdot}^{b_2c_1+b_1c_2}.$$

This clearly holds if the scalars are all equal to 1 or all equal to $-1$. Conversely, suppose that the above identity holds. Then:

- Setting $a_2 = b_2 = c_1 = d_1 = 0$ and $b_1 = c_2 = 1$ yields $q_{\star,\cdot} = q_{\diamond,\cdot} = q$ (say).
- Setting $a_2 = b_1 = c_2 = d_1 = 0$ and $b_2 = c_1 = 1$ yields $q_{\diamond,\star} = q_{\diamond,\cdot} = q$.
- Finally, setting $b_1 = b_2 = c_1 = c_2 = 0$ and $a_2 = d_1 = 1$ yields $q^2 = 1$.

This completes the proof.

### 7.3. Constructions of 3-monoidal categories

We now discuss two general constructions of 3-monoidal categories: from monoidal categories with products and coproducts, and from braided 2-monoidal categories. This complements the discussion in Section 7.2, where we constructed 3-monoidal categories from symmetric monoidal categories.

**7.3.1. Monoidal categories with products and coproducts.** Let $(\mathsf{C}, \star, J)$ be a monoidal category. Suppose that in the category $\mathsf{C}$, all finite products and coproducts exist. This yields the (co)cartesian monoidal categories $(\mathsf{C}, \amalg, I)$ and $(\mathsf{C}, \times, K)$ where $\amalg$ is constructed from the coproduct and $\times$ from the product, and $I$ is a initial object and $K$ is a terminal object. Further, by Example 6.19,

$$(\mathsf{C}, \amalg, I, \star, J), \quad (\mathsf{C}, \star, J, \times, K) \quad \text{and} \quad (\mathsf{C}, \amalg, I, \times, K)$$

are all 2-monoidal categories, with the interchange laws constructed from the universal property of the (co)product. Note that the 2-monoidal categories

$$(\mathsf{C}, \amalg, I, \amalg, I) \quad \text{and} \quad (\mathsf{C}, \times, K, \times, K)$$

are instances of this construction.

**Proposition 7.8.** *We have that*

$$(\mathsf{C}, \amalg, I, \amalg, I, \star, J), \quad (\mathsf{C}, \amalg, I, \star, J, \times, K) \quad \textit{and} \quad (\mathsf{C}, \star, J, \times, K, \times, K)$$

*are 3-monoidal categories.*

Proof. We only check that $(\mathsf{C}, \amalg, \star, \times)$ is a 3-monoidal category; the remaining two checks are similar.

The interchange axiom (7.3) holds, that is, the maps

$$\begin{array}{c} ((A_1 \times B_1) \star (A_2 \times B_2)) \amalg ((C_1 \times D_1) \star (C_2 \times D_2)) \\ \downarrow \\ ((A_1 \amalg C_1) \star (A_2 \amalg C_2)) \times ((B_1 \amalg D_1) \star (B_2 \amalg D_2)) \end{array}$$

obtained by following the two directions coincide. This map can be checked to be as follows. First note that from the universal property of the (co)product, such a map is equivalent to four maps

$$\begin{tikzcd} (A_1 \times B_1) \star (A_2 \times B_2) \arrow[r] \arrow[d] & (B_1 \amalg D_1) \star (B_2 \amalg D_2) \\ (A_1 \amalg C_1) \star (A_2 \amalg C_2) & (C_1 \times D_1) \star (C_2 \times D_2). \arrow[l] \arrow[u] \end{tikzcd}$$

The vertical map on the left is the composite

$$(A_1 \times B_1) \star (A_2 \times B_2) \to A_1 \star A_2 \to (A_1 \amalg C_1) \star (A_2 \amalg C_2).$$

In the notation of Section A.1, it is

$$\iota_{A_1}^{A_1 \amalg C_1} \pi_{A_1}^{A_1 \times B_1} \star \iota_{A_2}^{A_2 \amalg C_2} \pi_{A_2}^{A_2 \times B_2}.$$

The descriptions of the remaining three maps are similar.

Any object is canonically a bimonoid in $(\mathsf{C}, \amalg, \times)$; so in particular axiom (7.5) holds. Since $I$ is an initial object and $K$ is a terminal object, the remaining axioms hold as well. This shows that $(\mathsf{C}, \amalg, \star, \times)$ is a 3-monoidal category.

The above proof was direct. However, since the main check there was left to the reader, we provide an alternative proof below which is complete in all details.

In view of Proposition 7.3, part (ii), it is enough to show that

$$\star \colon (\mathsf{C} \times \mathsf{C}, \amalg, \times) \to (\mathsf{C}, \amalg, \times) \quad (A, B) \mapsto A \star B \qquad \mathcal{J} \colon \mathsf{I} \to (\mathsf{C}, \amalg, \times) \quad * \mapsto J$$

are bilax. For this, note that the one-arrow category $\mathsf{I}$ always has (co)products defined in the obvious manner, and if $\mathsf{C}$ has (co)products, then so does $\mathsf{C} \times \mathsf{C}$ by taking (co)products coordinatewise. So both functors above are of the form

$$(\mathsf{D}, \amalg, \times) \to (\mathsf{C}, \amalg, \times)$$

where the 2-monoidal structure is constructed from the coproduct and product. From Example 6.68, we know that such functors always carry a canonical bilax structure. This completes the proof. □

**7.3.2. Braided 2-monoidal categories.** Recall that a braided monoidal category can be viewed as a 2-monoidal category. Similarly, $\diamond$-braided and $\star$-braided 2-monoidal categories can be viewed as a 3-monoidal categories as follows.

A $\diamond$-braided 2-monoidal category $(\mathsf{C}, \diamond, \star)$ gives rise to a 3-monoidal category, namely,

$$(\mathsf{C}, \diamond, \diamond, \star).$$

The first two monoidal structures coincide. The interchange law for the first and third, as well as the second and third monoidal structures coincides with the interchange law of $(\mathsf{C}, \diamond, \star)$, while the interchange law for the first and second monoidal structure is constructed from the braiding.

The verification of the axioms offers no difficulty. For example, the interchange axiom (7.3) in pictorial notation takes the following form.

$$\begin{tikzcd}
\begin{array}{cc} A_1 & B_1 \\ \hline A_2 & B_2 \\ \hline C_1 & D_1 \\ \hline C_2 & D_2 \end{array} \arrow[r, "\zeta"] \arrow[d, "\beta"'] & \begin{array}{c|c} A_1 & B_1 \\ A_2 & B_2 \\ C_1 & D_1 \\ C_2 & D_2 \end{array} \arrow[d, "\beta\star\beta"] \\
\begin{array}{cc} A_1 & B_1 \\ \hline C_1 & D_1 \\ \hline A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \arrow[r, "\zeta"'] & \begin{array}{c|c} A_1 & B_1 \\ C_1 & D_1 \\ A_2 & B_2 \\ C_2 & D_2 \end{array}
\end{tikzcd}$$

The vertical maps are constructed from the braiding while the horizontal maps are constructed from the interchange law using (6.19). The commutativity of this diagram follows from that of the first diagram in (6.17) and naturality.

An alternative proof of the above result as well as a converse is given below.

**Proposition 7.9.** *Let $(\mathsf{C}, \diamond, \star)$ be a 2-monoidal category and let $(\mathsf{C}, \diamond)$ be braided. Then*

$$(\mathsf{C}, \diamond, \diamond, \star) \text{ is a 3-monoidal category} \iff (\mathsf{C}, \diamond, \star) \text{ is } \diamond\text{-braided.}$$

Proof. It follows from Propositions 6.6, 6.59 and 7.3 that both the above conditions are equivalent to the condition that the functors

$$\star \colon (\mathsf{C} \times \mathsf{C}, \diamond) \to (\mathsf{C}, \diamond) \quad \text{and} \quad \mathcal{J} \colon \mathsf{I} \to (\mathsf{C}, \diamond)$$

are braided lax. □

By passing to the opposite category, we deduce:

**Proposition 7.10.** *Let* $(\mathsf{C}, \diamond, \star)$ *be a* 2*-monoidal category and let* $(\mathsf{C}, \star)$ *be a braided monoidal category. Then*

$$(\mathsf{C}, \diamond, \star, \star) \textit{ is a } 3\textit{-monoidal category} \iff (\mathsf{C}, \diamond, \star) \textit{ is } \star\textit{-braided.}$$

In the above results, if we choose $\star$ to be the product or $\diamond$ to be the coproduct in $\mathsf{C}$ (whenever they exist), then we recover two of the 3-monoidal categories in Proposition 7.8.

## 7.4. Monoids in 3-monoidal categories

We consider four kinds of monoids in 3-monoidal categories: $(3,0)$-monoids, $(2,1)$-monoids, $(1,2)$-monoids and $(0,3)$-monoids. This is consistent with the notations introduced in Table 6.2.

**Definition 7.11.** A $(2,1)$-monoid in a 3-monoidal category $(\mathsf{C}, \diamond, \star, \cdot)$ is a tuple $(A, \mu, \nu, \Delta)$ where $A$ is an object in $\mathsf{C}$ such that

$$\begin{gathered}(A, \mu) \text{ is a monoid in } (\mathsf{C}, \diamond),\\ (A, \nu) \text{ is a monoid in } (\mathsf{C}, \star),\\ (A, \Delta) \text{ is a comonoid in } (\mathsf{C}, \cdot),\\ (A, \mu, \nu) \text{ is a double monoid in } (\mathsf{C}, \diamond, \star),\\ (A, \mu, \Delta) \text{ is a bimonoid in } (\mathsf{C}, \diamond, \cdot), \text{ and}\\ (A, \nu, \Delta) \text{ is a bimonoid in } (\mathsf{C}, \star, \cdot).\end{gathered}$$

The unit maps have been suppressed in the notation for simplicity.

**Definition 7.12.** Let $(A, \mu, \nu, \Delta)$ and $(A', \mu', \nu', \Delta')$ be $(2,1)$-monoids. A morphism of $(2,1)$-monoids between them is a map $A \to A'$ such that $(A, \mu) \to (A', \mu')$ and $(A, \nu) \to (A', \nu')$ are morphisms of monoids, and $(A, \Delta) \to (A', \Delta')$ is a morphism of comonoids.

The remaining kinds of monoids and morphisms between them are defined in a similar fashion. It is reasonable to refer to $(3,0)$-monoids as *triple monoids* and to $(0,3)$-monoids as *triple comonoids*.

**Example 7.13.** A commutative bimonoid in a $\diamond$-braided 2-monoidal category $(\mathsf{C}, \diamond, \star)$ is an example of a $(2,1)$-monoid in the 3-monoidal category $(\mathsf{C}, \diamond, \diamond, \star)$.

Similar statements hold for each of the other (co)commutative entries in Table 6.2.

For $i$ ranging between 0 and 3, let ${}^{3-i}\mathsf{Mon}^{i}(\mathsf{C})$ denote the category of $(i, 3-i)$-monoids in $\mathsf{C}$.

Recall that a bimonoid can be viewed as a monoid in a category of comonoids and viceversa. We now provide an interpretation along these lines for monoids in a 3-monoidal category. The basic observation is the following.

**Proposition 7.14.** *Let* $(\mathsf{C}, \diamond, \star, \cdot)$ *be a* 3*-monoidal category. Then*

$$(\mathsf{Mon}(\mathsf{C}, \diamond), \star, \cdot) \quad \textit{and} \quad (\mathsf{Comon}(\mathsf{C}, \cdot), \diamond, \star)$$

*are* 2*-monoidal categories.*

PROOF. We explain the first claim. By Proposition 6.35, $(\mathsf{Mon}(\mathsf{C},\diamond),\star)$ and $(\mathsf{Mon}(\mathsf{C},\diamond),\cdot)$ are both monoidal categories. In particular, if $A$ and $B$ be monoids in $(\mathsf{C},\diamond)$, then $A\star B$ and $A\cdot B$ are also monoids in $(\mathsf{C},\diamond)$.

To complete the proof, we need to check that the structure maps defining the 2-monoidal category $(\mathsf{C},\star,\cdot)$ are morphisms of monoids. For example, if $A$, $B$, $C$, and $D$ are monoids in $(\mathsf{C},\diamond)$, then the interchange law

$$(A\cdot B)\star(C\cdot D)\to(A\star C)\cdot(B\star D)$$

is a morphism of monoids. This follows from the interchange axiom (7.3) and (7.4). The remaining checks are similar. $\square$

*Warning.* Neither $\mathsf{Mon}(\mathsf{C},\star)$ nor $\mathsf{Comon}(\mathsf{C},\star)$ defines a 2-monoidal category with respect to $\diamond$ and $\cdot$.

Now we may combine the constructions of Propositions 6.35 and 7.14. Namely, start with a 3-monoidal category. Take monoids with respect to the first, or comonoids with respect to the last monoidal structure. Do the same on the resulting 2-monoidal categories. Finally, take monoids or comonoids in the resulting monoidal categories. It is not surprising that the result of these iterations yield the various types of monoids in the original 3-monoidal category. The precise result is given below.

**Proposition 7.15.** *There are canonical equivalences of categories*

$$\begin{aligned}
{}^0\mathsf{Mon}^3(\mathsf{C}) &\cong \mathsf{Mon}(\mathsf{Mon}(\mathsf{Mon}(\mathsf{C},\diamond),\star),\cdot)\\
{}^1\mathsf{Mon}^2(\mathsf{C}) &\cong \mathsf{Comon}(\mathsf{Mon}(\mathsf{Mon}(\mathsf{C},\diamond),\star),\cdot)\\
&\cong \mathsf{Mon}(\mathsf{Comon}(\mathsf{Mon}(\mathsf{C},\diamond),\cdot),\star)\\
&\cong \mathsf{Mon}(\mathsf{Mon}(\mathsf{Comon}(\mathsf{C},\cdot),\star),\diamond)\\
{}^2\mathsf{Mon}^1(\mathsf{C}) &\cong \mathsf{Mon}(\mathsf{Comon}(\mathsf{Comon}(\mathsf{C},\cdot),\star),\diamond)\\
&\cong \mathsf{Comon}(\mathsf{Mon}(\mathsf{Comon}(\mathsf{C},\cdot),\diamond),\star)\\
&\cong \mathsf{Comon}(\mathsf{Comon}(\mathsf{Mon}(\mathsf{C},\diamond),\cdot),\star)\\
{}^3\mathsf{Mon}^0(\mathsf{C}) &\cong \mathsf{Comon}(\mathsf{Comon}(\mathsf{Comon}(\mathsf{C},\cdot),\star),\diamond).
\end{aligned}$$

This result may be combined with Proposition 6.36 to give alternative descriptions of the categories of $(i,3-i)$-monoids. For example:

$$\begin{aligned}
{}^1\mathsf{Mon}^2(\mathsf{C}) &\cong \mathsf{Comon}(\mathsf{dMon}(\mathsf{C},\diamond,\star),\cdot)\\
&\cong \mathsf{Mon}(\mathsf{Bimon}(\mathsf{C},\diamond,\cdot),\star)\\
&\cong \mathsf{Bimon}(\mathsf{Mon}(\mathsf{C},\diamond),\star,\cdot)\\
&\cong \mathsf{dMon}(\mathsf{Comon}(\mathsf{C},\cdot),\diamond,\star).
\end{aligned}$$

## 7.5. Monoidal functors between 3-monoidal categories

We consider four kinds of functors between 3-monoidal categories. We call them

| lax-lax-lax | lax-lax-colax | lax-colax-colax | colax-colax-colax |
|---|---|---|---|
| $(3,0)$ | $(2,1)$ | $(1,2)$ | $(0,3)$. |

The notations extend those given in Figure 6.1. The type of functors are indexed by pairs $(i, j)$ of nonnegative integers such that $i + j = 3$. The first (second) coordinate indicates the number of lax (colax) structures that the functor carries. The lax structures always precede the colax structures in our convention.

**7.5.1. Definition.** The functors on 3-monoidal categories are defined in terms of functors on 2-monoidal categories in a straightforward manner without imposing any new axioms. Further, the definitions of $(3,0)$-, $(2,1)$-, $(1,2)$- and $(0,3)$-functors are all similar to one another. To avoid repetition, we explain $(2,1)$-functors only.

**Definition 7.16.** A functor

$$(\mathcal{F}, \varphi, \gamma, \psi) \colon (\mathsf{C}, \diamond, \star, \cdot) \to (\mathsf{D}, \diamond, \star, \cdot)$$

between 3-monoidal categories is *lax-lax-colax* or $(2,1)$ if

$$(\mathcal{F}, \varphi) \text{ is } \diamond\text{-lax}, \ \ (\mathcal{F}, \gamma) \text{ is } \star\text{-lax}, \ \text{ and } \ (\mathcal{F}, \psi) \text{ is } \cdot\text{-colax}$$

such that

$$\begin{aligned}
&(\mathcal{F}, \varphi, \gamma) \colon (\mathsf{C}, \diamond, \star) \to (\mathsf{D}, \diamond, \star) \text{ is lax-lax,}\\
&(\mathcal{F}, \varphi, \psi) \colon (\mathsf{C}, \diamond, \cdot) \to (\mathsf{D}, \diamond, \cdot) \text{ is lax-colax, and}\\
&(\mathcal{F}, \gamma, \psi) \colon (\mathsf{C}, \star, \cdot) \to (\mathsf{D}, \star, \cdot) \text{ is lax-colax.}
\end{aligned}$$

Recall from the theory of 2-monoidal categories that morphisms between bilax or double lax or double colax functors were defined in terms of morphisms of lax or colax functors without imposing any new axioms. The same feature continues to hold for morphisms between $(3,0)$-, $(2,1)$-, $(1,2)$- or $(0,3)$-functors. Again to avoid repetition, we explain morphisms between $(2,1)$-functors only.

**Definition 7.17.** Let $(\mathcal{F}, \varphi, \gamma, \psi)$ and $(\mathcal{G}, \varphi', \gamma', \psi')$ be lax-lax-colax functors between 3-monoidal categories $\mathsf{C}$ and $\mathsf{D}$. A morphism from $\mathcal{F}$ to $\mathcal{G}$ of lax-lax-colax functors is a natural transformation $\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$ such that $(\mathcal{F}, \varphi) \Rightarrow (\mathcal{G}, \varphi')$ and $(\mathcal{F}, \gamma) \Rightarrow (\mathcal{G}, \gamma')$ are morphisms of lax functors, and $(\mathcal{F}, \psi) \Rightarrow (\mathcal{G}, \psi)$ is a morphism of colax functors.

It is reasonable to refer to $(3,0)$-functors as *triple lax monoidal functors* and to $(0,3)$-functors as *triple colax monoidal functors*.

**7.5.2. Alternative descriptions.** We begin by providing an alternative description of lax-lax-colax functors.

Let $(\mathsf{C}, \diamond, \star, \cdot)$ and $(\mathsf{D}, \diamond, \star, \cdot)$ be 3-monoidal categories and suppose there is a functor $\mathcal{F}$ between them such that

$$(\mathcal{F}, \varphi, \psi) \colon (\mathsf{C}, \diamond, \cdot) \to (\mathsf{D}, \diamond, \cdot) \text{ is bilax, and } (\mathcal{F}, \gamma) \colon (\mathsf{C}, \star) \to (\mathsf{D}, \star) \text{ is lax.}$$

Consider the following diagrams.

$$\begin{tikzcd}
(\mathsf{C} \times \mathsf{C}, \diamond, \cdot) \arrow[r, "\star"] \arrow[d, "\mathcal{F} \times \mathcal{F}"'] & (\mathsf{C}, \diamond, \cdot) \arrow[d, "\mathcal{F}"] \\
(\mathsf{D} \times \mathsf{D}, \diamond, \cdot) \arrow[r, "\star"'] \arrow[ur, Rightarrow, "\gamma"'] & (\mathsf{D}, \diamond, \cdot)
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathsf{C}, \diamond, \cdot) \arrow[dr, "\mathcal{F}"] & \\
\mathsf{I} \arrow[ur, "\mathcal{J}"] \arrow[rr, "\mathcal{J}"'] \arrow[rr, phantom, shift left=4, "\Uparrow \gamma_0"] & & (\mathsf{D}, \diamond, \cdot)
\end{tikzcd}$$

By Proposition 7.3, part (ii), all functors involved in these diagrams are bilax.

**Proposition 7.18.** *In the above setup,*

$$(\mathcal{F}, \varphi, \gamma, \psi) \text{ is lax-lax-colax} \iff \gamma \text{ and } \gamma_0 \text{ are morphisms of bilax functors.}$$

PROOF. Apply Proposition 6.65 to obtain:

$$(\mathcal{F}, \varphi, \gamma) \text{ is lax-lax} \iff \gamma \text{ and } \gamma_0 \text{ are morphisms of lax functors.}$$
$$(\mathcal{F}, \gamma, \psi) \text{ is lax-colax} \iff \gamma \text{ and } \gamma_0 \text{ are morphisms of colax functors.}$$

The result follows. $\square$

We discuss a simple consequence. Let $\mathcal{F}$ be lax-lax-colax. Suppose $A$ and $B$ are bimonoids in $(\mathsf{C}, \diamond, \cdot)$. Then $A \star B$ is also a bimonoid in $(\mathsf{C}, \diamond, \cdot)$. Since $\mathcal{F}$ is bilax, it follows that

$$\mathcal{F}(A),\ \mathcal{F}(B),\ \mathcal{F}(A \star B) \quad \text{and} \quad \mathcal{F}(A) \star \mathcal{F}(B)$$

are all bimonoids in $(\mathsf{D}, \diamond, \cdot)$. Now applying Proposition 7.18, we deduce that

$$\gamma_{A,B} \colon \mathcal{F}(A) \star \mathcal{F}(B) \to \mathcal{F}(A \star B) \quad \text{and} \quad \gamma_0 \colon J \to \mathcal{F}(J)$$

are morphisms of bimonoids.

There is another similar description for lax-lax-colax functors which is as follows. Let $(\mathsf{C}, \diamond, \star, \cdot)$ and $(\mathsf{D}, \diamond, \star, \cdot)$ be 3-monoidal categories and suppose there is a functor $\mathcal{F}$ between them such that

$$(\mathcal{F}, \varphi, \gamma) \colon (\mathsf{C}, \diamond, \star) \to (\mathsf{D}, \diamond, \star) \text{ is double lax, and } (\mathcal{F}, \psi) \colon (\mathsf{C}, \cdot) \to (\mathsf{D}, \cdot) \text{ is colax.}$$

Consider the following diagrams.

$$\begin{tikzcd}
(\mathsf{C} \times \mathsf{C}, \diamond, \star) \arrow[r, "\cdot"] \arrow[d, "\mathcal{F} \times \mathcal{F}"'] & (\mathsf{C}, \diamond, \star) \arrow[d, "\mathcal{F}"] \arrow[dl, Rightarrow, "\psi"'] \\
(\mathsf{D} \times \mathsf{D}, \diamond, \star) \arrow[r, "\cdot"'] & (\mathsf{D}, \diamond, \star)
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathsf{C}, \diamond, \star) \arrow[dr, "\mathcal{F}"] \arrow[d, phantom, "\Downarrow \psi_0"] & \\
\mathsf{I} \arrow[ur, "\mathcal{K}"] \arrow[rr, "\mathcal{K}"'] & {} & (\mathsf{D}, \diamond, \star)
\end{tikzcd}$$

By Proposition 7.3, part (iii), all the functors involved are double lax.

**Proposition 7.19.** *In the above setup,*

$$(\mathcal{F}, \varphi, \gamma, \psi) \text{ is lax-lax-colax} \iff \psi \text{ and } \psi_0 \text{ are morphisms of double lax functors.}$$

**Remark 7.20.** One may naively think that there should be a similar third description for a lax-lax-colax functor which involves $\varphi$ and $\varphi_0$. However, this is *not* true, as one can check.

Since passing to the opposite category turns a lax-lax-colax functor into a lax-colax-colax functor, and viceversa, it follows that there are two similar descriptions for a lax-colax-colax functor as well.

The situation for lax-lax-lax and colax-colax-colax functors is different. They have only one description each.

These observations can be summarized in a unified manner: For any of the above types of functor, the rightmost lax structure and the leftmost colax structure yield one description each of the functor.

**7.5.3. 3-strong monoidal functors.** There is an obvious variation on the kind of functors discussed above: let one or more of the (co)lax structures be (co)strong. Thus we have functors which are lax-lax-costrong, or lax-strong-colax, or strong-strong-colax, and so on. As an extreme situation, we may consider functors where all (co)lax structures are (co)strong. From the discussion in Section 6.8.4, it follows that

strong-strong-strong, strong-strong-costrong,
strong-costrong-costrong, costrong-costrong-costrong

are equivalent notions. We call a functor of this kind a 3-*strong functor*. It can be simultaneously viewed as a $(3,0)$-, $(2,1)$-, a $(1,2)$- and a $(0,3)$-functor.

**Example 7.21.** A 3-strong functor between strong 3-monoidal categories is a familiar notion. Recall that a strong 3-monoidal category is equivalent to a symmetric monoidal category. Using the analysis of Example 6.64, it follows that a 3-strong functor between strong 3-monoidal categories is the same as a braided strong functor between symmetric monoidal categories.

**7.5.4. Examples.** We discuss monoidal functors related to the constructions of Section 7.3.

**Example 7.22.** Let $(\mathcal{F}, \gamma) : (\mathsf{C}, \star) \to (\mathsf{D}, \star)$ be a lax monoidal functor. Suppose that $\mathsf{C}$ and $\mathsf{D}$ have finite (co)products. Now for both $\mathsf{C}$ and $\mathsf{D}$, consider the 3-monoidal category of Proposition 7.8 in which the first monoidal structure is cocartesian and the third monoidal structure is cartesian. Then Example 6.68 shows that

$$(\mathcal{F}, \varphi, \gamma, \psi) \colon (\mathsf{C}, \amalg, \star, \times) \to (\mathsf{D}, \amalg, \star, \times)$$

is a lax-lax-colax functor with $\varphi$ and $\psi$ constructed from the universal properties of the (co)products.

Thus, in this situation, every lax monoidal functor carries a canonical lax-lax-colax monoidal structure. Similarly, every colax monoidal functor carries a canonical lax-colax-colax monoidal structure.

Similar statements can be made for the remaining two 3-monoidal categories of Proposition 7.8. For example, a lax functor $(\mathsf{C}, \star) \to (\mathsf{D}, \star)$ yields a lax-lax-lax functor

$$(\mathsf{C}, \amalg, \amalg, \star) \to (\mathsf{D}, \amalg, \amalg, \star)$$

and so forth.

**Example 7.23.** Recall from Proposition 7.9 that a $\diamond$-braided 2-monoidal category gives rise to a 3-monoidal category in which the first two monoidal structures are identical. In this situation, a $\diamond$-braided bilax functor between $\diamond$-braided 2-monoidal categories (Definition 6.61) gives rise to a lax-lax-colax functor between the corresponding 3-monoidal categories. Similarly, a $\diamond$-braided double (co)lax functor (Definition 6.62) gives rise to a (co)lax-(co)lax-(co)lax functor.

Similar statements can be made for functors between $\star$-braided 2-monoidal categories.

## 7.6. Higher monoidal categories

After 1-, 2- and 3-monoidal categories, we turn our attention to the general case. Higher monoidal categories are built out of these initial cases in a rather straightforward manner. Further motivation for the ideas discussed here is given in Section 7.9, where we explain how a $n$-monoidal category can be viewed as a pseudomonoid in a monoidal 2-category.

**7.6.1. Definition.** Let $\mathsf{C}$ be a category with $n$ monoidal structures, which are linearly ordered. We denote such a structure by

$$(\mathsf{C}, \diamond_1, I_1, \dots, \diamond_n, I_n)$$

where $\diamond_1, \diamond_2, \dots, \diamond_n$ are the monoidal structures and $I_1, I_2, \dots, I_n$ are the respective unit objects. Sometimes the latter are omitted and we simply write

$$(\mathsf{C}, \diamond_1, \dots, \diamond_n). \tag{7.10}$$

By restricting to any $k$ of these monoidal structures, we obtain a category with $k$ monoidal structures. As a shorthand, we write $\mathsf{C}_i$ for $(\mathsf{C}, \diamond_i)$, $\mathsf{C}_{ij}$ for $(\mathsf{C}, \diamond_i, \diamond_j)$, and so on.

**Definition 7.24.** A *$n$-monoidal category* for $n \geq 3$ is a category with $n$ linearly ordered monoidal structures as in (7.10). In other words,

$$\mathsf{C}_i \text{ is a monoidal category for each } 1 \leq i \leq n.$$

Further, we require that

$$\mathsf{C}_{ij} \text{ is a 2-monoidal category for each } 1 \leq i < j \leq n, \text{ and}$$
$$\mathsf{C}_{ijk} \text{ is a 3-monoidal category for each } 1 \leq i < j < k \leq n.$$

It is implicit (in the notation) that $\mathsf{C}_{ij}$ uses the structure of $\mathsf{C}_i$ and $\mathsf{C}_j$, and $\mathsf{C}_{ijk}$ uses the structure of $\mathsf{C}_{ij}$, $\mathsf{C}_{jk}$ and $\mathsf{C}_{ik}$.

Note that if $\mathsf{C}$ and $\mathsf{C}'$ are $n$-monoidal categories, then so is $\mathsf{C} \times \mathsf{C}'$.

**Remark 7.25.** A related notion of *iterated* or *$n$-fold monoidal categories* appears in [32, 132]. This is a more restrictive notion than ours; see Remark 6.2 for more details.

**Definition 7.26.** We say that a $n$-monoidal category is *strong* if all structure morphisms defining it are isomorphisms. Equivalently, a $n$-monoidal category is strong if all the 2-monoidal categories it contains are strong.

**7.6.2. Opposite and transposes.** We now extend the considerations of Sections 6.1.2 and 7.1.2.

Recall that if $\mathsf{C}$ is a 2- or 3-monoidal category, then so is the opposite category $\mathsf{C}^{\mathrm{op}}$. The monoidal structures of $\mathsf{C}^{\mathrm{op}}$ are same as those of $\mathsf{C}$ but they are written in the opposite order. The structure morphisms are defined by reversing arrows. It is easy to see that the same procedure works for any $n$-monoidal category. Thus, if $(\mathsf{C}, \diamond_1, \dots, \diamond_n)$ is a $n$-monoidal category, then so is $(\mathsf{C}^{\mathrm{op}}, \diamond_n, \dots, \diamond_1)$.

Let $\mathsf{C}$ be a $n$-monoidal category. Define a new monoidal structure $\tilde{\diamond}_i$ on $\mathsf{C}$ by

$$A \mathbin{\tilde{\diamond}_i} B := B \diamond_i A.$$

The category $(\mathsf{C}, \diamond_1, \dots, \tilde{\diamond}_i, \dots, \diamond_n)$ is also $n$-monoidal. We call it the *$\diamond_i$-transpose* of $\mathsf{C}$. This construction can be repeated on the other monoidal structures. In particular, we may take transpose of every monoidal structure. Thus,

$$(\mathsf{C}, \tilde{\diamond}_1, \dots, \tilde{\diamond}_n)$$

is a $n$-monoidal category. We call it the *transpose* of $\mathsf{C}$ and denote by $\mathsf{C}^t$.

**7.6.3. Examples.** Any symmetric monoidal category gives rise to a strong $n$-monoidal category with all $n$-monoidal structures identical to the given monoidal structure and all interchange laws constructed from the braiding. Further, it follows from the analysis in Section 7.2 that all strong $n$-monoidal categories arise in this manner. To summarize, a strong $n$-monoidal category is equivalent to a symmetric monoidal category.

A natural next step is to consider $n$-monoidal categories made up of two distinct monoidal structures. We now consider a specific context of this kind. This is a continuation of the discussion in Section 7.1.5.

Let $(\mathsf{C}, \diamond, \star)$ and $(\mathsf{C}, \star, \diamond)$ be 2-monoidal categories. Assume further that $(\mathsf{C}, \diamond)$ and $(\mathsf{C}, \star)$ are braided. This yields 2-monoidal categories $(\mathsf{C}, \diamond, \diamond)$ and $(\mathsf{C}, \star, \star)$. In this situation:

**Proposition 7.27.** *Suppose*

$$(\mathsf{C}, \diamond, \diamond, \star),\ (\mathsf{C}, \diamond, \star, \diamond),\ (\mathsf{C}, \star, \diamond, \diamond),\ (\mathsf{C}, \diamond, \star, \star),\ (\mathsf{C}, \star, \diamond, \star), \quad \textit{and} \quad (\mathsf{C}, \star, \star, \diamond)$$

*are 3-monoidal categories. Then any word of length $n$ in $\diamond$ and $\star$ turns $\mathsf{C}$ into a $n$-monoidal category.*

Proof. This follows from Proposition 7.4. □

## 7.7. Monoids in higher monoidal categories

We now consider monoids in any higher monoidal category. This unifies the $n = 1, 2$ and 3 cases that we have seen before. The $n = 1$ case is that of (co)monoids in a monoidal category, and the $n = 2$ case is that of double (co)monoids and bimonoids in a 2-monoidal category. The present discussion parallels the $n = 3$ case discussed in Section 7.4.

**7.7.1. Definition.** There are $n + 1$ different types of monoids in a $n$-monoidal category. It is natural to index them by pairs $(i, j)$ of non-negative integers such that $i+j = n$. Recall that this is precisely how we indexed monoids in a 3-monoidal category.

**Definition 7.28.** Let $(\mathsf{C}, \diamond_1, I_1, \dots, \diamond_n, I_n)$ be a $n$-monoidal category and let $0 \leq i \leq n$. A monoid of type $(i, n-i)$, or a $(i, n-i)$-*monoid* is an object $A$ with maps

$$\mu_j \colon A \diamond_j A \to A \qquad\qquad \Delta_k \colon A \to A \diamond_k A$$
$$\iota_j \colon I_j \to A \qquad\qquad \epsilon_k \colon A \to I_k$$

for $1 \leq j \leq i$ and $i + 1 \leq k \leq n$, such that

$$(A, \mu_j, \iota_j) \text{ is a monoid in } \mathsf{C}_j, \quad (A, \Delta_k, \epsilon_k) \text{ is a comonoid in } \mathsf{C}_k,$$

and

$(A, \mu_j, \iota_j, \mu_{j'}, \iota_{j'})$ is a double monoid in $C_{jj'}$ for $1 \leq j < j' \leq i$,

$(A, \mu_j, \iota_j, \Delta_k, \epsilon_k)$ is a bimonoid in $C_{jk}$ for $1 \leq j \leq i$ and $i + 1 \leq k \leq n$,

$(A, \Delta_k, \epsilon_k, \Delta_{k'}, \epsilon_{k'})$ is a double comonoid in $C_{kk'}$ for $i + 1 \leq k < k' \leq n$.

A morphism between two $(i, n-i)$-monoids is a morphism of the underlying monoids and comonoids.

For $n = 3$ and $i = 2$, the above specializes to Definitions 7.11 and 7.12 for $(2,1)$-monoids and morphisms between them.

Let ${}^{n-i}\mathsf{Mon}^{i}(\mathsf{C})$ denote the category of $(i, n-i)$-monoids in $\mathsf{C}$; the monoidal structures of $\mathsf{C}$ are not explicitly written in the notation. This is consistent with the notation for the category of $(i, 3-i)$-monoids introduced in Section 7.4. Let us also take note of categories of this kind for $n = 0, 1$ and $2$.

$$
\begin{gathered}
{}^{0}\mathsf{Mon}^{0}(\mathsf{C}) = \mathsf{C}, \quad {}^{0}\mathsf{Mon}^{1}(\mathsf{C}) = \mathsf{Mon}(\mathsf{C}), \quad {}^{1}\mathsf{Mon}^{0}(\mathsf{C}) = \mathsf{Comon}(\mathsf{C}), \\
{}^{1}\mathsf{Mon}^{1}(\mathsf{C}) = \mathsf{Bimon}(\mathsf{C}), \quad {}^{0}\mathsf{Mon}^{2}(\mathsf{C}) = \mathsf{dMon}(\mathsf{C}), \quad {}^{2}\mathsf{Mon}^{0}(\mathsf{C}) = \mathsf{dComon}(\mathsf{C}).
\end{gathered}
$$

In particular, a $(0,0)$-monoid is an object, a $(1,0)$-monoid is a monoid, a $(0,1)$-monoid is a comonoid, and so forth.

**7.7.2. Alternative descriptions.** Recall that monoids in a 2- or 3-monoidal category can be obtained by appropriate iterations of the monoid and comonoid constructions. We now explain how this procedure works in general. The basic observation is the following.

**Proposition 7.29.** *If $\mathsf{C}$ is a $n$-monoidal category, then*

$$
(\mathsf{Mon}(\mathsf{C}, \diamond_1), \diamond_2, \dots, \diamond_n) \qquad \textit{and} \qquad (\mathsf{Comon}(\mathsf{C}, \diamond_n), \diamond_1, \dots, \diamond_{n-1})
$$

*are $(n-1)$-monoidal categories.*

Proof. This follows from Proposition 7.14. □

The next basic observation is that these two constructions commute with each other. More precisely:

**Proposition 7.30.** *There are canonical equivalences of $(n-2)$-monoidal categories*

$$
\begin{aligned}
(\mathsf{Comon}(\mathsf{Mon}(\mathsf{C}, \diamond_1), \diamond_n), \diamond_2, \dots, \diamond_{n-1}) &\cong (\mathsf{Mon}(\mathsf{Comon}(\mathsf{C}, \diamond_n), \diamond_1), \diamond_2, \dots, \diamond_{n-1}) \\
&\cong (\mathsf{Bimon}(\mathsf{C}, \diamond_1, \diamond_n), \diamond_2, \dots, \diamond_{n-1}).
\end{aligned}
$$

Proof. This follows from the first claim in Proposition 6.36 and noting that the equivalences in that result are compatible with the remaining monoidal structures $\diamond_2, \dots, \diamond_{n-1}$. □

The equivalence in the above proposition is taken in the 2-category whose 0-cells are $(n-2)$-monoidal categories, 1-cells are $(n-2)$-strong functors (Section 7.8.2), and 2-cells are morphisms of $(n-2)$-strong functors. A similar remark applies to categorical equivalences appearing in the discussion below.

Let $\mathsf{C}$ be a $n$-monoidal category, and let $\mathsf{C}_{[i,j]}$ denote the $(n-j+i-1)$-monoidal category obtained from $\mathsf{C}$ by deleting the monoidal structures from $\diamond_i$ to $\diamond_j$ both inclusive. Do the monoid construction $i-1$ times in increasing order from the first structure to the $(i-1)$-st structure. Do the comonoid construction $n-j$ times in decreasing order from the last structure to the $(j+1)$-st structure. The monoid and constructions can be interleaved in any manner. The result will be a $(j-i+1)$-monoidal category whose objects are precisely $(i-1, n-j)$-monoids in $\mathsf{C}_{[i,j]}$. This follows from Definition 7.28 and Propositions 7.29 and 7.30.

This discussion is summarized in Propositions 7.31 and 7.32 below.

**Proposition 7.31.** *If* $\mathsf{C}$ *is a* $n$*-monoidal category, then*

$$({}^{n-j}\mathsf{Mon}^{i-1}(\mathsf{C}_{[i,j]}), \diamond_i, \dots, \diamond_j)$$

*is a* $(j-i+1)$*-monoidal category.*

In particular, if $A$ and $B$ are two $(i-1, n-j)$-monoids in $\mathsf{C}_{[i,j]}$, then so is $A \diamond_k B$ for $i \le k \le j$. The structure morphisms can be written down explicitly as in Section 6.5.4.

**Proposition 7.32.** *There are canonical equivalences of* $(j-i)$*-monoidal categories*

$$\begin{aligned}
&({}^{n-j}\mathsf{Mon}^{i}(\mathsf{C}_{[i+1,j]}), \diamond_{i+1}, \dots, \diamond_j) \\
&\qquad\cong (\mathsf{Mon}({}^{n-j}\mathsf{Mon}^{i-1}(\mathsf{C}_{[i,j]}), \diamond_i), \diamond_{i+1}, \dots, \diamond_j) \\
&\qquad\cong (\mathsf{Comon}({}^{n-j-1}\mathsf{Mon}^{i}(\mathsf{C}_{[i+1,j+1]}), \diamond_{j+1}), \diamond_{i+1}, \dots, \diamond_j).
\end{aligned}$$

An important special case of the above results is given below.

**Proposition 7.33.** *If* $\mathsf{C}$ *is a* $n$*-monoidal category, then*

$$({}^{n-i}\mathsf{Mon}^{i-1}(\mathsf{C}_{[i,i]}), \diamond_i)$$

*is a monoidal category. Further, there are canonical equivalences of categories*

$$\begin{aligned}
{}^{n-i}\mathsf{Mon}^{i}(\mathsf{C}) &\cong \mathsf{Mon}({}^{n-i}\mathsf{Mon}^{i-1}(\mathsf{C}_{[i,i]}), \diamond_i) \\
&\cong \mathsf{Comon}({}^{n-i-1}\mathsf{Mon}^{i}(\mathsf{C}_{[i+1,i+1]}), \diamond_{i+1}).
\end{aligned}$$

To summarize, the category of $(i, n-i)$-monoids in $\mathsf{C}$ is obtained from $\mathsf{C}$ by applying $i$ monoid and $(n-i)$ comonoid constructions. The monoid constructions are done in increasing order starting from the first structure, while the comonoid constructions are done independently in decreasing order starting from the last structure. Thus there are $\binom{n}{i}$ iterative interpretations of a $(i, n-i)$-monoid.

## 7.8. Monoidal functors between higher monoidal categories

There are $n+1$ different types of functors between $n$-monoidal categories. We call them

$$(n,0) \qquad (n-1,1) \qquad \cdots \qquad (i, n-i) \qquad \cdots \qquad (0,n).$$

We have already seen the $n = 1, 2$ and $3$ cases. For example, for monoidal categories, we have lax and colax functors, which in the above notation would be functors of type $(1,0)$ and $(0,1)$ respectively. For 2-monoidal categories, we have $(2,0)$- or double lax functors, $(1,1)$- or bilax functors, and $(0,2)$- or double colax functors (Figure 6.1). Functors between 3-monoidal categories were discussed in Section 7.5 and they serve as a model for the general definition.

**7.8.1. Definition and basic properties.** For a category $\mathsf{C}$ with $n$ monoidal structures, recall that $\mathsf{C}_i$ stands for $(\mathsf{C}, \diamond_i)$, $\mathsf{C}_{ij}$ stands for $(\mathsf{C}, \diamond_i, \diamond_j)$, and so on. We extend this notation to functors. In other words, for $\mathcal{F}$ a functor between $\mathsf{C}$ and $\mathsf{D}$, both categories with $n$ monoidal structures, $\mathcal{F}_i$ denotes either a lax or colax functor between $\mathsf{C}_i$ and $\mathsf{D}_i$ constructed from $\mathcal{F}$, $\mathcal{F}_{ij}$ denotes either a double lax or bilax or double colax functor between $\mathsf{C}_{ij}$ and $\mathsf{D}_{ij}$, and so forth.

**Definition 7.34.** Let $n \geq 3$ and $0 \leq m \leq n$. We say that a functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ between $n$-monoidal categories is of type $(m, n-m)$ if

$$\mathcal{F}_i\colon \mathsf{C}_i \to \mathsf{D}_i \qquad \text{is} \qquad \begin{cases} \text{lax} & \text{if } 1 \leq i \leq m \\ \text{colax} & \text{if } m+1 \leq i \leq n, \end{cases}$$

and

$$\mathcal{F}_{ij}\colon \mathsf{C}_{ij} \to \mathsf{D}_{ij} \qquad \text{is} \qquad \begin{cases} \text{lax-lax} & \text{if } 1 \leq i < j \leq m, \\ \text{lax-colax} & \text{if } 1 \leq i \leq m < j \leq n, \\ \text{colax-colax} & \text{if } m+1 \leq i < j \leq n. \end{cases}$$

It is implicit (in the notation) that $\mathcal{F}_{ij}$ uses the structure of $\mathcal{F}_i$ and $\mathcal{F}_j$.

**Definition 7.35.** Let $\mathcal{F}$ and $\mathcal{G}$ be $(m, n-m)$-functors between $n$-monoidal categories $\mathsf{C}$ and $\mathsf{D}$. A morphism from $\mathcal{F}$ to $\mathcal{G}$ of $(m, n-m)$-functors is a natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ that is a morphism of lax functors $\mathcal{F}_i \Rightarrow \mathcal{G}_i$ for $1 \leq i \leq m$ and a morphism of colax functors for $m+1 \leq i \leq n$.

For $n = 3$ and $m = 2$, the above specialize to Definitions 7.16 and 7.17 of $(2,1)$-functors and morphisms between them.

**Example 7.36.** Let $\mathcal{F}$ be a braided lax functor between symmetric monoidal categories $\mathsf{C}$ and $\mathsf{D}$. If we view $\mathsf{C}$ and $\mathsf{D}$ as $n$-monoidal categories in the manner discussed earlier, then $\mathcal{F}$ becomes a $(n, 0)$-functor between them. This is true for any $n$. Further, a morphism between $\mathcal{F}$ and $\mathcal{G}$ of braided lax functors translates to a morphism between the corresponding $(n, 0)$-functors.

Similarly, a braided colax functor between symmetric monoidal categories can be viewed as a $(0, n)$-functor, and a morphism of braided colax functors as a morphism of $(0, n)$-functors.

We now turn to some basic properties of higher monoidal functors. The first property is that monoidal functors of the same type can be composed, and these compositions are compatible with morphisms between them. This can be expressed as follows.

**Proposition 7.37.** *For $1 \leq m \leq n$ fixed, there is a 2-category whose 0-cells are $n$-monoidal categories, 1-cells are $(m, n-m)$-functors and 2-cells are morphisms between them.*

PROOF. This follows from Propositions 6.52 and 6.57. $\square$

Recall that the Cartesian product of $n$-monoidal categories is also a $n$-monoidal category. It is straightforward to check that this operation turns the 2-category of Proposition 7.37 into a monoidal 2-category (not to be confused with a 2-monoidal category). A part of the check involved is stated explicitly as a proposition below. It generalizes Proposition 3.6.

**Proposition 7.38.** *If $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{F}'\colon \mathsf{C}' \to \mathsf{D}'$ are monoidal functors both of the same type, then*

$$(\mathcal{F}, \mathcal{F}')\colon \mathsf{C} \times \mathsf{C}' \to \mathsf{D} \times \mathsf{D}'$$

*is also a monoidal functor of the same type.*

For any functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$, let $\mathcal{F}^{\mathrm{op}}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$ denote the same functor on the opposite categories. Recall that if $\mathsf{C}$ is a $n$-monoidal category, then so is $\mathsf{C}^{\mathrm{op}}$ with the order of the monoidal structures reversed.

**Proposition 7.39.** *Let* $\mathsf{C}$ *and* $\mathsf{D}$ *be* $n$*-monoidal categories. If* $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ *is a monoidal functor of type* $(i,j)$ *with* $i+j=n$*, then* $\mathcal{F}^{\mathrm{op}}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$ *is a monoidal functor of type* $(j,i)$*.*

This generalizes Proposition 3.7.

**7.8.2. $n$-strong monoidal functor.** Let $\mathcal{F}$ be a $(m, n-m)$-functor for which all $m$ lax structures are strong and all $n-m$ colax structures are costrong. Since strong and costrong are equivalent notions, $\mathcal{F}$ can be viewed as a $(i, n-i)$-functor for any $0 \le i \le n$. In this situation, we say that $\mathcal{F}$ is *$n$-strong*.

A 1-strong functor is the same as a strong, or equivalently, a costrong functor. A 2-strong functor is the same as a strong-strong, or strong-costrong, or costrong-costrong functor (the three notions are equivalent). This was discussed in Section 6.8.4. The case of 3-strong functors was discussed in Section 7.5.3.

Recall that a strong $n$-monoidal category is equivalent to a symmetric monoidal category. It follows that a $n$-strong functor between strong $n$-monoidal categories is the same as a braided strong functor between symmetric monoidal categories.

**7.8.3. Monoids and monoidal functors.** Higher monoids and higher monoidal functors relate to each other in two ways. First, a higher monoid is same as a higher monoidal functor from the one-arrow category. Second, a higher monoidal functor induces a functor between categories of higher monoids. The precise statements are given below. This generalizes the discussion in Section 3.4.

**Proposition 7.40.** *A* $(i,j)$*-monoid in* $\mathsf{C}$ *is same as a* $(i,j)$*-functor from the one-arrow category* $\mathsf{I}$ *to* $\mathsf{C}$*.*

Let $A$ be a $(i,j)$-monoid in $\mathsf{C}$. The corresponding $(i,j)$-functor, denoted $\mathcal{F}_A$, sends the unique object in $\mathsf{I}$ to $A$, and its structure morphisms are defined using the structure morphisms of $A$.

Recall that $\mathsf{C}_{[i,j]}$ denotes the $(n-j+i-1)$-monoidal category obtained from a $n$-monoidal category $\mathsf{C}$ by deleting the monoidal structures from $\diamond_i$ to $\diamond_j$ both inclusive.

**Proposition 7.41.** *Let* $\mathcal{F}$ *be a* $(m, n-m)$*-functor between* $\mathsf{C}$ *and* $\mathsf{D}$*. Further, let*

$$0 \le i-1 \le m \le j \le n.$$

*Then* $\mathcal{F}$ *induces a monoidal functor*

$$({}^{n-j}\mathsf{Mon}^{i-1}(\mathsf{C}_{[i,j]}), \diamond_i, \dots, \diamond_j) \longrightarrow ({}^{n-j}\mathsf{Mon}^{i-1}(\mathsf{D}_{[i,j]}), \diamond_i, \dots, \diamond_j) \tag{7.11}$$

*of type* $(m-i+1, j-m)$*.*

The categories in (7.11) are as constructed in Proposition 7.31. Setting $i-1 = m = j$ yields: A $(m, n-m)$-functor $\mathcal{F}$ between $n$-monoidal categories $\mathsf{C}$ and $\mathsf{D}$ induces a functor

$${}^{n-m}\mathsf{Mon}^m(\mathsf{C}) \longrightarrow {}^{n-m}\mathsf{Mon}^m(\mathsf{D})$$

between the categories of $(m, n-m)$-monoids.

**Proposition 7.42.** *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be* $(m, n-m)$*-functors between* $\mathsf{C}$ *and* $\mathsf{D}$*. A morphism* $\mathcal{F} \Rightarrow \mathcal{G}$ *of* $(m, n-m)$*-functors induces a morphism of* $(m-i+1, j-m)$*-functors between the categories in* (7.11)*.*

**Remark 7.43.** Propositions 7.31, 7.41 and 7.42 say that the 2-categories of Proposition 7.37, one for the indices $1 \le m \le n$ and the other for the indices $1 \le m-i+1 \le j-i+1$, are related by a 2-functor.

**7.8.4. Lower monoidal functors in a higher monoidal category.** Recall that a 3-monoidal category contains bilax and double (co)lax functors as a part of its structure. The latter are monoidal functors between 2-monoidal categories. In the same vein, any higher monoidal category contains monoidal functors involving fewer number of monoidal structures. Details follow.

Let $(\mathsf{C}, \diamond_1, \dots, \diamond_n)$ be a $n$-monoidal category. Recall that $\mathsf{C}_{[i,i]}$ stands for the $(n-1)$-monoidal category obtained from $\mathsf{C}$ by deleting the $i$-th monoidal structure $\diamond_i$. In this situation:

**Proposition 7.44.** *For* $1 \le i \le n$,

$$\diamond_i \colon (\mathsf{C} \times \mathsf{C})_{[i,i]} \to \mathsf{C}_{[i,i]} \quad (A, B) \mapsto A \diamond_i B \qquad\qquad \mathcal{I}_i \colon \mathsf{I} \to \mathsf{C}_{[i,i]} \quad * \mapsto I_i$$

*are functors of* $(n-1)$*-monoidal categories of type* $(i-1, n-i)$.

In view of Proposition 7.40, the second statement can be rephrased by saying that the unit object $I_i$ is a monoid of type $(i-1, n-i)$ in $\mathsf{C}_{[i,i]}$.

**7.8.5. An iterative description of higher monoidal functors.** We now give an iterative description of a higher monoidal functor. This generalizes the discussion of Section 7.5.2.

Let $\mathsf{C}$ and $\mathsf{D}$ be $n$-monoidal categories and suppose there is a functor $\mathcal{F}$ between them such that

$$\mathcal{F} \colon \mathsf{C}_{[i,i]} \to \mathsf{D}_{[i,i]} \text{ is of type } (i-1, n-i), \text{ and } (\mathcal{F}, \gamma) \colon (\mathsf{C}, \diamond_i) \to (\mathsf{D}, \diamond_i) \text{ is lax.}$$

Consider the following diagrams.

$$\begin{tikzcd}
(\mathsf{C} \times \mathsf{C})_{[i,i]} \arrow[r, "\diamond_i"] \arrow[d, "\mathcal{F} \times \mathcal{F}"'] & \mathsf{C}_{[i,i]} \arrow[d, "\mathcal{F}"] \\
(\mathsf{D} \times \mathsf{D})_{[i,i]} \arrow[r, "\diamond_i"'] \arrow[ur, Rightarrow, "\gamma"'] & \mathsf{D}_{[i,i]}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathsf{C}_{[i,i]} \arrow[dr, "\mathcal{F}"] & \\
\mathsf{I} \arrow[ur, "\mathcal{I}_i"] \arrow[rr, "\mathcal{I}_i"'] \arrow[rr, phantom, shift left=4, "\Uparrow \gamma_0"] & & \mathsf{D}_{[i,i]}
\end{tikzcd}$$

By Proposition 7.44, all functors involved in these diagrams are of type $(i-1, n-i)$.

**Proposition 7.45.** *In the above setup,*

$$\mathcal{F} \text{ is of type } (i, n-i) \iff \gamma \text{ and } \gamma_0 \text{ are morphisms of } (i-1, n-i)\text{-functors.}$$

A similar statement can be given by replacing the lax structure $\gamma$ by a colax structure.

**7.8.6. Adjunctions of monoidal functors.** Recall the notion of lax-lax and colax-colax adjunctions from Section 3.9.2. We now extend these notions to higher monoidal functors.

Let $\mathsf{C}$ and $\mathsf{D}$ be $n$-monoidal categories and let

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

be an adjunction.

**Definition 7.46.** Let $\mathcal{F}$ and $\mathcal{G}$ be functors of type $(m, n-m)$. We say that they form an adjunction of type $(m, n-m)$ if the unit and counit of the adjunction are morphisms of $(m, n-m)$-functors.

A $(1,0)$-functor is same as a lax functor. Further, a $(1,0)$-adjunction is the same as a lax-lax adjunction. Similarly, a $(0,1)$-adjunction is the same as a colax-colax adjunction.

**Proposition 7.47.** *If $\mathcal{F}$ and $\mathcal{G}$ form a $(m, n-m)$-adjunction between $\mathsf{C}$ and $\mathsf{D}$, then they induce a $(m-i+1, j-m)$-adjunction between the categories in (7.11).*

Proof. This follows from Remark 7.43. $\square$

**Example 7.48.** Let $\mathsf{C}$ and $\mathsf{D}$ be symmetric monoidal categories and let $(\mathcal{F}, \mathcal{G})$ be a braided lax-lax adjunction between them. By viewing $\mathsf{C}$ and $\mathsf{D}$ as $n$-monoidal categories, it follows from the discussion in Example 7.36 that $(\mathcal{F}, \mathcal{G})$ is an adjunction of type $(n, 0)$ between them.

Now consider the diagrams

$$\mathsf{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{D}, \quad \mathsf{Mon}(\mathsf{C}) \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{Mon}(\mathsf{D}), \quad \text{and} \quad \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}) \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{Mon}^{\mathsf{co}}(\mathsf{D}).$$

If the first adjunction is braided lax-lax, then Proposition 7.47 along with the above discussion implies that the second and the third adjunctions are also braided lax-lax.

Similar statements apply with lax replaced by colax.

## 7.9. Higher monoidal categories viewed as pseudomonoids

We began this chapter with 3-monoidal categories. These were defined as categories with three monoidal structures equipped with certain structure maps satisfying a list of axioms. It is natural to ask: How do we know that the list of axioms for this definition is complete? In other words, is there a systematic procedure which will give us the list of axioms, rather than our having to guess it? The same question can be asked about higher monoidal categories and functors between them.

The goal of this section is to answer the latter question.

**7.9.1. Iterations of the lax and colax constructions.** Monoidal 2-categories hold the key to the problem. The first step in the solution was taken in Section 6.11 where we showed that 2-monoidal categories and the three types of functors between them can be understood in terms of the lax and colax constructions on monoidal 2-categories. Recall that these constructions are denoted by $\mathsf{l}(-)$ and $\mathsf{c}(-)$.

We claim that by iterating these constructions further, one can systematically construct higher monoidal categories and the different types of functors between them. In other words, Figure 6.2 can be extended indefinitely leading to a categorical version of Pascal's triangle. This is shown in Figure 7.1. A precise statement is given in Proposition 7.49.

The triangle is divided into rows. We count from the top starting with zero. The $n$-th row of the triangle has $(n+1)$ entries. We index them by $(i,j)$ with $i+j=n$ and $j$ running from 0 to $n$ increasing from left to right. The 2-category appearing in the $(i,j)$ entry is as follows: 0-cells are $n$-monoidal categories, 1-cells are $(i,j)$-functors, and 2-cells are morphisms of $(i,j)$-functors. We refer to this

$$\begin{tikzcd}[column sep=small]
 & & & \mathsf{Cat} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & \mathsf{lCat} \arrow[dl, dash] \arrow[dr, dash] & & \mathsf{cCat} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \mathsf{llCat} \arrow[dl, dash] \arrow[dr, dash] & & \mathsf{lcCat} \arrow[dl, dash] \arrow[dr, dash] & & \mathsf{ccCat} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathsf{lllCat} \arrow[d, dash, dashed] & & \mathsf{llcCat} \arrow[d, dash, dashed] & & \mathsf{lccCat} \arrow[d, dash, dashed] & & \mathsf{cccCat} \arrow[d, dash, dashed] \\
 {} & & {} & & {} & & {}
\end{tikzcd}$$

FIGURE 7.1. The lax-colax hierarchy of higher monoidal categories.

2-category as $\mathsf{Cat}(i, j)$. The 2-category $\mathsf{Cat}(0, 0)$ is $\mathsf{Cat}$. Its 0-cells are 0-monoidal categories, or simply categories, 1-cells are $(0, 0)$-functors, or simply functors, and 2-cells are morphisms of functors, or natural transformations.

**Proposition 7.49.** *The lax construction applied to the* 2*-category* $\mathsf{Cat}(i, j)$ *yields the* 2*-category* $\mathsf{Cat}(i + 1, j)$ *while the colax construction yields* $\mathsf{Cat}(i, j + 1)$*.*

The proof has an inductive flavor. Accordingly, we start at the top and work our way down. The first two iterations were understood in Propositions 6.72 and 6.75. The first step of the third iteration is as follows.

**Proposition 7.50.** *A pseudomonoid in* $\mathsf{llCat}$ *or* $\mathsf{lcCat}$ *or* $\mathsf{ccCat}$ *is the same as a* 3*-monoidal category.*

PROOF. The proof proceeds along the same lines as Proposition 6.73, so we will be brief. Let us first consider a pseudomonoid in $\mathsf{lcCat}$. To start with, we require an object in $\mathsf{lcCat}$. This is a 2-monoidal category, which for definiteness, we call $(\mathsf{C}, \diamond, \cdot)$. Next, we require bilax functors

$$\star \colon (\mathsf{C} \times \mathsf{C}, \diamond, \cdot) \to (\mathsf{C}, \diamond, \cdot) \qquad \text{and} \qquad \mathcal{J} \colon \mathsf{I} \to (\mathsf{C}, \diamond, \cdot),$$

where $\mathsf{I}$ is the one-arrow category. This gives us the missing structure morphisms, such as the interchange law for $\diamond$ and $\star$, as well as all the 3-monoidal category axioms. This is the content of item (ii) of Proposition 7.3. The remaining requirements ensure that $(\mathsf{C}, \diamond, \star)$ and $(\mathsf{C}, \star, \cdot)$ are also 2-monoidal categories. It is clear that for this part, one is essentially reusing the arguments of Proposition 6.73.

For a pseudomonoid in $\mathsf{llCat}$, we start with $(\mathsf{C}, \diamond, \star)$ while for a pseudomonoid in $\mathsf{ccCat}$, we start with $(\mathsf{C}, \star, \cdot)$. These choices ensure uniformity of notation. The proof then proceeds the same way; we make use of the remaining two items of Proposition 7.3. $\square$

The further cases of Proposition 7.49 are fairly straightforward and there are no new axioms left to uncover. One of the ingredients in the proof is Proposition 7.45.

**Remark 7.51.** In Pascal's triangle, there are $\binom{n}{i}$ paths from $(0, 0)$ to $(i, n-i)$. This then is the number of ways to define $\mathsf{Cat}(i, n - i)$ iteratively. It is also the number of ways to define a $(i, n - i)$ functor iteratively. For example, there is only way to

$$\begin{gathered}\mathsf{Cat} = \text{(Categories, functors, natural transformations)}\\ \downarrow \\ \text{(Monoidal categories, strong functors, –)}\\ \downarrow \\ \text{(Braided monoidal categories, braided strong functors, –)}\\ \downarrow \\ \text{(Symmetric monoidal categories, braided strong functors, –)}\end{gathered}$$

FIGURE 7.2. The strong hierarchy of higher monoidal categories.

define a double (co)lax functor but two ways to define a bilax functor iteratively (Proposition 6.65). The analogous remark for $(i, n-i)$-monoids was made at the end of Section 7.7; the $n = 3$ is shown in full detail in Proposition 7.15.

Note that all 2-categories in the $n$-th row of the triangle have the same 0-cells, namely $n$-monoidal categories. It follows that a $n$-monoidal category can be interpreted as a pseudomonoid in $n$ different ways. This was shown explicitly for $n = 2$ and $n = 3$ in Propositions 6.73 and 7.50.

**7.9.2. Iterations of the strong construction.** Recall that there is a strong version of the lax and colax constructions on monoidal 2-categories. Joyal and Street [184, Remark 5.1] studied the iterations of the strong construction on $\mathsf{Cat}$. The result is shown in Figure 7.2. One sees that braided monoidal categories emerge from this construction but not bilax functors. Another noteworthy feature is that the iterations stabilize and one does not see anything new after symmetric monoidal categories. For related facts, see [28, Section V] and [32].

From our present point of view, Figure 7.2 may be regarded as a degenerate version of Figure 7.1 in which all entries in a given row get identified. This is substantiated by the results of Sections 6.3 and 7.2.

## 7.10. Contragredience for higher monoidal categories

In this section, we generalize the contragredient construction of Section 6.12 to higher monoidal categories. The generalization is rather straightforward and we present it here for completeness.

**7.10.1. Contravariant monoidal functors.** Let $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ be a contravariant functor. Now let $\mathsf{C}$ and $\mathsf{D}$ be $n$-monoidal categories. We say that $\mathcal{F}$ is *contravariant $n$-strong* if

$$\mathcal{F} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}, \qquad \text{or equivalently,} \qquad \mathcal{F} \colon \mathsf{C} \to \mathsf{D}^{\mathrm{op}}$$

is $n$-strong. Now let $n = 2m$. We say that $\mathcal{F}$ is a *contravariant $(m, m)$-functor* if

$$\mathcal{F} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}, \qquad \text{or equivalently,} \qquad \mathcal{F} \colon \mathsf{C} \to \mathsf{D}^{\mathrm{op}}$$

is of type $(m, m)$. (The equivalence used in the second definition follows from Proposition 7.39.)

**7.10.2. Contragredient of higher monoidal categories.** Now let

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "*"] & \mathsf{C}' \arrow[l, bend left, "*"] \end{tikzcd}$$

be a contravariant adjoint equivalence of categories, as in (3.44).

Let, say $(\mathsf{C}, \diamond, \star)$, be a 2-monoidal category. In Section 6.12, we explained how the above functors can be used to turn $\mathsf{C}'$ into a 2-monoidal category. We denoted it by $(\mathsf{C}', \star^\vee, \diamond^\vee)$ and called it the contragredient of $\mathsf{C}$.

Now, let $(\mathsf{C}, \diamond, \star, \cdot)$ be a 3-monoidal category. By applying the previous construction to each of the 2-monoidal categories

$$(\mathsf{C}, \diamond, \star), \quad (\mathsf{C}, \diamond, \cdot) \quad \text{and} \quad (\mathsf{C}, \star, \cdot)$$

yields the 2-monoidal categories

$$(\mathsf{C}', \star^\vee, \diamond^\vee), \quad (\mathsf{C}', \cdot^\vee, \diamond^\vee) \quad \text{and} \quad (\mathsf{C}', \cdot^\vee, \star^\vee).$$

These when put together yield a 3-monoidal category

$$(\mathsf{C}', \cdot^\vee, \star^\vee, \diamond^\vee).$$

We call this the *contragredient* of $(\mathsf{C}, \diamond, \star, \cdot)$. It is now clear this construction can be applied to any higher monoidal category.

**Proposition 7.52.** *The functors*

$$\begin{tikzcd} (\mathsf{C}, \diamond_1, \dots, \diamond_n) \arrow[r, bend left, "*"] & (\mathsf{C}', \diamond_n^\vee, \dots, \diamond_1^\vee) \arrow[l, bend left, "*"] \end{tikzcd}$$

*are contravariant $n$-strong.*

The proof is straightforward.

**7.10.3. Contragredient of monoidal functors.** Consider the situation

$$\mathcal{F}^\vee \colon \mathsf{C} \xrightarrow{\ *\ } \mathsf{C}' \xrightarrow{\ \mathcal{F}\ } \mathsf{D}' \xrightarrow{\ *\ } \mathsf{D}$$

as in (3.45). For a natural transformation $\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$, let $\theta^\vee \colon \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee$ denote the induced natural transformation.

**Proposition 7.53.** *Let $\mathsf{C}'$ and $\mathsf{D}'$ be $n$-monoidal categories. If $\mathcal{F} \colon \mathsf{C}' \to \mathsf{D}'$ is a monoidal functor of type $(m, n-m)$, then $\mathcal{F}^\vee \colon \mathsf{C} \to \mathsf{D}$ is a monoidal functor of type $(n-m, m)$.*

*Further, if $\theta \colon \mathcal{F} \Rightarrow \mathcal{G}$ is a morphism of $(m, n-m)$-functors, then $\theta^\vee \colon \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee$ is a morphism of $(n-m, m)$-functors.*

We elaborate further on the above construction; the $n = 2$ case was explained in Proposition 6.80. The categories $\mathsf{C}$ and $\mathsf{D}$ are given the $n$-monoidal structures contragredient to those of $\mathsf{C}'$ and $\mathsf{D}'$. The $m$ colax structures of $\mathcal{F}^\vee$ are constructed from the $m$ lax structures of $\mathcal{F}$, as in Proposition 3.102. Similar statement holds for the $n-m$ lax structures of $\mathcal{F}^\vee$. The proof makes use of Proposition 7.52.

**7.10.4. Self-duality.** We now work in the situation where $\mathsf{C} = \mathsf{C}'$ and where this category is equipped with a self-adjoint $*$ functor.

**Definition 7.54.** A $n$-monoidal category $(\mathsf{C}, \diamond_1, \dots, \diamond_n)$ is *self-dual* if

$$\mathrm{id} \colon (\mathsf{C}, \diamond_1, \dots, \diamond_n) \to (\mathsf{C}, \diamond_n^\vee, \dots, \diamond_1^\vee)$$

is a $n$-strong equivalence.

**Definition 7.55.** Let $\mathsf{C}$ and $\mathsf{D}$ be self-dual $2m$-monoidal categories. A $(m, m)$-functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ is *self-dual* if $\mathcal{F}^\vee \cong \mathcal{F}$ as $(m, m)$-functors.

**Proposition 7.56.** *A self-dual $(m, m)$-functor induces a self-dual functor on the corresponding categories of $(m, m)$-monoids. In particular, it preserves self-dual $(m, m)$-monoids.*

The proof of the first claim is straightforward. The second claim follows from the first by Proposition 3.107.

**Definition 7.57.** Let $\mathsf{C}$ and $\mathsf{D}$ be self-dual $2m$-monoidal categories, and let $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ be a $(m, m)$-functor. A natural transformation $\theta : \mathcal{F} \Rightarrow \mathcal{F}^\vee$ of $(m, m)$-functors is *self-dual* if $\theta^\vee \cong \theta$.

# Part II

# Hopf Monoids in Species

CHAPTER 8

# Monoidal Structures on Species

We now enter the world of species. The category of species may be regarded as an analogue of the category of graded vector spaces, the former possessing more structure than the latter. Graded vector spaces were discussed in detail in Chapter 2. Standard material on species is reviewed in Section 8.1; this includes definitions of the Hadamard, Cauchy and substitution tensor products on species. Of particular importance for us is the Cauchy product and any reference to monoids in species is with respect to this product. In Sections 8.2–8.5 we discuss monoids and Hopf monoids in species, along with a first few examples.

We also consider species with special properties or with additional structure. These include set species, linearized species, and species with restrictions in Section 8.7, connected and positive species in Section 8.9, and species with up-down operators in Section 8.12. Coradical filtrations of positive comonoids are discussed in Section 8.10. In addition to the tensor products, there are a number of interesting operations on species. We discuss duality of species in Section 8.6, and derivatives of species in Section 8.11.

In Section 8.8, we interpret bimonoids in species as bilax monoidal functors. This is different from the interpretation of a bimonoid (in any braided monoidal category) as a bilax functor from the one-arrow category (Section 3.4.1). Further, this interpretation is specific to species, the corresponding result for graded bialgebras does not hold. We will see a few instances of this kind in this chapter.

The dual of a species allows us to define the dual of a Hopf monoid. In Section 8.13 we make use of an interchange law between the Hadamard and Cauchy products to give a construction of self-dual Hopf monoids in species. We also reformulate this result in the language of 2- and 3-monoidal categories.

## 8.1. Species

Good treatments on species can be found in the original work of Joyal [181] or the book by Bergeron, Labelle and Leroux [40], whose notation for the most part we follow. Kelly [197] gave an early detailed discussion of the basic operations on the category of species. In some form, *espèces de structures* appear in the work of Ehresmann [115, Chapter II].

**8.1.1. Definition.** Let $\Bbbk$ be a field of arbitrary characteristic. Consider the following categories:

- $\mathsf{Set}$, whose objects are arbitrary sets and whose morphisms are arbitrary maps between sets;
- $\mathsf{Set}^{\times}$, whose objects are finite sets and whose morphisms are bijections between finite sets;

- $\mathsf{Vec}$, whose objects are vector spaces over $\Bbbk$ and whose morphisms are linear maps between vector spaces.

**Definition 8.1.** A *set species* is a functor

$$\mathsf{Set}^{\times} \longrightarrow \mathsf{Set}.$$

A *vector species* is a functor

$$\mathsf{Set}^{\times} \longrightarrow \mathsf{Vec}.$$

A morphism between species $\mathbf{p}$ and $\mathbf{q}$ is a natural transformation between the functors $\mathbf{p}$ and $\mathbf{q}$.

More generally, one can consider species with values on an arbitrary category $\mathsf{C}$, as in Example A.13. We are mainly interested in vector species, and we shall refer to them simply as *species*.

We write $\mathsf{Sp}$ for the category of species. Given a species $\mathbf{p}$, we denote the image of a finite set $I$ by $\mathbf{p}[I]$ and say that $\mathbf{p}[I]$ is the space of $\mathbf{p}$-*structures* on the set $I$, or the $I$-*component* of $\mathbf{p}$.

Thus, a species consists of a family of vector spaces $\mathbf{p}[I]$, one for each finite set $I$, together with linear maps

$$\mathbf{p}[\sigma]\colon \mathbf{p}[I] \to \mathbf{p}[J],$$

one for each bijection $\sigma\colon I \to J$, such that

$$\mathbf{p}[\mathrm{id}_I] = \mathrm{id}_{\mathbf{p}[I]} \quad \text{and} \quad \mathbf{p}[\tau\sigma] = \mathbf{p}[\tau]\mathbf{p}[\sigma]$$

whenever $I \xrightarrow{\sigma} J \xrightarrow{\tau} K$ are composable bijections. It follows that the map $\mathbf{p}[\sigma]$ is an isomorphism with inverse $\mathbf{p}[\sigma^{-1}]$.

Similarly, a morphism of species $f\colon \mathbf{p} \to \mathbf{q}$ consists of a family of linear maps

$$f_I\colon \mathbf{p}[I] \to \mathbf{q}[I],$$

one for each finite set $I$, such that for each bijection $\sigma\colon I \to J$, the diagram

$$\begin{tikzcd}
\mathbf{p}[I] \arrow[r, "f_I"] \arrow[d, "\mathbf{p}[\sigma]"'] & \mathbf{q}[I] \arrow[d, "\mathbf{q}[\sigma]"] \\
\mathbf{p}[J] \arrow[r, "f_J"'] & \mathbf{q}[J]
\end{tikzcd} \tag{8.1}$$

commutes. The map $f_I$ is the $I$-*component* of $f$. When $\mathbf{p}$ is understood, we let

$$\mathrm{id}_I\colon \mathbf{p}[I] \to \mathbf{p}[I]$$

denote the identity of $\mathbf{p}[I]$. These are the components of the identity map of $\mathbf{p}$.

**Definition 8.2.** A species $\mathbf{p}$ is *finite-dimensional* if for every finite set $I$, the space $\mathbf{p}[I]$ is finite-dimensional. It is of *finite support* if $\mathbf{p}[I] = 0$ for all but finitely many cardinalities $|I|$.

Most, but not all, of the concrete examples of species discussed in this monograph are finite-dimensional. Few are of finite support.

*Warning.* We make no notational distinction between the category of all species and the full subcategory of finite-dimensional species. The notation $\mathsf{Sp}$ refers to either one or the other depending on the context.

We write $[n]$ as a shorthand for $\{1, 2, \dots, n\}$ and $\mathbf{p}[n]$ for $\mathbf{p}[\{1, 2, \dots, n\}]$. In particular, $[0] = \emptyset$.

Let $\mathrm{S}_n$ be the symmetric group on $n$ letters; $\mathrm{S}_0$ is the group with one element. Each element $\pi$ of $\mathrm{S}_n$ induces a map $\mathbf{p}[\pi]\colon \mathbf{p}[n] \to \mathbf{p}[n]$. This turns $\mathbf{p}[n]$ into a left $\mathrm{S}_n$-module. We denote the action of $\pi \in \mathrm{S}_n$ on $x \in \mathbf{p}[n]$ by $\mathbf{p}[\pi](x)$ or, when there is no risk of confusion, simply by $\pi(x)$.

Up to isomorphism, the species $\mathbf{p}$ can be recovered from the sequence of spaces

$$\mathbf{p}[0], \mathbf{p}[1], \mathbf{p}[2], \dots,$$

together with the structure of $\mathrm{S}_n$-module on each $\mathbf{p}[n]$; see [40, Exercise 1.1.6]. From this point of view, a morphism from the species $\mathbf{p}$ to the species $\mathbf{q}$ is a family of morphisms $\mathbf{p}[n] \to \mathbf{q}[n]$ of $\mathrm{S}_n$-modules, one for each $n$.

**Example 8.3.** We give two basic examples of species. They will accompany us throughout the rest of Part II as they illustrate our main constructions.

- exponential species: $\mathbf{E}[I] := \Bbbk$ for all $I$. We let $*_I$ denote the element $1 \in \Bbbk = \mathbf{E}[I]$.
- linear order species: $\mathbf{L}[I]$ is the $\Bbbk$-span of the set of linear (total) orders on $I$. By convention, $\mathbf{L}[\emptyset] = \Bbbk$. We write

  $$l = l^1|l^2|\cdots|l^n \in \mathbf{L}[I]$$

  to denote the linear order $l$ on the set $I = \{l^1, \dots, l^n\}$ for which

  $$l^1 < \cdots < l^n.$$

Many more examples can be found in [40, Chapter 1.1] and throughout this monograph, particularly in Chapters 12 and 13.

**8.1.2. Monoidal structures.** There are a variety of monoidal structures on the category of species $\mathsf{Sp}$. One of the simplest is given by the *addition operation* on species. It is defined by

$$\begin{aligned}(\mathbf{p}+\mathbf{q})[I] &:= \mathbf{p}[I] \oplus \mathbf{q}[I],\\ (\mathbf{p}+\mathbf{q})[\sigma] &:= \mathbf{p}[\sigma] \oplus \mathbf{q}[\sigma]\end{aligned} \tag{8.2}$$

on a finite set $I$ and a bijection $\sigma\colon I \to J$. This defines a functor $\mathbf{p}+\mathbf{q}\colon \mathsf{Set}^\times \to \mathsf{Vec}$, that is, a species. This operation gives rise to a monoidal category $(\mathsf{Sp}, +)$, with the *zero species* serving as the unit object. This species, which we denote simply by $\mathbf{0}$, has

$$\mathbf{0}[I] = 0$$

for every finite set $I$.

**Remark 8.4.** The species $\mathbf{p}+\mathbf{q}$ is both the categorical product and the categorical coproduct of $\mathbf{p}$ and $\mathbf{q}$ in $\mathsf{Sp}$. This follows from Proposition A.10, item (iii) since direct sum is the categorical (co)product in $\mathsf{Vec}$ and it is a special case of (co)limit.

The other monoidal structures on species that we consider are defined below and summarized in Table 8.1. The corresponding analogues for the category of graded vector spaces can be found in Table 2.1.

The units are as follows. The exponential species $\mathbf{E}$ is defined in Example 8.3. The species $\mathbf{1}$ and $\mathbf{X}$, characteristic of the empty set and singletons respectively,

TABLE 8.1. Monoidal structures on species.

| Name | Tensor product | Unit |
|---|---|---|
| Cauchy | $\cdot$ | $\mathbf{1}$ |
| Hadamard | $\times$ | $\mathbf{E}$ |
| Substitution | $\circ$ | $\mathbf{X}$ |

are defined by

$$\mathbf{1}[I] := \begin{cases} \Bbbk & \text{if } I \text{ is empty,} \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{X}[I] := \begin{cases} \Bbbk & \text{if } I \text{ is a singleton,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.3}$$

One can define *decorated versions* of the above species as follows. For any vector space $V$, let

$$\mathbf{1}_V[I] := \begin{cases} V & \text{if } I \text{ is empty,} \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{X}_V[I] := \begin{cases} V & \text{if } I \text{ is a singleton,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.4}$$

The decorated version of the exponential species is defined by

$$\mathbf{E}_V[I] := V^{\otimes I}, \tag{8.5}$$

where the right-hand side is the unordered tensor product of $V$ over $I$. Letting $V = \Bbbk$ recovers the previous definitions.

**Definition 8.5.** Let $\mathbf{p}$ and $\mathbf{q}$ be two species. Define new species $\mathbf{p} \cdot \mathbf{q}$, $\mathbf{p} \times \mathbf{q}$, and $\mathbf{p} \circ \mathbf{q}$ as follows. For any finite set $I$,

$$(\mathbf{p} \cdot \mathbf{q})[I] := \bigoplus_{I = S \sqcup T} \mathbf{p}[S] \otimes \mathbf{q}[T]; \tag{8.6}$$

$$(\mathbf{p} \times \mathbf{q})[I] := \mathbf{p}[I] \otimes \mathbf{q}[I]; \tag{8.7}$$

$$(\mathbf{p} \circ \mathbf{q})[I] := \bigoplus_{X \vdash I} \mathbf{p}[X] \otimes \Big( \bigotimes_{S \in X} \mathbf{q}[S] \Big). \tag{8.8}$$

We refer to these operations as the *Cauchy* product, the *Hadamard* product, and the *substitution* of species, in the order (8.6)–(8.8), by analogy with the operations for graded vector spaces defined in Section 2.1.1. The following remarks complement their definition.

- In the Cauchy product (8.6), the sum is over all ordered decompositions of $I$ into disjoint subsets $S$ and $T$. Thus, if $|I| = n$, there are $2^n$ terms in the direct sum.
- A bijection $\sigma\colon I \to J$ gives rise to maps
$$\mathbf{p}[S] \otimes \mathbf{q}[T] \xrightarrow{\mathbf{p}[\sigma|_S] \otimes \mathbf{p}[\sigma|_T]} \mathbf{p}[\sigma(S)] \otimes \mathbf{q}[\sigma(T)],$$
where $\sigma|_S$ denotes the restriction of $\sigma$ to $S$. This turns $\mathbf{p} \cdot \mathbf{q}$ into a functor $\mathsf{Set}^\times \to \mathsf{Vec}$, that is, a species. A similar remark applies to the other operations.
- In the substitution product (8.8), the sum is over all partitions $X$ of $I$, that is, collections of disjoint nonempty subsets of $I$ whose union is $I$. There is more information on set partitions in Section 10.1.2. This definition of $\mathbf{p} \circ \mathbf{q}$ applies only when $\mathbf{q}[\emptyset] = 0$ and we will use it only in this situation.

The general definition of the substitution product (when $\mathbf{q}[\emptyset]$ is arbitrary) is discussed fully in Section B.4.

Each of the operations (8.6)–(8.8) defines a monoidal structure on the category of species, with units as shown in Table 8.1. The resulting monoidal categories are denoted

$$(\mathsf{Sp}, \cdot), \quad (\mathsf{Sp}, \times) \quad \text{and} \quad (\mathsf{Sp}, \circ).$$

*Warning.* To obtain a monoidal category using the unit $\mathbf{X}$ of (8.3) and the substitution product as defined in (8.8), one must either restrict to the category of positive species (Section B.1), or use the whole category of species but modify the definition of (8.8) as done in Section B.4.

The monoidal categories $(\mathsf{Sp}, \cdot)$ and $(\mathsf{Sp}, \times)$ are braided, and in fact symmetric. The symmetry $\mathbf{p} \cdot \mathbf{q} \to \mathbf{q} \cdot \mathbf{p}$ has components

$$\mathbf{p}[S] \otimes \mathbf{q}[T] \to \mathbf{q}[T] \otimes \mathbf{p}[S], \qquad x \otimes y \mapsto y \otimes x.$$

The symmetry $\mathbf{p} \times \mathbf{q} \to \mathbf{q} \times \mathbf{p}$ is defined similarly. The substitution operation is not braided.

The operations (8.6)–(8.8) can be defined for set species as well. We return to this point in Section 8.7.1.

**8.1.3. Monoids and comonoids.** The notions of monoid and comonoid in a general monoidal category are recalled in Definition 1.9. For the monoidal categories of species discussed above, they are as follows.

A monoid in $(\mathsf{Sp}, \circ)$ is an *operad*, while a comonoid in $(\mathsf{Sp}, \circ)$ is a *cooperad*. We point out a technicality. There is a variant of the substitution product, and it is comonoids with respect to this variant that we call cooperads. These ideas are discussed in full detail in Appendix B.

A monoid in $(\mathsf{Sp}, \times)$ is a functor $\mathsf{Set}^{\times} \to \mathsf{Alg}$, where $\mathsf{Alg}$ is the category of algebras over $\Bbbk$. Similarly, a comonoid in $(\mathsf{Sp}, \times)$ is a functor $\mathsf{Set}^{\times} \to \mathsf{Coalg}$, where $\mathsf{Coalg}$ is the category of coalgebras over $\Bbbk$.

Monoids and comonoids in $(\mathsf{Sp}, \cdot)$ are of central importance to our work. We do not reserve a special name for these objects, but sometimes refer to them simply as *(co)monoids in species*. We begin their study in Section 8.2.

An equivalent notion to that of monoids in $(\mathsf{Sp}, \cdot)$ was first introduced by Barratt [33], who called them *twisted algebras*. The term "twisted" appears in various contexts and is frequently given other meanings in the literature. For this reason, we do not employ this terminology.

Monoids in $(\mathsf{Sp}, \cdot)$ (or twisted algebras) appear in the work of Joyal [182], Fresse [136, Section 1.2.10], Stover [346] and in the work of Patras with Livernet, Reutenauer, and Schocker [234, 291, 292, 293]. The relevance of these objects to recent work relating Hopf algebras to combinatorics was first pointed out in [291], and independently by the authors of this monograph.

## 8.2. Monoids and comonoids in species

From now on we focus our attention on $(\mathsf{Sp}, \cdot)$, the monoidal category of species under the Cauchy product. In this section we discuss the notion of (co)monoid in $(\mathsf{Sp}, \cdot)$ in explicit terms. We employ the notations of Table 8.2 to denote various related categories.

TABLE 8.2. Categories of (co)monoids in species.

| Category | Description |
|---|---|
| $\mathsf{Mon}(\mathsf{Sp}, \cdot)$ | Monoids |
| $\mathsf{Comon}(\mathsf{Sp}, \cdot)$ | Comonoids |
| $\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}, \cdot, \beta)$ | Commutative monoids |
| ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{Sp}, \cdot, \beta)$ | Cocommutative comonoids |

It will often be understood that the Cauchy product is the monoidal structure under consideration. In this case, we may simply write $\mathsf{Mon}(\mathsf{Sp})$ for the category of monoids in $(\mathsf{Sp}, \cdot)$, and similarly for the other categories.

**8.2.1. Monoids.** We make the notion of monoid in $(\mathsf{Sp}, \cdot)$ explicit. A monoid in $(\mathsf{Sp}, \cdot)$ is a species $\mathbf{p}$ together with morphisms of species

$$\mu\colon \mathbf{p} \cdot \mathbf{p} \to \mathbf{p} \qquad \text{and} \qquad \iota\colon \mathbf{1} \to \mathbf{p}$$

which are associative and unital in the sense of Definition 1.9. The product $\mu$ entails one linear map for each finite set $I$

$$\bigoplus_{I=S\sqcup T} \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I].$$

This in turn consists of a map

$$\mu_{S,T}\colon \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I] \tag{8.9}$$

for each decomposition $I = S \sqcup T$. These are the *components* of the product. The unit $\iota$ consists of a single linear map

$$\iota_\emptyset\colon \Bbbk \to \mathbf{p}[\emptyset],$$

the other components necessarily being 0. The following diagrams must commute: for each bijection $\sigma\colon I \to J$ and each decomposition $I = S \sqcup T$ of a finite set $I$ into disjoint subsets $S$ and $T$,

$$\begin{tikzcd}
\mathbf{p}[S] \otimes \mathbf{p}[T] \arrow[r, "\mu_{S,T}"] \arrow[d, "\mathbf{p}[\sigma|_S] \otimes \mathbf{p}[\sigma|_T]"'] & \mathbf{p}[I] \arrow[d, "\mathbf{p}[\sigma]"] \\
\mathbf{p}[\sigma(S)] \otimes \mathbf{p}[\sigma(T)] \arrow[r, "\mu_{\sigma(S),\sigma(T)}"'] & \mathbf{p}[J];
\end{tikzcd} \tag{8.10}$$

for each decomposition $I = R \sqcup S \sqcup T$,

$$\begin{tikzcd}
\mathbf{p}[R] \otimes \mathbf{p}[S] \otimes \mathbf{p}[T] \arrow[r, "\mathrm{id}_R \otimes \mu_{S,T}"] \arrow[d, "\mu_{R,S} \otimes \mathrm{id}_T"'] & \mathbf{p}[R] \otimes \mathbf{p}[S \sqcup T] \arrow[d, "\mu_{R,S\sqcup T}"] \\
\mathbf{p}[R \sqcup S] \otimes \mathbf{p}[T] \arrow[r, "\mu_{R\sqcup S,T}"'] & \mathbf{p}[R \sqcup S \sqcup T];
\end{tikzcd} \tag{8.11}$$

finally, for each $I$,

(8.12)
$$\begin{tikzcd}
\mathbf{p}[I] & \mathbf{p}[\emptyset]\otimes\mathbf{p}[I] \arrow[l, "\mu_{\emptyset,I}"'] \\
& \Bbbk\otimes\mathbf{p}[I] \arrow[ul, equal] \arrow[u, "\iota_\emptyset\otimes\mathrm{id}_I"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{p}[I]\otimes\mathbf{p}[\emptyset] \arrow[r, "\mu_{I,\emptyset}"] & \mathbf{p}[I] \\
\mathbf{p}[I]\otimes\Bbbk. \arrow[u, "\mathrm{id}_I\otimes\iota_\emptyset"] \arrow[ur, equal]
\end{tikzcd}$$

We now make the notion of a commutative monoid explicit (Definition 1.17). Let

(8.13)
$$\beta_{S,T}\colon \mathbf{p}[S]\otimes\mathbf{q}[T]\to\mathbf{q}[T]\otimes\mathbf{p}[S], \qquad x\otimes y\mapsto y\otimes x$$

be the components of the braiding.

A monoid $(\mathbf{p},\mu,\iota)$ in $(\mathsf{Sp},\cdot)$ is commutative if diagram

(8.14)
$$\begin{tikzcd}
\mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[rr, "\beta_{S,T}"] \arrow[dr, "\mu_{S,T}"'] & & \mathbf{p}[T]\otimes\mathbf{p}[S] \arrow[dl, "\mu_{T,S}"] \\
& \mathbf{p}[I] &
\end{tikzcd}$$

commutes, for all decompositions $I=S\sqcup T$.

Morphisms of monoids in monoidal categories are discussed in Section 1.2.1. Let

$$f\colon(\mathbf{p},\mu,\iota)\to(\mathbf{p}',\mu',\iota')$$

be a morphism of monoids in $(\mathsf{Sp},\cdot)$. Explicitly, it consists of a map

$$f_I\colon\mathbf{p}[I]\to\mathbf{p}'[I]$$

for each finite set $I$, satisfying (8.1) and such that the following diagrams commute.

(8.15)
$$\begin{tikzcd}
\mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[r, "f_S\otimes f_T"] \arrow[d, "\mu_{S,T}"'] & \mathbf{p}'[S]\otimes\mathbf{p}'[T] \arrow[d, "\mu'_{S,T}"] \\
\mathbf{p}[I] \arrow[r, "f_I"'] & \mathbf{p}'[I]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{p}[\emptyset] \arrow[rr, "f_\emptyset"] & & \mathbf{p}'[\emptyset] \\
& \Bbbk \arrow[ul, "\iota_\emptyset"] \arrow[ur, "\iota'_\emptyset"'] &
\end{tikzcd}$$

**8.2.2. Comonoids.** A comonoid in $(\mathsf{Sp},\cdot)$ is a species $\mathbf{p}$ together with morphisms of species

$$\Delta\colon\mathbf{p}\to\mathbf{p}\cdot\mathbf{p} \qquad\text{and}\qquad \epsilon\colon\mathbf{p}\to\mathbf{1}$$

which are coassociative and counital. Let

(8.16)
$$\Delta_{S,T}\colon\mathbf{p}[I]\to\mathbf{p}[S]\otimes\mathbf{p}[T]$$

be the components of the coproduct; there is one for each disjoint decomposition $I=S\sqcup T$. Let

$$\epsilon_\emptyset\colon\mathbf{p}[\emptyset]\to\Bbbk$$

be the only nonzero component of the counit. Replacing $\mu_{S,T}$ by $\Delta_{S,T}$ and $\iota_\emptyset$ by $\epsilon_\emptyset$ (and reversing those arrows) in diagrams (8.10)–(8.15) makes the notion of comonoid (cocommutative comonoid, morphism of comonoids) in $(\mathsf{Sp}, \cdot)$ explicit. For instance, coassociativity is expressed by the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{p}[I] \arrow[r, "\Delta_{R\sqcup S,T}"] \arrow[d, "\Delta_{R,S\sqcup T}"'] & \mathbf{p}[R \sqcup S] \otimes \mathbf{p}[T] \arrow[d, "\Delta_{R,S}\otimes \mathrm{id}_T"] \\
\mathbf{p}[R] \otimes \mathbf{p}[S \sqcup T] \arrow[r, "\mathrm{id}_R \otimes \Delta_{S,T}"'] & \mathbf{p}[R] \otimes \mathbf{p}[S] \otimes \mathbf{p}[T]
\end{tikzcd}$$

for each decomposition $I = R \sqcup S \sqcup T$.

**Remark 8.6.** If $\mathbf{p}$ is a (co)monoid in $(\mathsf{Sp}, \cdot)$, then $\mathbf{p}[\emptyset]$ is a (co)algebra, whose structures maps are the $\emptyset$-components of the structure maps of $\mathbf{p}$. Indeed, replacing all finite sets for the empty set in axioms (8.11) and (8.12) yields the monoid axioms for $\mathbf{p}[\emptyset]$.

In addition, a (co)algebra structure on a vector space $V$ is the same as a (co)monoid structure on the species $\mathbf{1}_V$ defined in (8.4).

Analogous statements hold for (co)commutative (co)monoids and Lie monoids (Definition 1.25).

**8.2.3. Modules, comodules, and cohomology.** Recall the notions of (bi)module over a monoid and (bi)comodule over a comonoid from Section 1.2.3. Let us briefly consider a bicomodule $\mathbf{m}$ over a comonoid $\mathbf{c}$ in $(\mathsf{Sp}, \cdot)$. The structure maps

$$\chi^1 : \mathbf{m} \to \mathbf{c} \cdot \mathbf{m} \quad \text{and} \quad \chi^2 : \mathbf{m} \to \mathbf{m} \cdot \mathbf{c}$$

consist of components

$$\chi^1_{S,T} : \mathbf{m}[I] \to \mathbf{c}[S] \otimes \mathbf{m}[T] \quad \text{and} \quad \chi^2_{S,T} : \mathbf{m}[I] \to \mathbf{m}[S] \otimes \mathbf{c}[T]$$

for each decomposition $I = S \sqcup T$. The bicomodule axioms translate into the commutativity of certain diagrams, including the following.

$$\begin{tikzcd}
\mathbf{m}[I] \arrow[r, "\chi^2_{R\sqcup S,T}"] \arrow[d, "\chi^1_{R,S\sqcup T}"'] & \mathbf{m}[R \sqcup S] \otimes \mathbf{c}[T] \arrow[d, "\chi^1_{R,S}\otimes \mathrm{id}_T"] \\
\mathbf{c}[R] \otimes \mathbf{m}[S \sqcup T] \arrow[r, "\mathrm{id}_R \otimes \chi^2_{S,T}"'] & \mathbf{c}[R] \otimes \mathbf{m}[S] \otimes \mathbf{c}[T]
\end{tikzcd}$$

The remaining axioms take similar forms.

*Cohomology* of a (co)monoid with coefficients in a bi(co)module can be defined in the same manner as for (co)algebras (Section 2.7.6). Indeed, let $\mathbf{c}$ and $\mathbf{m}$ be as above. We define

$$C^n(\mathbf{c}, \mathbf{m}) := \mathrm{Hom}_{\mathsf{Sp}}(\mathbf{m}, \mathbf{c}^{\cdot n})$$

and

$$d^n : C^n(\mathbf{c}, \mathbf{m}) \to C^{n+1}(\mathbf{c}, \mathbf{m})$$

by

$$d^n(f) := (\mathrm{id} \cdot f)\chi^1 + \sum_{i=1}^{n} (-1)^i (\mathrm{id}^{\cdot(i-1)} \cdot \Delta \cdot \mathrm{id}^{\cdot(n-i)}) f + (-1)^{n+1} (f \cdot \mathrm{id})\chi^2.$$

TABLE 8.3. Categories of bimonoids in species.

| Category | Description |
|---|---|
| $\mathsf{Bimon}(\mathsf{Sp}, \cdot, \beta)$ | Bimonoids |
| $\mathsf{Hopf}(\mathsf{Sp}, \cdot, \beta)$ | Hopf monoids |

An element $f \in C^n(\mathbf{c}, \mathbf{m})$ has components

$$f_{S_1,\ldots,S_n} \colon \mathbf{m}[I] \to \mathbf{c}[S_1] \otimes \cdots \otimes \mathbf{c}[S_n],$$

one for each decomposition $I = S_1 \sqcup \cdots \sqcup S_n$. The components of the differential of $f$ are as follows. Given a decomposition $I = S_1 \sqcup \cdots \sqcup S_{n+1}$, we have

$$\begin{aligned} d^n(f)&_{S_1,\ldots,S_{n+1}} \\ &= (\mathrm{id}_{S_1} \otimes f_{S_2,\ldots,S_{n+1}})\chi^1_{S_1,S_2\sqcup\cdots\sqcup S_{n+1}} \\ &\quad + \sum_{i=1}^{n} (-1)^i (\mathrm{id}_{S_1} \otimes \cdots \otimes \mathrm{id}_{S_{i-1}} \otimes \Delta_{S_i,S_{i+1}} \otimes \mathrm{id}_{S_{i+2}} \otimes \cdots \otimes \mathrm{id}_{S_{n+1}}) \\ &\qquad\qquad f_{S_1,\ldots,S_{i-1},S_i\sqcup S_{i+1},S_{i+2},\ldots,S_{n+1}} \\ &\quad + (-1)^{n+1}(f_{S_1,\ldots,S_n} \otimes \mathrm{id}_{S_{n+1}})\chi^2_{S_1\sqcup\cdots\sqcup S_n,S_{n+1}}. \end{aligned} \tag{8.17}$$

## 8.3. Bimonoids and Hopf monoids in species

We discuss bimonoids and Hopf monoids in species, along with examples. These notions are defined in the general context of braided monoidal categories in Definitions 1.10 and 1.15. We employ the notations of Table 8.3 (and more generally of Table 1.1) to denote various related categories.

The Cauchy product and the braiding will often be understood from the context. In that case, we may simply write $\mathsf{Bimon}(\mathsf{Sp})$ and $\mathsf{Hopf}(\mathsf{Sp})$ for the category of bimonoids and Hopf monoids in $(\mathsf{Sp}, \cdot, \beta)$, and similarly for other related categories.

**8.3.1. Bimonoids.** A bimonoid in $(\mathsf{Sp}, \cdot, \beta)$ is a species $\mathbf{h}$ together with a monoid structure $(\mu, \iota)$ and a comonoid structure $(\Delta, \epsilon)$ such that $\Delta$ and $\epsilon$ are morphisms of monoids, or equivalently, $\mu$ and $\iota$ are morphisms of comonoids. Below we make these conditions explicit in terms of the maps $\mu_{S,T}$, $\Delta_{S,T}$ and $\beta_{S,T}$ of (8.9), (8.16) and (8.13).

**Lemma 8.7.** *Let $S \sqcup T = I = S' \sqcup T'$ be two decompositions of a finite set $I$. Then there are unique subsets $A, B, C$, and $D$ of $I$ such that*

$$S = A \sqcup B, \quad T = C \sqcup D, \quad S' = A \sqcup C, \quad T' = B \sqcup D.$$

PROOF. The only choice is $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, $D = T \cap T'$. $\square$

Figure 8.1 illustrates the situation.

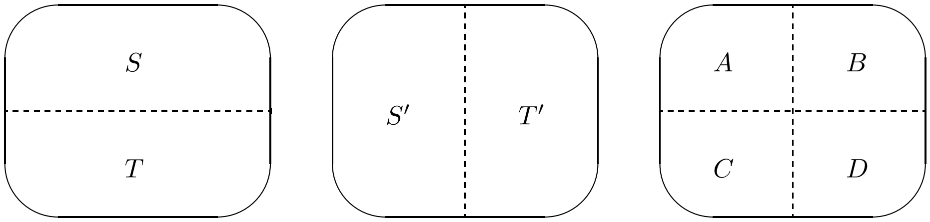



FIGURE 8.1. The sets of Lemma 8.7.

The compatibility conditions in Definition 1.10 take the following explicit form. First, for any pair of decompositions $S \sqcup T = I = S' \sqcup T'$ of a finite set $I$, diagram

$$(8.18)\qquad \begin{tikzcd}
\mathbf{p}[A]\otimes\mathbf{p}[B]\otimes\mathbf{p}[C]\otimes\mathbf{p}[D] \arrow[rr, "\mathrm{id}_A\otimes\beta_{B,C}\otimes\mathrm{id}_D"] & & \mathbf{p}[A]\otimes\mathbf{p}[C]\otimes\mathbf{p}[B]\otimes\mathbf{p}[D] \arrow[d, "\mu_{A,C}\otimes\mu_{B,D}"] \\
\mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[u, "\Delta_{A,B}\otimes\Delta_{C,D}"] \arrow[r, "\mu_{S,T}"'] & \mathbf{p}[I] \arrow[r, "\Delta_{S',T'}"'] & \mathbf{p}[S']\otimes\mathbf{p}[T']
\end{tikzcd}$$

must commute, where $A$, $B$, $C$, and $D$ are as in Lemma 8.7. In addition, diagrams

$$(8.19)\qquad \begin{tikzcd}
\mathbf{p}[\emptyset]\otimes\mathbf{p}[\emptyset] \arrow[r, "\epsilon_\emptyset\otimes\epsilon_\emptyset"] \arrow[d, "\mu_{\emptyset,\emptyset}"'] & \Bbbk\otimes\Bbbk \arrow[d, equal] \\
\mathbf{p}[\emptyset] \arrow[r, "\epsilon_\emptyset"'] & \Bbbk
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \arrow[r, "\iota_\emptyset"] \arrow[d, equal] & \mathbf{p}[\emptyset] \arrow[d, "\Delta_{\emptyset,\emptyset}"] \\
\Bbbk\otimes\Bbbk \arrow[r, "\iota_\emptyset\otimes\iota_\emptyset"'] & \mathbf{p}[\emptyset]\otimes\mathbf{p}[\emptyset]
\end{tikzcd}$$

$$(8.20)\qquad \begin{tikzcd}
& \mathbf{p}[\emptyset] \arrow[dr, "\epsilon_\emptyset"] & \\
\Bbbk \arrow[ur, "\iota_\emptyset"] \arrow[rr, equal] & & \Bbbk
\end{tikzcd}$$

must commute as well.

These diagrams are respectively equivalent to those in (1.9), (1.10) and (1.11). The assertion regarding diagram (1.9) deserves argument: going around the top of this diagram we encounter the composite

$$\begin{tikzcd}
\bigoplus \mathbf{p}[A]\otimes\mathbf{p}[B]\otimes\mathbf{p}[C]\otimes\mathbf{p}[D] \arrow[r, "\mathrm{id}_A\otimes\beta_{B,C}\otimes\mathrm{id}_D"] & \bigoplus \mathbf{p}[A]\otimes\mathbf{p}[C]\otimes\mathbf{p}[B]\otimes\mathbf{p}[D] \arrow[d, "\bigoplus \mu_{A,C}\otimes\mu_{B,D}"] \\
\mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[u, "\bigoplus \Delta_{A,B}\otimes\Delta_{C,D}"] & \bigoplus \mathbf{p}[A\sqcup C]\otimes\mathbf{p}[B\sqcup D].
\end{tikzcd}$$

The first two direct sums are over all $A$, $B$, $C$, and $D$ such that $S = A \sqcup B$ and $T = C \sqcup D$. The third direct sum is over all $A$, $B$, $C$, and $D$ which in addition to these conditions satisfy $S' = A \sqcup C$ and $T' = B \sqcup D$. According to Lemma 8.7 there is only one such choice, so our claim is justified.

**Remark 8.8.** For the first time here we encounter an important difference between bimonoids in species and graded bialgebras. Namely, diagram (1.9) dictates the following compatibility condition for the components of the structure maps of a graded bialgebra:

$$\Delta_{n,m}(x \cdot y) = \sum_{a,b,c,d} \Delta_{a,b}(x)\Delta_{c,d}(y),$$

where

$$n + m = |x| + |y|$$

and $a$, $b$, $c$, and $d$ are related to $n$, $m$, and the degrees of $x$ and $y$ by

$$a + c = n, \quad b + d = m, \quad a + b = |x|, \quad c + d = |y|.$$

In contrast to the situation of Lemma 8.7, these four equations do not determine $a$, $b$, $c$, and $d$ uniquely. This dissimilarity between bimonoids in species and graded bialgebras will persist in a number of constructions and the reason will always be the failure of Lemma 8.7 when sets are replaced by numbers; see Remarks 8.36 and 8.65.

**8.3.2. Hopf monoids.** A Hopf monoid in $(\mathsf{Sp}, \cdot, \beta)$ is a bimonoid along with a map $\mathrm{s}\colon \mathbf{h} \to \mathbf{h}$ (the antipode) which is the inverse of the identity map in the convolution algebra $\mathrm{Hom}(\mathbf{h}, \mathbf{h})$; see Definitions 1.13 and 1.15. Explicitly, this requires the existence of a linear map

$$\mathrm{s}_I \colon \mathbf{h}[I] \to \mathbf{h}[I]$$

for each finite set $I$, commuting with bijections, and such that for each nonempty set $I$ the composites

$$\mathbf{h}[I] \xrightarrow{\oplus\Delta_{S,T}} \bigoplus_{S\sqcup T=I} \mathbf{h}[S]\otimes\mathbf{h}[T] \xrightarrow{\mathrm{id}_S\otimes\mathrm{s}_T} \bigoplus_{S\sqcup T=I} \mathbf{h}[S]\otimes\mathbf{h}[T] \xrightarrow{\oplus\mu_{S,T}} \mathbf{h}[I] \tag{8.21}$$

$$\mathbf{h}[I] \xrightarrow{\oplus\Delta_{S,T}} \bigoplus_{S\sqcup T=I} \mathbf{h}[S]\otimes\mathbf{h}[T] \xrightarrow{\mathrm{s}_S\otimes\mathrm{id}_T} \bigoplus_{S\sqcup T=I} \mathbf{h}[S]\otimes\mathbf{h}[T] \xrightarrow{\oplus\mu_{S,T}} \mathbf{h}[I] \tag{8.22}$$

are zero, and for which the following diagrams commute

$$\begin{tikzcd} \mathbf{h}[\emptyset]\otimes\mathbf{h}[\emptyset] \arrow[rr, "\mathrm{id}_\emptyset\otimes\mathrm{s}_\emptyset"] & & \mathbf{h}[\emptyset]\otimes\mathbf{h}[\emptyset] \arrow[d, "\mu_{\emptyset,\emptyset}"] \\ \mathbf{h}[\emptyset] \arrow[u, "\Delta_{\emptyset,\emptyset}"] \arrow[r, "\epsilon_\emptyset"'] & \Bbbk \arrow[r, "\iota_\emptyset"'] & \mathbf{h}[\emptyset] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{h}[\emptyset]\otimes\mathbf{h}[\emptyset] \arrow[rr, "\mathrm{s}_\emptyset\otimes\mathrm{id}_\emptyset"] & & \mathbf{h}[\emptyset]\otimes\mathbf{h}[\emptyset] \arrow[d, "\mu_{\emptyset,\emptyset}"] \\ \mathbf{h}[\emptyset] \arrow[u, "\Delta_{\emptyset,\emptyset}"] \arrow[r, "\epsilon_\emptyset"'] & \Bbbk \arrow[r, "\iota_\emptyset"'] & \mathbf{h}[\emptyset]. \end{tikzcd} \tag{8.23}$$

These conditions are equivalent to diagrams (1.13).

General results on Hopf monoids (Section 1.2) apply to Hopf monoids in species. In particular, when viewed as a map $\mathrm{s}\colon \mathbf{h} \to \mathbf{h}^{\mathrm{op,cop}}$, the antipode is a morphism of Hopf monoids (Proposition 1.22); if $\mathbf{h}$ is (co)commutative, then $\mathrm{s}^2 = \mathrm{id}$.

**Remark 8.9.** As in Remark 8.6, if $\mathbf{h}$ is a (bi, Hopf) monoid in species, then $\mathbf{h}[\emptyset]$ is a (bi, Hopf) algebra, whose structures maps are the $\emptyset$-components of the structure maps of $\mathbf{h}$. In particular, if $\mathrm{s}$ is the antipode of $\mathbf{h}$, then the map $\mathrm{s}_\emptyset$ is the antipode of the Hopf algebra $\mathbf{h}[\emptyset]$, in view of axioms (8.23). Also, a (bi, Hopf) algebra structure on a vector space $V$ is the same as a (bi, Hopf) monoid structure on the species $\mathbf{1}_V$ defined in (8.4).

The following result states that a Hopf monoid $\mathbf{h}$ is equivalent to a bimonoid $\mathbf{h}$ for which $\mathbf{h}[\emptyset]$ is a Hopf algebra.

**Proposition 8.10.** *Let* $\mathbf{h}$ *be a bimonoid in species.*

(i) *Suppose* $\mathbf{h}$ *is a Hopf monoid with antipode* $\mathrm{S}$. *Then* $\mathbf{h}[\emptyset]$ *is a Hopf algebra with antipode* $\mathrm{S}_\emptyset$.

(ii) *Suppose* $\mathbf{h}[\emptyset]$ *is a Hopf algebra and let* $\mathrm{S}_0$ *denote its antipode. Then* $\mathbf{h}$ *is a Hopf monoid with antipode* $\mathrm{S}$ *given by*

$$\mathrm{S}_\emptyset := \mathrm{S}_0$$

*and*

$$\mathrm{S}_I := \sum_{\substack{S_1\sqcup\cdots\sqcup S_k=I\\ S_i\neq\emptyset\ \ k\geq 1}} (-1)^k \mu_{\emptyset,S_1,\emptyset,\ldots,\emptyset,S_k,\emptyset}\big(\mathrm{S}_0 \otimes \mathrm{id}_{S_1} \otimes \mathrm{S}_0 \otimes \cdots \otimes \mathrm{S}_0 \otimes \mathrm{id}_{S_k} \otimes \mathrm{S}_0\big) \Delta_{\emptyset,S_1,\emptyset,\ldots,\emptyset,S_k,\emptyset} \tag{8.24}$$

*for any nonempty finite set* $I$.

The sum is over all ordered decompositions of $I$ into *nonempty* subsets $S_i$. Notice this makes the sum finite ($k$ can be at most $|I|$). The map

$$\mu_{S_1,\ldots,S_k} \colon \mathbf{h}[S_1]\otimes\cdots\otimes\mathbf{h}[S_k] \to \mathbf{h}[I] \tag{8.25}$$

is a component of the $k-1$ iteration of the product of $\mathbf{h}$. These are well-defined by associativity (8.11). Similarly, the map

$$\Delta_{S_1,\ldots,S_k} \colon \mathbf{h}[I] \to \mathbf{h}[S_1]\otimes\cdots\otimes\mathbf{h}[S_k] \tag{8.26}$$

is a component of the iterated coproduct. Similar remarks apply to the maps

$$\mu_{\emptyset,S_1,\emptyset,\ldots,\emptyset,S_k,\emptyset} \colon \mathbf{h}[\emptyset]\otimes\mathbf{h}[S_1]\otimes\mathbf{h}[\emptyset]\otimes\cdots\otimes\mathbf{h}[\emptyset]\otimes\mathbf{h}[S_k]\otimes\mathbf{h}[\emptyset] \to \mathbf{h}[I]$$

and

$$\Delta_{\emptyset,S_1,\emptyset,\ldots,\emptyset,S_k,\emptyset} \colon \mathbf{h}[I] \to \mathbf{h}[\emptyset]\otimes\mathbf{h}[S_1]\otimes\mathbf{h}[\emptyset]\otimes\cdots\otimes\mathbf{h}[\emptyset]\otimes\mathbf{h}[S_k]\otimes\mathbf{h}[\emptyset].$$

PROOF. Part (i) was discussed in Remark 8.9. To prove (ii), we verify the antipode axioms directly. Axioms (8.23) hold by hypothesis. Below we check (8.21); axiom (8.22) holds by symmetry.

Choose a decomposition $I = S \sqcup T$. Using (8.24) for $\mathrm{S}_T$ we have that

$$\begin{aligned}
&\mu_{S,T}(\mathrm{id}_S \otimes \mathrm{S}_T)\Delta_{S,T}\\
&\quad= \sum_{\substack{T_1\sqcup\cdots\sqcup T_k=T\\ T_i\neq\emptyset\ \ k\geq 1}} (-1)^k \mu_{S,T}\big(\mathrm{id}_S \otimes \mu_{\emptyset,T_1,\emptyset,\ldots,\emptyset,T_k,\emptyset}\big)\big(\mathrm{id}_S \otimes \mathrm{S}_0 \otimes \mathrm{id}_{T_1} \otimes \mathrm{S}_0 \otimes \cdots \otimes \mathrm{S}_0 \otimes \mathrm{id}_{T_k} \otimes \mathrm{S}_0\big)\\
&\qquad\qquad\big(\mathrm{id}_S \otimes \Delta_{\emptyset,T_1,\emptyset,\ldots,\emptyset,T_k,\emptyset}\big)\Delta_{S,T}\\
&\quad= \sum_{\substack{T_1\sqcup\cdots\sqcup T_k=T\\ T_i\neq\emptyset\ \ k\geq 1}} (-1)^k \mu_{S,\emptyset,T_1,\emptyset,\ldots,\emptyset,T_k,\emptyset}\big(\mathrm{id}_S \otimes \mathrm{S}_0 \otimes \mathrm{id}_{T_1} \otimes \mathrm{S}_0 \otimes \cdots \otimes \mathrm{S}_0 \otimes \mathrm{id}_{T_k} \otimes \mathrm{S}_0\big) \Delta_{S,\emptyset,T_1,\emptyset,\ldots,\emptyset,T_k,\emptyset}.
\end{aligned}$$

We used associativity and coassociativity.

Suppose now that $S = \emptyset$ (and $T = I$). The previous formula gives us

$$\begin{aligned}
&\mu_{\emptyset,I}(\mathrm{id}_\emptyset \otimes \mathrm{s}_I)\Delta_{\emptyset,I} \\
&\quad = \sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k \mu_{\emptyset,\emptyset,S_1,\emptyset,\dots,\emptyset,S_k,\emptyset} \big(\mathrm{id}_\emptyset \otimes \mathrm{s}_0 \otimes \mathrm{id}_{S_1} \otimes \mathrm{s}_0 \otimes \cdots \otimes \mathrm{s}_0 \otimes \mathrm{id}_{S_k} \otimes \mathrm{s}_0\big) \Delta_{\emptyset,\emptyset,S_1,\emptyset,\dots,\emptyset,S_k,\emptyset} \\
&\quad = \sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k \mu_{\emptyset,S_1,\emptyset,\dots,\emptyset,S_k,\emptyset} \Big(\big(\mu_{\emptyset,\emptyset}(\mathrm{id}_\emptyset \otimes \mathrm{s}_0)\Delta_{\emptyset,\emptyset}\big) \otimes \big(\mathrm{id}_{S_1} \otimes \mathrm{s}_0 \otimes \cdots \otimes \mathrm{s}_0 \otimes \mathrm{id}_{S_k} \otimes \mathrm{s}_0\big)\Big) \Delta_{\emptyset,S_1,\emptyset,\dots,\emptyset,S_k,\emptyset} \\
&\quad = \sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k \mu_{\emptyset,S_1,\emptyset,\dots,\emptyset,S_k,\emptyset} \Big(\big(\iota_\emptyset \epsilon_\emptyset\big) \otimes \big(\mathrm{id}_{S_1} \otimes \mathrm{s}_0 \otimes \cdots \otimes \mathrm{s}_0 \otimes \mathrm{id}_{S_k} \otimes \mathrm{s}_0\big)\Big) \Delta_{\emptyset,S_1,\emptyset,\dots,\emptyset,S_k,\emptyset} \\
&\quad = \sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k \mu_{S_1,\emptyset,\dots,\emptyset,S_k,\emptyset} \big(\mathrm{id}_{S_1} \otimes \mathrm{s}_0 \otimes \cdots \otimes \mathrm{s}_0 \otimes \mathrm{id}_{S_k} \otimes \mathrm{s}_0\big) \Delta_{S_1,\emptyset,\dots,\emptyset,S_k,\emptyset}.
\end{aligned}$$

We used (co)associativity, the antipode axiom for $\mathrm{s}_0$, and (co)unitality.

Finally, in the sum

$$\sum_{S \sqcup T = I} \mu_{S,T}(\mathrm{id}_S \otimes \mathrm{s}_T)\Delta_{S,T},$$

consider the terms for which $S \neq \emptyset$ and the term for which $S = \emptyset$. Using the previously obtained expressions for $\mu_{S,T}(\mathrm{id}_S \otimes \mathrm{s}_T)\Delta_{S,T}$ and $\mu_{\emptyset,I}(\mathrm{id}_\emptyset \otimes \mathrm{s}_I)\Delta_{\emptyset,I}$ we see that all terms cancel. More precisely, the summand corresponding to $(T_1, \dots, T_k)$ in the former cancels with the term corresponding to $(S, T_1, \dots, T_k)$ in the latter. Thus, the sum is zero and axiom (8.21) holds. □

The first cases ($|I| \leq 2$) of (8.24) are as follows.

$$\begin{aligned}
\mathrm{s}_{\{a\}} &= -\, \mu_{\emptyset,\{a\},\emptyset}\big(\mathrm{s}_\emptyset \otimes \mathrm{id}_{\{a\}} \otimes \mathrm{s}_\emptyset\big)\Delta_{\emptyset,\{a\},\emptyset}, \\
\mathrm{s}_{\{a,b\}} &= -\, \mu_{\emptyset,\{a,b\},\emptyset}\big(\mathrm{s}_\emptyset \otimes \mathrm{id}_{\{a,b\}} \otimes \mathrm{s}_\emptyset\big)\Delta_{\emptyset,\{a,b\},\emptyset} \\
&\quad + \mu_{\emptyset,\{a\},\emptyset,\{b\},\emptyset}\big(\mathrm{s}_\emptyset \otimes \mathrm{id}_{\{a\}} \otimes \mathrm{s}_\emptyset \otimes \mathrm{id}_{\{b\}} \otimes \mathrm{s}_\emptyset\big)\Delta_{\emptyset,\{a\},\emptyset,\{b\},\emptyset} \\
&\quad + \mu_{\emptyset,\{b\},\emptyset,\{a\},\emptyset}\big(\mathrm{s}_\emptyset \otimes \mathrm{id}_{\{b\}} \otimes \mathrm{s}_\emptyset \otimes \mathrm{id}_{\{a\}} \otimes \mathrm{s}_\emptyset\big)\Delta_{\emptyset,\{b\},\emptyset,\{a\},\emptyset}.
\end{aligned}$$

## 8.4. Antipode formulas for connected bimonoids

Connected bimonoids in species are analogous to graded connected bialgebras (Section 2.3.2). We define them in Section 8.4.1. Connected species are studied separately later, in Section 8.9. Connected bimonoids are necessarily Hopf monoids. For this special class of Hopf monoids there are explicit and recursive formulas for the antipode which we discuss in Sections 8.4.2 and 8.4.3.

### 8.4.1. Connected bimonoids.

**Proposition 8.11.** *Let* $\mathbf{h}$ *be a bimonoid in species. The following are equivalent statements.*

(i) $\dim \mathbf{h}[\emptyset] = 1$.

(ii) *The composite* $\mathbf{h}[\emptyset] \xrightarrow{\epsilon_\emptyset} \Bbbk \xrightarrow{\iota_\emptyset} \mathbf{h}[\emptyset]$ *is the identity.*

(iii) *$\iota_\emptyset$ and $\epsilon_\emptyset$ define inverse isomorphisms of bialgebras $\Bbbk \cong \mathbf{h}[\emptyset]$.*

Proof. The equivalence between (i) and (ii) holds in view of (8.20). The equivalence with (iii) follows from diagrams (8.19). $\square$

**Definition 8.12.** A *connected bimonoid* is a bimonoid in species that verifies the conditions in Proposition 8.11.

A connected bimonoid is necessarily a Hopf monoid. This result can be deduced in a number of ways. It follows from either Proposition 8.10, or from the antipode formulas given in Propositions 8.13 or 8.14 below. It also appears in [346, Proposition 4.4].

### 8.4.2. Takeuchi's antipode formula.

**Proposition 8.13.** *Let $\mathbf{h}$ be a connected bimonoid. For any nonempty finite set $I$, the $I$-component of the antipode is*

$$\mathrm{s}_I = \sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k \mu_{S_1,\ldots,S_k} \Delta_{S_1,\ldots,S_k}. \tag{8.27}$$

The sum is over all ordered decompositions of $I$ into *nonempty* subsets $S_i$. The maps $\mu_{S_1,\ldots,S_k}$ and $\Delta_{S_1,\ldots,S_k}$ are as in (8.25) and (8.26). By definition, both maps $\mu_I$ and $\Delta_I$ (the case $k = 1$ of the preceding) are the identity of $\mathbf{h}[I]$.

Proof. This result is a special case of Proposition 8.11. Indeed, since $\mathbf{h}$ is connected, the bialgebra $\mathbf{h}[\emptyset] = \Bbbk$ is a Hopf algebra with antipode equal to $\mathrm{id}_\Bbbk$. In addition, axioms (8.12) imply that the iterated product $\mu_{\emptyset,S_1,\emptyset,\ldots,\emptyset,S_k,\emptyset}$ identifies with $\mu_{S_1,\ldots,S_k}$, and similarly for the iterated coproducts. Thus, formula (8.24) becomes (8.27).

We provide another proof. The map $\iota\epsilon - \mathrm{id} \colon \mathbf{h} \to \mathbf{h}$ is locally nilpotent in the convolution algebra $\mathrm{Hom}(\mathbf{h}, \mathbf{h})$, hence id is invertible and its inverse is

$$\sum_{k \geq 0} (\iota\epsilon - \mathrm{id})^{*k}.$$

The $\emptyset$-component of the map $\iota\epsilon - \mathrm{id}$ is zero; therefore, the right-hand side of (8.27) is the $I$-component of the previous sum. $\square$

The preceding result is analogous to Takeuchi's expression for the antipode of a connected Hopf algebra (2.55), and for this reason we refer to it as *Takeuchi's antipode formula*.

Many examples of Hopf monoids are discussed in this monograph. They are usually connected, so (8.27) applies. The formula yields an expression for the antipode as an alternating sum in which many cancellations often take place. By contrast, we are often interested in an explicit formula for the structure constants of the antipode on a given basis. Obtaining such a formula requires understanding of these cancellations; this is often a challenging combinatorial problem. This problem will be solved for many but not all of the examples discussed in this monograph. Often, we will employ Takeuchi's formula as the first step in the derivation of such explicit antipode formulas.

### 8.4.3. Milnor and Moore's antipode formulas.

**Proposition 8.14.** *Let* $\mathbf{h}$ *be a connected bimonoid. Define maps* $\mathrm{s}_I$ *and* $\mathrm{s}'_I$ *by induction on the cardinality of* $I$ *as follows. Let*

$$\mathrm{s}_\emptyset = \mathrm{s}'_\emptyset$$

*be the identity of* $\mathbf{h}[\emptyset] = \Bbbk$, *and for* $|I| > 0$,

$$\mathrm{s}_I := -\sum_{\substack{S \sqcup T = I \\ T \neq I}} \mu_{S,T}(\mathrm{id}_S \otimes \mathrm{s}_T)\Delta_{S,T}, \tag{8.28}$$

$$\mathrm{s}'_I := -\sum_{\substack{S \sqcup T = I \\ S \neq I}} \mu_{S,T}(\mathrm{s}'_S \otimes \mathrm{id}_T)\Delta_{S,T}. \tag{8.29}$$

*Then*

$$\mathrm{s} = \mathrm{s}'$$

*and this map is the antipode of* $\mathbf{h}$.

Proof. By construction, $\mathrm{s}$ satisfies (8.21) and is therefore a right inverse of id in $\mathrm{Hom}(\mathbf{h}, \mathbf{h})$. Similarly, $\mathrm{s}'$ is the left inverse of id, and then $\mathrm{s} = \mathrm{s}'$ is the antipode. $\square$

Proposition 8.14 provides two recursive formulas for the components of the antipode of a connected Hopf monoid. It is analogous to the result of Milnor and Moore for the antipode of a graded connected Hopf algebra given in (2.56). For this reason, we refer to (8.28) and (8.29) as *Milnor and Moore's antipode formulas.* Note, however, that these formulas demand the same work as directly verifying the antipode axioms (8.21)–(8.23).

## 8.5. The simplest Hopf monoids

We define a Hopf monoid structure on the exponential species and on the species of linear orders (Example 8.3). The dual Hopf monoids are discussed in Section 8.6. These are simple but important examples. They are the basic building blocks for the more elaborate Hopf monoids studied in Chapter 12. Additional interesting examples are discussed in Chapter 13.

We define the Hopf monoid structure maps using the notation of Section 8.2. We fix a decomposition $I = S \sqcup T$ and describe the components

$$\mathbf{h}[S] \otimes \mathbf{h}[T] \xrightarrow{\mu_{S,T}} \mathbf{h}[I] \qquad \text{and} \qquad \mathbf{h}[I] \xrightarrow{\Delta_{S,T}} \mathbf{h}[S] \otimes \mathbf{h}[T]$$

of the product and coproduct in each case. The unit and counit are determined by the maps

$$\Bbbk \xrightarrow{\iota_\emptyset} \mathbf{h}[\emptyset] \xrightarrow{\epsilon_\emptyset} \Bbbk.$$

The Hopf monoids below are connected (Definition 8.12), so $\iota_\emptyset$ and $\epsilon_\emptyset$ are the inverse bijections that identify $1 \in \Bbbk$ with the distinguished basis element of $\mathbf{h}[\emptyset]$, while $\mathrm{s}_\emptyset$ is the identity of $\mathbf{h}[\emptyset]$. For this reason, we do not mention the unit or counit structure maps. The existence of the antipode is guaranteed by Proposition 8.10. We provide explicit formulas for the antipode.

**Example 8.15.** For the exponential species $\mathbf{E}$, the product and coproduct simply identify the basis elements on each side:

$$\begin{array}{ccc} \mathbf{E}[S] \otimes \mathbf{E}[T] \to \mathbf{E}[I] & \qquad & \mathbf{E}[I] \to \mathbf{E}[S] \otimes \mathbf{E}[T] \\ *_S \otimes *_T \mapsto *_I & & *_I \mapsto *_S \otimes *_T. \end{array}$$

The antipode is given by

$$\mathrm{s}(*_I) = (-1)^{|I|} *_I .$$

This can be verified in various ways. One may directly check (8.21)–(8.23). One may also start by noting that for a singleton set $I$, these conditions give $\mathrm{s}(*_I) = -(*_I)$. The fact that the antipode is a morphism of monoids $\mathrm{s}\colon \mathbf{h} \to \mathbf{h}^{\mathrm{op}}$ then gives the general formula.

Finally, it may also be derived from Takeuchi's formula (8.27) by using

$$\sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k = (-1)^{|I|},$$

or equivalently

$$\sum_{\substack{s_1 + \cdots + s_k = n \\ s_i \geq 1 \;\; k \geq 1}} \binom{n}{s_1, \ldots, s_k} (-1)^k = (-1)^n.$$

This is a well-known identity. It provides the reduced Euler characteristic of the Coxeter complex of type $A_{n-1}$, which is a sphere of dimension $n-2$. See Section 10.3 for information on the Coxeter complex.

**Example 8.16.** For the species $\mathbf{L}$ of linear orders, the product and coproduct are:

$$\begin{array}{ccc} \mathbf{L}[S] \otimes \mathbf{L}[T] \to \mathbf{L}[I] & \qquad & \mathbf{L}[I] \to \mathbf{L}[S] \otimes \mathbf{L}[T] \\ l_1 \otimes l_2 \mapsto l_1 \cdot l_2 & & l \mapsto l|_S \otimes l|_T. \end{array}$$

The linear order $l_1 \cdot l_2$ is the *ordinal sum* or *concatenation* of the linear orders $l_1$ and $l_2$: it is the linear order on $I$ whose restrictions to $S$ and $T$ are $l_1$ and $l_2$, and in which the elements of $S$ precede the elements of $T$. In other words, if $l_1 = l_1^1|\cdots|l_1^s$ and $l_2 = l_2^1|\cdots|l_2^t$, then $l_1 \cdot l_2 = l_1^1|\cdots|l_1^s|l_2^1|\cdots|l_2^t$. The linear order $l|_S$ is the restriction of the linear order $l$ on $I$ to the subset $S$. We refer to the coproduct of $\mathbf{L}$ as *deshuffling*.

For example, for the decomposition $I = \{l, a, k\} \sqcup \{s, h, m, i\}$,

$$l|a|k \otimes s|h|m|i \mapsto l|a|k|s|h|m|i, \qquad m|i|k|s|h|l|a \mapsto k|l|a \otimes m|i|s|h.$$

The antipode is given by

$$\mathrm{s}(l) = (-1)^{|I|}\, \overline{l},$$

where $\overline{l}$ is the linear order on $I$ obtained by reversing the linear order $l$. As for $\mathbf{E}$, this can be verified in various ways. One may directly check (8.21)–(8.23). One may also start by noting that for a singleton set $I$, these conditions give $\mathrm{s}(*_I) = -(*_I)$. The fact that the antipode is a morphism of monoids $\mathrm{s}\colon \mathbf{h} \to \mathbf{h}^{\mathrm{op}}$ then gives the general formula. Finally, it may also be derived from Takeuchi's formula; we discuss this below.

Applying Takeuchi's formula (8.27) to $\mathbf{L}$ yields

$$\mathrm{s}(l) = \sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k\, l|_{S_1} \cdots l|_{S_k}.$$

The right-hand side involves concatenations of restrictions of the linear order $l$. To reconcile this with our earlier formula, one has to show that for any pair of linear orders $l$ and $l'$ on $I$, we have

$$\sum_{\substack{(S_1,\dots,S_k)\\ k\geq 1}} (-1)^k = \begin{cases} (-1)^{|I|} & \text{if } l' = \bar{l} \\ 0 & \text{otherwise,} \end{cases} \tag{8.30}$$

where the sum if over all ordered decompositions $I = S_1 \sqcup \cdots \sqcup S_k$ into nonempty subsets for which $l|_{S_1} \cdots l|_{S_k} = l'$. Such decompositions are partially ordered by refinement and in this manner they form a Boolean poset. They are decompositions into intervals of $l'$, and on each interval the orders $l$ and $l'$ must agree. The thinnest one is always the decomposition into singleton intervals. If $l' \neq \bar{l}$, then there are longer intervals of $l'$ on which $l$ and $l'$ agree. The result then follows by inclusion-exclusion.

The map $\pi\colon \mathbf{L} \to \mathbf{E}$ given by

$$\mathbf{L}[I] \xrightarrow{\pi_I} \mathbf{E}[I], \qquad l \mapsto *_I \tag{8.31}$$

for every linear order $l \in \mathbf{L}[I]$, is a morphism of Hopf monoids.

The Hopf monoid $\mathbf{E}$ is both commutative and cocommutative. The Hopf monoid $\mathbf{L}$ is cocommutative but not commutative. Moreover, $\mathbf{E}$ is the *free commutative monoid on one generator* and $\mathbf{L}$ is the *free monoid on one generator*. Here, "one generator" is understood as the unit species $\mathbf{X}$. In this sense, beyond the trivial Hopf monoid $\mathbf{1}$, $\mathbf{E}$ and $\mathbf{L}$ are the simplest examples of Hopf monoids.

The precise meaning of freeness and cofreeness is discussed in detail in Chapter 11. The universal properties of $\mathbf{E}$ and $\mathbf{L}$ are given in Examples 11.11, 11.15 and 11.28.

**Example 8.17.** Consider the Cauchy product of the exponential species with itself,

$$\mathbf{E}^{\cdot 2} := \mathbf{E} \cdot \mathbf{E}.$$

According to (8.6), there is one basis element in $\mathbf{E}^{\cdot 2}[I]$ for each decomposition $I = S \sqcup T$, or equivalently, for each subset $S$ of $I$. For this reason, we refer to $\mathbf{E}^{\cdot 2}$ as the species of subsets, and we regard $\mathbf{E}^{\cdot 2}[I]$ as the vector space with basis the set of all subsets of $I$.

Since the monoidal category $(\mathsf{Sp}, \cdot)$ is symmetric, the Cauchy product of two Hopf monoids is again a Hopf monoid (Section 1.2.7). Therefore, there is a Hopf monoid structure on the species of subsets. Explicitly, the structure maps are

$$\begin{aligned} \mathbf{E}^{\cdot 2}[S] \otimes \mathbf{E}^{\cdot 2}[T] &\to \mathbf{E}^{\cdot 2}[I] \\ S' \otimes T' &\mapsto S' \sqcup T' \end{aligned} \qquad\qquad \begin{aligned} \mathbf{E}^{\cdot 2}[I] &\to \mathbf{E}^{\cdot 2}[S] \otimes \mathbf{E}^{\cdot 2}[T] \\ I' &\mapsto (I' \cap S) \otimes (I' \cap T) \end{aligned}$$

where $S'$, $T'$ and $I'$ denote subsets of $S$, $T$ and $I$, respectively.

The antipode of $\mathbf{E}^{\cdot 2}$ is

$$\mathrm{s}(I') = (-1)^{|I|} I'$$

for any subset $I'$ of $I$.

The Hopf monoid structure on $\mathbf{L}^{\cdot 2}$ can be similarly described in terms of dispositions into two blocks (Section 10.1.3). Higher products $\mathbf{E}^{\cdot k}$ and $\mathbf{L}^{\cdot k}$ and mixed products such as $\mathbf{E} \cdot \mathbf{L}$ yield additional examples of Hopf monoids.

**Example 8.18.** Let $\mathbf{E}_V$ be the species of (8.5). If $B$ is a basis of $V$, then the set of all functions $I \to B$ is a basis of $V^{\otimes I}$. Let $I = S \sqcup T$ be a decomposition. The species $\mathbf{E}_V$ is a Hopf monoid with product

$$\mathbf{E}_V[S] \otimes \mathbf{E}_V[T] \to \mathbf{E}_V[I] \qquad f_1 \otimes f_2 \mapsto f,$$

where $f \in \mathbf{E}_V[I]$ is the function such that $f|_S = f_1$ and $f|_T = f_2$, and coproduct

$$\mathbf{E}_V[I] \to \mathbf{E}_V[S] \otimes \mathbf{E}_V[T] \qquad f \mapsto f|_S \otimes f|_T.$$

The antipode is given by

$$\mathrm{s}_I(f) = (-1)^{|I|} f \qquad \text{for } f \in \mathbf{E}_V[I].$$

Note that $\mathbf{E}_{\Bbbk} = \mathbf{E}$, the Hopf monoid defined in Example 8.15. If $V$ is two-dimensional, then $\mathbf{E}_V \cong \mathbf{E}^{\cdot 2}$, the Hopf monoid of Example 8.17. If $(e_1, e_2)$ is an ordered basis for $V$, then the isomorphism sends $f\colon I \to \{e_1, e_2\}$ to the subset $f^{-1}(\{e_1\})$.

The Hopf monoid $\mathbf{E}_V$ is discussed in [33] and [291, Section 4] and referred to as the *tensor algebra*, in view of the fact that $\mathbf{E}_V[n] = V^{\otimes n}$.

## 8.6. Duality in species

We now discuss the monoidal properties of the duality functor on species along with examples. The analogous discussion for graded vector spaces is given in Section 2.1.4. A general framework for duality is given in Section 3.10; the duality functor on species is an example of the $*$ functor (3.44).

### 8.6.1. The duality functor.

**Definition 8.19.** For a species $\mathbf{q}$, define the *contragredient* or *dual species* $\mathbf{q}^*$ by

$$\mathbf{q}^*[I] := \mathbf{q}[I]^*,$$

where $\mathbf{q}[I]^*$ denotes the dual vector space of $\mathbf{q}[I]$.

A bijection $I \to J$ induces an invertible linear map $\mathbf{q}^*[J] \to \mathbf{q}^*[I]$, whose inverse gives a map $\mathbf{q}^*[I] \to \mathbf{q}^*[J]$. This turns $\mathbf{q}^*$ into a functor $\mathsf{Set}^{\times} \to \mathsf{Vec}$, that is, a species. Moreover, a map $\mathbf{p} \to \mathbf{q}$ of species induces a map $\mathbf{q}^* \to \mathbf{p}^*$ of species. This defines a functor

$$(-)^*\colon \mathsf{Sp}^{\mathrm{op}} \to \mathsf{Sp}$$

called the duality functor.

On finite-dimensional species, the duality functor is an adjoint equivalence. In particular, it is an involution.

For a species $\mathbf{q}$ arising as the linearization of a set species, the vector space $\mathbf{q}[I]$ has a canonical basis. For the dual species $\mathbf{q}^*$, we use superscript $*$ to denote the dual basis elements of $\mathbf{q}[I]^*$. When $\mathbf{q}$ is finite-dimensional, the map $\mathbf{q} \to \mathbf{q}^*$ which sends a basis element to the corresponding dual basis element is an isomorphism of species.

**8.6.2. Interaction with monoidal structures.** In this section we assume that all species are finite-dimensional.

The Hadamard product on species plays a special role in the context of duality. Namely, there are canonical morphisms of species

$$\mathbf{E} \to \mathbf{m} \times \mathbf{m}^* \qquad \text{and} \qquad \mathbf{m}^* \times \mathbf{m} \to \mathbf{E} \tag{8.32}$$

that turn $(\mathsf{Sp}, \times, *)$ into a monoidal category with duals. As a consequence:

**Proposition 8.20.** *For any species* $\mathbf{p}$ *and* $\mathbf{q}$, *there are natural isomorphisms*

$$(\mathbf{p} \times \mathbf{q})^* \cong \mathbf{p}^* \times \mathbf{q}^* \qquad \textit{and} \qquad (\mathbf{p}^*)^* \cong \mathbf{p}.$$

It follows that

$$(-)^* \colon (\mathsf{Sp}^{\mathrm{op}}, \times, \beta^{\mathrm{op}}) \to (\mathsf{Sp}, \times, \beta)$$

is a bistrong monoidal functor.

The duality functor also behaves well with respect to the Cauchy product. We have canonical isomorphisms

$$(\mathbf{p} \cdot \mathbf{q})^* \cong \mathbf{p}^* \cdot \mathbf{q}^*$$

which turn

$$(-)^* \colon (\mathsf{Sp}^{\mathrm{op}}, \cdot, \beta^{\mathrm{op}}) \to (\mathsf{Sp}, \cdot, \beta)$$

into a bistrong monoidal functor. We also employ the terminology that $(-)^*$ is a contravariant bistrong monoidal functor on $(\mathsf{Sp}, \cdot)$. Therefore by Proposition 3.50, $(-)^*$ maps monoids in $(\mathsf{Sp}, \cdot)$ to comonoids in $(\mathsf{Sp}, \cdot)$ and viceversa, and Hopf monoids to Hopf monoids preserving antipodes. If $\mathbf{h}$ is a Hopf monoid, the resulting Hopf monoid $\mathbf{h}^*$ is called the dual of $\mathbf{h}$.

**Definition 8.21.** A Hopf monoid $\mathbf{h}$ in $(\mathsf{Sp}, \cdot)$ is *self-dual* if $\mathbf{h} \cong \mathbf{h}^*$ as Hopf monoids.

We now discuss the duals of the Hopf monoids in Examples 8.15, 8.16 and 8.18.

**Example 8.22.** The Hopf monoid $\mathbf{E}$ (the exponential species) is self-dual. The isomorphism $\mathbf{E} \to \mathbf{E}^*$ is given by

$$\mathbf{E}[I] \to \mathbf{E}^*[I], \qquad *_I \mapsto (*_I)^*,$$

where $*_I$ is the basis element of $\mathbf{E}[I]$. Note that the map does not involve any coefficient; hence self-duality of $\mathbf{E}$ holds over a field of any characteristic.

**Example 8.23.** Let $V$ be a finite-dimensional vector space. The previous example generalizes as follows:

$$(\mathbf{E}_V)^* = \mathbf{E}_{V^*}.$$

This holds in any characteristic. Since $V \cong V^*$, $\mathbf{E}_V$ is self-dual (though noncanonically). If, in addition, $V$ is equipped with a basis, then the self-duality is canonical.

**Example 8.24.** Fix a decomposition $I = S \sqcup T$. The product for the dual $\mathbf{L}^*$ of the Hopf monoid of linear orders is given by

$$\begin{aligned} \mathbf{L}^*[S] \otimes \mathbf{L}^*[T] &\to \mathbf{L}^*[I] \\ l_1^* \otimes l_2^* &\mapsto \sum_{\text{shuffles}} l^*. \end{aligned}$$

The sum is over all *shuffles* $l$ of $l_1$ and $l_2$. These are extensions of the linear orders $l_1$ on $S$ and $l_2$ on $T$ to a linear order $l$ on $I$.

The coproduct is given by

$$\begin{aligned} \mathbf{L}^*[I] &\to \mathbf{L}^*[S] \otimes \mathbf{L}^*[T] \\ l^* &\mapsto \begin{cases} (l|_S)^* \otimes (l|_T)^* & \text{if } S \text{ is an initial segment of } l, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The subset $S$ is an initial segment of $l$ if all its elements precede the elements of $T$ according to $l$. We also say in this case that $T$ is a final segment of $I$. Note that in this case $l$ is the concatenation of $l|_S$ and $l|_T$. We refer to the coproduct of $\mathbf{L}^*$ as *deconcatenation*.

The Hopf monoid $\mathbf{L}^*$ is commutative but not cocommutative.

The dual $\pi^* \colon \mathbf{E}^* \to \mathbf{L}^*$ of the morphism (8.31) is given by

$$\mathbf{E}^*[I] \xrightarrow{\pi_I^*} \mathbf{L}^*[I], \qquad (*_I)^* \mapsto \sum l^*, \tag{8.33}$$

where the sum is over all linear orders on $I$.

One can check that the map

$$\mathbf{L} \to \mathbf{L}^* \qquad\qquad l' \mapsto \sum l^*,$$

where the sum is over all linear orders $l$ on $I$, is a morphism of Hopf monoids. It is clear that on any $I$-component, the image of this map is one-dimensional and hence it is far from being an isomorphism. In fact, it factors through the morphism (8.31) to yield the following commutative diagram of Hopf monoids.

$$\begin{tikzcd} \mathbf{L} \arrow[r] \arrow[d, "\pi"'] & \mathbf{L}^* \\ \mathbf{E} \arrow[r, "\mathrm{id}"'] & \mathbf{E}^* \arrow[u, "\pi^*"'] \end{tikzcd} \tag{8.34}$$

## 8.7. Set species and linearized species

Recall the notion of set species from Definition 8.1. Consider the *linearization functor*

$$\Bbbk(-) \colon \mathsf{Set} \longrightarrow \mathsf{Vec},$$

which sends a set to the vector space with basis the given set. Composing a set species P with the linearization functor gives a (vector) species, which we denote $\Bbbk$P. A *linearized species* is a species $\mathbf{p}$ of the form $\Bbbk$P for some set species P. We will often follow a similar notational convention when dealing with linearized species. For instance, there are set species E and L (defined below) whose linearizations are the species $\mathbf{E}$ and $\mathbf{L}$ of Example 8.3.

In this section we focus on set species and the associated linearized (vector) species; particularly on linearized (co, bi)-monoids in $(\mathsf{Sp}, \cdot)$. Linearized species exhibit certain remarkable features. We highlight a couple of them.

- A construction of Méndez defines a canonical partial order on each set $(\mathrm{E} \cdot \mathrm{M})[I]$ for any linearized connected monoid $\mathbf{m} = \Bbbk\mathrm{M}$. This is reviewed in Section 8.7.6.
- If in addition $\mathbf{m}$ is a comodule-monoid, then $(\mathrm{E} \cdot \mathrm{M})[I]$ carries the structure of a left regular band. This is explained in Section 8.7.7.

Linearized comonoids and bimonoids play a central role in the cohomology theory of species. This is discussed later in Section 9.6.

**8.7.1. Monoidal structures on set species.** In Section 8.1.2 we discussed a number of monoidal structures on the category of (vector) species. The operations (8.6)–(8.8) can also be defined for set species, as follows.

Let $\mathrm{P}$ and $\mathrm{Q}$ be two set species. New species $\mathrm{P}\cdot\mathrm{Q}$, $\mathrm{P}\times\mathrm{Q}$, and $\mathrm{P}\circ\mathrm{Q}$ are defined by

$$(\mathrm{P}\cdot\mathrm{Q})[I] := \coprod_{I=S\sqcup T} \mathrm{P}[S]\times\mathrm{Q}[T], \tag{8.35}$$

$$(\mathrm{P}\times\mathrm{Q})[I] := \mathrm{P}[I]\times\mathrm{Q}[I], \tag{8.36}$$

$$(\mathrm{P}\circ\mathrm{Q})[I] := \coprod_{X\vdash I} \mathrm{P}[X]\times\Bigl(\prod_{S\in X}\mathrm{Q}[S]\Bigr), \tag{8.37}$$

on any finite set $I$. In the right-hand sides, $\times$ denotes the Cartesian product of sets.

We employ the same terminology for these operations as for the analogous operations for (vector) species: Cauchy, Hadamard, and substitution. Similar remarks to those following Definition 8.5 apply in this situation. For instance, the definition of $\mathrm{P}\circ\mathrm{Q}$ applies only when $\mathrm{Q}[\emptyset]=\emptyset$.

Each operation defines a monoidal structure on the category of set species. The unit objects 1, E, and X are defined by

$$1[I] := \begin{cases}\{*\} & \text{if } I \text{ is empty},\\ 0 & \text{otherwise},\end{cases} \qquad \mathrm{E}[I] := \{*\}, \qquad \mathrm{X}[I] := \begin{cases}\{*\} & \text{if } I \text{ is a singleton},\\ 0 & \text{otherwise}.\end{cases}$$

Comparing with Definition 8.5 we see that

$$\Bbbk(\mathbf{p}\cdot\mathbf{q}) \cong \Bbbk\mathbf{p}\cdot\Bbbk\mathbf{q}, \qquad \Bbbk(\mathbf{p}\times\mathbf{q}) \cong \Bbbk\mathbf{p}\times\Bbbk\mathbf{q}, \qquad \Bbbk(\mathbf{p}\circ\mathbf{q}) \cong \Bbbk\mathbf{p}\circ\Bbbk\mathbf{q}. \tag{8.38}$$

Also,

$$\Bbbk 1 = \mathbf{1}, \qquad \Bbbk\mathrm{E} = \mathbf{E}, \qquad \Bbbk\mathrm{X} = \mathbf{X}.$$

Thus, the linearization functor is strong monoidal for each operation.

**Remark 8.25.** The Hadamard product is the categorical product in the category of set species. This follows from the dual of item (iii) in Proposition A.10, since Cartesian product is the categorical product in the category of sets. Explicitly, given set species $\mathrm{P}_1$ and $\mathrm{P}_2$, let

$$p_i\colon \mathrm{P}_1\times\mathrm{P}_2 \to \mathrm{P}_i,\ \ i=1,2,$$

be the morphisms of set species whose $I$-component is the canonical projection $\mathrm{P}_1[I]\times\mathrm{P}_2[I]\to\mathrm{P}_i[I]$. Then, given a set species $\mathrm{Q}$ and morphisms of species

$$f_i\colon \mathrm{Q}\to\mathrm{P}_i,$$

there is a unique morphism

$$f\colon \mathrm{Q}\to\mathrm{P}_1\times\mathrm{P}_2$$

such that

$$\begin{tikzcd} & \mathrm{Q} \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\ \mathrm{P}_1 & \mathrm{P} \arrow[l, "p_1"] \arrow[r, "p_2"'] & \mathrm{P}_2 \end{tikzcd}$$

commutes. The components of $f$ are

$$\mathrm{Q}[I] \to \mathrm{P}_1[I] \times \mathrm{P}_2[I], \qquad x \mapsto \big(f_1(x), f_2(x)\big).$$

Addition of set species is defined by

$$(\mathrm{P}_1 + \mathrm{P}_2)[I] := \mathrm{P}_1[I] \amalg \mathrm{P}_2[I],$$

where $S \amalg T$ denotes the disjoint union of two arbitrary sets $S$ and $T$. Addition is the coproduct in the category of set species. Compare with Remark 8.4.

**8.7.2. Linearized monoids and comonoids.** Let P be a set species and $\mathbf{p} := \Bbbk\mathrm{P}$ its linearization.

Consider the monoidal category of set species under the Cauchy product (8.35) and assume that P is a monoid therein. Then $\mathbf{p}$ acquires a structure of monoid in $(\mathsf{Sp}, \cdot)$ by linearization; in this situation we say that $\mathbf{p}$ is a *linearized monoid*.

Equivalently, $\mathbf{p}$ is a monoid in $(\mathsf{Sp}, \cdot)$ in such a way that the structure maps of $\mathbf{p}$ preserve the basis species P: for any decomposition $I = S \sqcup T$, the maps

$$\mu_{S,T} \colon \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I] \quad \text{and} \quad \iota_{\emptyset} \colon \Bbbk \to \mathbf{p}[\emptyset]$$

are the linearization of maps

$$\mathrm{P}[S] \times \mathrm{P}[T] \to \mathrm{P}[I] \quad \text{and} \quad \{*\} \to \mathrm{P}[\emptyset].$$

Given $x \in \mathrm{P}[S]$ and $y \in \mathrm{P}[T]$, let

$$x \cdot y \in \mathrm{P}[I] \tag{8.39}$$

denote the image of $x \otimes y$ under the multiplication $\mu_{S,T} \colon \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I]$. Also, let $1 \in \mathrm{P}[\emptyset]$ denote the image of $1 \in \Bbbk$ under the unit map $\iota_{\emptyset} \colon \Bbbk \to \mathbf{p}[\emptyset]$ (or equivalently of $*$ under $\{*\} \to \mathrm{P}[\emptyset]$).

The monoid axioms (8.11)–(8.12) then acquire the familiar form

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{8.40}$$

for all decompositions $I = R \sqcup S \sqcup T$ and $x \in \mathrm{P}[R]$, $y \in \mathrm{P}[S]$, $z \in \mathrm{P}[T]$, and

$$x \cdot 1 = x = 1 \cdot x \tag{8.41}$$

for all $x \in \mathrm{P}[I]$.

In particular, $\mathrm{P}[\emptyset]$ is an ordinary monoid, and $\mathbf{p}[\emptyset] = \Bbbk\mathrm{P}[\emptyset]$ is its monoid algebra.

The situation for comonoids is slightly different. There are no nontrivial comonoids in the category of set species, since there are no maps to the unit object other than from species concentrated on the empty set. There is nevertheless a companion notion for comonoids to that of linearized monoids.

We say that a comonoid $\mathbf{p}$ in $(\mathsf{Sp}, \cdot)$ is *linearized* if the species $\mathbf{p}$ is linearized ($\mathbf{p} = \Bbbk\mathrm{P}$) and the structure maps of $\mathbf{p}$ preserve the basis species P. More precisely, for any decomposition $I = S \sqcup T$, the maps

$$\Delta_{S,T} \colon \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T] \quad \text{and} \quad \epsilon_{\emptyset} \colon \mathbf{p}[\emptyset] \to \Bbbk$$

are the linearization of maps

$$\mathrm{P}[I] \to \mathrm{P}[S] \times \mathrm{P}[T] \quad \text{and} \quad \mathrm{P}[\emptyset] \to \{*\}.$$

Given $x \in \mathrm{P}[I]$, we write

$$\Delta_{S,T}(x) =: (x|_S, x/_S). \tag{8.42}$$

We say that $x|_S \in \mathrm{P}[S]$ and $x/_S \in \mathrm{P}[T]$ are the *restriction of $x$ to $S$* and *contraction of $S$ from $x$*, respectively. The terminology is motivated by the example of matroids (Section 13.8).

Coassociativity is equivalent to the identities

$$(x|_{R\sqcup S})|_R = x|_R, \quad (x|_{R\sqcup S})/_R = (x/_R)|_S, \quad x/_{R\sqcup S} = (x/_R)/_S, \tag{8.43}$$

for any decomposition $I = R \sqcup S \sqcup T$ and $x \in \mathrm{P}[I]$. Counitality is equivalent to

$$x|_I = x = x/_\emptyset \tag{8.44}$$

for any $x \in \mathrm{P}[I]$.

In particular, it follows from (8.44) that for $x \in \mathrm{P}[\emptyset]$ we have $\Delta_{\emptyset,\emptyset}(x) = x \otimes x$. Thus, $\mathbf{p}[\emptyset] = \Bbbk\mathrm{P}[\emptyset]$ is the usual coalgebra of a set (Example 3.52).

**8.7.3. Linearized bimonoids.** Continue to assume that $\mathbf{p} = \Bbbk\mathrm{P}$ is a linearized species. We say it is a *linearized bimonoid* if $\mathbf{p}$ is a bimonoid in $(\mathsf{Sp}, \cdot, \beta)$ and the structure maps preserve the basis species P. The compatibility axiom (8.18) takes the following form. Let $I = S \sqcup T = S' \sqcup T'$ be two decompositions, and $A, B, C, D$ the resulting intersections, as in Lemma 8.7. Then we must have

$$x|_A \cdot y|_C = (x \cdot y)|_{S'} \quad \text{and} \quad x/_A \cdot y/_C = (x \cdot y)/_{S'} \tag{8.45}$$

for all $x \in \mathrm{P}[S]$ and $y \in \mathrm{P}[T]$. The unit conditions (8.19)–(8.20) are all automatic; in particular, the compatibility between $\Delta_{\emptyset,\emptyset}$ and $\iota_\emptyset$ follows from (8.44).

A number of examples of bimonoids discussed in this monograph are linearized, including those in Section 8.5. The dual (Section 8.6) of a linearized bimonoid is in general not linearized.

**8.7.4. Linearized comodules.** Let $\mathbf{p} = \Bbbk\mathrm{P}$ be a linearized comonoid. Let $\mathbf{m} = \Bbbk\mathrm{M}$ be a linearized species and assume that $\mathbf{m}$ is a right $\mathbf{p}$-comodule (Section 8.2.3) with structure map

$$\chi\colon \mathbf{m} \to \mathbf{m} \cdot \mathbf{p}.$$

We say that the comodule $\mathbf{m}$ is linearized if for each decomposition $I = S \sqcup T$, the component $\chi_{S,T}$ is the linearization of a map

$$\mathrm{M}[I] \to \mathrm{M}[S] \times \mathrm{P}[T].$$

As in (8.42), we write

$$\chi_{S,T}(x) =: (x|_S, x/_S),$$

with $x|S \in \mathrm{M}[S]$ and $x/S \in \mathrm{P}[T]$. With this notation, coassociativity of $\chi$ is equivalent to the conditions in (8.43) and counitality is equivalent to $x|_I = x$ for all $x \in \mathrm{M}[I]$.

Suppose now that $\mathbf{p}$ is a linearized bimonoid. In this case, the category of right $\mathbf{p}$-comodules is monoidal (Section 1.2.3). Assume that $\mathbf{m}$ is a monoid in this category, and that all its structure is linearized. We say in this case that $\mathbf{m}$ is a *linearized right comodule-monoid*. The compatibility between the monoid and comodule structures of $\mathbf{m}$ is then expressed by the conditions in (8.45).

**8.7.5. Connected linearized species.** Connected species and connected (co, bi)-monoids are discussed later in Section 8.9.1. In the linearized setting, one can say the following.

A linearized species $\mathbf{p} = \Bbbk\mathrm{P}$ is *connected* precisely if $\mathrm{P}[\emptyset]$ is a singleton. If in addition $\mathbf{p}$ is a linearized (co, bi)-monoid, then it is automatically connected as a (co, bi)-monoid.

**8.7.6. A partial order associated to a linearized connected monoid.** We review an interesting construction of Méndez which appears in his thesis [269] and in his work with Yang [271]. The language of this monograph allows us to formulate this construction in simpler terms (in those references, the notion of monoid in species is not used).

Let $\mathbf{m} = \Bbbk\mathrm{M}$ be a linearized monoid. Assume in addition that $\mathbf{m}$ is connected. Let $I$ be a finite set. Consider the set

$$(\mathrm{E}\cdot\mathrm{M})[I] = \coprod_{I=S\sqcup T} \mathrm{E}[S]\times\mathrm{M}[T] = \coprod_{I=S\sqcup T} \mathrm{M}[T].$$

We view the elements of this set as pairs $(x, T)$ where $T \subseteq I$ and $x \in \mathrm{M}[T]$.

We define a partial order on $(\mathrm{E}\cdot\mathrm{M})[I]$ as follows. Given $x_1 \in \mathrm{M}[T_1]$ and $x_2 \in \mathrm{M}[T_2]$, we say that

$$(x_1, T_1) \leq (x_2, T_2) \quad \text{if } T_1 \subseteq T_2 \text{ and there is } y \in \mathrm{M}[T_2 \setminus T_1] \text{ such that } x_1 \cdot y = x_2. \tag{8.46}$$

(We used notation (8.39) for the product of the monoid $\mathbf{m}$.) Briefly,

$$(x_1, T_1) \leq (x_2, T_2) \text{ if } x_1 \text{ is a left divisor of } x_2.$$

This is indeed a partial order. Transitivity follows from (8.40), and antisymmetry from (8.41) plus connectedness.

For the simplest example, consider the exponential species $\mathbf{E} = \Bbbk\mathrm{E}$ (Example 8.15). Since there is only one structure on each finite set, the elements of $(\mathrm{E}\cdot\mathrm{M})[I]$ can be identified with subsets of $I$. The partial order (8.46) is inclusion. Thus, $(\mathrm{E}\cdot\mathrm{M})[I]$ is the Boolean poset $2^I$.

We will not dwell on this topic in this monograph, except for the discussion of a related construction in Section 8.7.7 below. For other examples and applications, see [269, 271].

There is a similar partial order to (8.46) defined in terms of right divisibility. Left and right divisibility coincide if the monoid $\mathbf{m}$ is commutative.

**8.7.7. An LRB associated to a linearized connected comodule-monoid.** A (unital) *left regular band* is an ordinary monoid $(\Sigma, *)$ such that

$$x * y * x = x * y \tag{8.47}$$

for all $x, y \in \Sigma$.

The origin of left regular bands (LRBs) can be traced to Schützenberger [324] and Klein-Barmen [201]. This notion plays a prominent role in the work of Brown on random walks [70, 71]. More information can be found in [154, 294, 295] and [12, Chapter 2].

Suppose now that $\mathbf{p}$ is a linearized bimonoid and $\mathbf{m}$ is a linearized right $\mathbf{p}$-comodule-monoid. We define a product $*$ on the set $(\mathrm{E}\cdot\mathrm{M})[I]$ as follows. Given $x_1 \in \mathrm{M}[T_1]$ and $x_2 \in \mathrm{M}[T_2]$, we set

$$(x_1, T_1) * (x_2, T_2) := \big(x_1 \cdot (x_2|_{T_2\setminus T_1}), T_1 \cup T_2\big). \tag{8.48}$$

Above, we used notations (8.42) for the comodule structure map

$$\mathrm{M}[T_2] \to \mathrm{M}[T_2 \setminus T_1] \times \mathrm{P}[T_1 \cap T_2],$$
$$x_2 \mapsto (x_2|_{T_2\setminus T_1}, x_2/_{T_2\setminus T_1})$$

and (8.39) for the product map

$$\begin{aligned}\mathrm{M}[T_1] \times \mathrm{M}[T_2 \setminus T_1] &\to \mathrm{M}[T_1 \sqcup (T_2 \setminus T_1)] = \mathrm{M}[T_1 \cup T_2],\\ (x_1, x_2|_{T_2\setminus T_1}) &\mapsto x_1 \cdot (x_2|_{T_2\setminus T_1}).\end{aligned}$$

**Proposition 8.26.** *In this situation, $\big((\mathrm{E}\cdot\mathrm{M})[I], *\big)$ is an ordinary monoid with unit element $(1, \emptyset)$. If in addition $\mathbf{m}$ is connected, then $\big((\mathrm{E}\cdot\mathrm{M})[I], *\big)$ is an LRB.*

PROOF. We check associativity. We have

$$\big((x_1, T_1) * (x_2, T_2)\big) * (x_3, T_3) = \big((x_1 \cdot x_2|_{T_2\setminus T_1}) \cdot x_3|_{T_3\setminus T_1\cup T_2}, T_1 \cup T_2 \cup T_3\big).$$

On the other hand,

$$\begin{aligned}(x_1, T_1) * \big((x_2, T_2) * (x_3, T_3)\big) &= (x_1, T_1) * (x_2 \cdot x_3|_{T_3\setminus T_2}, T_2 \cup T_3)\\ &= (x_1 \cdot (x_2 \cdot x_3|_{T_3\setminus T_2})|_{T_2\cup T_3\setminus T_1}, T_1 \cup T_2 \cup T_3).\end{aligned}$$

By (8.45) and (8.43),

$$(x_2 \cdot x_3|_{T_3\setminus T_2})|_{T_2\cup T_3\setminus T_1} = (x_2|_{T_2\setminus T_1}) \cdot (x_3|_{T_3\setminus T_2}|_{T_3\setminus T_1\cup T_2}) = (x_2|_{T_2\setminus T_1}) \cdot (x_3|_{T_3\setminus T_1\cup T_2}).$$

Together with (8.40), this implies associativity as needed.

We check unitality. We have

$$(1, \emptyset) * (x, I) = (1 \cdot x|_I, I) = (x, I)$$

and

$$(x, I) * (1, \emptyset) = (x \cdot 1|_\emptyset, I) = (x, I),$$

in both cases by (8.44) and (8.41).

Finally, we check the LRB axiom (8.47), assuming connectedness. The first calculation above implies

$$(x_1, T_1) * (x_2, T_2) * (x_1, T_1) = (x_1 \cdot x_2|_{T_2\setminus T_1} \cdot x_1|_{T_1\setminus T_1\cup T_2}, T_1 \cup T_2 \cup T_1).$$

But

$$x_1|_{T_1\setminus T_1\cup T_2} = x_1|_\emptyset = 1,$$

since $\mathbf{m}$ is connected. Hence,

$$(x_1, T_1) * (x_2, T_2) * (x_1, T_1) = (x_1 \cdot x_2|_{T_2\setminus T_1}), T_1 \cup T_2) = (x_1, T_1) * (x_2, T_2)$$

as needed. □

Associated to any LRB $(\Sigma, *)$ there is a partial order on the set $\Sigma$ defined by [70, Section 2.2]

$$x \le y \quad \text{if} \quad x * y = y. \tag{8.49}$$

We now check that when applied to the LRB of Proposition 8.26, this construction recovers the partial order of Section 8.7.6.

**Proposition 8.27.** *Suppose $\mathbf{m}$ is a linearized connected right $\mathbf{p}$-comodule-monoid. The partial order (8.49) associated to the LRB $\big((\mathrm{E}\cdot\mathrm{M})[I], *\big)$ coincides with the partial order (8.46) associated to the connected monoid $\mathbf{m}$.*

PROOF. Suppose $(x_1, T_1) \leq (x_2, T_2)$ in (8.49). Then

$$(x_2, T_2) = (x_1, T_1) * (x_2, T_2) = \big(x_1 \cdot (x_2|_{T_2 \setminus T_1}), T_1 \cup T_2\big).$$

Hence,

$$T_1 \subseteq T_2 \quad \text{and} \quad x_2 = x_1 \cdot (x_2|_{T_2 \setminus T_1}).$$

In particular, $x_1$ is a left divisor of $x_2$ and $(x_1, T_1) \leq (x_2, T_2)$ in (8.46).

Conversely, suppose $(x_1, T_1) \leq (x_2, T_2)$ in (8.46). Then $T_1 \subseteq T_2$ and $x_1 \cdot y = x_2$ for some $y \in \mathrm{M}[T_2 \setminus T_1]$. By (8.44),

$$y|_{T_2 \setminus T_1} = y.$$

Hence,

$$(x_1, T_1) * (y, T_2 \setminus T_1) = (x_2, T_2),$$

and by (8.47),

$$(x_1, T_1) * (x_2, T_2) = (x_1, T_1) * (x_1, T_1) * (y, T_2 \setminus T_1) = (x_1, T_1) * (y, T_2 \setminus T_1) = (x_2, T_2).$$

Thus, $(x_1, T_1) \leq (x_2, T_2)$ in (8.49). $\square$

**Example 8.28.** Consider the species of linear orders $\mathbf{L} = \Bbbk\mathrm{L}$. It is a linearized connected bimonoid (Example 8.16). Any bimonoid may be viewed as a (right) comodule-monoid over itself (the comodule structure map is the coproduct) and we do so with $\mathbf{L}$. Let us view a linear order $l = l^1 | \cdots | l^n$ as a word without repeated letters. The associated LRB $\big((\mathrm{E} \cdot \mathrm{L})[I], *\big)$ can be described as follows. Its elements are words whose letters are some of the elements of $I$ and which do not contain repeated letters. Given two such words $l_1$ and $l_2$, their product is obtained by first removing all letters in $l_1$ from $l_2$ and then concatenating the resulting subword of $l_2$ at the end of $l_1$.

It follows that $\big((\mathrm{E} \cdot \mathrm{L})[I], *\big)$ is the *free LRB* on the set $I$ [70, Example 1.3].

We do not pursue this topic any further in this monograph.

**8.7.8. Species with restrictions and linearized comonoids.** A *species with restrictions* is a contravariant functor from the category of finite sets with injections as morphisms to the category of sets.

This terminology is due to Schmitt [322]. Equivalently, a species with restrictions is a presheaf on finite sets. Given a species with restrictions P, a finite set $V$ and a subset $U$, let

$$\rho_{V,U} \colon \mathrm{P}[V] \to \mathrm{P}[U]$$

denote the image under the functor P of the inclusion $U \hookrightarrow V$. We refer to it as a restriction map. These maps satisfy the presheaf axioms

$$\rho_{V,U}\rho_{W,V} = \rho_{W,U} \qquad \text{and} \qquad \rho_{U,U} = \mathrm{id}_U \tag{8.50}$$

for any finite sets $W \supseteq V \supseteq U$.

Since an arbitrary injection equals a bijection followed by an inclusion, a species with restrictions is equivalent to an ordinary species equipped with restriction maps satisfying (8.50).

Let P be a species with restrictions and $\mathbf{p} = \Bbbk\mathrm{P}$ its linearization. Given a decomposition $I = S \sqcup T$, define a linear map

$$\Delta_{S,T} \colon \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T] \qquad \text{by} \qquad x \mapsto \rho_{I,S}(x) \otimes \rho_{I,T}(x)$$

for $x \in \mathrm{P}[I]$. Define $\epsilon_\emptyset \colon \mathbf{p}[\emptyset] \to \Bbbk$ by sending all elements of $\mathrm{P}[\emptyset]$ to 1. With this structure, $\mathbf{p}$ is a linearized comonoid in $(\mathsf{Sp}, \cdot)$. Indeed, the comonoid axioms (8.43) and (8.44) follow from the presheaf axioms (8.50). Moreover, the comonoid $\mathbf{p}$ is cocommutative.

We refer to the above as *Schmitt's comonoid construction* (see Remark 8.34).

Conversely, suppose that $\mathbf{p} = \Bbbk\mathrm{P}$ is a linearized comonoid in $(\mathsf{Sp}, \cdot)$. Then, using the notation (8.42), we may define restriction maps on $\mathbf{p}$ either by

$$\rho^{(1)}_{V,U} \colon \mathbf{p}[V] \to \mathbf{p}[U] \quad \text{by} \quad x \mapsto x|_U,$$

or

$$\rho^{(2)}_{V,U} \colon \mathbf{p}[V] \to \mathbf{p}[U] \quad \text{by} \quad x \mapsto x/_{V\setminus U},$$

for $x \in \mathrm{P}[V]$. Both $\rho^{(1)}$ and $\rho^{(2)}$ turn $\mathbf{p}$ into a species with restrictions: the presheaf axioms (8.50) follow from (8.43) and (8.44).

If the comonoid $\mathbf{p}$ is cocommutative, then $\rho^{(1)} = \rho^{(2)}$. In this case, (8.43) and (8.44) are equivalent to the presheaf axioms (8.50).

**Proposition 8.29.** *The following categories are equivalent.*

(i) *The category of species with restrictions.*
(ii) *The category of linearized cocommutative comonoids.*
(iii) *The category of linearized right* $\mathbf{E}$*-comodules.*
(iv) *The category of linearized left* $\mathbf{E}$*-comodules.*

Proof. The equivalence between (i) and (ii) follows from the preceding discussion. The equivalence between (i) and either (iii) or (iv) follows along similar lines. For instance, starting with a species P with restrictions, we define maps

$$\chi_{S,T} \colon \mathrm{P}[I] \to \mathrm{P}[S] \times \mathrm{E}[T], \qquad x \mapsto \big(\rho_{I,S}(x), *_T\big).$$

These turn $\mathbf{p} = \Bbbk\mathrm{P}$ into a linearized right $\mathbf{E}$-comodule, and any such structure is necessarily of this form. $\square$

A similar discussion leads to the following result.

**Proposition 8.30.** *The following categories are equivalent.*

(i) *The category of linearized comonoids.*
(ii) *The category of linearized* $\mathbf{E}$*-bicomodules.*

**8.7.9. Monoids with restrictions and linearized bimonoids.** Let $\mathsf{Spr}$ denote the category of species with restrictions. The Cauchy product of two species with restrictions is defined as for set species (8.35). It is again a species with restrictions by defining

$$\rho_{V,U} \colon (\mathrm{P}\cdot\mathrm{Q})[V] \to (\mathrm{P}\cdot\mathrm{Q})[U]$$

as follows. For each decomposition $V = S \sqcup T$, on the subset $\mathrm{P}[S]\times\mathrm{Q}[T] \subseteq (\mathrm{P}\cdot\mathrm{Q})[V]$, $\rho_{V,U}$ is the map

$$\mathrm{P}[S] \times \mathrm{Q}[T] \xrightarrow{\rho_{S,S\cap U} \times \rho_{T,T\cap U}} \mathrm{P}[S\cap U] \times \mathrm{Q}[T \cap U] \subseteq (\mathrm{P}\cdot\mathrm{Q})[U].$$

This defines the monoidal category $(\mathsf{Spr}, \cdot)$; the unit object is the set species 1.

A monoid P in $(\mathsf{Spr}, \cdot)$ is a monoid in the category of set species with respect to the Cauchy product, equipped in addition with restriction maps such that for each $U \subseteq V = S \sqcup T$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{P}[S] \times \mathrm{P}[T] \arrow[r, "\rho_{S,S\cap U} \times \rho_{T,T\cap U}"] \arrow[d, "\mu_{S,T}"'] & \mathrm{P}[S \cap U] \times \mathrm{P}[T \cap U] \arrow[d, "\mu_{S\cap U, T\cap U}"] \\
\mathrm{P}[V] \arrow[r, "\rho_{V,U}"'] & \mathrm{P}[U]
\end{tikzcd} \tag{8.51}$$

This generalizes Schmitt's notion of *coherent exponential species with restrictions* [322, Section 3.3].

Let $\mathbf{p}$ be the linearization of a monoid P in $(\mathsf{Spr}, \cdot)$. Since P is a species with restrictions, $\mathbf{p}$ is a linearized cocommutative comonoid. Since a monoid in set species with respect to the Cauchy product is the same as a linearized monoid, $\mathbf{p}$ is a linearized monoid. In addition, the commutativity of (8.51) is equivalent to the compatibility axioms (8.45). We thus have the first part of the following result.

**Proposition 8.31.** *The following categories are equivalent.*

(i) *The category of monoids in* $(\mathsf{Spr}, \cdot)$.
(ii) *The category of linearized cocommutative bimonoids.*
(iii) *The category of linearized right* $\mathbf{E}$*-comodule-monoids.*
(iv) *The category of linearized left* $\mathbf{E}$*-comodule-monoids.*

Proof. The equivalence between (i) and (ii) follows from the preceding discussion, and the remaining follow as in Proposition 8.29. □

Similarly, we have the companion result to Proposition 8.30.

**Proposition 8.32.** *The following categories are equivalent.*

(i) *The category of linearized bimonoids.*
(ii) *The category of linearized* $\mathbf{E}$*-bicomodule-monoids.*

**Example 8.33.** We know that $\mathbf{E} = \Bbbk\mathrm{E}$ and $\mathbf{L} = \Bbbk\mathrm{L}$ are linearized cocommutative bimonoids. Hence by Proposition 8.31, E and L are monoids in $(\mathsf{Spr}, \cdot)$. The restriction maps are:

$$\rho_{V,U} \colon \mathrm{E}[V] \xrightarrow{\cong} \mathrm{E}[U] \qquad *_V \mapsto *_U \qquad \text{and} \qquad \rho_{V,U} \colon \mathrm{L}[V] \to \mathrm{L}[U] \qquad l \mapsto l|_U,$$

where $l|_U$ denotes the restriction of a linear order $l$ on $V$ to the subset $U$. One can check directly that the products on E and L satisfy (8.51).

**Remark 8.34.** In [322, Section 3], Schmitt gave an interesting construction of coalgebras and bialgebras from certain species. Our constructions in Propositions 8.29 and 8.31 are a reformulation of Schmitt's. The link to the original construction is made later in Section 17.5.6.

## 8.8. Bimonoids as bilax monoidal functors

We now turn to some general properties of bimonoids in $(\mathsf{Sp}, \cdot)$. They are simple but somewhat surprising in that they are specific to species; there are no analogous properties satisfied by graded bialgebras. They are based on a fruitful observation made to us by Chase. It involves the notion of bilax monoidal functors, which is discussed in Section 3.1.

**8.8.1. Bimonoids as bilax monoidal functors.** View the category $\mathsf{Set}^{\times}$ of finite sets and bijections as a monoidal category under disjoint union. We use the notation $S \amalg T$ to distinguish it from $S \sqcup T$ which is used when $S$ and $T$ are given to us as disjoint subsets of a given set. View the category $\mathsf{Vec}$ of vector spaces as a monoidal category under the ordinary tensor product $\otimes$. Both categories are symmetric.

Let $\mathbf{p}$ be a bimonoid in $(\mathsf{Sp}, \cdot)$ with structure maps $\mu$ and $\Delta$. The species $\mathbf{p}$ is a functor $\mathsf{Set}^{\times} \to \mathsf{Vec}$ and we may use the components of $\mu$ and $\Delta$ to define natural transformations

$$\mathbf{p}[S] \otimes \mathbf{p}[T] \xrightarrow{\varphi_{S,T} := \mu_{S,T}} \mathbf{p}[S \amalg T] \quad \text{and} \quad \mathbf{p}[S \amalg T] \xrightarrow{\psi_{S,T} := \Delta_{S,T}} \mathbf{p}[S] \otimes \mathbf{p}[T].$$

Here we use (8.9) and (8.16) with $I = S \amalg T$ and we view $S$ and $T$ as disjoint subsets of $I$. Similarly, we use $\iota$ and $\epsilon$ to define

$$\Bbbk \xrightarrow{\varphi_0 := \iota_\emptyset} \mathbf{p}[\emptyset] \qquad \text{and} \qquad \mathbf{p}[\emptyset] \xrightarrow{\psi_0 := \epsilon_\emptyset} \Bbbk.$$

The explicit description of the monoid axioms for $(\mathbf{p}, \mu, \iota)$ given in Section 8.2.1 shows that they are equivalent to the statement that $(\mathbf{p}, \varphi, \varphi_0) \colon (\mathsf{Set}^{\times}, \amalg) \to (\mathsf{Vec}, \otimes)$ is a lax monoidal functor. Similarly, the comonoid axioms for $(\mathbf{p}, \Delta, \epsilon)$ are equivalent to the statement that $(\mathbf{p}, \psi, \psi_0)$ is a colax monoidal functor $(\mathsf{Set}^{\times}, \amalg) \to (\mathsf{Vec}, \otimes)$. The work in Section 8.3.1 shows that the corresponding statement for bimonoids and bilax monoidal functors is also true (though less evident; see Remark 8.36 below). We state this next.

**Proposition 8.35 (Chase).** *Let $\mathbf{p} \colon \mathsf{Set}^{\times} \to \mathsf{Vec}$ be a species. The above constructions define equivalences between monoid (comonoid, bimonoid) structures in $(\mathsf{Sp}, \cdot)$ on the species $\mathbf{p}$ and lax (colax, bilax) monoidal structures on the functor*

$$\mathbf{p} \colon (\mathsf{Set}^{\times}, \amalg) \to (\mathsf{Vec}, \otimes). \tag{8.52}$$

*Moreover, the bimonoid is connected if and only if the corresponding bilax monoidal functor is normal.*

**Remark 8.36.** There are similar equivalences between graded algebras (coalgebras) and certain lax (colax) monoidal functors, but not for graded bialgebras. This is explained in detail in Example 3.18 and corresponds to the different behavior between graded bialgebras and bimonoids in species discussed in Remark 8.8. Ultimately, the reason for the failure for graded bialgebras is that Lemma 8.7 does not hold if one replaces sets by numbers.

Let $\mathrm{P}$ be a set species and $\mathbf{p} = \Bbbk\mathrm{P}$ its linearization. Recall from Section 8.7.2 that $\mathbf{p}$ is a linearized monoid if and only if $\mathrm{P}$ is a monoid in the monoidal category of set species. On the other hand, the analogous statements for linearized comonoids and linearized bimonoids do not hold.

There is an analogue of Proposition 8.35 for set species which allows us to view linearized (co, bi)monoids in a uniform manner and without reference to vector spaces. We view $\mathsf{Set}^{\times}$ as a monoidal category under disjoint union, as above, and $\mathsf{Set}$ as a monoidal category under Cartesian product.

**Proposition 8.37.** *Let $\mathrm{P}$ be a set species. Then $\mathbf{p}$ is a linearized monoid (comonoid, bimonoid) if and only if*

$$\mathrm{P} \colon (\mathsf{Set}^{\times}, \amalg) \to (\mathsf{Set}, \times)$$

*is a lax (colax, bilax) monoidal functor. Moreover, the linearized bimonoid is connected if and only if the corresponding bilax monoidal functor is normal.*

The proof is as for Proposition 8.35.

**8.8.2. Properties of bimonoids.** We can now use general results on bilax monoidal functors to deduce properties of bimonoids in species that do not have a parallel for graded bialgebras.

**Corollary 8.38.** *Let* $\mathbf{p}$ *be a connected bimonoid in* $(\mathsf{Sp}, \cdot)$*. The following diagrams involving the components of the product and coproduct of* $\mathbf{p}$ *are commutative.*

(i) *For any decomposition* $I = S \sqcup T$,

$$\begin{tikzcd}[column sep=small] & \mathbf{p}[I] \arrow[dr, "\Delta_{S,T}"] & \\ \mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[ur, "\mu_{S,T}"] \arrow[rr, "\mathrm{id}_S\otimes\mathrm{id}_T"'] & & \mathbf{p}[S]\otimes\mathbf{p}[T] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathbf{p}[I] \arrow[dr, "\Delta_{T,S}"] & \\ \mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[ur, "\mu_{S,T}"] \arrow[rr, "\beta_{S,T}"'] & & \mathbf{p}[T]\otimes\mathbf{p}[S] \end{tikzcd}$$

(ii) *For any decomposition* $I = R \sqcup S \sqcup T$,

$$\begin{tikzcd} \mathbf{p}[R]\otimes\mathbf{p}[S\sqcup T] \arrow[r, "\mathrm{id}_R\otimes\Delta_{S,T}"] \arrow[d, "\mu_{R,S\sqcup T}"'] & \mathbf{p}[R]\otimes\mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[d, "\mu_{R,S}\otimes\mathrm{id}_T"] \\ \mathbf{p}[R\sqcup S\sqcup T] \arrow[r, "\Delta_{R\sqcup S,T}"'] & \mathbf{p}[R\sqcup S]\otimes\mathbf{p}[T] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{p}[R\sqcup S]\otimes\mathbf{p}[T] \arrow[r, "\Delta_{R,S}\otimes\mathrm{id}_T"] \arrow[d, "\mu_{R\sqcup S,T}"'] & \mathbf{p}[R]\otimes\mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[d, "\mathrm{id}_R\otimes\mu_{S,T}"] \\ \mathbf{p}[R\sqcup S\sqcup T] \arrow[r, "\Delta_{R,S\sqcup T}"'] & \mathbf{p}[R]\otimes\mathbf{p}[S\sqcup T] \end{tikzcd}$$

PROOF. This is Proposition 3.41 applied to the bilax monoidal functor (8.52). This functor is normal because $\mathbf{p}$ is connected (Proposition 8.35). □

The diagrams in Corollary 8.38 can also be deduced directly from those in Sections 8.2.1 and 8.3.1 expressing the definition of monoid, comonoid and bimonoid in terms of components. For example, for the proof of the first diagram in (i), we set $B = C = \emptyset$ in (8.18) and proceed from there. This is essentially unwinding the proof of Proposition 3.41.

**Example 8.39.** For the bimonoid $\mathbf{L}$ of linear orders (Example 8.16), the properties of Corollary 8.38 boil down to the following relations between concatenation and restriction of linear orders:

$$(l_1 \cdot l_2)|_S = l_1, \qquad (l_1 \cdot l_2)|_T = l_2$$

for $l_1 \in \mathbf{L}[S]$ and $l_2 \in \mathbf{L}[T]$, and more generally

$$(l_1 \cdot l_2)|_{R\sqcup S} = l_1 \cdot (l_2|_S), \qquad (l_1 \cdot l_2)|_T = l_2|_T$$

for $l_1 \in \mathbf{L}[R]$ and $l_2 \in \mathbf{L}[S \sqcup T]$, and

$$(l_1 \cdot l_2)|_R = l_1|_R, \qquad (l_1 \cdot l_2)|_{S\sqcup T} = (l_1|_S) \cdot l_2$$

for $l_1 \in \mathbf{L}[R \sqcup S]$ and $l_2 \in \mathbf{L}[T]$.

The reader may enjoy verifying the properties of Corollary 8.38 for the other (connected) bimonoids in Chapters 12 and 13.

**8.8.3. Hopf monoids and Hopf lax functors.** Let $\mathbf{h}$ be a bimonoid in $(\mathsf{Sp}, \cdot)$ and

$$\mathbf{h}\colon (\mathsf{Set}^{\times}, \amalg) \to (\mathsf{Vec}, \otimes)$$

be the corresponding bilax monoidal functor. For each finite set $I$, let $\mathrm{s}_I\colon \mathbf{h}[I] \to \mathbf{h}[I]$ be a linear map, commuting with bijections.

The condition for $\mathbf{h}$ to be a Hopf monoid with antipode $\mathrm{s}$ is different from the one for $\mathbf{h}$ to be a Hopf lax functor with antipode $\mathrm{s}$ (Definition 3.54). To see this, fix a decomposition $(R, S, T)$ of $I$. Then the composite map

$$\begin{tikzcd}
\mathbf{h}[R]\otimes\mathbf{h}[S]\otimes\mathbf{h}[T] \arrow[r, "\mathrm{id}_R\otimes \mathrm{s}_S\otimes \mathrm{id}_T"] & \mathbf{h}[R]\otimes\mathbf{h}[S]\otimes\mathbf{h}[T] \arrow[d, "\mu_{R,S,T}"] \\
\mathbf{h}[I] \arrow[u, "\Delta_{R,S,T}"] & \mathbf{h}[I]
\end{tikzcd}$$

is identity in the latter case by axiom (3.29). In the former case however, it is the identity only after summing over all decompositions $(R, S, T)$ of $I$.

**Proposition 8.40.** *Under the correspondence in Proposition 8.35, a Hopf lax functor gives rise to a Hopf monoid.*

PROOF. If $\mathbf{h}$ is Hopf lax, then (3.32) implies that $\mathbf{h}[\emptyset]$ is a Hopf algebra (since $\emptyset$ is the unit object in $(\mathsf{Set}^{\times}, \amalg)$). Hence by Proposition 8.10, it follows that $\mathbf{h}$ is a Hopf monoid (Of course, the two antipodes will be different). $\square$

As an example, $\mathbf{E}$ is a Hopf lax functor with antipode $\mathrm{s}(*_I) = *_I$. Hence it is also a Hopf monoid but the antipode is given by $\mathrm{s}(*_I) = (-1)^{|I|} *_I$.

The converse to Proposition 8.40 does not hold. Indeed, let $\mathbf{h}$ be any Hopf monoid for which $\Delta_{R,S,T}$ is not injective (this holds for most of our examples; take $\mathbf{L}$ for a concrete example). Then the composite in the above diagram cannot be the identity and hence $\mathbf{h}$ is not Hopf lax.

## 8.9. Connected and positive species

This section collects a number of simple but technical observations about the categories of connected and positive species and monoids and Hopf monoids therein. The concept of bistrong monoidal functors (Section 3.6) is relevant to this discussion.

**8.9.1. Connected species.** A species $\mathbf{q}$ is *connected* if there is a specified isomorphism

$$\mathbf{q}[\emptyset] \xrightarrow{\cong} \Bbbk. \tag{8.53}$$

*Warning.* Our use of the term connected differs from the one by Fresse [137, Section 1.2.1].

A morphism of connected species is a morphism of species $\mathbf{q}_1 \to \mathbf{q}_2$ such that

$$\begin{tikzcd} \mathbf{q}_1[\emptyset] \arrow[rr] \arrow[dr, "\cong"'] & & \mathbf{q}_2[\emptyset] \arrow[dl, "\cong"] \\ & \Bbbk & \end{tikzcd}$$

commutes. Let $\mathsf{Sp}^{\mathsf{o}}$ be the category of connected species. If $\mathbf{p}$ and $\mathbf{q}$ are connected, then so is $\mathbf{p} \cdot \mathbf{q}$, with (8.53) given by

$$(\mathbf{p} \cdot \mathbf{q})[\emptyset] = \mathbf{p}[\emptyset] \otimes \mathbf{q}[\emptyset] \xrightarrow{\cong} \Bbbk \otimes \Bbbk \cong \Bbbk.$$

Further, the species $\mathbf{1}$ is connected in the obvious manner. This yields a monoidal category $(\mathsf{Sp}^{\mathsf{o}}, \cdot)$.

**Definition 8.41.** A *connected monoid* (*comonoid*, *bimonoid*) is a monoid (comonoid, bimonoid) in $(\mathsf{Sp}^{\mathsf{o}}, \cdot)$.

It follows that a connected (co)monoid is the same as a (co)monoid in species whose (co)unit map is an isomorphism. For a connected monoid, the isomorphism (8.53) coincides with $\iota_\emptyset^{-1}$, and for a connected comonoid with $\epsilon_\emptyset$. For a connected bimonoid, it coincides with both $\epsilon_\emptyset$ and $\iota_\emptyset^{-1}$. This shows that the above definition of connected bimonoid agrees with Definition 8.12.

There is a functor $\mathsf{Sp}^{\mathsf{o}} \to \mathsf{Sp}$ which forgets the isomorphism (8.53). It is bistrong with respect to the Cauchy product and hence preserves monoids, comonoids, and Hopf monoids. We denote the induced functors on these categories by *inc*. As mentioned above, on these categories the isomorphism (8.53) is the counit of the comonoid (or the inverse of the unit of the monoid). Therefore, *inc* identifies the category of connected monoids (comonoids, Hopf monoids) with a full subcategory of the category of monoids (comonoids, Hopf monoids).

Consider the functor $(-)^{\mathrm{o}} \colon \mathsf{Sp} \to \mathsf{Sp}^{\mathsf{o}}$ which sends $\mathbf{q}$ to $\mathbf{q}^{\mathrm{o}}$ where

$$\mathbf{q}^{\mathrm{o}}[I] := \begin{cases} \Bbbk & \text{if } I = \emptyset, \\ \mathbf{q}[I] & \text{otherwise,} \end{cases} \tag{8.54}$$

with the isomorphism (8.53) being the identity. This functor preserves monoids, comonoids, and Hopf monoids. For instance, if $\mathbf{q}$ is a monoid in $(\mathsf{Sp}, \cdot)$, then the components of the product of $\mathbf{q}^{\mathrm{o}}$ are: If $S$ and $T$ are nonempty, then $\mu^{\mathrm{o}}_{S,T} := \mu_{S,T}$, while $\mu^{\mathrm{o}}_{\emptyset,I}$ and $\mu^{\mathrm{o}}_{I,\emptyset}$ are the composites

$$\Bbbk \otimes \mathbf{q}[I] \xrightarrow{\iota_\emptyset \otimes \mathrm{id}_I} \mathbf{q}[\emptyset] \otimes \mathbf{q}[I] \xrightarrow{\mu_{\emptyset,I}} \mathbf{q}[I] \quad \text{and} \quad \mathbf{q}[I] \otimes \Bbbk \xrightarrow{\mathrm{id}_I \otimes \iota_\emptyset} \mathbf{q}[I] \otimes \mathbf{q}[\emptyset] \xrightarrow{\mu_{I,\emptyset}} \mathbf{q}[I].$$

The functors $(-)^{\mathrm{o}}$ and *inc* are not adjoint. However, the functors they induce at the level of monoids, comonoids, and Hopf monoids are adjoint, as indicated by the following diagrams, in which the functors above the arrows are left adjoint to the functors below the arrows:

$$\begin{tikzcd} \mathsf{Mon}(\mathsf{Sp}^{\mathsf{o}}, \cdot) \arrow[r, bend left, "inc"] & \mathsf{Mon}(\mathsf{Sp}, \cdot), \arrow[l, bend left, "(-)^{\mathrm{o}}"] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Comon}(\mathsf{Sp}, \cdot) \arrow[r, bend left, "(-)^{\mathrm{o}}"] & \mathsf{Comon}(\mathsf{Sp}^{\mathsf{o}}, \cdot), \arrow[l, bend left, "inc"] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}}, \cdot, \beta) \arrow[r, bend left, "inc"] & \mathsf{Hopf}(\mathsf{Sp}, \cdot, \beta), \arrow[l, bend left, "(-)^{\mathrm{o}}"] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Hopf}(\mathsf{Sp}, \cdot, \beta) \arrow[r, bend left, "(-)^{\mathrm{o}}"] & \mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}}, \cdot, \beta). \arrow[l, bend left, "inc"] \end{tikzcd}$$

The verification of the adjunctions is straightforward.

**8.9.2. Positive species.** A species $\mathbf{q}$ is *positive* if it satisfies $\mathbf{q}[\emptyset] = 0$. Let $\mathsf{Sp}_+$ be the category of positive species. It is a full subcategory of $\mathsf{Sp}$. The Cauchy product of two positive species is again positive; however, $\mathbf{1}$ is not a positive species. Thus, the category $(\mathsf{Sp}_+, \cdot)$ is a monoidal subcategory of $(\mathsf{Sp}, \cdot)$, but without a unit object. This only allows us to define nonunital monoids and noncounital comonoids in $(\mathsf{Sp}_+, \cdot)$ (Section 1.2.1).

The following standard construction allows us to transform $(\mathsf{Sp}_+, \cdot)$ into a unital monoidal category. Let

$$\mathbf{p} \odot \mathbf{q} := \mathbf{p} \cdot \mathbf{q} + \mathbf{p} + \mathbf{q}. \tag{8.55}$$

We call this the *modified Cauchy product.* Then $(\mathsf{Sp}_+, \odot, \beta)$ is a symmetric monoidal category, with the zero species as the unit object.

The identity functor induces equivalences of categories

$$\mathsf{Mon}(\mathsf{Sp}_+, \odot) \cong \mathsf{Mon}(\mathsf{Sp}_+, \cdot) \qquad \text{and} \qquad \mathsf{Comon}(\mathsf{Sp}_+, \odot) \cong \mathsf{Comon}(\mathsf{Sp}_+, \cdot),$$

where $\mathsf{Mon}(\mathsf{Sp}_+, \odot)$ stands for the category of unital monoids in $(\mathsf{Sp}_+, \odot)$ while $\mathsf{Mon}(\mathsf{Sp}_+, \cdot)$ stands for the category of nonunital monoids in $(\mathsf{Sp}_+, \cdot)$, and similarly for comonoids. In view of the above equivalences, we define:

**Definition 8.42.** A *positive (co)monoid* is a (co)monoid in $(\mathsf{Sp}_+, \odot)$, or equivalently, a non(co)unital (co)monoid in $(\mathsf{Sp}_+, \cdot)$.

A *positive bimonoid (Hopf monoid)* is a bimonoid (Hopf monoid) in $(\mathsf{Sp}_+, \odot, \beta)$.

We write $\mathsf{Mon}(\mathsf{Sp}_+)$ and $\mathsf{Comon}(\mathsf{Sp}_+)$ for the category of positive monoids and positive comonoids. If the context requires us to be more specific about the viewpoint, then we either provide the tensor product $\cdot$ or $\odot$ or say in words that we are dealing with the Cauchy or modified Cauchy product, as may be the case.

Consider the functor $(-)_+ \colon \mathsf{Sp} \to \mathsf{Sp}_+$ which sends $\mathbf{q}$ to $\mathbf{q}_+$ where

$$\mathbf{q}_+[I] := \begin{cases} 0 & \text{if } I = \emptyset, \\ \mathbf{q}[I] & \text{otherwise.} \end{cases} \tag{8.56}$$

Let *inc* denote the inclusion functor $\mathsf{Sp}_+ \to \mathsf{Sp}$. The following are pairs of adjoint functors, the functors above the arrows being left adjoints to the functors below the arrows:

$$\begin{tikzcd} \mathsf{Sp}_+ \arrow[r, bend left, "inc"] & \mathsf{Sp} \arrow[l, bend left, "(-)_+"] \arrow[r, bend left, "(-)_+"] & \mathsf{Sp}_+ . \arrow[l, bend left, "inc"] \end{tikzcd} \tag{8.57}$$

In other words, $(-)_+$ and *inc* are adjoint on both sides.

The functor $(-)_+$ preserves monoids and comonoids. For instance, if $\mathbf{h}$ is a comonoid in $\mathsf{Sp}$ with coproduct $\Delta$, then $\mathbf{h}_+$ is a noncounital comonoid in $\mathsf{Sp}_+$ with coproduct

$$\Delta_+ \colon \mathbf{h}_+ \to \mathbf{h}_+ \cdot \mathbf{h}_+ \tag{8.58}$$

being the morphism of species whose components are the components $\Delta_{S,T}$ of the coproduct $\Delta$ of $\mathbf{h}$, where $S$ and $T$ are nonempty. We refer to $\Delta_+$ as the *positive part* of the coproduct.

We may view this construction more formally as follows. First, note that for any nonempty finite set $I$ we have

$$(\mathbf{p}\cdot\mathbf{q})_+[I] = (\mathbf{p}_+\cdot\mathbf{q}_+)[I] \oplus (\mathbf{p}_+[I]\otimes\mathbf{q}[\emptyset]) \oplus (\mathbf{p}[\emptyset]\otimes\mathbf{q}_+[I]). \tag{8.59}$$

This gives rise to two canonical maps, one from $(\mathbf{p}\cdot\mathbf{q})_+[I]$ to $(\mathbf{p}_+\cdot\mathbf{q}_+)[I]$ and the other one back (the projection and the inclusion). These maps turn

$$(-)_+ \colon (\mathsf{Sp},\cdot) \to (\mathsf{Sp}_+,\cdot)$$

into a bilax monoidal functor, so the (nonunital version of the) results of Section 3.4.3 can be applied. Here we view both monoidal categories $(\mathsf{Sp},\cdot)$ and $(\mathsf{Sp}_+,\cdot)$ as nonunital.

The functor

$$\mathit{inc}\colon (\mathsf{Sp}_+,\cdot) \to (\mathsf{Sp},\cdot)$$

is strong monoidal. Proposition 3.94 then implies the existence of a lax and a colax structure on its two-sided adjoint functor $(-)_+$. It is easy to see that these coincide with the structures of the preceding paragraph.

**8.9.3. Interaction between connected and positive species.** Let $\mathsf{Sp}_+ \to \mathsf{Sp}^\circ$ be the functor obtained by restricting the functor $(-)^\circ$ of Section 8.9.1 to the subcategory $\mathsf{Sp}_+$. This restricted functor is also denoted $(-)^\circ$. Similarly, we restrict the functor $(-)_+$ of Section 8.9.2 to $\mathsf{Sp}^\circ$ and obtain $(-)_+\colon \mathsf{Sp}^\circ \to \mathsf{Sp}_+$.

**Proposition 8.43.** *The following is a bistrong adjoint equivalence.*

$$\begin{tikzcd} (\mathsf{Sp}^\circ,\cdot) \arrow[r, bend left, "(-)_+"] & (\mathsf{Sp}_+,\odot) \arrow[l, bend left, "(-)^\circ"] \end{tikzcd}$$

Proof. First observe that the above is an adjoint equivalence. Now let $\mathbf{p}$ and $\mathbf{q}$ be connected species. In this case, (8.59) shows that there is a natural isomorphism

$$(\mathbf{p}\cdot\mathbf{q})_+ \cong \mathbf{p}_+\odot\mathbf{q}_+. \tag{8.60}$$

Similarly, for positive species $\mathbf{p}$ and $\mathbf{q}$, there is a natural isomorphism

$$(\mathbf{1}+\mathbf{p})\cdot(\mathbf{1}+\mathbf{q}) \cong \mathbf{1}+\mathbf{p}\odot\mathbf{q}.$$

It follows that $(-)_+$ and $(-)^\circ$ are bistrong monoidal functors. One checks that the unit and counit of the adjunction are isomorphisms of bistrong functors and the result follows. □

The above functors induce equivalences of categories as follows:

$$\begin{gathered} \mathsf{Mon}(\mathsf{Sp}^\circ,\cdot) \cong \mathsf{Mon}(\mathsf{Sp}_+,\odot),\\ \mathsf{Comon}(\mathsf{Sp}^\circ,\cdot) \cong \mathsf{Comon}(\mathsf{Sp}_+,\odot),\\ \mathsf{Bimon}(\mathsf{Sp}^\circ,\cdot,\beta) \cong \mathsf{Bimon}(\mathsf{Sp}_+,\odot,\beta),\\ \mathsf{Hopf}(\mathsf{Sp}^\circ,\cdot,\beta) \cong \mathsf{Hopf}(\mathsf{Sp}_+,\odot,\beta). \end{gathered} \tag{8.61}$$

In particular, we have:

**Proposition 8.44.** *A connected (co, bi, Hopf) monoid is equivalent to a positive (co, bi, Hopf) monoid.*

## 8.10. Primitive elements and the coradical filtration

Associated to any positive comonoid there is a canonical filtration. Primitive elements form the first step in this filtration.

Let $(\mathbf{q}, \Delta)$ be a positive comonoid. The *primitive elements* of $\mathbf{q}$ is the positive species defined by

$$\mathcal{P}(\mathbf{q}) := \ker \Delta.$$

The functor $\mathcal{P}$ is left adjoint to the functor *inc* which views a positive species as a positive comonoid with zero coproduct:

$$\begin{tikzcd} \mathsf{Sp}_+ \arrow[r, bend left, "inc"] & \mathsf{Comon}(\mathsf{Sp}_+). \arrow[l, bend left, "\mathcal{P}"] \end{tikzcd}$$

Consider more generally the positive species

$$\mathcal{P}^{(k)}(\mathbf{q}) := \ker \Delta^{(k)},$$

for $k \geq 1$, where $\Delta^{(k)}$ is the $k$-fold iteration of the coproduct. This is the $k$-th term of the *coradical filtration* of $\mathbf{q}$. Each $\mathcal{P}^{(k)}(\mathbf{q})$ is by definition a subspecies of $\mathbf{q}$.

**Proposition 8.45.** *For any positive comonoid* $\mathbf{q}$, *we have*

$$\mathcal{P}^{(1)}(\mathbf{q}) \subseteq \mathcal{P}^{(2)}(\mathbf{q}) \subseteq \cdots \subseteq \mathbf{q}, \tag{8.62}$$

$$\bigcup_{k\geq 1} \mathcal{P}^{(k)}(\mathbf{q}) = \mathbf{q}, \tag{8.63}$$

$$\Delta^{(k-1)}\big(\mathcal{P}^{(k)}(\mathbf{q})\big) \subseteq \mathcal{P}(\mathbf{q})^{\cdot k}. \tag{8.64}$$

Proof. Coassociativity implies (8.62). Since $\mathbf{q}$ is positive, the only components $\Delta_{S,T}$ of the coproduct which may be nonzero are those for which $S$ and $T$ are nonempty. It follows that $\Delta^{(k-1)}$ vanishes on $\mathbf{q}[I]$ as soon as $k > |I|$, and (8.63) follows.

By coassociativity,

$$\Big(\mathrm{id}^{\cdot(i-1)} \cdot \Delta \cdot \mathrm{id}^{\cdot(k-i)}\Big)\, \Delta^{(k-1)}(\mathcal{P}^{(k)}(\mathbf{q})) = \Delta^{(k)}\big(\mathcal{P}^{(k)}(\mathbf{q})\big) = 0,$$

for $1 \leq i \leq k$. Thus,

$$\Delta^{(k-1)}(\mathcal{P}^{(k)}(\mathbf{q})) \subseteq \bigcap_{i=1}^{k} \ker \Big(\mathrm{id}^{\cdot(i-1)} \cdot \Delta \cdot \mathrm{id}^{\cdot(k-i)}\Big) = \bigcap_{i=1}^{k} \mathbf{q}^{\cdot(i-1)} \cdot \mathcal{P}(\mathbf{q}) \cdot \mathbf{q}^{\cdot(k-i)} = \mathcal{P}(\mathbf{q})^{\cdot k}$$

as needed. □

The following is a useful result.

**Proposition 8.46.** *Let* $f\colon \mathbf{q} \to \mathbf{p}$ *be a morphism of positive comonoids. Then*

$$f\big(\mathcal{P}^{(k)}(\mathbf{q})\big) \subseteq \mathcal{P}^{(k)}(\mathbf{p}).$$

*In addition, if the restriction* $f\colon \mathcal{P}(\mathbf{q}) \to \mathcal{P}(\mathbf{p})$ *is injective, then* $f\colon \mathbf{q} \to \mathbf{p}$ *is injective.*

Proof. The first assertion holds since $f$ commutes with $\Delta^{(k)}$. For the second, we proceed by induction. Choose $z \in \ker(f)$. By (8.63), there exists $k \geq 1$ such that $z \in \mathcal{P}^{(k)}(\mathbf{q})$. If $k = 1$, then $z = 0$ by hypothesis. Suppose $k \geq 2$. We have

$$f^{\cdot k}\big(\Delta^{(k-1)}(z)\big) = \Delta^{(k-1)}\left(f(z)\right) = 0.$$

Thus,

$$\Delta^{(k-1)}(z) \in \ker(f^{\cdot k}) = \sum_{i=1}^{k} \mathbf{q}^{\cdot(i-1)} \cdot \ker(f) \cdot \mathbf{q}^{\cdot(k-i)}.$$

Combining this with (8.64) we obtain

$$\Delta^{(k-1)}(z) \in \sum_{i=1}^{k} \mathcal{P}(\mathbf{q})^{\cdot(i-1)} \cdot \big(\ker(f) \cap \mathcal{P}(\mathbf{q})\big) \cdot \mathcal{P}(\mathbf{q})^{\cdot(k-i)} = 0.$$

Hence $z \in \mathcal{P}^{(k-1)}(\mathbf{q})$. By induction hypothesis, $z = 0$. □

In view of the equivalence between positive comonoids and connected comonoids (Proposition 8.44), the above notions can be formulated for connected comonoids. If $(\mathbf{q}, \Delta, \epsilon)$ is such a comonoid, we set

$$\mathcal{P}^{(k)}(\mathbf{q}) = \mathcal{P}^{(k)}(\mathbf{q}_+) = \ker \Delta_+^{(k)},$$

where $\Delta_+$ is the positive part of the coproduct (8.58). In this context, the $k$-th term of the coradical filtration is

$$\mathbf{q}[\emptyset] \oplus \mathcal{P}^{(k)}(\mathbf{q}).$$

**Remark 8.47.** For some of the above arguments we made use of the fact that we are working over a field. The analogous results for connected coalgebras go back to [274, Proposition 3.9] and [300, Appendix B.3]. The former reference works over more general commutative rings. For information on the coradical filtration of not necessarily connected coalgebras, see [279, Chapter 5].

## 8.11. Derivatives and internal Hom

The Cauchy product in the category of species admits an internal Hom, which is described in terms of derivatives. We briefly discuss these notions in this section.

**8.11.1. The derivatives of a species.** The notion of derivative of a species appears in [181, §2.3] and [40, Section 1.4]. We recall this notion and extend it to higher derivatives. It is the analogue of the operator on graded vector spaces, which shifts the grading of each component by 1.

**Definition 8.48.** The *derivative* of a species $\mathbf{p}$ is the species $\mathbf{p}'$ defined by

$$\mathbf{p}'[I] := \mathbf{p}[I^+] \qquad \text{where } I^+ := I \sqcup \{*_I\}.$$

Here, as elsewhere in the monograph, $*_I$ denotes a new element (an element not in $I$) canonically associated to $I$, which for definiteness we may take to be $*_I := I$.

Given a bijection $\sigma\colon I \to J$, we let $\sigma^+\colon I^+ \to J^+$ be

$$\sigma^+(i) := \sigma(i) \text{ for } i \in I \quad \text{and} \quad \sigma(*_I) := *_J,$$

and define

$$\mathbf{p}'[\sigma] := \mathbf{p}[\sigma^+].$$

**Convention 8.49.** We view $[n]^+ = [1+n]$ by shifting all elements of $[n]$ up by 1 and identifying $*_{[n]}$ with 1. With this convention,

$$\mathbf{p}'[n] = \mathbf{p}[1+n].$$

The action of $\sigma \in \mathrm{S}_n$ on $\mathbf{p}[n+1]$ is by means of $\mathbf{p}[\sigma^+]$, where $\sigma^+ \in \mathrm{S}_{n+1}$ is the permutation given by

$$\sigma^+(1) = 1 \qquad \text{and} \qquad \sigma^+(i+1) = \sigma(i) + 1 \tag{8.65}$$

for $1 \le i \le n$.

Derivation $\mathbf{p} \mapsto \mathbf{p}'$ is a functor on the category of species: given a morphism $\mathbf{p} \to \mathbf{q}$, its derivative $\mathbf{p}' \to \mathbf{q}'$ is defined by means of the diagram below.

$$\begin{tikzcd}
\mathbf{p}'[I] \arrow[r, dashed] \arrow[d, equal] & \mathbf{q}'[I] \arrow[d, equal] \\
\mathbf{p}[I^+] \arrow[r] & \mathbf{q}[I^+]
\end{tikzcd}$$

It behaves as follows with respect to the Cauchy product:

$$(\mathbf{p} \cdot \mathbf{q})' = \mathbf{p} \cdot \mathbf{q}' + \mathbf{p}' \cdot \mathbf{q}, \qquad \mathbf{1}' = \mathbf{0} \tag{8.66}$$

(see Section 8.1.2 for the relevant definitions). The former follows from the fact that a decomposition of $I^+$ into two disjoint subsets is of the form $S \sqcup T^+$ or of the form $S^+ \sqcup T$, where $I = S \sqcup T$ is a decomposition of $I$.

Derivation preserves duality of species (Section 8.6); we have

$$(\mathbf{p}^*)' = (\mathbf{p}')^*. \tag{8.67}$$

Indeed, on a finite set $I$, both species evaluate to $\mathbf{p}[I^+]^*$.

We now discuss higher derivatives.

**Definition 8.50.** Let $X$ be a finite set. The $X$-*derivative* of a species $\mathbf{p}$ is the species $\mathbf{p}^{[X]}$ defined by

$$\mathbf{p}^{[X]}[I] := \mathbf{p}[X \amalg I].$$

Here $X \amalg I$ denotes the disjoint union of $X$ and $I$, which for definiteness we may take to be

$$X \amalg I := (X \times \{0\}) \cup (I \times \{1\}).$$

Higher derivatives satisfy the following properties:

$$\begin{aligned}
(\mathbf{p}^{[X]})^{[Y]} &\cong \mathbf{p}^{[X \amalg Y]}, \\
(\mathbf{p} \cdot \mathbf{q})^{[X]} &\cong \sum_{S \sqcup T = X} \mathbf{p}^{[S]} \cdot \mathbf{q}^{[T]}, \\
\mathbf{E}^{[X]} &\cong \mathbf{E}, \\
\mathbf{1}^{[X]} &\cong \begin{cases} \mathbf{1} & \text{if } X \text{ is empty,} \\ \mathbf{0} & \text{otherwise.} \end{cases}
\end{aligned}$$

Here, $\mathbf{p}, \mathbf{q}$ are arbitrary species, $X, Y$ are finite sets, and $\mathbf{E}$ is the exponential species (Example 8.3). The notion of derivative is recovered by taking $X$ to be a singleton set.

**8.11.2. Internal Hom for the Cauchy product.** Recall the notion of internal Hom for monoidal categories from Section 1.3.

For species $\mathbf{p}_1$ and $\mathbf{p}_2$, let $\mathcal{H}^{\cdot}(\mathbf{p}_1, \mathbf{p}_2)$ denote the species defined by

$$\mathcal{H}^{\cdot}(\mathbf{p}_1, \mathbf{p}_2)[I] := \mathrm{Hom}_{\mathsf{Sp}}(\mathbf{p}_1, \mathbf{p}_2^{[I]}),$$

where $\mathbf{p}_2^{[I]}$ is the $I$-derivative of $\mathbf{p}_2$. The right-hand side is the vector space of morphisms from $\mathbf{p}_1$ to $\mathbf{p}_2^{[I]}$ in the category $\mathsf{Sp}$. Note that this is isomorphic to

$$\prod_{n \geq 0} \mathrm{Hom}_{\mathrm{S}_n}\big(\mathbf{p}_1[n], \mathbf{p}_2[i+n]\big)$$

if $i := |I|$. The action of $\mathrm{S}_n$ on $\mathbf{p}_2[i+n]$ is dictated by Convention 8.49. It is as follows. $\mathrm{S}_n$ acts on $[i+n]$ by fixing the first $i$ letters and acting on the subset $[i+1, 1+n]$ via conjugation with the order-preserving bijection $[n] \to [i+1, 1+n]$.

This defines a functor

$$\mathcal{H}^{\cdot} : \mathsf{Sp}^{\mathrm{op}} \times \mathsf{Sp} \to \mathsf{Sp} \qquad\qquad (\mathbf{p}_1, \mathbf{p}_2) \mapsto \mathcal{H}^{\cdot}(\mathbf{p}_1, \mathbf{p}_2)$$

where $\mathsf{Sp}^{\mathrm{op}}$ denotes the opposite of the category $\mathsf{Sp}$.

**Proposition 8.51.** *For any species* $\mathbf{p}$, $\mathbf{m}$, *and* $\mathbf{n}$, *there is a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{Sp}}(\mathbf{p} \cdot \mathbf{m}, \mathbf{n}) \cong \mathrm{Hom}_{\mathsf{Sp}}\big(\mathbf{p}, \mathcal{H}^{\cdot}(\mathbf{m}, \mathbf{n})\big).$$

The proof is straightforward. The above result states that $\mathcal{H}^{\cdot}$ is the internal Hom for the monoidal category $(\mathsf{Sp}, \cdot)$. A similar discussion is given by Kelly [197, Equation (2.6)].

## 8.12. Species with up-down operators

We now discuss up-down operators on species. These are analogues of the creation-annihilation operators for graded vector spaces discussed in Section 2.8.

**8.12.1. Up-down operators.** For a species $\mathbf{p}$, let $\mathbf{p}'$ denote the derivative of $\mathbf{p}$ as explained in Section 8.11.1.

**Definition 8.52.** A species with *up operators* is a species $\mathbf{p}$ with a morphism of species

$$u \colon \mathbf{p} \to \mathbf{p}'.$$

A species with *down operators* is a species $\mathbf{p}$ with a morphism of species

$$d \colon \mathbf{p}' \to \mathbf{p}.$$

A morphism of species with up operators $(\mathbf{p}, u) \to (\mathbf{q}, v)$ is a morphism of species $\mathbf{p} \to \mathbf{q}$ that intertwines $u$ with $v$. This defines the category $\mathsf{Sp}^{\mathsf{u}}$ of species with up operators. The categories $\mathsf{Sp}_{\mathsf{d}}$ of species with down operators and $\mathsf{Sp}_{\mathsf{d}}^{\mathsf{u}}$ of species with up *and* down operators are defined similarly.

**8.12.2. The Cauchy product.** Define the Cauchy product of two species with up operators $(\mathbf{p}, u)$ and $(\mathbf{q}, v)$ to be $(\mathbf{p}\cdot\mathbf{q}, w)$ where

$$w\colon \mathbf{p}\cdot\mathbf{q} \xrightarrow{u\cdot\mathrm{id}+\mathrm{id}\cdot v} \mathbf{p}'\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{q}' = (\mathbf{p}\cdot\mathbf{q})'. \tag{8.68}$$

This turns $\mathsf{Sp}^{\mathsf{u}}$ into a symmetric monoidal category, which we denote by $(\mathsf{Sp}^{\mathsf{u}}, \cdot, \beta)$. The unit object is the species $\mathbf{1}$, equipped with the unique (zero) map to $\mathbf{1}' = \mathbf{0}$.

For any scalar $q \in \Bbbk$, one can deform the above monoidal structure by defining $w$ as follows.

$$\mathbf{p}[S]\otimes\mathbf{q}[T] \xrightarrow{u\otimes\mathrm{id}_T + q^{|S|}\mathrm{id}_S\otimes v} \mathbf{p}[S^+]\otimes\mathbf{q}[T]\oplus\mathbf{p}[S]\otimes\mathbf{q}[T^+].$$

To write this in compact form, for any species $\mathbf{p}$, let

$$\tau_q\colon \mathbf{p}\to\mathbf{p}$$

be defined by

$$\mathbf{p}[I]\to\mathbf{p}[I] \qquad x\mapsto q^{|I|}x. \tag{8.69}$$

The definition of $w$ can now be rewritten as

$$w = u\cdot\mathrm{id}+\tau_q\cdot v. \tag{8.70}$$

This defines a monoidal category which we denote by $(\mathsf{Sp}^{\mathsf{u}}, \cdot_q)$. Only the cases $q = \pm 1$ yield braided monoidal categories and these are in fact symmetric.

The above constructions also work for the categories $\mathsf{Sp}_{\mathsf{d}}$ and $\mathsf{Sp}^{\mathsf{u}}_{\mathsf{d}}$.

**8.12.3. Duality.** Suppose $(\mathbf{p}, u)$ is a species with up operators. Then, by (8.67), we have

$$u^*\colon (\mathbf{p}^*)'\to\mathbf{p}^*$$

and $(\mathbf{p}^*, u^*)$ is a species with down operators. Similarly, the dual of a species with down operators is a species with up operators. On finite-dimensional species, this defines inverse strong monoidal contravariant functors

$$(\mathsf{Sp}^{\mathsf{u}}, \cdot_q) \underset{*}{\overset{*}{\rightleftarrows}} (\mathsf{Sp}_{\mathsf{d}}, \cdot_q).$$

One can avoid the contravariant usage by replacing $\mathsf{Sp}^{\mathsf{u}}$ by its opposite category.

**Remark 8.53.** Suppose $(\mathbf{p}, u, d)$ is a species with up and down operators. The composites

$$du\colon \mathbf{p}\to\mathbf{p} \qquad \text{and} \qquad ud\colon \mathbf{p}'\to\mathbf{p}'$$

are morphisms of species, so for each bijection between finite sets $\sigma\colon I\to J$ we have commutative diagrams

$$\begin{tikzcd} \mathbf{p}[I] \arrow[r, "du"] \arrow[d, "\mathbf{p}[\sigma]"'] & \mathbf{p}[I] \arrow[d, "\mathbf{p}[\sigma]"] \\ \mathbf{p}[J] \arrow[r, "du"'] & \mathbf{p}[J] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{p}[I^+] \arrow[r, "ud"] \arrow[d, "\mathbf{p}[\sigma^+]"'] & \mathbf{p}[I^+] \arrow[d, "\mathbf{p}[\sigma^+]"] \\ \mathbf{p}[J^+] \arrow[r, "ud"'] & \mathbf{p}[J^+]. \end{tikzcd}$$

In particular, the maps

$$du\colon \mathbf{p}[n]\to\mathbf{p}[n] \qquad \text{and} \qquad ud\colon \mathbf{p}[n+1]\to\mathbf{p}[n+1]$$

are both $\mathrm{S}_n$-equivariant. The action of $\mathrm{S}_n$ on $[n+1]$ is described in Convention 8.49.

**8.12.4. Derivations and coderivations.** Let $(\mathbf{p}, \mu, \iota)$ be a monoid in $(\mathsf{Sp}, \cdot)$. An *up derivation* of $\mathbf{p}$ is an up operator $u\colon \mathbf{p} \to \mathbf{p}'$ such that the following diagrams commute.

$$\begin{tikzcd} \mathbf{p}\cdot\mathbf{p} \arrow[r, "u\cdot\mathrm{id}+\mathrm{id}\cdot u"] \arrow[d, "\mu"'] & (\mathbf{p}\cdot\mathbf{p})' \arrow[d, "\mu'"] \\ \mathbf{p} \arrow[r, "u"'] & \mathbf{p}' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{p} \arrow[r, "u"] & \mathbf{p}' \\ \mathbf{1} \arrow[u, "\iota"] \arrow[ur, "0"'] & \end{tikzcd} \tag{8.71}$$

In fact, the commutativity of the first diagram implies the commutativity of the second; so it need not be a part of the definition.

Dually, let $(\mathbf{p}, \Delta, \epsilon)$ be a comonoid in $(\mathsf{Sp}, \cdot)$. An *up coderivation* of $\mathbf{p}$ is an up operator $u\colon \mathbf{p} \to \mathbf{p}'$ such that the following diagrams commute.

$$\begin{tikzcd} \mathbf{p} \arrow[r, "u"] \arrow[d, "\Delta"'] & \mathbf{p}' \arrow[d, "\Delta'"] \\ \mathbf{p}\cdot\mathbf{p} \arrow[r, "u\cdot\mathrm{id}+\mathrm{id}\cdot u"'] & (\mathbf{p}\cdot\mathbf{p})' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{p} \arrow[r, "u"] \arrow[dr, "0"'] & \mathbf{p}' \arrow[d, "\epsilon'"] \\ & \mathbf{1}' \end{tikzcd} \tag{8.72}$$

The second diagram commutes trivially since $\mathbf{1}' = \mathbf{0}$, the zero species; so it need not be a part of the definition.

Down (co)derivations are defined similarly. More generally, one defines a $q$-version of both up and down (co)derivations by replacing

$$u\cdot\mathrm{id}+\mathrm{id}\cdot u \quad \text{by} \quad u\cdot\mathrm{id}+\tau_q\cdot u, \qquad \text{and} \qquad d\cdot\mathrm{id}+\mathrm{id}\cdot d \quad \text{by} \quad d\cdot\mathrm{id}+\tau_q\cdot d$$

in the corresponding definitions, with $\tau_q$ as in (8.69).

**Remark 8.54.** Up and down (co)derivations on (co)monoids in species are the analogues of (co)derivations of degree 1 and $-1$ respectively on graded vector spaces. Keeping this in mind, one expects (co)derivations on (co)monoids in species of higher degrees. This can be done, however it is not required for our present purposes, so we omit it.

**8.12.5. Monoids and comonoids with up or down operators.** We have seen that a (co)monoid in $\mathsf{gVec}^{\mathsf{c}}$ is a graded (co)algebra equipped with a derivation of degree 1. (Co)monoids in $\mathsf{Sp}^{\mathsf{u}}$ admit a similar description. More precisely:

A monoid in $(\mathsf{Sp}^{\mathsf{u}}, \cdot)$ is a monoid $(\mathbf{p}, \mu, \iota)$ in species equipped with an up derivation $u\colon \mathbf{p} \to \mathbf{p}'$, that is, diagram (8.71) commutes.

A comonoid in $(\mathsf{Sp}^{\mathsf{u}}, \cdot)$ is a comonoid $(\mathbf{p}, \Delta, \epsilon)$ in species equipped with an up coderivation $u\colon \mathbf{p} \to \mathbf{p}'$, that is, diagram (8.72) commutes.

Monoids and comonoids in $(\mathsf{Sp}_{\mathsf{d}}, \cdot)$ and $(\mathsf{Sp}^{\mathsf{u}}_{\mathsf{d}}, \cdot)$ admit similar descriptions. For the deformed Cauchy product $\cdot_q$, one replaces (co)derivations by $q$-(co)derivations.

Duality exchanges monoids and comonoids and up and down operators.

**Example 8.55.** The exponential species (Example 8.3) is a species with up-down operators. Indeed, there is an isomorphism of species

$$\mathbf{E}' \cong \mathbf{E}$$

which for each finite set $I$ identifies the basis element $*_{I^+}$ of $\mathbf{E}'[I] = \mathbf{E}[I^+]$ with the basis element $*_I$ of $\mathbf{E}[I]$. We let $u\colon \mathbf{E} \to \mathbf{E}'$ and $d\colon \mathbf{E}' \to \mathbf{E}$ be the identity maps.

Recall that $\mathbf{E}$ is a Hopf monoid in species (Example 8.15). Note that $d$ is a down derivation but $u$ is not an up derivation; the first diagram below commutes but the second does not.

$$\begin{tikzcd}
(\mathbf{E}\cdot\mathbf{E})' \arrow[r, "d\cdot\mathrm{id}+\mathrm{id}\cdot d"] \arrow[d, "\mu'"'] & \mathbf{E}\cdot\mathbf{E} \arrow[d, "\mu"] \\
\mathbf{E}' \arrow[r, "d"'] & \mathbf{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{E}\cdot\mathbf{E} \arrow[r, "u\cdot\mathrm{id}+\mathrm{id}\cdot u"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\text{No}"] & (\mathbf{E}\cdot\mathbf{E})' \arrow[d, "\mu'"] \\
\mathbf{E} \arrow[r, "u"'] & \mathbf{E}'
\end{tikzcd}$$

For the second diagram, the image of a basis element $*_S \otimes *_T$ under $u\mu$ is $*_{I^+}$, but the image under $\mu'(u\cdot\mathrm{id}+\mathrm{id}\cdot u)$ is 2 times $*_{I^+}$.

Similarly, $u$ is an up coderivation but $d$ is not a down coderivation. Thus, $(\mathbf{E}, d)$ is a monoid in $\mathsf{Sp}_{\mathsf{d}}$ and $(\mathbf{E}, u)$ is a comonoid in $\mathsf{Sp}^{\mathsf{u}}$, but we cannot view $\mathbf{E}$ as a bimonoid in either category.

**Example 8.56.** Consider now the species $\mathbf{L}$ of linear orders (Example 8.3). We turn $\mathbf{L}$ into a species with up-down operators by defining

$$\mathbf{L}[I] \underset{d}{\overset{u}{\rightleftarrows}} \mathbf{L}[I^+]$$

by

$$u(l^1|l^2|\cdots|l^n) := *_I|l^1|l^2|\cdots|l^n \quad \text{for } l^1|l^2|\cdots|l^n \in \mathbf{L}[I]$$

and

$$d(l^1|l^2|\cdots|l^n|l^{n+1}) := \begin{cases} l^2|\cdots|l^n|l^{n+1} & \text{if } l^1 = *_I, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } l^1|l^2|\cdots|l^n|l^{n+1} \in \mathbf{L}[I^+].$$

In other words, $u$ extends $l$ to a linear order on $I^+$ by adding the new element $*_I$ as the minimum, while $d$ removes this element from a linear order on $I^+$ if it is the minimum, otherwise it sends the order to 0.

One can check that $(\mathbf{L}, u)$ is a comonoid in $(\mathsf{Sp}^{\mathsf{u}}, \cdot)$ but not a monoid. By using the same $d$ as above for $\mathbf{L}^*$, it follows by duality that $(\mathbf{L}^*, d)$ is a monoid in $(\mathsf{Sp}_{\mathsf{d}}, \cdot)$.

Deformations of these examples are given in Sections 9.3 and 9.5.5.

## 8.13. The Hadamard product and an interchange law on species

We study the interplay between the Cauchy and Hadamard product on species and derive constructions of Hopf monoids and self-dual Hopf monoids as an application.

**8.13.1. The Hadamard product as a bilax monoidal functor.** Consider the functor

$$(-\times-)\colon \mathsf{Sp}\times\mathsf{Sp} \to \mathsf{Sp} \qquad (\mathbf{p}_1, \mathbf{p}_2) \mapsto \mathbf{p}_1 \times \mathbf{p}_2,$$

where $\mathbf{p}_1 \times \mathbf{p}_2$ is the Hadamard product on species (8.7). Let $(\mathsf{Sp}\times\mathsf{Sp}, \cdot)$ be the monoidal category obtained by taking the Cartesian product of $(\mathsf{Sp}, \cdot)$ with itself. We proceed to turn the Hadamard functor into a bilax monoidal functor.

Define maps

$$\begin{tikzcd}
(\mathbf{p}_1\times\mathbf{p}_2)\cdot(\mathbf{q}_1\times\mathbf{q}_2) \arrow[r, bend left, "\varphi"] & (\mathbf{p}_1\cdot\mathbf{q}_1)\times(\mathbf{p}_2\cdot\mathbf{q}_2) \arrow[l, bend left, "\psi"]
\end{tikzcd}$$

as follows. For any finite set $I$, we define

$$
\begin{tikzcd}
\displaystyle\bigoplus_{I=S\sqcup T} (\mathbf{p}_1[S]\otimes\mathbf{p}_2[S])\otimes(\mathbf{q}_1[T]\otimes\mathbf{q}_2[T]) \arrow[d, bend left, "\varphi"] \\
\Big(\displaystyle\bigoplus_{I=S_1\sqcup T_1}\mathbf{p}_1[S_1]\otimes\mathbf{q}_1[T_1]\Big)\otimes\Big(\displaystyle\bigoplus_{I=S_2\sqcup T_2}\mathbf{p}_2[S_2]\otimes\mathbf{q}_2[T_2]\Big) \arrow[u, bend left, "\psi"]
\end{tikzcd}
\tag{8.73}
$$

where $\varphi$ is the natural embedding given by switching the middle factors and $\psi$ is the surjection which is identity if $S_1 = S_2$ and $T_1 = T_2$ and zero otherwise.

In addition, let

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, bend left, "\varphi_0"] & \mathbf{1}\times\mathbf{1} \arrow[l, bend left, "\psi_0"]
\end{tikzcd}
\tag{8.74}
$$

be the obvious isomorphisms.

The fact that $\varphi$ and $\psi$ endow the Hadamard functor with a bilax monoidal structure is more subtle than one may expect, so we give a detailed proof below. Indeed, the corresponding result for graded vector spaces fails; see Remark 8.65.

**Lemma 8.57.** *Let $S_1, S_2, T_1, T_2, U_1, U_2$ and $V_1, V_2$ be subsets of a set such that*

$$S_1 \sqcup T_1 = S_2 \sqcup T_2, \qquad U_1 \sqcup V_1 = U_2 \sqcup V_2$$

*and let these two sets be disjoint from each other. Then*

$$S_1 \sqcup U_1 = S_2 \sqcup U_2,\ T_1 \sqcup V_1 = T_2 \sqcup V_2 \iff S_1 = S_2,\ T_1 = T_2,\ U_1 = U_2,\ V_1 = V_2.$$

This result is a reformulation of Lemma 8.7.

**Proposition 8.58.** *The Hadamard functor*

$$(\times, \varphi, \psi)\colon (\mathsf{Sp}\times\mathsf{Sp}, \cdot) \to (\mathsf{Sp}, \cdot)$$

*is a normal braided bilax monoidal functor.*

Proof. It is straightforward to check that the functor $(\times, \varphi)$ is braided lax and that $(\times, \psi)$ is braided colax. For clarity of notation, we write $\mathcal{F}$ instead of $\times$ for the rest of the proof. To show that $\mathcal{F}$ is bilax, we first check that it satisfies the braiding axiom (3.11). It takes the following form.

$$
\begin{tikzcd}
\mathcal{F}(\mathbf{p}_1\cdot\mathbf{q}_1, \mathbf{p}_2\cdot\mathbf{q}_2)\cdot\mathcal{F}(\mathbf{r}_1\cdot\mathbf{s}_1, \mathbf{r}_2\cdot\mathbf{s}_2) \arrow[r, "\psi\cdot\psi"] \arrow[d, "\varphi"] & \mathcal{F}(\mathbf{p}_1,\mathbf{p}_2)\cdot\mathcal{F}(\mathbf{q}_1,\mathbf{q}_2)\cdot\mathcal{F}(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathcal{F}(\mathbf{s}_1,\mathbf{s}_2) \arrow[d, "\mathrm{id}\cdot\beta\cdot\mathrm{id}"] \\
\mathcal{F}(\mathbf{p}_1\cdot\mathbf{q}_1\cdot\mathbf{r}_1\cdot\mathbf{s}_1, \mathbf{p}_2\cdot\mathbf{q}_2\cdot\mathbf{r}_2\cdot\mathbf{s}_2) \arrow[d, "\mathcal{F}(\mathrm{id}\cdot\beta\cdot\mathrm{id})"'] & \mathcal{F}(\mathbf{p}_1,\mathbf{p}_2)\cdot\mathcal{F}(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathcal{F}(\mathbf{q}_1,\mathbf{q}_2)\cdot\mathcal{F}(\mathbf{s}_1,\mathbf{s}_2) \arrow[d, "\varphi\cdot\varphi"] \\
\mathcal{F}(\mathbf{p}_1\cdot\mathbf{r}_1\cdot\mathbf{q}_1\cdot\mathbf{s}_1, \mathbf{p}_2\cdot\mathbf{r}_2\cdot\mathbf{q}_2\cdot\mathbf{s}_2) \arrow[r, "\psi"'] & \mathcal{F}(\mathbf{p}_1\cdot\mathbf{r}_1,\mathbf{p}_2\cdot\mathbf{r}_2)\cdot\mathcal{F}(\mathbf{q}_1\cdot\mathbf{s}_1,\mathbf{q}_2\cdot\mathbf{s}_2)
\end{tikzcd}
\tag{8.75}
$$

For definiteness, let us start in the component

$$\mathbf{p}_1[S_1] \otimes \mathbf{q}_1[T_1] \otimes \mathbf{p}_2[S_2] \otimes \mathbf{q}_2[T_2] \otimes \mathbf{r}_1[U_1] \otimes \mathbf{s}_1[V_1] \otimes \mathbf{r}_2[U_2] \otimes \mathbf{s}_2[V_2],$$

where the sets $S_1$, $S_2$, etc., satisfy the hypothesis of Lemma 8.57. Following along the two directions in (8.75), we note that this component survives in the end precisely when the two equivalent conditions in the conclusion of Lemma 8.57 hold. Hence (8.75) commutes. We remark that the vertical maps in the diagram play a passive role in the check.

The fact that $\mathcal{F}$ satisfies the unitality axioms (3.12) and (3.13) is straightforward and we conclude that $\mathcal{F}$ is bilax. Normality is clear since $\varphi_0$ and $\psi_0$ are inverse isomorphisms. $\square$

**Corollary 8.59.** *If $\mathbf{h}_1$ and $\mathbf{h}_2$ are bimonoids (Hopf monoids) in $(\mathsf{Sp}, \cdot)$, then so is their Hadamard product $\mathbf{h}_1 \times \mathbf{h}_2$. Further, if $\mathbf{h}_1$ and $\mathbf{h}_2$ are (co)commutative, then so is $\mathbf{h}_1 \times \mathbf{h}_2$.*

Proof. Note that a bimonoid in $(\mathsf{Sp} \times \mathsf{Sp}, \cdot)$ consists of a pair of bimonoids in $(\mathsf{Sp}, \cdot)$. Since the Hadamard functor is bilax, it preserves bimonoids by Proposition 3.31. Hence, if $\mathbf{h}_1$ and $\mathbf{h}_2$ are bimonoids in $(\mathsf{Sp}, \cdot)$, then so is $\mathbf{h}_1 \times \mathbf{h}_2$. For the case of Hopf monoids, we use Proposition 8.10 and the observation that if $\mathbf{h}_1[\emptyset]$ and $\mathbf{h}_2[\emptyset]$ are Hopf algebras, then so is $(\mathbf{h}_1 \times \mathbf{h}_2)[\emptyset] = \mathbf{h}_1[\emptyset] \otimes \mathbf{h}_2[\emptyset]$. This proves the first claim.

Since the Hadamard functor is braided (co)lax, it preserves (co)commutativity by Proposition 3.37. This implies the second claim. $\square$

The product and coproduct in $\mathbf{h}_1 \times \mathbf{h}_2$ involve the maps $\varphi$ and $\psi$, as well as the products and coproducts of $\mathbf{h}_1$ and $\mathbf{h}_2$; see Proposition 3.31. On the other hand, the Hadamard functor is neither bistrong nor Hopf lax; the latter follows from the former by Proposition 3.60. Hence the antipode of $\mathbf{h}_1 \times \mathbf{h}_2$ is not directly related to the antipodes of $\mathbf{h}_1$ and $\mathbf{h}_2$.

**8.13.2. Self-duality of the Hadamard functor.** Let us restrict the Hadamard functor to finite-dimensional species. Consider its contragredient functor (Section 3.10). It is clear that

$$\varphi^\vee = \psi \qquad \text{and} \qquad \psi^\vee = \varphi, \tag{8.76}$$

with $\varphi^\vee$ and $\psi^\vee$ defined as in Proposition 3.102 (and with the canonical identification of the double dual of a finite-dimensional vector space with itself). Therefore, by Definition 3.105, we conclude:

**Proposition 8.60.** *On finite-dimensional species, the Hadamard functor is self-dual. Namely,*

$$(\times, \varphi, \psi)^\vee \cong (\times, \varphi, \psi). \tag{8.77}$$

We now state a couple of straightforward consequences.

**Corollary 8.61.** *If $\mathbf{h}$ is a finite-dimensional Hopf monoid, then $\mathbf{h}^* \times \mathbf{h}$ is a self-dual Hopf monoid.*

Proof. We first note that $\mathbf{h}^*$ is a Hopf monoid, and hence by Corollary 8.59, $\mathbf{h}^* \times \mathbf{h}$ is a Hopf monoid. It is the image under the Hadamard functor of $(\mathbf{h}^*, \mathbf{h})$

which is a Hopf monoid in $\mathsf{Sp} \times \mathsf{Sp}$. To show that it is self-dual, we note that $(\mathbf{h}^*, \mathbf{h})$ is self-dual by

$$(\mathbf{h}^*, \mathbf{h})^* \cong (\mathbf{h}, \mathbf{h}^*) \cong (\mathbf{h}^*, \mathbf{h}),$$

the second isomorphism given by switching the factors. Hence the result follows from Propositions 3.107 and 8.60. $\square$

An important example of the above construction is given by the Hopf monoid $\mathbf{L} \times \mathbf{L}^*$, which is studied in detail in Section 12.3.

**Corollary 8.62.** *If $\mathbf{h}_1$ and $\mathbf{h}_2$ are finite-dimensional Hopf monoids, then*

$$(\mathbf{h}_1 \times \mathbf{h}_2)^* \cong \mathbf{h}_1^* \times \mathbf{h}_2^* \qquad \text{and} \qquad (\mathbf{h}_1^*)^* \cong \mathbf{h}_1$$

*as Hopf monoids.*

**8.13.3. Internal Hom for the Hadamard product.** The Hadamard product admits an internal Hom (Section 1.3). We describe this functor and show that it is bilax monoidal.

For vector spaces $V$ and $W$, let $\mathrm{Hom}_{\mathsf{Vec}}(V, W)$ denote the space of linear maps from $V$ to $W$, and let

$$\mathrm{End}_{\mathsf{Vec}}(V) := \mathrm{Hom}_{\mathsf{Vec}}(V, V).$$

For species $\mathbf{p}_1$ and $\mathbf{p}_2$, let $\mathcal{H}^\times(\mathbf{p}_1, \mathbf{p}_2)$ denote the species defined as follows. For any finite set $I$,

$$\mathcal{H}^\times(\mathbf{p}_1, \mathbf{p}_2)[I] := \mathrm{Hom}_{\mathsf{Vec}}(\mathbf{p}_1[I], \mathbf{p}_2[I]),$$

and for any bijection $\sigma \colon I \to J$,

$$\mathcal{H}^\times(\mathbf{p}_1, \mathbf{p}_2)[\sigma]$$

is the map

$$\mathrm{Hom}_{\mathsf{Vec}}(\mathbf{p}_1[I], \mathbf{p}_2[I]) \xrightarrow{\mathrm{Hom}_{\mathsf{Vec}}(\mathbf{p}_1[\sigma^{-1}], \mathbf{p}_2[\sigma])} \mathrm{Hom}_{\mathsf{Vec}}(\mathbf{p}_1[J], \mathbf{p}_2[J]).$$

This defines a functor

$$\mathcal{H}^\times : \mathsf{Sp}^{\mathrm{op}} \times \mathsf{Sp} \to \mathsf{Sp} \qquad\qquad (\mathbf{p}_1, \mathbf{p}_2) \mapsto \mathcal{H}^\times(\mathbf{p}_1, \mathbf{p}_2)$$

where $\mathsf{Sp}^{\mathrm{op}}$ denotes the opposite of the category $\mathsf{Sp}$.

**Proposition 8.63.** *For any species $\mathbf{p}$, $\mathbf{m}$, and $\mathbf{n}$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{Sp}}(\mathbf{p} \times \mathbf{m}, \mathbf{n}) \;\cong\; \mathrm{Hom}_{\mathsf{Sp}}\big(\mathbf{p}, \mathcal{H}^\times(\mathbf{m}, \mathbf{n})\big).$$

The proof is straightforward. The above result states that $\mathcal{H}^\times$ is the internal Hom for the monoidal category $(\mathsf{Sp}, \times)$.

There is a natural map

$$\mathbf{m}^* \times \mathbf{n} \to \mathcal{H}^\times(\mathbf{m}, \mathbf{n}), \tag{8.78}$$

which is an isomorphism if the species $\mathbf{m}$ is finite-dimensional.

We now endow $\mathcal{H}^\times$ with a bilax structure. Let maps

$$\mathcal{H}^\times(\mathbf{p}_1, \mathbf{p}_2) \cdot \mathcal{H}^\times(\mathbf{q}_1, \mathbf{q}_2) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{H}^\times(\mathbf{p}_1 \cdot \mathbf{q}_1, \mathbf{p}_2 \cdot \mathbf{q}_2) \tag{8.79}$$

be defined by mimicking (8.73). Let the maps $\varphi_0$ and $\psi_0$ between $\mathbf{1}$ and $\mathcal{H}^\times(\mathbf{1}, \mathbf{1})$ be the obvious isomorphisms.

**Proposition 8.64.** *The functor* $(\mathcal{H}^\times, \varphi, \psi)\colon (\mathsf{Sp}^{\mathrm{op}} \times \mathsf{Sp}, \cdot) \to (\mathsf{Sp}, \cdot)$ *is a normal braided bilax monoidal functor.*

PROOF. The proof of Proposition 8.58 applies, with $\mathcal{F} = \mathcal{H}^\times$. □

It follows that if $\mathbf{m}$ and $\mathbf{n}$ are Hopf monoids, then so is $\mathcal{H}^\times(\mathbf{m}, \mathbf{n})$.

The map (8.78) yields a morphism of bilax monoidal functors

$$(-)^* \times (-) \ \to\ \mathcal{H}^\times(-,-),$$

where the left-hand term is viewed as a composite of bilax monoidal functors. In particular, if $\mathbf{m}$ and $\mathbf{n}$ are Hopf monoids, then (8.78) is a morphism of Hopf monoids (an isomorphism if $\mathbf{m}$ is finite-dimensional).

For a species $\mathbf{p}$, let

$$\mathcal{E}^\times(\mathbf{p}) := \mathcal{H}^\times(\mathbf{p}, \mathbf{p}). \tag{8.80}$$

There is a natural map

$$\mathbf{p}^* \times \mathbf{p} \to \mathcal{E}^\times(\mathbf{p}),$$

which is an isomorphism if $\mathbf{p}$ is finite-dimensional.

It follows from Propositions 1.28 and 8.63 that $\mathcal{E}^\times(\mathbf{p})$ is a monoid in $(\mathsf{Sp}, \times)$. Explicitly,

$$\mathcal{E}^\times(\mathbf{p})[I] = \mathrm{End}_{\mathsf{Vec}}(\mathbf{p}[I]),$$

is an algebra under composition for each $I$. It follows from Proposition 8.64 that if $\mathbf{p}$ is a Hopf monoid, then so is $\mathcal{E}^\times(\mathbf{p})$.

**Remark 8.65.** Consider the Hadamard and internal Hom functors for the category of graded vector spaces

$$(- \times -)\colon \mathsf{gVec} \times \mathsf{gVec} \to \mathsf{gVec} \qquad \text{and} \qquad \mathcal{H}^\times\colon \mathsf{gVec}^{\mathrm{op}} \times \mathsf{gVec} \to \mathsf{gVec}.$$

They are respectively defined in Sections 2.1.1 and 2.1.5. Analogues of the structure maps $\varphi$ and $\psi$ exist for both functors; they can be defined along the lines of (8.73). For the Hadamard functor, they are explicitly discussed in Example 6.22. Both functors are lax and colax with respect to the Cauchy product on $\mathsf{gVec}$; however, they are not bilax. The reason for this can be again traced back to the failure of Lemmas 8.7 or 8.57 when sets are replaced by numbers, as in Remark 8.8.

**8.13.4. The functor $\mathbf{m} \times (-)$.** Let $\mathsf{I}$ be the one-arrow category. Consider the composite of functors

$$\mathsf{Sp} \xrightarrow{\ \cong\ } \mathsf{I} \times \mathsf{Sp} \xrightarrow{\mathcal{F}_{\mathbf{m}} \times \mathrm{id}} \mathsf{Sp} \times \mathsf{Sp} \xrightarrow{\ \times\ } \mathsf{Sp},$$

where the functor $\mathcal{F}_{\mathbf{m}}$ sends the unique object of $\mathsf{I}$ to $\mathbf{m}$ as in Section 3.4.1. We denote the above composite by $\mathbf{m} \times (-)$ since it sends $\mathbf{p}$ to $\mathbf{m} \times \mathbf{p}$.

**Proposition 8.66.** *If* $\mathbf{m}$ *is a monoid (comonoid, bimonoid) in species, then the functor* $\mathbf{m} \times (-)$ *is lax (colax, bilax). Further, if* $\mathbf{m}$ *is (co)commutative, then* $\mathbf{m} \times (-)$ *is braided (co)lax.*

PROOF. Let $\mathbf{m}$ be a monoid. Then the functor $\mathcal{F}_{\mathbf{m}}$ is lax with structure morphisms as in (3.22). Since $\mathbf{m} \times (-)$ is the composite of lax functors, it is also lax by Theorem 3.21.

The proof when $\mathbf{m}$ is a comonoid or a bimonoid or (co)commutative is similar. □

Let $\mathbf{m}$ be a finite-dimensional species. The existence of an internal Hom for the Hadamard product says that the functor $\mathbf{m} \times (-)$ has a right adjoint and by (8.78), it is given by $\mathbf{m}^* \times (-)$. Observe that the unit and counit of this adjunction are defined using (8.32). If $\mathbf{m}$ is a comonoid, then by Proposition 8.66, the former functor is colax while the latter functor is lax. In this setting, we have the following result.

**Proposition 8.67.** *For a finite-dimensional* (*cocommutative*) *comonoid* $\mathbf{m}$, *the adjunction*

$$(\mathsf{Sp}, \cdot) \underset{\mathbf{m}^* \times (-)}{\overset{\mathbf{m} \times (-)}{\rightleftarrows}} (\mathsf{Sp}, \cdot) \tag{8.81}$$

*is* (*braided*) *colax-lax. Further, the adjunction is self-dual* (*Definition* 3.109).

Proof. The diagrams in (3.41) reduce to the following.

$$\begin{tikzcd}
\mathbf{m}[I] \otimes \mathbf{m}^*[S] \otimes \mathbf{m}^*[T] \arrow[r, "\mathrm{id}_I \otimes \mu_{S,T}"] \arrow[d, "\Delta_{S,T} \otimes \mathrm{id}_S \otimes \mathrm{id}_T"'] & \mathbf{m}[I] \otimes \mathbf{m}^*[I] \arrow[d] \\
\mathbf{m}[S] \otimes \mathbf{m}[T] \otimes \mathbf{m}^*[S] \otimes \mathbf{m}^*[T] \arrow[r] & \Bbbk
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{m}[\emptyset] \otimes \Bbbk \arrow[r, "\mathrm{id}_\emptyset \otimes \iota_\emptyset"] \arrow[d, "\epsilon_\emptyset \otimes \mathrm{id}"'] & \mathbf{m}[\emptyset] \otimes \mathbf{m}^*[\emptyset] \arrow[d] \\
\Bbbk \otimes \Bbbk \arrow[r, "\cong"'] & \Bbbk
\end{tikzcd}$$

The unlabeled maps are induced by the canonical map $V \otimes V^* \to \Bbbk$ for a vector space $V$. It is clear that the diagrams commute and hence the adjunction is colax-lax.

If $\mathbf{m}$ is cocommutative, then the functors in question are braided colax and braided lax respectively, and hence the adjunction is braided colax-lax.

Since the contragredient of $\mathcal{F}_{\mathbf{m}}$ is $\mathcal{F}_{\mathbf{m}^*}$ and the Hadamard functor is self-dual (8.77), it follows that

$$\big(\mathbf{m} \times (-)\big)^{\vee} \cong \mathbf{m}^* \times (-) \tag{8.82}$$

as lax functors. It follows that the adjunction is self-dual. $\square$

**8.13.5. Cauchy and Hadamard as a 2-monoidal category.** The interaction between the Cauchy and Hadamard products is best understood in the language of 2-monoidal categories (Section 6.1).

The maps in (8.73) provide interchange laws between the Cauchy and Hadamard products in both directions. Further, on finite-dimensional species, they are contragredients of each other (8.76). This can be expressed as follows.

**Proposition 8.68.** *With the structure maps* (8.73) *and* (8.74),

$$(\mathsf{Sp}, \cdot, \times) \qquad \textit{and} \qquad (\mathsf{Sp}, \times, \cdot)$$

*are braided* 2*-monoidal categories. The subcategories of finite-dimensional species are contragredients of each other.*

The corresponding result for graded vector spaces is given in Example 6.22. To prove the above result, one needs to verify the axioms in Definitions 6.1 and 6.5. This is straightforward. Propositions 6.4 and 6.6 then imply that the Hadamard functor is both braided lax and braided colax. This is part of the claim made in Proposition 8.58. To understand how the rest of the claim in that proposition fits in, we need to go one step further and invoke 3-monoidal categories (Section 7.1) as follows.

**Proposition 8.69.** *We have that* $(\mathsf{Sp}, \cdot, \times, \cdot)$ *is a* 3*-monoidal category. The subcategory of finite-dimensional species is self-dual.*

The self-duality follows from (8.76). To prove the rest, one needs to check the list of axioms for a 3-monoidal category given in Definition 7.1. By Proposition 7.3, this is equivalent to checking that the Hadamard functor is bilax and $\mathbf{E}$ is a bimonoid, both for the Cauchy product. The former is the content of Proposition 8.58, while the latter was one of the first examples of bimonoids in species (Example 8.15).

**8.13.6. Cauchy and Hadamard on species with restrictions.** The Cauchy and Hadamard products also give rise to a 2-monoidal structure on the category of species with restrictions (Section 8.7.8).

The Hadamard product, denoted $\times$, is defined for species with restrictions as it is for set species (8.36). This defines the monoidal category $(\mathsf{Spr}, \times)$; the unit object is the exponential set species $\mathrm{E}$ (the restriction maps are as in Example 8.33).

**Proposition 8.70.** *We have that* $(\mathsf{Spr}, \cdot, \times)$ *is a braided* 2*-monoidal category.*

As for set species (Remark 8.25), the Hadamard product for species with restrictions coincides with the categorical product. Hence the above result can be deduced from Example 6.19.

We now provide an alternative approach to Schmitt's comonoid construction (Section 8.7.8).

Observe that a species with restrictions can be turned into a species by forgetting the restriction maps and linearizing. This defines a functor

$$\mathcal{F} \colon \mathsf{Spr} \to \mathsf{Sp}.$$

We proceed to turn this into a bilax monoidal functor $(\mathsf{Spr}, \cdot, \times) \to (\mathsf{Sp}, \cdot, \cdot)$. Define

$$\varphi_{\mathrm{P},\mathrm{Q}} \colon \mathcal{F}(\mathrm{P}) \cdot \mathcal{F}(\mathrm{Q}) \to \mathcal{F}(\mathrm{P} \cdot \mathrm{Q}) \qquad \text{and} \qquad \varphi_0 \colon \mathbf{1} \to \mathcal{F}(1)$$

to be the identity morphisms. Next, we define

$$\psi_{\mathrm{P},\mathrm{Q}} \colon \mathcal{F}(\mathrm{P} \times \mathrm{Q}) \to \mathcal{F}(\mathrm{P}) \cdot \mathcal{F}(\mathrm{Q}) \qquad \text{and} \qquad \psi_0 \colon \mathcal{F}(\mathrm{E}) \to \mathbf{1}$$

as follows. The first morphism is defined by adding the linearization of the maps

$$(\rho_{I,S}, \rho_{I,T}) \colon \mathrm{P}[I] \times \mathrm{Q}[I] \to \mathrm{P}[S] \times \mathrm{Q}[T]$$

over all decompositions $(S, T)$ of $I$. The second morphism is defined using the identification $\Bbbk\mathrm{E}[\emptyset] = \mathbf{1}[\emptyset]$.

**Proposition 8.71.** *The functor*

$$(\mathcal{F}, \varphi, \psi) \colon (\mathsf{Spr}, \cdot, \times) \to (\mathsf{Sp}, \cdot, \cdot) \tag{8.83}$$

*is braided bilax.*

The proof is straightforward. Recall that in the source category, $\times$ coincides with the categorical product. Therefore by Example 6.42, a cocommutative comonoid with respect to $\times$ is the same as an object, and a cocommutative bimonoid with respect to $(\cdot, \times)$ is the same as a monoid with respect to $\cdot$.

A braided colax functor preserves cocommutative comonoids, and a bilax functor preserves bimonoids (Propositions 3.31 and 3.37). Thus, using the observation in the preceding paragraph, the functor (8.83) induces functors

$$\mathsf{Spr} \to {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{Sp}, \cdot) \qquad \text{and} \qquad \mathsf{Mon}(\mathsf{Spr}, \cdot) \to {}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{Sp}, \cdot, \cdot).$$

These functors are those entering in the equivalences described in Propositions 8.29 and 8.31; in particular, the first one is the construction of Schmitt which associates a cocommutative comonoid to a species with restrictions.

**8.13.7. Pointing and the species of elements.** Let $\mathbf{e}$ be the species defined by

$$\mathbf{e}[I] = \Bbbk I,$$

the vector space with basis the elements of $I$. This is the species *of elements* [40, Section 1.1.3]. It is positive ($\mathbf{e}[\emptyset] = 0$) and self-dual:

$$\mathbf{e} \cong \mathbf{e}^*.$$

Given a species $\mathbf{p}$, its *pointing* is the species defined by

$$\mathbf{p}^\bullet := \mathbf{p} \times \mathbf{e}.$$

Pointing satisfies the following properties:

$$\begin{aligned} \mathbf{1}^\bullet &\cong \mathbf{0}, \\ \mathbf{E}^\bullet &\cong \mathbf{e}, \\ (\mathbf{p} \cdot \mathbf{q})^\bullet &\cong \mathbf{p}^\bullet \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{q}^\bullet, \\ \mathbf{p}^\bullet &\cong \mathbf{X} \cdot \mathbf{p}', \\ (\mathbf{p}^*)^\bullet &\cong (\mathbf{p}^\bullet)^*. \end{aligned}$$

In Section B.6.3 we relate pointing to the substitution operation (8.8).

CHAPTER 9

# Deformations of Hopf Monoids

One can perform a one-parameter deformation of the braiding on the category of species equipped with the Cauchy product. The parallel of this feature for graded vector spaces was explained in Section 2.3. This allows us to define a $q$-Hopf monoid which is the species analogue of a $q$-Hopf algebra. Letting $q = 1$ recovers the notion of Hopf monoid studied in Chapter 8.

Many ideas and constructions pertaining to bimonoids and Hopf monoids carry over to this deformed setting: We define connected and positive $q$-bimonoids and show that these are equivalent notions. We elaborate on the case of 0-bimonoids which is of independent interest. The duality and Hadamard functors are defined the same way as before. The former continues to be bistrong and the latter continues to be braided bilax (for suitable choices of the braiding parameters).

In addition to the above, the deformed theory has some new things to offer as well. We construct the signature functor whose main property is to send a $q$-Hopf monoid to a $(-q)$-Hopf monoid. The parity shift explains why this functor was not visible in the undeformed theory.

The main examples we provide are those of the signed exponential species denoted $\mathbf{E}^-$ and a one-parameter deformation of the linear order species denoted $\mathbf{L}_q$. The former is indeed the value of the signature functor on the exponential species $\mathbf{E}$.

We also introduce a cohomology theory for linearized comonoids in species. Our main emphasis is on the fact that a 2-cocycle on such a comonoid can be used to construct a deformation of that comonoid. As an example, we define two interesting 2-cocycles on $\mathbf{L}$, namely the Schubert and descent cocycles. The former is related to the Schubert statistic (2.13) and the latter to the descent statistic on permutations. We further show that $\mathbf{L}_q$ is precisely the deformation of $\mathbf{L}$ corresponding to the Schubert cocycle.

## 9.1. $q$-Hopf monoids

In this section, $q \in \Bbbk$ denotes a fixed scalar, possibly zero.

**9.1.1. A family of braidings on species.** We endow the monoidal category $(\mathsf{Sp}, \cdot)$ of species with a twist map

$$\beta_q \colon \mathbf{p} \cdot \mathbf{q} \to \mathbf{q} \cdot \mathbf{p}$$

that depends on $q$. The components of this map are as follows.

$$(\beta_q)_{S,T} \colon \mathbf{p}[S] \otimes \mathbf{q}[T] \to \mathbf{q}[T] \otimes \mathbf{p}[S], \qquad x \otimes y \mapsto q^{|S||T|} y \otimes x. \tag{9.1}$$

Note that $\beta_1 = \beta$ as given in (8.13). If $q$ is nonzero, then $\beta_q$ defines a braiding. The inverse braiding is $\beta_{q^{-1}}$, so $\beta_q$ is a symmetry if and only if $q = \pm 1$. Now consider the case $q = 0$. Note that $\beta_0$ is not invertible, hence it is not a braiding. However, it is a lax braiding.

The duality functor on the category of finite-dimensional species continues to be a bistrong monoidal functor

$$(-)^* : (\mathsf{Sp}^{\mathrm{op}}, \cdot, \beta_q^{\mathrm{op}}) \to (\mathsf{Sp}, \cdot, \beta_q)$$

and it continues to play an important role in the general theory.

**9.1.2. $q$-Hopf monoids.** A *$q$-bimonoid* is defined to be a bimonoid in the lax braided monoidal category $(\mathsf{Sp}, \cdot, \beta_q)$. A *$q$-Hopf monoid* is defined similarly as a Hopf monoid in $(\mathsf{Sp}, \cdot, \beta_q)$. (Recall that Hopf monoids can be defined in any lax braided monoidal category; this allows us to consider 0-bimonoids and 0-Hopf monoids and treat them on a par with the rest.)

On the other hand, the concept of a monoid or a comonoid does not involve the braiding, so a (co)monoid in $(\mathsf{Sp}, \cdot, \beta_q)$ is simply a (co)monoid in species.

We denote the category of $q$-bimonoids either by $\mathsf{Bimon}(\mathsf{Sp}, \cdot, \beta_q)$ or by $q$-$\mathsf{Bimon}(\mathsf{Sp})$, depending on convenience of the context. Similarly, we denote the category of $q$-Hopf monoids either by $\mathsf{Hopf}(\mathsf{Sp}, \cdot, \beta_q)$ or by $q$-$\mathsf{Hopf}(\mathsf{Sp})$. We employ similar notations for related categories.

**9.1.3. Connected and positive $q$-bimonoids.** Recall the symmetric monoidal categories $(\mathsf{Sp}^\circ, \cdot, \beta)$ and $(\mathsf{Sp}_+, \odot, \beta)$ of connected and positive species (Section 8.9). In both cases, one may replace $\beta$ by $\beta_q$ to obtain lax braided monoidal categories. For the former, one essentially uses the twist map $\beta_q$ for the Cauchy product, while for the latter,

$$\beta_q : \mathbf{p} + \mathbf{q} + \mathbf{p} \cdot \mathbf{q} \to \mathbf{q} + \mathbf{p} + \mathbf{q} \cdot \mathbf{p}$$

interchanges the first two terms and uses the twist map $\beta_q$ for the Cauchy product on the third term.

A *connected $q$-bimonoid* is defined to be a bimonoid in $(\mathsf{Sp}^\circ, \cdot, \beta_q)$. A *positive $q$-bimonoid* is defined to be a bimonoid in $(\mathsf{Sp}_+, \odot, \beta_q)$.

We have seen that positive and connected bimonoids (or Hopf monoids) are equivalent notions (Proposition 8.44). This generalizes to the deformed setting. More precisely, as a generalization of (8.61), there are equivalences of categories

$$\begin{aligned} \mathsf{Bimon}(\mathsf{Sp}^\circ, \cdot, \beta_q) &\cong \mathsf{Bimon}(\mathsf{Sp}_+, \odot, \beta_q), \\ \mathsf{Hopf}(\mathsf{Sp}^\circ, \cdot, \beta_q) &\cong \mathsf{Hopf}(\mathsf{Sp}_+, \odot, \beta_q). \end{aligned} \tag{9.2}$$

As for bimonoids (Proposition 8.10), a $q$-bimonoid for which the $\emptyset$-component is a Hopf algebra is automatically a $q$-Hopf monoid. In particular, a connected $q$-bimonoid, that is, a $q$-bimonoid whose $\emptyset$-component is of dimension 1, is always a $q$-Hopf monoid. In this case, Takeuchi's formula (8.27) for the antipode is still valid.

## 9.2. Connected 0-bimonoids

Positive (or equivalently, connected) $q$-bimonoids are of independent interest when $q = 0$. In this section, we discuss these objects in explicit terms. These 0-bimonoids are similar to the 0-bialgebras of Section 2.3.6.

This section is largely independent of the rest of the chapter.

**9.2.1. A diagrammatic characterization.** The compatibility conditions for a connected 0-bimonoid has features common with as well as distinct from those for a usual bimonoid. These conditions are provided in the result below which gives an explicit characterization of connected 0-bimonoids.

**Proposition 9.1.** *A connected* 0*-bimonoid is a* 5*-tuple* $(\mathbf{h}, \mu, \iota, \Delta, \epsilon)$ *such that* $\mathbf{h}$ *is a connected species,* $(\mathbf{h}, \mu, \iota)$ *is a connected monoid,* $(\mathbf{h}, \Delta, \epsilon)$ *is a connected comonoid, and the following diagrams commute.*

$$\begin{tikzcd} & \mathbf{h}[\emptyset] \arrow[dr, "\epsilon_\emptyset", "\cong"'] & \\ \Bbbk \arrow[ur, "\iota_\emptyset", "\cong"'] \arrow[rr, "\mathrm{id}"'] & & \Bbbk. \end{tikzcd} \tag{9.3}$$

*For any decomposition* $I = S \sqcup T$ *into subsets,*

$$\begin{tikzcd} & \mathbf{h}[I] \arrow[dr, "\Delta_{S,T}"] & \\ \mathbf{h}[S] \otimes \mathbf{h}[T] \arrow[ur, "\mu_{S,T}"] \arrow[rr, "\mathrm{id}_S \otimes \mathrm{id}_T"'] & & \mathbf{h}[S] \otimes \mathbf{h}[T]. \end{tikzcd} \tag{9.4}$$

*For any pair of decompositions* $I = S \sqcup T = S' \sqcup T'$ *such that neither* $S \subseteq S'$ *nor* $S' \subseteq S$*, or equivalently neither* $T \subseteq T'$ *nor* $T' \subseteq T$*,*

$$\begin{tikzcd} & \mathbf{h}[I] \arrow[dr, "\Delta_{S',T'}"] & \\ \mathbf{h}[S] \otimes \mathbf{h}[T] \arrow[ur, "\mu_{S,T}"] \arrow[rr, "0"'] & & \mathbf{h}[S'] \otimes \mathbf{h}[T']. \end{tikzcd} \tag{9.5}$$

*For any decomposition* $I = R \sqcup S \sqcup T$*,*

$$\begin{tikzcd} \mathbf{h}[R] \otimes \mathbf{h}[S \sqcup T] \arrow[r, "\mathrm{id}_R \otimes \Delta_{S,T}"] \arrow[d, "\mu_{R, S \sqcup T}"'] & \mathbf{h}[R] \otimes \mathbf{h}[S] \otimes \mathbf{h}[T] \arrow[d, "\mu_{R,S} \otimes \mathrm{id}_T"] \\ \mathbf{h}[R \sqcup S \sqcup T] \arrow[r, "\Delta_{R \sqcup S, T}"'] & \mathbf{h}[R \sqcup S] \otimes \mathbf{h}[T], \end{tikzcd} \tag{9.6}$$

$$\begin{tikzcd} \mathbf{h}[R \sqcup S] \otimes \mathbf{h}[T] \arrow[r, "\Delta_{R,S} \otimes \mathrm{id}_T"] \arrow[d, "\mu_{R \sqcup S, T}"'] & \mathbf{h}[R] \otimes \mathbf{h}[S] \otimes \mathbf{h}[T] \arrow[d, "\mathrm{id}_R \otimes \mu_{S,T}"] \\ \mathbf{h}[R \sqcup S \sqcup T] \arrow[r, "\Delta_{R, S \sqcup T}"'] & \mathbf{h}[R] \otimes \mathbf{h}[S \sqcup T]. \end{tikzcd} \tag{9.7}$$

PROOF. The compatibility conditions for a $q$-bimonoid are the same as those for an ordinary bimonoid (Section 8.3.1), inserting the braiding $\beta_q$ where appropriate.

Diagram (9.3) is the compatibility condition (8.20). This implies that $\iota_\emptyset$ and $\epsilon_\emptyset$ are inverse maps, given that $\mathbf{h}$ is connected. The remaining compatibility conditions (8.19) follow from the (co)monoid axioms.

It remains to consider the compatibility condition (8.18) involving the lax braiding $\beta_0$. Consider the situation of Lemma 8.7: we choose two arbitrary decompositions $I = S \sqcup T = S' \sqcup T'$ of a nonempty set $I$ and let $A, B, C, D$ denote the resulting intersections, as in Figure 8.1. According to (9.1), the map $(\beta_0)_{B,C}$ is 0 unless $B = \emptyset$ or $C = \emptyset$, in which case it is an identity.

It follows that if $B \neq \emptyset$ and $C \neq \emptyset$, then $\Delta_{S',T'}\mu_{S,T} = 0$. This happens precisely when neither $S \subseteq S'$ nor $S' \subseteq S$, thus (9.5).

If both $B = \emptyset$ and $C = \emptyset$, then all maps along the top of diagram (8.18) are identities, and therefore $\Delta_{S',T'}\mu_{S,T} = \mathrm{id}$. But this happens precisely when $S = S'$ and $T = T'$, resulting in (9.3).

If $B = \emptyset$ but $C \neq \emptyset$, then renaming $(S, C, D)$ to $(R, S, T)$, diagram (8.18) becomes (9.6). Similarly, the case $B \neq \emptyset$, $C = \emptyset$ of diagram (8.18) is equivalent to (9.7). □

The following is an immediate consequence of Proposition 9.1, in view of the equivalence between connected and positive $q$-bimonoids.

**Proposition 9.2.** *A positive* 0*-bimonoid is a triple* $(\mathbf{h}, \mu, \Delta)$ *such that* $\mathbf{h}$ *is a positive species,* $(\mathbf{h}, \mu)$ *is a positive monoid,* $(\mathbf{h}, \Delta)$ *is a positive comonoid, and for all decompositions of a nonempty set* $I$ *into nonempty subsets as in Proposition* 9.1*, diagrams* (9.4)*–*(9.7) *commute.*

**Remark 9.3.** There is an important overlap between the conditions in Proposition 9.1 and those in Corollary 8.38. This corollary states that axioms (9.4), (9.6), and (9.7) all hold for an ordinary connected bimonoid. Axiom (9.3) does too. It is axiom (9.5) that prevents an ordinary bimonoid from becoming a 0-bimonoid. An illustrative example is given in Section 9.5.3.

As for $q$-Hopf monoids, connected 0-bimonoids are always 0-Hopf monoids. A structure theorem for connected 0-Hopf monoids is given in Section 11.10.3.

**9.2.2. An alternative characterization.** There is an alternative way to characterize 0-bimonoids, which is parallel to the description of 0-bialgebras given in Propositions 2.11 and 2.12. Indeed, the compatibility condition for a positive 0-bimonoid can be written as follows. For each decomposition $I = S \sqcup T$ of a nonempty finite set $I$ into nonempty subsets,

$$\Delta(ab) = ab_{(1)} \otimes b_{(2)} + a_{(1)} \otimes a_{(2)}b + a \otimes b, \tag{9.8}$$

where $a \in \mathbf{h}[S]$, $b \in \mathbf{h}[T]$, the product $\mu(a, b)$ is denoted by $ab$, and Sweedler's notation is used for the coproduct: $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

This follows as for 0-bialgebras, examining the compatibility condition (1.9) in terms of the Cauchy product of positive species and the lax braiding $\beta_0$. It can also be directly related to the conditions in Proposition 9.2 as follows. Considering the components of $\mu$ and $\Delta$, the equality (9.8) is equivalent to a set of equalities, one for each pair of decompositions $I = S \sqcup T = S' \sqcup T'$ into nonempty subsets. As in the proof of Proposition 9.1, there are four possible cases, according to whether the intersections $S \cap T'$ and $S' \cap T$ are empty or not.

The term $a \otimes b$ contributes to the component for which both intersections are empty and is responsible for (9.4). The terms of the form $ab_{(1)} \otimes b_{(2)}$ contribute to the components for which $S \cap T' = \emptyset$ and $S' \cap T \neq \emptyset$, and are responsible for (9.6). Similarly, the terms of the form $a_{(1)} \otimes a_{(2)}b$ correspond to (9.7). There are no terms in the right-hand side of (9.8) contributing to the components for which neither intersection is empty; this yields (9.5).

Similarly, the compatibility condition for a connected 0-bimonoid is

$$\Delta(ab) = ab_{(1)} \otimes b_{(2)} + a_{(1)} \otimes a_{(2)}b - a \otimes b,$$

with $a$ and $b$ as above, for each arbitrary decomposition $I = S \sqcup T$ of a finite set $I$.

Recall the species of primitive elements from Section 8.10. When restricted to primitive elements, the product of a positive 0-bimonoid splits the coproduct.

**Lemma 9.4.** *Let* $(\mathbf{h}, \mu, \Delta)$ *be a positive* 0*-bimonoid. Then, for any* $k \geq 0$,

$$\Delta^{(k)}\mu^{(k)} = \mathrm{id}$$

*on* $\mathcal{P}(\mathbf{h})^{\cdot k}$.

PROOF. This follows from (9.8), by induction. □

## 9.3. The signed exponential species

Recall the Hopf monoid associated to the exponential species $\mathbf{E}$ (Example 8.15). In this section, we construct its signed analogue. It is an example of a $(-1)$-Hopf monoid. It displays the same features as $\mathbf{E}$ but in the world of $(-1)$-Hopf monoids.

**9.3.1. The signed exponential species.** Consider the species defined by

$$\mathbf{E}^{-}[I] := \mathrm{Det}(\Bbbk I) \tag{9.9}$$

where $\mathrm{Det}(V)$ denotes the highest exterior power of a finite-dimensional vector space $V$, and $\Bbbk I$ denotes the vector space over $\Bbbk$ with basis $I$. We call it the *signed exponential species*. The Det notation appears in the work of Ginzburg and Kapranov [146, Section (3.2.0)]. Note that the element

$$l^1 \wedge l^2 \wedge \cdots \wedge l^i \in \mathbf{E}^{-}[I]$$

provides a basis for $\mathbf{E}^{-}[I]$ for any linear order $l^1|\cdots|l^i$ on $I$. The basis elements corresponding to two linear orders differ at most by a sign. Note that $\mathbf{E}^{-}[n]$ is the sign representation of $\mathrm{S}_n$.

**9.3.2. $(-1)$-Hopf monoid.** We now turn $\mathbf{E}^{-}$ into a $(-1)$-Hopf monoid. The coproduct is given by

$$\begin{gathered}\mathbf{E}^{-}[I] \to \mathbf{E}^{-}[S] \otimes \mathbf{E}^{-}[T]\\ l^1 \wedge \cdots \wedge l^i \mapsto (-1)^{\mathrm{sch}_{S,T}(l)}\,(l^{i_1} \wedge \cdots \wedge l^{i_s}) \otimes (l^{j_1} \wedge \cdots \wedge l^{j_t})\end{gathered}$$

where $\{i_1 < \cdots < i_s\} = S$, and $\{j_1 < \cdots < j_t\} = T$, and $l = l^1|\cdots|l^i$, and $\mathrm{sch}_{S,T}(l)$ is the Schubert cocycle defined later in (9.12). We note that the coproduct is well-defined (independent of which linear order on $I$ is used).

The product is given by

$$\begin{gathered}\mathbf{E}^{-}[S] \otimes \mathbf{E}^{-}[T] \to \mathbf{E}^{-}[I]\\ (l^1 \wedge \cdots \wedge l^s) \otimes (m^1 \wedge \cdots \wedge m^t) \mapsto l^1 \wedge \cdots \wedge l^s \wedge m^1 \wedge \cdots \wedge m^t,\end{gathered}$$

where $S = \{l^1, \ldots, l^s\}$ and $T = \{m^1, \ldots, m^t\}$.

For example,

$$\begin{aligned}u \wedge m \wedge a \mapsto 1 \otimes (u \wedge m \wedge a) + u \otimes (m \wedge a) - m \otimes (u \wedge a) + a \otimes (u \wedge m)\\ + (u \wedge m) \otimes a - (u \wedge a) \otimes m + (m \wedge a) \otimes u + (u \wedge m \wedge a) \otimes 1.\end{aligned}$$

$$(l \wedge a \wedge k) \otimes (s \wedge h \wedge m \wedge i) \mapsto l \wedge a \wedge k \wedge s \wedge h \wedge m \wedge i.$$

It is routine to check that $\mathbf{E}^-$ is a $(-1)$-Hopf monoid; we call it the signed partner of the exponential species $\mathbf{E}$. The antipode is given by

$$\mathrm{s}\colon \mathbf{E}^-[I] \to \mathbf{E}^-[I] \qquad\qquad x \mapsto (-1)^{|I|}\, x.$$

For example,

$$\mathrm{s}(s \wedge h \wedge i \wedge v \wedge a) = -(s \wedge h \wedge i \wedge v \wedge a).$$

Further, $\mathbf{E}^-$ is commutative, cocommutative and self-dual (over any field), just as $\mathbf{E}$.

**9.3.3. Up-down operators.** Recall that the exponential species carries up-down operators (Example 8.55). The signed exponential species is also a species with up-down operators:

$$\begin{aligned} d\colon \mathbf{E}^-[I^+] &\to \mathbf{E}^-[I] & \qquad u\colon \mathbf{E}^-[I] &\to \mathbf{E}^-[I^+] \\ *_I \wedge (-) &\mapsto (-) & (-) &\mapsto *_I \wedge (-) \end{aligned}$$

In other words, the special element $*_I$ should be inserted in front while going up, and brought to the front and deleted while going down.

Let the deformation $\cdot_{-1}$ be as defined in (8.70). Analogous to the case of the exponential species, one can check that $(\mathbf{E}^-, d)$ is a monoid in $(\mathsf{Sp}_\mathsf{d}, \cdot_{-1})$, and dually $(\mathbf{E}^-, u)$ is a comonoid in $(\mathsf{Sp}^\mathsf{u}, \cdot_{-1})$.

## 9.4. The Hadamard and signature functors

Recall that the Hadamard product of two bimonoids is again a bimonoid (Section 8.13.1). In this section, we note that the Hadamard product of a $p$-bimonoid and a $q$-bimonoid is a $pq$-bimonoid and discuss some related results.

The consideration of $q$-bimonoids opens up a new possibility: a functor to transform a $q$-bimonoid into a $(-q)$-bimonoid. We call this the signature functor, see Corollary 9.10.

Throughout this section, $p$ and $q$ are fixed scalars, possibly zero.

**9.4.1. The Hadamard functor.** Recall that using an interchange law on species, we constructed a bilax monoidal functor

$$(\times, \varphi, \psi)\colon (\mathsf{Sp} \times \mathsf{Sp}, \cdot, \beta) \to (\mathsf{Sp}, \cdot, \beta)$$

(Proposition 8.58). We call this the Hadamard functor since it sends a pair $(\mathbf{p}_1, \mathbf{p}_2)$ to their Hadamard product $\mathbf{p}_1 \times \mathbf{p}_2$. We now show that the Hadamard functor continues to be bilax provided the braidings on species are deformed appropriately.

**Proposition 9.5.** *The functor*

$$(\times, \varphi, \psi)\colon (\mathsf{Sp} \times \mathsf{Sp}, \cdot, \beta_p \times \beta_q) \to (\mathsf{Sp}, \cdot, \beta_{pq})$$

*is a normal braided bilax monoidal functor. On finite-dimensional species, it is self-dual.*

Proof. We follow the notation in the proof of Proposition 8.58. The main diagram to check for commutativity is (8.75); the two occurrences of $\beta$ in that diagram are now replaced by $\beta_{pq}$ and $\beta_p \times \beta_q$. From Proposition 8.58, this new diagram commutes up to a power of $p$ and $q$. Following the two sides of the diagram, one checks that the powers of $p$ and $q$ are both $tu$ where $t = |T_1| = |T_2|$ and

$u = |U_1| = |U_2|$. The remaining diagrams are checked similarly. This shows that the Hadamard functor is bilax. The fact that it is braided boils down to the identity

$$(pq)^{|S|\,|T|} = p^{|S|\,|T|} q^{|S|\,|T|}.$$

Since the structure maps $\varphi$ and $\psi$ are the same as before, the normal property and self-duality follow from Propositions 8.58 and 8.60. □

We record some consequences; they provide $q$-analogues to Corollaries 8.59, 8.61 and 8.62.

**Corollary 9.6.** *If $\mathbf{h}_1$ is a $p$-bimonoid and $\mathbf{h}_2$ is a $q$-bimonoid, then $\mathbf{h}_1 \times \mathbf{h}_2$ is a $pq$-bimonoid. Further, if $\mathbf{h}_1$ and $\mathbf{h}_2$ are (co)commutative, then so is $\mathbf{h}_1 \times \mathbf{h}_2$.*

The same statement holds for Hopf monoids.

**Corollary 9.7.** *If $\mathbf{h}$ is a finite-dimensional $q$-Hopf monoid, then $\mathbf{h}^* \times \mathbf{h}$ is a self-dual $q^2$-Hopf monoid.*

**Corollary 9.8.** *If $\mathbf{h}_1$ and $\mathbf{h}_2$ are finite-dimensional $p$- and $q$-Hopf monoids respectively, then*

$$(\mathbf{h}_1 \times \mathbf{h}_2)^* \cong \mathbf{h}_1^* \times \mathbf{h}_2^*$$

*as $pq$-Hopf monoids.*

**9.4.2. The signature functor on species.** In Section 9.3, we discussed the signed analogue $\mathbf{E}^-$ of the exponential species $\mathbf{E}$. We now show that the situation is quite general. Namely, there is a correspondence between $p$-Hopf monoids and $(-p)$-Hopf monoids.

Define the *signature functor* to be the functor $(-)^- \colon \mathsf{Sp} \to \mathsf{Sp}$ that sends a species $\mathbf{p}$ to the species

$$\mathbf{p}^- := \mathbf{p} \times \mathbf{E}^-. \tag{9.10}$$

Since $\mathbf{E}$ is the unit for the Hadamard product, the notation $\mathbf{E}^-$, which gets defined twice, is unambiguous. We refer to $\mathbf{p}^-$ as the *signed partner* of $\mathbf{p}$.

The signature functor is an involution:

$$(\mathbf{p}^-)^- \cong \mathbf{p}.$$

This is because $\mathbf{E}^- \times \mathbf{E}^- \cong \mathbf{E}$.

**Proposition 9.9.** *The signature functor $(-)^-$ gives rise to a bistrong monoidal functor*

$$(\mathsf{Sp}, \cdot, \beta_p) \to (\mathsf{Sp}, \cdot, \beta_{-p}).$$

Proof. The signature functor is bilax since $\mathbf{E}^-$ is a $(-1)$-Hopf monoid, and the Hadamard functor is bilax (Proposition 9.5). Further, each component of the product and coproduct of $\mathbf{E}^-$ is bijective; so the structure morphisms of the signature functor are invertible, and hence it is bistrong. □

This establishes a correspondence between $p$-Hopf monoids and $(-p)$-Hopf monoids:

**Corollary 9.10.** *The signature functor $(-)^-$ takes a $p$-Hopf monoid to a $(-p)$-Hopf monoid. Further, the transformation $\big((-)^-\big)^- \Rightarrow \mathrm{id}$ is an isomorphism of bilax monoidal functors.*

The duality and the signature functors commute (up to an isomorphism of bilax functors). This follows from the self-duality of $\mathbf{E}^-$ and Corollary 9.8. As a consequence:

**Proposition 9.11.** *On finite-dimensional species, the signature functor is self-dual.*

**Proposition 9.12.** *The following diagram commutes* (*up to isomorphism*) *as bilax monoidal functors.*

$$\begin{tikzcd}
(\mathsf{Sp}\times\mathsf{Sp},\cdot,\beta_p\times\beta_q) \arrow[r, "\times"] \arrow[d, "(\mathrm{id},(-)^-)"'] & (\mathsf{Sp},\cdot,\beta_{pq}) \arrow[d, "(-)^-"] \\
(\mathsf{Sp}\times\mathsf{Sp},\cdot,\beta_p\times\beta_{-q}) \arrow[r, "\times"'] & (\mathsf{Sp},\cdot,\beta_{-pq})
\end{tikzcd}$$

Proof. Starting with say $(\mathbf{p}_1,\mathbf{p}_2)$, following the two directions, one ends up with $(\mathbf{p}_1\times\mathbf{p}_2)\times\mathbf{E}^-$ and $\mathbf{p}_1\times(\mathbf{p}_2\times\mathbf{E}^-)$. We identify these via the associativity constraint in $(\mathsf{Sp},\times)$. The commutativity of the following diagram implies that the lax structures match up.

$$\begin{tikzcd}
(\mathbf{p}_1\times\mathbf{p}_2\times\mathbf{E}^-)\cdot(\mathbf{q}_1\times\mathbf{q}_2\times\mathbf{E}^-) \arrow[r] \arrow[d] & \big((\mathbf{p}_1\times\mathbf{p}_2)\cdot(\mathbf{q}_1\times\mathbf{q}_2)\big)\times\mathbf{E}^- \arrow[d] \\
(\mathbf{p}_1\cdot\mathbf{q}_1)\times\big((\mathbf{p}_2\times\mathbf{E}^-)\cdot(\mathbf{q}_2\times\mathbf{E}^-)\big) \arrow[r] & (\mathbf{p}_1\cdot\mathbf{q}_1)\times(\mathbf{p}_2\cdot\mathbf{q}_2)\times\mathbf{E}^-.
\end{tikzcd}$$

The maps are defined using the lax structure of the Hadamard functor and the monoid structure of $\mathbf{E}^-$.

For the colax structures, one verifies the commutativity of the above diagram with the arrows reversed. This is again straightforward. □

## 9.5. The $q$-Hopf monoids of linear orders

Recall from Examples 8.16 and 8.24 the Hopf monoids $\mathbf{L}$ and $\mathbf{L}^*$ based on linear orders. In this section, we show that these Hopf monoids admit one-parameter deformations. We denote them by $\mathbf{L}_q$ and $\mathbf{L}_q^*$. These are examples of $q$-Hopf monoids and dual to each other.

**9.5.1. The Schubert cocycle.** Let $\mathrm{L}[I]$ denote the set of linear orders on a finite set $I$. Given $l\in\mathrm{L}[I]$ and a decomposition $I=S\sqcup T$, let

$$\mathrm{Sch}_{S,T}(l) := \{(i,j)\in S\times T \mid i>j \text{ according to } l\} \tag{9.11}$$

and

$$\mathrm{sch}_{S,T}(l) := |\mathrm{Sch}_{S,T}(l)|. \tag{9.12}$$

For instance, if

$$l = s|h|i|v|a, \quad S=\{i,s,a\}, \quad T=\{v,h\},$$

then

$$\mathrm{Sch}_{S,T}(l) = \{(i,h),(a,h),(a,v)\} \quad \text{and} \quad \mathrm{sch}_{S,T}(l)=3.$$

This is a reformulation of the Schubert statistic (2.13). If $I = [n]$ and $l = 1|\cdots|n$, then

$$\operatorname{Sch}_{S,T}(l) = \operatorname{Sch}_n(S) \quad \text{and} \quad \operatorname{sch}_{S,T}(l) = \operatorname{sch}_n(S). \tag{9.13}$$

We view $\operatorname{sch}_{S,T}$ as an integer-valued function on $\mathrm{L}[I]$ and refer to the family of maps $\operatorname{sch}_{S,T}$ as the *Schubert cocycle*.

It satisfies the following properties. These are analogues of (2.14)–(2.18), and can be deduced from them.

$$\operatorname{sch}_{I,\emptyset}(l) = \operatorname{sch}_{\emptyset,I}(l) = 0. \tag{9.14}$$

$$\operatorname{sch}_{S,T}(l) + \operatorname{sch}_{T,S}(l) = |S|\,|T|. \tag{9.15}$$

$$\operatorname{sch}_{S,T}(l) = \operatorname{sch}_{T,S}(\bar{l}), \tag{9.16}$$

where $\bar{l}$ denotes the linear order opposite to $l$.

For any decomposition $I = R \sqcup S \sqcup T$, and for any linear order $l$ on $I$,

$$\operatorname{sch}_{R,S\sqcup T}(l) + \operatorname{sch}_{S,T}(l|_{S\sqcup T}) = \operatorname{sch}_{R\sqcup S,T}(l) + \operatorname{sch}_{R,S}(l|_{R\sqcup S}). \tag{9.17}$$

This is the cocycle condition.

Consider a pair of decompositions $I = S \sqcup T = S' \sqcup T'$ and let $A$, $B$, $C$, and $D$ be the resulting intersections, as in Lemma 8.7. Then

$$\operatorname{sch}_{S',T'}(l \cdot m) = \operatorname{sch}_{A,B}(l) + \operatorname{sch}_{C,D}(m) + |B||C|, \tag{9.18}$$

for any linear order $l$ on $S$, and linear order $m$ on $T$. This is the multiplicative property of the cocycle.

Cocycles and related notions are explained in more depth in Section 9.6.

### 9.5.2. $q$-Hopf monoid.

**Definition 9.13.** For the species $\mathbf{L}_q$ of linear orders, the product and coproduct are:

$$\begin{aligned} \mathbf{L}_q[S] \otimes \mathbf{L}_q[T] &\to \mathbf{L}_q[I] \\ l_1 \otimes l_2 &\mapsto l_1 \cdot l_2 \end{aligned} \qquad\qquad \begin{aligned} \mathbf{L}_q[I] &\to \mathbf{L}_q[S] \otimes \mathbf{L}_q[T] \\ l &\mapsto q^{\operatorname{sch}_{S,T}(l)}\, l|_S \otimes l|_T, \end{aligned}$$

where $\operatorname{sch}_{S,T}(l)$ is the Schubert cocycle (9.12).

For example,

$$l|a|k \otimes s|h|m|i \mapsto l|a|k|s|h|m|i,$$

$$\begin{aligned} u|m|a \mapsto 1 \otimes u|m|a + u \otimes m|a + q(m \otimes u|a) + q^2(a \otimes u|m) \\ + u|m \otimes a + q(u|a \otimes m) + q^2(m|a \otimes u) + u|m|a \otimes 1. \end{aligned}$$

A comparison with Example 8.16 shows that the product is as before, while the coproduct is deformed by a power of $q$. We need to check that $\mathbf{L}_q$ is a $q$-Hopf monoid. It is clear that the necessary diagrams commute up to a power of $q$. Checking that the powers of $q$ work out correctly boils down to basic properties of the Schubert cocycle. For example, the coassociativity of the coproduct follows from the cocycle condition (9.17). The compatibility between the product and coproduct follows from (9.18).

**Proposition 9.14.** *The antipode of $\mathbf{L}_q$ is given by*

$$\mathrm{s}_I(l) = (-1)^{|I|} q^{\binom{|I|}{2}}\, \bar{l},$$

*where $\bar{l}$ is the reversal of the linear order $l$ on a set $I$.*

Proof. This formula can be most easily derived by direct verification of axioms (8.21)–(8.23), or by reducing to the case of singletons, as in Example 8.16. We provide details on how Takeuchi's formula (8.27) also yields the result.

The formula gives

$$\mathrm{s}_I(l) = \sum_{\substack{S_1 \sqcup \cdots \sqcup S_k = I \\ S_i \neq \emptyset \;\; k \geq 1}} (-1)^k q^{\mathrm{sch}_{S_1,\ldots,S_k}(l)}\, l|_{S_1} \cdots l|_{S_k},$$

where $\mathrm{sch}_{S_1,\ldots,S_k}(l)$ is inductively defined by

$$\mathrm{sch}_I(l) := 0,$$

$$\mathrm{sch}_{S,T}(l) \text{ is the Schubert cocycle (9.12)},$$

$$\mathrm{sch}_{S_1,\ldots,S_k}(l) := \mathrm{sch}_{S_1,\ldots,S_{k-1}}(l|_{S_1 \sqcup \cdots \sqcup S_{k-1}}) + \mathrm{sch}_{S_1 \sqcup \cdots \sqcup S_{k-1}, S_k}(l).$$

Equivalently, $\mathrm{sch}_{S_1,\ldots,S_k}(l)$ is the number of pairs $(i,j) \in I^2$ such that $i > j$ according to $l$ and there are $h < k$ with $i \in S_h$ and $j \in S_k$.

The result follows once we show that for any pair of linear orders $l$ and $l'$ on $I$, we have

$$\sum_{\substack{(S_1,\ldots,S_k) \\ k \geq 1}} (-1)^k q^{\mathrm{sch}_{S_1,\ldots,S_k}(l)} = \begin{cases} (-1)^{|I|} q^{\binom{|I|}{2}} & \text{if } l' = \bar{l} \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all ordered decompositions $I = S_1 \sqcup \cdots \sqcup S_k$ into nonempty intervals of $l'$ on which $l$ and $l'$ agree.

To prove the above claim, we begin by noting that the term involving $q$ in the left-hand side only depends on $l$ and $l'$. So it can be pulled out of the sum. The claim now follows by applying (8.30). $\square$

For example,

$$\mathrm{s}(u|m|a) = -q^3 (a|m|u).$$

Let $\mathbf{L}_q^*$ be the $q$-Hopf monoid dual to $\mathbf{L}_q$. Its product, coproduct, and antipode can be written down by dualizing the above formulas.

**9.5.3. 0-Hopf monoid.** We now briefly discuss the connected 0-Hopf monoid $\mathbf{L}_0$ (set $q = 0$ in the above discussion). The product is concatenation of linear orders

$$\mathbf{L}_0[S] \otimes \mathbf{L}_0[T] \to \mathbf{L}_0[I], \qquad l_1 \otimes l_2 \mapsto l_1 \cdot l_2.$$

Note that the Schubert statistic $\mathrm{sch}_{S,T}(l)$ is 0 precisely when $S$ is an initial segment of $l$. The coproduct of $\mathbf{L}_0$ is therefore deconcatenation:

$$\mathbf{L}_0[I] \to \mathbf{L}_0[S] \otimes \mathbf{L}_0[T], \qquad l \mapsto \begin{cases} l|_S \otimes l|_T & \text{if } S \text{ is an initial segment of } l, \\ 0 & \text{otherwise.} \end{cases}$$

We comment on the conditions of Proposition 9.1. Consider decompositions $I = S \sqcup T = S' \sqcup T'$. If $l$ is the concatenation of linear orders $l_1$ on $S$ and $l_2$ on $T$, and $S'$ is an initial segment of $l$, then necessarily $S \subseteq S'$ or $S' \subseteq S$. This is why (9.5) holds. Note this axiom fails for the bimonoid $\mathbf{L}$, for which the coproduct is restriction.

The remaining conditions in Proposition 9.1 express further compatibilities between concatenation and deconcatenation of linear orders which are easily verified.

It follows from Proposition 9.14 that the components of the antipode of $\mathbf{L}_0$ are simply

$$(\mathrm{s}_0)_I = \begin{cases} \mathrm{id} & \text{if } I = \emptyset, \\ -\mathrm{id} & \text{if } I \text{ is a singleton}, \\ 0 & \text{otherwise.} \end{cases} \tag{9.19}$$

It is interesting to note that $\mathbf{L}_0$ is self-dual: the map

$$\mathbf{L}_0 \to (\mathbf{L}_0)^*, \qquad l \mapsto l^*$$

is an isomorphism of 0-Hopf monoids. This fact will be generalized later in Proposition 12.6.

**9.5.4. The signed partners.** There is an isomorphism

$$\mathbf{L}_q \times \mathbf{E}^- \to \mathbf{L}_{-q} \qquad (l^1|\cdots|l^i) \otimes (l^1 \wedge \cdots \wedge l^i) \mapsto l^1|\cdots|l^i.$$

of $(-q)$-Hopf monoids. The same result holds with $\mathbf{L}_q^*$ instead of $\mathbf{L}_q$. It follows that the signed partners of $\mathbf{L}_q$ and $\mathbf{L}_q^*$ are:

$$\mathbf{L}_q^- \cong \mathbf{L}_{-q} \qquad \text{and} \qquad (\mathbf{L}_q^*)^- \cong \mathbf{L}_{-q}^*. \tag{9.20}$$

In particular, $\mathbf{L}$ and $\mathbf{L}_{-1}$ are signed partners.

Now applying the signature functor to the morphism $\mathbf{L} \to \mathbf{E}$ of (8.31) yields the morphism

$$\pi_{-1} \colon \mathbf{L}_{-1} \to \mathbf{E}^- \qquad l^1|\cdots|l^i \mapsto l^1 \wedge \cdots \wedge l^i.$$

of $(-1)$-Hopf monoids. More generally, applying the signature functor to (8.34) yields the following commutative diagram.

$$\begin{tikzcd} \mathbf{L}_{-1} \arrow[r] \arrow[d, "\pi_{-1}"'] & \mathbf{L}_{-1}^* \\ \mathbf{E}^- \arrow[r, "\mathrm{id}"'] & (\mathbf{E}^-)^* \arrow[u, "\pi_{-1}^*"'] \end{tikzcd} \tag{9.21}$$

The top horizontal map is given by

$$\mathbf{L}_{-1} \to \mathbf{L}_{-1}^* \qquad l' \mapsto \sum (-1)^{\mathrm{dist}(l',l)} l^*,$$

where the sum is over all linear orders $l$ on $I$, and $\mathrm{dist}(l', l)$ is as defined in (10.27). It counts the numbers of adjacent transpositions necessary to go from $l'$ to $l$. For example,

$$u|m|a \mapsto u|m|a - m|u|a - u|a|m + m|a|u + a|u|m - a|m|u.$$

**9.5.5. Up-down operators.** In Example 8.56, we saw that the linear order species is a species with up-down operators, and further $(\mathbf{L}, u)$ is a comonoid in $(\mathsf{Sp}^{\mathsf{u}}, \cdot)$. In the present context, this fact can be generalized as follows. Consider the deformation $\cdot_q$ defined in (8.70). Then, $(\mathbf{L}_q, u)$ is a comonoid in $(\mathsf{Sp}^{\mathsf{u}}, \cdot_q)$, and dually $(\mathbf{L}_q^*, d)$ is a monoid in $(\mathsf{Sp}_{\mathsf{d}}, \cdot_q)$.

## 9.6. Cohomology of linearized comonoids in species

In this section, we lay out the basics of a cohomology theory of linearized comonoids in species, with emphasis on the deformation corresponding to a 2-cocycle. This theory is related to the cohomology of the comonoid $\mathbf{E}$ with coefficients in a bicomodule. More general constructions are possible, but we do not consider them.

Examples of cocycles and the associated deformations are given in Section 9.7 and Chapter 13.

Throughout Part II, we are assuming that $\Bbbk$ is a field. In this section however, we prefer to work in greater generality and let $\Bbbk$ be a commutative ring. Accordingly, vector species take values in the category of $\Bbbk$-modules, and the linearization of a set species is performed over the ring $\Bbbk$.

Let $\mathbb{Z}$ denote the group of integers under addition and $\mathbb{N}$ the set of nonnegative integers. We also work with an arbitrary abelian group $\mathbb{A}$. Throughout this section, we assume that $\mathbf{p} := \Bbbk\mathrm{P}$ is a linearized comonoid, as in Section 8.7.2.

**9.6.1. Low dimensional cocycles.** Let $\mathbf{p} = \Bbbk\mathrm{P}$ be as above. A 1-*cochain* on $\mathbf{p}$ is a family $\alpha$ of (set) maps

$$\alpha_I \colon \mathrm{P}[I] \to \mathbb{Z},$$

one for each finite set $I$, which is natural in $I$. In other words, for any bijection $\sigma\colon I \to J$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{P}[I] \arrow[d, "{\mathrm{P}[\sigma]}"'] \arrow[dr, "\alpha_I"] & \\ \mathrm{P}[J] \arrow[r, "\alpha_J"'] & \mathbb{Z}. \end{tikzcd}$$

The 1-cochain $\alpha$ is *normal* if

$$\alpha_\emptyset(x) = 0 \tag{9.22}$$

for any $x \in \mathrm{P}[\emptyset]$. It is a 1-*cocycle* if

$$\alpha_I(x) = \alpha_S(x|_S) + \alpha_T(x/_S) \tag{9.23}$$

for any $x \in \mathrm{P}[I]$ and any decomposition $I = S \sqcup T$.

A 2-*cochain* on $\mathbf{p}$ is a family $\gamma$ of (set) maps

$$\gamma_{S,T} \colon \mathrm{P}[I] \to \mathbb{Z},$$

one for each decomposition $I = S \sqcup T$, which is natural in $I$. In other words, for any bijection $\sigma\colon I \to J$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{P}[I] \arrow[d, "{\mathrm{P}[\sigma]}"'] \arrow[dr, "\gamma_{S,T}"] & \\ \mathrm{P}[J] \arrow[r, "\gamma_{\sigma(S),\sigma(T)}"'] & \mathbb{Z}. \end{tikzcd}$$

The 2-cochain $\gamma$ is *normal* if

$$\gamma_{I,\emptyset}(x) = \gamma_{\emptyset,I}(x) = 0 \tag{9.24}$$

for any $x \in \mathrm{P}[I]$. It is a 2-*cocycle* if

$$\gamma_{R,S\sqcup T}(x) + \gamma_{S,T}(x/_R) = \gamma_{R\sqcup S,T}(x) + \gamma_{R,S}(x|_{R\sqcup S}) \tag{9.25}$$

for any $x \in \mathrm{P}[I]$ and any decomposition $I = R \sqcup S \sqcup T$.

Let $\alpha$ be a 1-cochain. Consider the 2-cochain $d(\alpha)$ given by

$$d(\alpha)_{S,T}(x) := \alpha_S(x|_S) + \alpha_T(x/_S) - \alpha_I(x). \tag{9.26}$$

**Lemma 9.15.** *The 2-cochain $d(\alpha)$ is in fact a 2-cocycle. If $\alpha$ is normal, then so is $d(\alpha)$.*

PROOF. The left-hand side of (9.25) is

$$\begin{aligned} &d(\alpha)_{R,S\sqcup T}(x) + d(\alpha)_{S,T}(x/_R) \\ &\quad = \alpha_R(x|_R) + \alpha_{S\sqcup T}(x/_R) - \alpha_I(x) + \alpha_S\big((x/_R)|_S\big) + \alpha_T\big((x/R)/S\big) - \alpha_{S\sqcup T}(x/_R). \end{aligned}$$

In view of (8.43), this equals

$$\begin{aligned} \alpha_R\big((x|_{R\sqcup S})_R\big) - \alpha_I(x) + \alpha_S\big((x|_{R\sqcup S})/_R\big) + \alpha_T(x/_{R\sqcup S}) \\ = \alpha_{R\sqcup S}(x|_{R\sqcup S}) + \alpha_R\big((x|_{R\sqcup S})_R\big) - \alpha_I(x) + \alpha_S\big((x|_{R\sqcup S})/_R\big) \\ + \alpha_T(x/_{R\sqcup S}) - \alpha_{R\sqcup S}(x|_{R\sqcup S}) \\ = d(\alpha)_{R\sqcup S,T}(x) + d(\alpha)_{R,S}(x|_{R\sqcup S}). \end{aligned}$$

□

The 2-cocycles of the form $d(\alpha)$ for some 1-cochain $\alpha$ are called 2-*coboundaries*.

Let $C^1(\mathbf{p}, \mathbb{Z})$ denote the set of 1-cochains on $\mathbf{p}$. It is a group under pointwise addition: given 1-cochains $\alpha$ and $\beta$, we set

$$(\alpha + \beta)_I(x) := \alpha_I(x) + \beta_I(x)$$

for all $x \in \mathrm{P}[I]$. Similarly, the set $C^2(\mathbf{p}, \mathbb{Z})$ of 2-cochains on $\mathbf{p}$ is a group under addition, and the map

$$d \colon C^1(\mathbf{p}, \mathbb{Z}) \to C^2(\mathbf{p}, \mathbb{Z})$$

defined by (9.26) is a morphism of groups.

Let $Z^k(\mathbf{p}, \mathbb{Z})$ denote the set of $k$-cocycles, $k = 1, 2$, and $B^2(\mathbf{p}, \mathbb{Z})$ the set of 2-coboundaries. The set $Z^1(\mathbf{p}, \mathbb{Z})$ is the kernel of $d$ and the set $B^2(\mathbf{p}, \mathbb{Z})$ is the image. By Lemma 9.15, $B^2(\mathbf{p}, \mathbb{Z}) \subseteq Z^2(\mathbf{p}, \mathbb{Z})$.

The *second cohomology group of* $\mathbf{p}$ is the quotient

$$H^2(\mathbf{p}, \mathbb{Z}) := Z^2(\mathbf{p}, \mathbb{Z})/B^2(\mathbf{p}, \mathbb{Z}). \tag{9.27}$$

Two 2-cocycles are *cohomologous* if they differ by a 2-coboundary; that is, if they have the same image in the second cohomology group.

**9.6.2. Comonoid deformations.** Let $\mathbf{p} = \Bbbk\mathrm{P}$ be a linearized comonoid, as above. Fix a scalar $q \in \Bbbk$ and a normal 2-cocycle $\gamma$ on $\mathbf{p}$. Assume either that $\gamma$ takes values in $\mathbb{N}$, or $q$ is invertible in $\Bbbk$. In this situation, we may define a map

$$\Delta_\gamma \colon \mathbf{p} \to \mathbf{p} \cdot \mathbf{p}$$

as follows. Given a decomposition $I = S \sqcup T$, we set

$$(\Delta_\gamma)_{S,T} \colon \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T], \quad x \mapsto q^{\gamma_{S,T}(x)} x|_S \otimes x/_S, \tag{9.28}$$

for any $x \in \mathrm{P}[I]$ (and extend by linearity).

The dependence of $\Delta_\gamma$ on $q$ is not reflected in the notation. However, in contexts where $\gamma$ is understood, we may write $\Delta_q$ instead of $\Delta_\gamma$.

**Proposition 9.16.** *In the above situation, $(\mathbf{p}, \Delta_\gamma, \epsilon)$ is a comonoid in $(\mathsf{Sp}, \cdot)$.*

PROOF. Coassociativity of $\Delta_\gamma$ follows from that of $\Delta$ plus (9.25). Counitality follows similarly from (9.24). $\square$

We say that the coproduct $\Delta_\gamma$ is a *deformation* of $\Delta$.

Let $\gamma_1$ and $\gamma_2$ be two cohomologous normal 2-cocycles, and $\alpha$ a normal 1-cochain such that

$$\gamma_2 - \gamma_1 = d(\alpha).$$

Assume either that $\alpha$ takes values in $\mathbb{N}$, or $q$ is invertible in $\Bbbk$. In this situation, we may define a map $f_\alpha \colon \mathbf{p} \to \mathbf{p}$ by

$$(f_\alpha)_I \colon \mathbf{p}[I] \to \mathbf{p}[I], \quad x \mapsto q^{\alpha_I(x)}x, \tag{9.29}$$

for any $x \in \mathrm{P}[I]$ (and extend by linearity).

**Proposition 9.17.** *The map $f_\alpha$ is a morphism of comonoids*

$$(\mathbf{p}, \Delta_{\gamma_1}, \epsilon) \to (\mathbf{p}, \Delta_{\gamma_2}, \epsilon).$$

*If $q$ is invertible in $\Bbbk$, then $f_\alpha$ is an isomorphism.*

PROOF. The hypothesis and (9.26) give

$$(\gamma_2)_{S,T}(x) - (\gamma_1)_{S,T}(x) = \alpha_S(x|_S) + \alpha_T(x/_S) - \alpha_I(x).$$

which implies the commutativity of

$$\begin{tikzcd} \mathbf{p}[I] \arrow[r, "(\Delta_{\gamma_1})_{S,T}"] \arrow[d, "(f_\alpha)_I"'] & \mathbf{p}[S] \otimes \mathbf{p}[T] \arrow[d, "(f_\alpha)_S \otimes (f_\alpha)_T"] \\ \mathbf{p}[I] \arrow[r, "(\Delta_{\gamma_2})_{S,T}"'] & \mathbf{p}[S] \otimes \mathbf{p}[T]. \end{tikzcd}$$

Thus, $f_\alpha$ preserves coproducts. Similarly, $f_\alpha$ preserves counits by (9.22). $\square$

Suppose that $\gamma$ is a 2-coboundary. The coproduct corresponding to the trivial cocycle (the zero map) is just the original coproduct $\Delta$. Therefore, if $q$ is invertible, the deformed comonoid $(\mathbf{p}, \Delta_\gamma, \epsilon)$ is isomorphic to the undeformed comonoid $(\mathbf{p}, \Delta, \epsilon)$. If $q$ is not invertible (and $\gamma$ and $\alpha$ take values in $\mathbb{N}$), we only obtain a morphism

$$f_\alpha \colon (\mathbf{p}, \Delta, \epsilon) \to (\mathbf{p}, \Delta_\gamma, \epsilon).$$

If $q = 0$, then the coproduct $\Delta_\gamma$ takes the form

$$x \mapsto \begin{cases} x|S \otimes x/S & \text{if } \gamma_{S,T}(x) = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{9.30}$$

and the morphism $f_\alpha$ is

$$x \mapsto \begin{cases} x & \text{if } \alpha_I(x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**9.6.3. Multiplicative cocycles.** We assume now that $\mathbf{p} = \Bbbk\mathrm{P}$ is a linearized bimonoid in $(\mathsf{Sp}, \cdot, \beta)$, as in Section 8.7.3.

We say that a 1-cochain $\alpha$ on the linearized bimonoid $\mathbf{p}$ is *multiplicative* if

$$\alpha_I(x \cdot y) = \alpha_S(x) + \alpha_T(y), \tag{9.31}$$

for any decomposition $I = S \sqcup T$, $x \in \mathrm{P}[S]$, $y \in \mathrm{P}[T]$.

**Lemma 9.18.** *Let $\alpha$ be a multiplicative 1-cochain on a linearized bimonoid $\mathbf{p}$.*

(i) *The cochain $\alpha$ satisfies the unit condition*

$$\alpha_{\emptyset}(1) = 0. \tag{9.32}$$

(ii) *If $\mathbf{p}$ is connected, then $\alpha$ is normal.*

PROOF. Choosing $I = \emptyset$, and $x = y = 1$ in (9.31), we deduce (9.32). If $\mathbf{p}$ is connected, then $\mathrm{P}[\emptyset] = \{1\}$, and normality (9.22) boils down to (9.32). $\square$

Fix an integer $m \in \mathbb{Z}$. Consider a pair of decompositions $I = S \sqcup T = S' \sqcup T'$ and let $A$, $B$, $C$, and $D$ be the resulting intersections, as in Lemma 8.7. We say that a 2-cochain $\gamma$ on the bimonoid $\mathbf{p}$ is *multiplicative of twist $m$* if

$$\gamma_{S',T'}(x \cdot y) = \gamma_{A,B}(x) + \gamma_{C,D}(y) + m \cdot |B| \cdot |C|, \tag{9.33}$$

for any pair of decompositions $I = S \sqcup T = S' \sqcup T'$ as above, $x \in \mathrm{P}[S]$, $y \in \mathrm{P}[T]$.

We note a few consequences of the axioms.

**Lemma 9.19.** *Let $\gamma$ be a multiplicative 2-cochain of twist $m$ on $\mathbf{p}$.*

(i) *The cochain $\gamma$ satisfies the unit condition*

$$\gamma_{\emptyset,\emptyset}(1) = 0. \tag{9.34}$$

(ii) *If $\mathbf{p}$ is connected, the cochain $\gamma$ is normal.*

(iii) *If $\gamma$ is normal, then*

$$\gamma_{S,T}(x \cdot y) = 0 \quad \textit{and} \quad \gamma_{T,S}(x \cdot y) = m \cdot |S| \cdot |T| \tag{9.35}$$

*for any $x \in \mathrm{P}[S]$, $y \in \mathrm{P}[T]$.*

PROOF. First, choosing $I = \emptyset$ and $x = y = 1$ in (9.33) we deduce (9.34).

Assume now that $\mathbf{p}$ is connected. Then $\mathrm{P}[\emptyset] = \{1\}$, so

$$x/_I = 1 = x|_{\emptyset}$$

for any $x \in \mathrm{P}[I]$. Choosing $R = I$, $S = T = \emptyset$ in (9.25) we obtain

$$\gamma_{I,\emptyset}(x) + \gamma_{\emptyset,\emptyset}(x/_I) = \gamma_{I,\emptyset}(x) + \gamma_{I,\emptyset}(x|_I).$$

It follows, in view of (8.44) and (9.34), that $\gamma_{I,\emptyset}(x) = 0$. Similarly, choosing $R = S = \emptyset$, $T = I$ in (9.25) we obtain $\gamma_{\emptyset,I}(x) = 0$. Thus, $\gamma$ is normal.

Finally, assume that $\gamma$ is normal. Choose $S = S'$ and $T = T'$ in (9.33). Then $A = S$, $B = C = \emptyset$, $D = T$, and using (9.24) we deduce the first equality in (9.35). Similarly, choosing $S = T'$ and $T = S'$ we deduce the second. $\square$

Let $k = 1, 2$. A multiplicative $k$-cocycle is a multiplicative $k$-cochain that is a cocycle.

The set of multiplicative $k$-cochains is a subgroup of the group $C^k(\mathbf{p}, \mathbb{Z})$ of all cochains. We denote it by $C^k_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z})$. Similarly let $Z^k_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z})$ denote the subgroup of multiplicative $k$-cocycles of the group $Z^k(\mathbf{p}, \mathbb{Z})$ of all cocycles. Observe that the map

$$C^2_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z}) \to \mathbb{Z}, \quad \gamma \mapsto m$$

which sends a multiplicative 2-cochain to its twist, is a morphism of groups.

The map $d\colon C^1(\mathbf{p}, \mathbb{Z}) \to C^2(\mathbf{p}, \mathbb{Z})$ (9.26) restricts to multiplicative cochains, as follows.

**Lemma 9.20.** *If the 1-cochain $\alpha$ is multiplicative, then the 2-coboundary $d(\alpha)$ is multiplicative of twist 0.*

The proof is straightforward.

The image of $C^1_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z})$ under $d$ is a subgroup of

$$Z^2_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z}) \cap B^2(\mathbf{p}, \mathbb{Z})$$

denoted $B^2_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z})$, the multiplicative 2-coboundaries. This leads to the quotient group

$$H^2_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z}) := Z^2_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z})/B^2_{\mathrm{mul}}(\mathbf{p}, \mathbb{Z}). \tag{9.36}$$

which maps canonically to the cohomology group $H^2(\mathbf{p}, \mathbb{Z})$ defined in (9.27).

In Theorem 9.27, we determine this group for the bimonoid of linear orders $\mathbf{L}$.

**9.6.4. Bimonoid deformations.** We continue to assume that $\mathbf{p} = \Bbbk \mathrm{P}$ is a linearized bimonoid in $(\mathsf{Sp}, \cdot, \beta)$.

Let $\gamma$ be a normal multiplicative 2-cocycle of twist $m$ on $\mathbf{p}$.

Fix a scalar $q \in \Bbbk$. Assume either that $q$ is invertible, or that $\gamma$ takes values in $\mathbb{N}$ and $m \in \mathbb{N}$. Consider the deformed comonoid $(\mathbf{p}, \Delta_\gamma, \epsilon)$ defined in Section 9.6.2.

**Proposition 9.21.** *Let $\gamma$ be a normal 2-cocycle on the linearized bimonoid $(\mathbf{p}, \mu, \iota, \Delta, \epsilon)$. If $\gamma$ is multiplicative of twist $m$, then $(\mathbf{p}, \mu, \iota, \Delta_\gamma, \epsilon)$ is a bimonoid in $(\mathsf{Sp}, \cdot, \beta_{q^m})$.*

PROOF. The compatibility axiom (8.18) relating $\mu$, $\Delta_\gamma$ and the braiding $\beta_{q^m}$ follows from the corresponding axiom for $\mu$ and $\Delta$ plus (9.33).

The unit condition (9.34) (or the stronger condition (9.24)) implies $(\Delta_\gamma)_{\emptyset,\emptyset}(1) = 1 \otimes 1$. This guarantees the compatibility between $\Delta_\gamma$ and $\iota$ in (8.19). ☐

In particular, if $\gamma$ is multiplicative of twist $m = 0$, then the deformed bimonoid is still an ordinary bimonoid (though no longer linearized), and if $\gamma$ is of twist $m = 1$, then the deformed bimonoid is a $q$-bimonoid.

Let $\alpha$ be a 1-cochain and $f_\alpha\colon \mathbf{p} \to \mathbf{p}$ the map defined in (9.29). Suppose $\gamma_1$ and $\gamma_2$ are two cohomologous 2-cocycles which differ by $d(\alpha)$, as in Proposition 9.17.

**Proposition 9.22.** *Suppose $\gamma_1$ and $\gamma_2$ are both normal and multiplicative of twist $m$. Suppose $\alpha$ is multiplicative and normal. Then the map $f_\alpha$ is a morphism of $q^m$-bimonoids*

$$(\mathbf{p}, \mu, \iota, \Delta_{\gamma_1}, \epsilon) \to (\mathbf{p}, \mu, \iota, \Delta_{\gamma_2}, \epsilon).$$

*If $q$ is invertible in $\Bbbk$, then $f_\alpha$ is an isomorphism.*

PROOF. We know that $f_\alpha$ is a morphism of comonoids from Proposition 9.17. Equation (9.31) implies that it preserves products, and (9.32) that it preserves units. ☐

**9.6.5. Abelian group coefficients.** The discussion in the preceding sections can be carried out for an arbitrary abelian group $\mathbb{A}$ in place of the group of integers $\mathbb{Z}$. All cocycle conditions such as (9.23) and (9.25) are understood as equalities in $\mathbb{A}$. The twist $m$ intervening in the definition of multiplicative cocycle is now an element of $\mathbb{A}$, and the term $m \cdot |B| \cdot |C|$ in (9.33) is

$$\underbrace{m + \cdots + m}_{|B| \cdot |C|}$$

(repeated addition in the group $\mathbb{A}$).

This leads to the second cohomology group of a linearized comonoid $\mathbf{p}$ with coefficients on $\mathbb{A}$

$$H^2(\mathbf{p}, \mathbb{A}) := Z^2(\mathbf{p}, \mathbb{A})/B^2(\mathbf{p}, \mathbb{A})$$

and its multiplicative version

$$H^2_{\mathrm{mul}}(\mathbf{p}, \mathbb{A}).$$

Let $\chi\colon \mathbb{A} \to \Bbbk^\times$ be a *character* on the group $\mathbb{A}$. In other words, $\chi$ is a morphism of groups from $\mathbb{A}$ to the group of invertible elements in $\Bbbk$. In this situation, given a normal 2-cocycle $\gamma$ on $\mathbf{p}$ with values on $\mathbb{A}$, we may deform the coproduct of $\mathbf{p}$ as follows. Given a decomposition $I = S \sqcup T$, we set

$$(\Delta_\gamma)_{S,T}\colon \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T], \quad x \mapsto \chi\big(\gamma_{S,T}(x)\big) x|_S \otimes x/_S,$$

for any $x \in \mathrm{P}[I]$. The morphism associated to a 1-cochain $\alpha$ on $\mathbf{p}$ with values on $\mathbb{A}$ is similarly defined by

$$(f_\alpha)_I\colon \mathbf{p}[I] \to \mathbf{p}[I], \quad x \mapsto \chi\big(\alpha_I(x)\big)x.$$

The results of Sections 9.6.2 and 9.6.4 continue to hold.

Let $q \in \Bbbk^\times$ be an invertible scalar. The $q$-deformations considered in Sections 9.6.2 and 9.6.4 correspond to the case when $\mathbb{A} = \mathbb{Z}$ and $\chi(i) = q^i$ for any integer $i$.

**9.6.6. Higher cohomology groups.** Let $\mathbf{p} = \Bbbk\mathrm{P}$ be a linearized comonoid in species. We will make use of the notions of restriction and contraction introduced in (8.42).

We proceed to define a sequence of categories $\{\mathsf{B}_k(\mathbf{p})\}_{k\geq 0}$. Given $k \in \mathbb{N}$, the objects of $\mathsf{B}_k(\mathbf{p})$ are sequences

$$(I, x, S_1, \ldots, S_k)$$

where $I$ is a finite set, $x \in \mathrm{P}[I]$, and $S_1 \sqcup \cdots \sqcup S_k = I$ (in the terminology of Section 10.1.2, $(S_1, \ldots, S_k)$ is a weak composition of $I$ into $k$ blocks). The morphisms

$$(I, x, S_1, \ldots, S_k) \to (J, y, T_1, \ldots, T_k)$$

of $\mathsf{B}_k(\mathbf{p})$ are bijections $\sigma\colon I \to J$ such that

$$\mathrm{P}[\sigma](x) = y \quad \text{and} \quad \sigma(S_i) = T_i \text{ for all } i.$$

We now turn the sequence $\{\mathsf{B}_k(\mathbf{p})\}_{k\geq 0}$ into a simplicial category (Section 5.1.1). Take $i = 0, \ldots, k$. Define functors

$$d_i\colon \mathsf{B}_k(\mathbf{p}) \to \mathsf{B}_{k-1}(\mathbf{p}) \qquad \text{and} \qquad s_i\colon \mathsf{B}_k(\mathbf{p}) \to \mathsf{B}_{k+1}(\mathbf{p})$$

as follows. Given $x \in \mathrm{P}[I]$ and $S_1 \sqcup \cdots \sqcup S_k = I$,

$$d_i(I, x, S_1, \dots, S_k) := \begin{cases} (I \setminus S_1, x/S_1, S_2 \dots, S_k) & \text{if } i = 0, \\ (I, x, S_1, \dots, S_i \sqcup S_{i+1}, \dots, S_k) & \text{if } 0 < i < k, \\ (I \setminus S_k, x|_{S_1 \sqcup \cdots \sqcup S_{k-1}}, S_1, \dots, S_{k-1}) & \text{if } i = k, \end{cases} \tag{9.37}$$

$$s_i(I, x, S_1, \dots, S_k) := (I, x, S_1, \dots, S_i, \emptyset, S_{i+1}, \dots, S_k). \tag{9.38}$$

On a morphism $\sigma \colon (I, x, S_1, \dots, S_k) \to (J, y, T_1, \dots, T_k)$, we set

$$d_i(\sigma) := \begin{cases} \sigma|_{I \setminus S_1} & \text{if } i = 0, \\ \sigma & \text{if } 0 < i < k, \\ \sigma|_{I \setminus S_k} & \text{if } i = k, \end{cases}$$
$$s_i(\sigma) := \sigma.$$

These are well-defined by naturality of restriction and contraction (which in turn follow from naturality of the coproduct of $\mathbf{p}$).

**Lemma 9.23.** *The functors $d_i$ and $s_i$ satisfy the simplicial relations* (5.4).

PROOF. Consider the relations $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq k$. The cases in which $(i, j)$ equals $(k-1, k)$, $(0, k)$ and $(0, 1)$ are respectively equivalent to each of the equalities in (8.42). The other cases are straightforward.

The relations $d_k s_k = \mathrm{id} = d_0 s_0$ are equivalent to (8.44). All other simplicial relations are straightforward. □

Therefore by Lemma 5.2, we have a simplicial category $\{\mathsf{B}_k(\mathbf{p})\}_{k \geq 0}$.

Let $\mathbb{A}$ be an abelian group. Consider the associated *discrete* category in which the objects are the elements of $\mathbb{A}$ and the only morphisms are the identities. We define

$$C^k(\mathbf{p}, \mathbb{A}) := \mathrm{Hom}_{\mathsf{Cat}}(\mathsf{B}_k(\mathbf{p}), \mathbb{A}).$$

In other words, the elements of $C^k(\mathbf{p}, \mathbb{A})$ are functors from the category $\mathsf{B}_k(\mathbf{p})$ to the discrete category on $\mathbb{A}$. The set $C^k(\mathbf{p}, \mathbb{A})$ is a group under pointwise addition. We refer to $C^k(\mathbf{p}, \mathbb{A})$ as the group of *$k$-cochains* on $\mathbf{p}$.

Given a functor $\mathsf{B}_k(\mathbf{p}) \to \mathbb{A}$, let

$$\gamma_{S_1, \dots, S_k}(x) \in \mathbb{A}$$

denote the image of the object $(I, x, S_1, \dots, S_k)$ of $\mathsf{B}_k(\mathbf{p})$. Functoriality boils down to the following condition: for any bijection $\sigma \colon I \to J$ we must have a commutative diagram

$$\begin{tikzcd} \mathrm{P}[I] \arrow[d, "{\mathrm{P}[\sigma]}"'] \arrow[dr, "{\gamma_{S_1,\dots,S_k}}"] & \\ \mathrm{P}[J] \arrow[r, "{\gamma_{\sigma(S_1),\dots,\sigma(S_k)}}"'] & \mathbb{A}. \end{tikzcd} \tag{9.39}$$

Thus, a $k$-cochain is a family of maps $\gamma_{S_1, \dots, S_k} \colon \mathrm{P}[I] \to \mathbb{A}$, one for each decomposition $I = S_1 \sqcup \cdots \sqcup S_k$, which is natural in $I$ in the sense that the above diagrams commute. In particular, low dimensional cochains are as defined in Section 9.6.1.

Let $(\mathsf{Ab}, \otimes)$ denote the category of abelian groups. The functor

$$\mathrm{Hom}_{\mathsf{Cat}}(-, \mathbb{A}) \colon \mathsf{Cat} \to \mathsf{Ab}$$

is contravariant. Therefore, the sequence $\{C^k(\mathbf{p}, \mathbb{A})\}_{k\geq 0}$ is a cosimplicial abelian group. The associated cochain complex defines the cohomology of $\mathbf{p}$ with coefficients in $\mathbb{A}$. The low dimensional structure is as described in Section 9.6.1.

When $\mathbf{p}$ is a linearized bimonoid, multiplicative versions of these cohomology groups can also be defined, extending the notions of Section 9.6.3. The details are omitted.

We thank Paulo Lima-Filho for useful comments regarding this construction.

**9.6.7. Connection with cohomology with coefficients in a bicomodule.** Let $\mathbf{c}$ be a comonoid in $(\mathsf{Sp}, \cdot)$ (not necessarily linearized) and $\mathbf{m}$ a $\mathbf{c}$-bicomodule. Recall the cochain complex $C(\mathbf{c}, \mathbf{m})$ from Section 8.2.3.

Let us now set $\Bbbk = \mathbb{Z}$. Let $\mathbf{E}$ be the exponential species with its usual comonoid structure (Example 8.15). Suppose that $\mathbf{p} = \mathbb{Z}\mathrm{P}$ is a linearized comonoid. Recall that such a structure is equivalent to a linearized $\mathbf{E}$-bicomodule structure on $\mathbf{p}$ (Proposition 8.30). Therefore, the cochain complex $C(\mathbf{E}, \mathbf{p})$ is defined.

According to the definition in Section 8.2.3, the $\mathbb{Z}$-module $C^n(\mathbf{E}, \mathbf{p})$ consists of morphisms of species

$$f \colon \mathbf{p} \to \mathbf{E}^{\cdot n}.$$

Note that

$$\mathbf{E}^{\cdot n}[I] = \bigoplus_{S_1 \sqcup \cdots \sqcup S_n = I} \mathbb{Z}.$$

Hence, such a morphism $f$ is equivalent to a family of maps

$$f_{S_1,\dots,S_n} \colon \mathrm{P}[I] \to \mathbb{Z}$$

satisfying the naturality condition (9.39). Thus, the space $C^n(\mathbf{p}, \mathbb{Z})$ defined in Section 9.6.6 coincides with the space $C^n(\mathbf{E}, \mathbf{p})$ defined in Section 8.2.3. Moreover, a comparison of (8.17) and (9.37) shows that the differential is the same for both cochain complexes.

In conclusion, the cohomology groups of the $\mathbb{Z}$-linearized comonoid $\mathbf{p}$ with coefficients in $\mathbb{Z}$ from Section 9.6.6 coincide with the cohomology groups of the comonoid $\mathbf{E}$ with coefficients in the bimodule $\mathbf{p}$ from Section 8.2.3.

## 9.7. The Schubert and descent cocycles

Consider the Hopf monoid of linear orders $\mathbf{L}$ from Example 8.16. It is a linearized bimonoid. Thus, we may speak of (multiplicative) cocycles on $\mathbf{L}$ (Section 9.6). Note that the notions of restriction and contraction defined in (8.42) both coincide with ordinary restriction of linear orders (as defined in Example 8.16).

In this section, we discuss two important 2-cocycles on $\mathbf{L}$: the Schubert cocycle and the descent cocycle.

**9.7.1. Uniqueness of the Schubert cocycle.** Consider the Schubert cocycle defined in Section 9.5.1. Note that the family of maps $\mathrm{sch}_{S,T} \colon \mathrm{L}[I] \to \mathbb{N}$ is natural. Further:

**Proposition 9.24.** *The Schubert cocycle defines a normal* 2*-cocycle on the comonoid* $\mathbf{L}$. *Moreover, this cocycle is multiplicative of twist* 1.

Proof. Normality (9.24) follows from (9.14), the cocycle condition (9.25) follows from (9.17), and multiplicativity (9.33) (with $m = 1$) follows from (9.18). $\square$

We point out that this cocycle is not a coboundary. Indeed, since $\mathbf{L}$ is cocommutative, any 2-coboundary $d(\alpha)$ satisfies $d(\alpha)_{S,T} = d(\alpha)_{T,S}$, in view of (9.26). However, $\mathrm{sch}_{S,T} \neq \mathrm{sch}_{T,S}$; see (9.16) in this regard.

We now proceed to establish an important uniqueness property of the Schubert cocycle: up to scalar multiples, it is the unique multiplicative cocycle on $\mathbf{L}$ (Theorem 9.27).

First, we determine the group of multiplicative 1-cochains (9.31) on the comonoid $\mathbf{L}$. Given an integer $k \in \mathbb{Z}$, let

$$(\alpha_k)_I(l) := |I| \cdot k$$

for any linear order $l$ on a finite set $I$.

**Proposition 9.25.** *The family of maps $\alpha_k$ is a multiplicative 1-cochain on $\mathbf{L}$. Moreover, the map*

$$\mathbb{Z} \to C^1_{mul}(\mathbf{L}, \mathbb{Z}), \quad k \mapsto \alpha_k$$

*is an isomorphism of groups.*

Proof. The first assertion is clear. To prove the converse, let $\alpha$ denote an arbitrary multiplicative 1-cochain on $\mathbf{L}$. On the empty set, $\alpha$ and $\alpha_k$ agree by (9.32). Let $\{i\}$ be a singleton, and define $k := \alpha_{\{i\}}(*_i)$, where $*_i$ denotes the unique linear order on $\{i\}$. This is independent of the choice of the singleton, by naturality of $\alpha$. Now let $I$ be an arbitrary nonempty set and $l = l^1|\cdots|l^n$ a linear order on $I$. Then $l$ is the concatenation of $*_{l^1}, \ldots, *_{l^n}$, and by (9.31),

$$\alpha_I(l) = \alpha_{\{l^1\}}(*_{l^1}) + \cdots + \alpha_{\{l^n\}}(*_{l^n}) = |I| \cdot k.$$

Thus, $\alpha = \alpha_k$ as needed. $\square$

Next, we show that there are no nontrivial multiplicative 2-coboundaries on $\mathbf{L}$.

**Proposition 9.26.** *Any multiplicative 1-cochain on $\mathbf{L}$ is a 1-cocycle. Thus,*

$$B^2_{mul}(\mathbf{L}, \mathbb{Z}) = 0 \quad \text{and} \quad H^2_{mul}(\mathbf{L}, \mathbb{Z}) = Z^2_{mul}(\mathbf{L}, \mathbb{Z}).$$

Proof. Let $\alpha_k$ be a multiplicative 1-cochain. Given a linear order $l$ on $I$ and a decomposition $I = S \sqcup T$, we have

$$(\alpha_k)_S(l|_S) + (\alpha_k)_T(l|_T) = |S| \cdot k + |T| \cdot k = |I| \cdot k = (\alpha_k)_I(l).$$

Thus, (9.23) holds and $\alpha_k$ is a 1-cocycle. $\square$

**Theorem 9.27.** *Let $\gamma$ be a multiplicative 2-cocycle on $\mathbf{L}$ of twist $m$. Then*

$$\gamma = m \cdot \mathrm{sch}$$

*where* sch *is the Schubert cocycle. Thus,*

$$H^2_{mul}(\mathbf{L}, \mathbb{Z}) = Z^2_{mul}(\mathbf{L}, \mathbb{Z}) \cong \mathbb{Z}$$

*under the map $m \mapsto m \cdot \mathrm{sch}$.*

Proof. Let $I = S \sqcup T$. We show $\gamma_{S,T} = m \cdot \mathrm{sch}_{S,T}$ by induction on $|I|$.

If $|I| \leq 1$, then at least one of $S$ and $T$ is empty, and

$$\gamma_{S,T} = 0 = \mathrm{sch}_{S,T}$$

since both cocycles are normal: $\gamma$ by Lemma 9.19 (since $\mathbf{L}$ is connected) and sch by Proposition 9.24.

Suppose $|I| \geq 2$. Let $l$ be a linear order on $I$. Choose any proper, nonempty, initial segment $S'$ of $l$, and let $T'$ be its complement. Then both $|S'|$ and $|T'|$ are smaller than $|I|$, and $l$ is the concatenation of $l|_{S'}$ with $l|_{T'}$:

$$l = l|_{S'} \cdot l|_{T'}.$$

Let $A, B, C, D$ be the intersections of $S$ and $T$ with $S'$ and $T'$, as in Lemma 8.7. Since $\gamma$ is multiplicative of twist $m$ (9.33), we have

$$\gamma_{S,T}(l) = \gamma_{A,C}(l|_{S'}) + \gamma_{B,D}(l|_{T'}) + m \cdot |B| \cdot |C|$$

(the roles of $S, T$ and $S', T'$ here are reversed from their roles in (9.33)).

By induction hypothesis, and since sch is multiplicative of twist 1, the above sum equals

$$m \cdot \mathrm{sch}_{A,C}(l|_{S'}) + m \cdot \mathrm{sch}_{B,D}(l|_{T'}) + m \cdot |B| \cdot |C| = m \cdot \mathrm{sch}_{S,T}(l).$$

Thus, $\gamma_{S,T}(l) = m \cdot \mathrm{sch}_{S,T}(l)$ as needed.

The assertion about the cohomology group now follows from Proposition 9.26. □

Let $q \in \Bbbk$ be an arbitrary scalar (not necessarily invertible). According to Propositions 9.21 and 9.24, we may deform the bimonoid $\mathbf{L}$ using the Schubert cocycle and obtain a $q$-bimonoid. The result is in fact a $q$-Hopf monoid, by connectedness. It follows from (9.28) that this $q$-Hopf monoid is indeed $\mathbf{L}_q$ (Definition 9.13). Theorem 9.27 implies that there are no other such deformations, up to reparametrizations $q \mapsto q^m$.

**9.7.2. The descent cocycle.** Given a linear order $l$ on $I$ and a decomposition $I = S \sqcup T$, let

$$\mathrm{D}_{S,T}(l) := \{(i,j) \in S \times T \mid i \text{ immediately succeeds } j \text{ according to } l\} \tag{9.40}$$

and

$$\mathrm{d}_{S,T}(l) := |\mathrm{D}_{S,T}(l)|. \tag{9.41}$$

For instance, if

$$l = s|h|i|v|a, \quad S = \{i, s, a\}, \quad T = \{v, h\},$$

then

$$\mathrm{D}_{S,T}(l) = \{(i,h), (a,v)\} \quad \text{and} \quad \mathrm{d}_{S,T}(l) = 2.$$

The statistic d admits the following description in terms of lattice paths: if we walk east when we read an element of $S$ and north when we read an element of $T$, then $\mathrm{d}_{S,T}(l)$ is the number of right turns made when reading the elements of $l$ from left to right. See Figure 9.1 for an illustration.

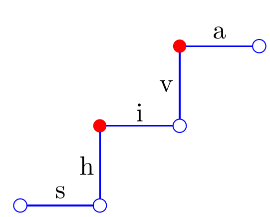



FIGURE 9.1. The descent statistic as the right turns in a lattice path.

**Proposition 9.28.** *The family of maps* $\mathrm{d}_{S,T} \colon \mathrm{L}[I] \to \mathbb{N}$ *defines a normal* 2*-cocycle on the comonoid* $\mathbf{L}$*.*

PROOF. Let $l$ be a linear order on $I = R \sqcup S \sqcup T$. Let $(i,j) \in I^2$ be such that $i$ immediately succeeds $j$. The pair is counted by

$$\mathrm{d}_{R,S\sqcup T}(l) + \mathrm{d}_{S,T}(l|_R)$$

if $i$ is in $R$ and $j$ is in $S$ or $T$, or if $i$ is in $S$ and $j$ is in $T$. Similarly, $(i,j)$ is counted by

$$\mathrm{d}_{R\sqcup S,T}(l) + \mathrm{d}_{R,S}(l|_{R\sqcup S})$$

if $i$ is in $R$ or $S$ and $j$ is in $T$, or $i$ is in $R$ and $j$ is in $S$. Thus, both counts are equal and the cocycle condition (9.25) holds. Normality (9.24) is clear. □

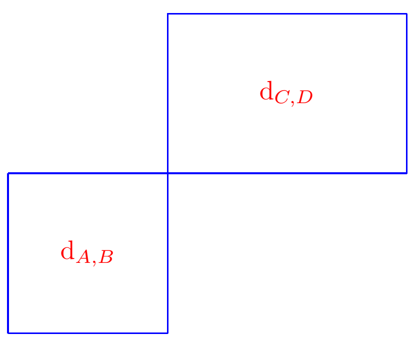
Suppose $I = [n]$ and $l = 1|\cdots|n$. Given $S \sqcup T = [n]$, let $\zeta \in \mathrm{S}_n$ be the corresponding shuffle as in (2.26). Then

$$\mathrm{d}_{S,T}(l) = \mathrm{des}(\zeta^{-1}), \tag{9.42}$$

the number of descents of the permutation $\zeta$ (Section 10.7.1). By naturality, the computation of $\mathrm{d}_{S,T}$ can always be reduced to this situation.

We refer to the family of maps $\mathrm{d}_{S,T}$ as the *descent cocycle*. As the Schubert cocycle, the descent cocycle is not symmetric: $\mathrm{d}_{S,T} \neq \mathrm{d}_{T,S}$, hence it is not a coboundary.

The descent cocycle is not multiplicative. This can be understood in terms of lattice paths. Consider decompositions $I = S \sqcup T = S' \sqcup T'$ and sets $A, B, C, D$ as in Lemma 8.7. The concatenation of linear orders $l_1$ on $S$ and $l_2$ on $T$ yields a path which is the concatenation of the paths corresponding to $l_1$ and $l_2$. The steps east correspond to $S' = A \sqcup C$ and the steps north to $T' = B \sqcup D$, as schematized below.




The total number of right turns is either the sum of the number of right turns of each smaller path, or this number plus 1, in the case when the first path ends north and the second starts east. This last possibility violates condition (9.33). The descent cocycle may be used to deform the comonoid structure of $\mathbf{L}$, but since it is not multiplicative, the resulting coproduct is not compatible with the monoid structure of $\mathbf{L}$. We do not consider this deformation in this monograph.

CHAPTER 10

# The Coxeter Complex of Type $A$

Coxeter groups play an important role in many areas of mathematics. A concise introduction to Coxeter groups particularly relevant to the ideas presented here is given in [12, Chapter 1]. Supplementary material can be found in the books by Abramenko and Brown [3], Davis [89], Grove and Benson [156], Humphreys [174], Björner and Brenti [51] or Bourbaki [62]. Material related to the general context of hyperplane arrangements and oriented matroids can be found in [52, 257, 344, 286].

In this chapter, we only deal with the symmetric group, which is the Coxeter group of type $A$. This is because our main interest here is to tie Coxeter theory to species. The theory for the symmetric group can be understood explicitly; this makes our exposition fairly self-contained. We begin with a discussion of a number of combinatorial structures that play a fundamental role here and elsewhere in the monograph (Section 10.1). In Sections 10.2, 10.3, 10.4 and 10.5, we review some standard material, namely, the braid arrangement, faces and flats therein, the Coxeter complex of type $A$, Tits projection maps, the gallery metric, and the gate property. Of particular importance is a monoid structure carried by the set of faces. It is defined in terms of the projection maps and lifts the lattice structure of the set of flats. Sections 10.6 and 10.7 deal with shuffles (and their geometric meaning), and the descent and global descent maps. The action of faces on chambers by multiplication yields an embedding of the algebra of faces in the endomorphism algebra of the space of chambers. In Section 10.8 we explain how this relates to the notion of descents and Solomon's descent algebra.

Section 10.9 deals with directed faces and directed flats. Just as faces and flats carry a monoid structure, directed faces and directed flats carry a dimonoid structure. The inter-relationships between these algebraic objects are studied in Section 10.10. In Section 10.11, we discuss the break and join maps. These are natural companions to the projection maps and together they explain how various combinatorial and geometric objects compose and decompose. These ideas will be used to construct a number of Hopf monoids in Chapter 12.

In Section 10.12, we discuss a weighted version of the gallery metric. The starting data is an integer square matrix $A$ of size $r$. Letting $A = [1]$ recovers the usual gallery metric. Interesting distinctions occur when $A$ is symmetric or antisymmetric. We relate them to the unoriented and oriented cases which occur in integration theory. In Section 10.13, we return to the (weighted) Schubert statistic of Section 2.2 and relate it to the ideas of this chapter by interpreting it in terms of the (weighted) gallery metric. In Sections 10.14 and 10.15, we define some interesting bilinear forms on faces, directed faces and chambers (maximal faces) and study conditions under which they are nondegenerate. We will see later in Chapter 12 that the nondegeneracy of these forms implies (among other things) the self-duality of related Hopf monoids.

## 10.1. Partitions and compositions

Partitions and compositions are basic combinatorial structures. They play an important role in the theory of species. In this section we review these and some related structures.

**10.1.1. Partitions and compositions of a number.** Let $n$ be a nonnegative integer. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$ is a finite sequence of positive integers such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of $n$ is a finite sequence of positive integers such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = n.$$

If the numbers $\alpha_i$ are allowed to be nonnegative, we say that $\alpha$ is a *weak composition* of $n$.

We write $\lambda \vdash n$ and $\alpha \vDash n$ to indicate that $\lambda$ is a partition of $n$ and $\alpha$ a composition of $n$. The numbers $\lambda_i$ and $\alpha_i$ are the *parts* of $\lambda$ and $\alpha$. The empty sequence is the only partition (and composition) of 0; it has no parts.

Fix a positive integer $k$. The number of partitions of $n$ into $k$ parts is denoted $p_k(n)$. The generating function is

$$\sum_{n\geq 1} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

The number of compositions of $n$ into $k$ parts is the binomial coefficient $\binom{n-1}{k-1}$, with generating function

$$\sum_{n\geq 1} \binom{n-1}{k-1} x^n = \frac{x^k}{(1-x)^k}.$$

For $n \geq k \geq 1$, there is a bijection between compositions of $n$ into $k$ parts and subsets of $[n-1]$ of cardinality $k-1$ given by

$$(\alpha_1, \alpha_2, \ldots, \alpha_k) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}. \tag{10.1}$$

**10.1.2. Partitions and compositions of a set.** Let $I$ be a finite set. A *partition* $X$ of $I$ is an unordered collection $X$ of disjoint nonempty subsets of $I$ such that

$$I = \bigcup_{S\in X} S.$$

A *composition* of $I$ is an ordered sequence $F = (F^1, \ldots, F^k)$ of disjoint nonempty subsets of $I$ such that

$$I = \bigcup_{i=1}^{k} F^i.$$

If the subsets $F^i$ are allowed to be empty, we say that $F$ is a *weak composition* of $I$.

When confusion with compositions and partitions of numbers may arise, we may use the terms *set compositions* and *set partitions*. The subsets $S$ of $I$ which belong to $X$ and the subsets $F^i$ in the sequence $F$ are the *blocks* or *parts* of $X$ and $F$, respectively. We agree that there is only one composition and one partition of the empty set (with no blocks). We write $X \vdash I$ and $F \vDash I$ to indicate that $X$ is a partition of $I$ and $F$ a composition of $I$. We often write $F = F^1|\cdots|F^k$ instead of $F = (F^1, \dots, F^k)$. For partitions, we may choose an arbitrary ordering of its blocks and write $X = \{X^1, \dots, X^k\}$.

*Decomposition* is just another term for weak composition. In contexts where we are interested in compositions as a combinatorial structure, we stick to the latter terminology. In other contexts, we prefer to speak of decompositions $(S^1, \dots, S^k)$ of a finite set $I$ and write

$$I = S^1 \sqcup \cdots \sqcup S^k.$$

Sometimes, for emphasis, we may add that the decomposition is disjoint and ordered, even though this is always assumed when using this notation.

Decompositions are called *partages* by Joyal [181, Section 2.1]. Set compositions are often called *ordered set partitions*, *preferential arrangements* [202, Exercise 5.3.1.3], [341, Example 3.15.10], or *ballots* [40].

Fix a positive integer $k$. The number of partitions of $[n]$ into $k$ blocks is denoted $S(n,k)$ and called the *Stirling number of the second kind*. The generating function is

$$\sum_{n\geq 1} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

The number of compositions of $[n]$ into $k$ blocks is $k!S(n,k)$, with generating function

$$\sum_{n\geq 1} k!S(n,k)x^n = \frac{x}{1-x}\cdot\frac{2x}{1-2x}\cdots\frac{kx}{1-kx}.$$

Given a surjective function $f\colon I \twoheadrightarrow [k]$, the sequence of fibers $f^{-1}(1)|\cdots|f^{-1}(k)$ is a composition of $I$ into $k$ blocks. This sets up a bijective correspondence between compositions of $I$ into $k$ blocks and surjective functions $I \twoheadrightarrow [k]$.

**10.1.3. Linear partitions and linear compositions of a set.** Let $I$ be a finite set. A *linear partition* (*composition*) of $I$ is a partition (composition) of $I$ together with a linear order on each of its blocks. A *disposition* of $I$ is a weak composition of $I$ with a linear order on each block.

The terminology used here is that of Rota *et al* [180, 57]. Linear partitions are also called *partitions into ordered blocks*. In [12, Section 5.4.2] we used *fully nested set partition* (*composition*) for linear partition (composition).

We extend the notation employed for linear orders (Example 8.3) to these structures as follows. To specify a linear partition, we give the set of blocks and display the order in each block by listing the elements from left to right in increasing order, separated by bars. To specify a linear composition, we further indicate the order among blocks by listing them from left to right separated by long bars. For example,

$$\{n|a, i|r, k|h|s\} \quad \text{and} \quad a|n\Big|r|i\Big|s|k|h$$

are respectively a linear partition and a linear composition of $\{k, r, i, s, h, n, a\}$. The former is equal to $\{i|r, n|a, k|h|s\}$ but not to $\{a|n, i|r, s|h|k\}$.

Note that a linear composition may be equivalently described by a pair $(F, C)$, where $F$ is a composition of $I$ and $C$ is a linear order on $I$ which refines $F$, or even as a linear order on $I$ together with a composition of $|I|$. For example,

$$(na|ri|ksh, a|n|r|i|s|k|h) \qquad \text{and} \qquad \big(a|n|r|i|s|k|h, (2,2,3)\big)$$

both correspond to the linear composition $a|n\big|r|i\big|s|k|h$ of $\{k,r,i,s,h,n,a\}$.

It follows that the number of linear compositions of $[n]$ into $k$ blocks and the number of linear partitions of $[n]$ into $k$ blocks are respectively

$$n!\binom{n-1}{k-1} \quad \text{and} \quad \frac{n!}{k!}\binom{n-1}{k-1}.$$

The latter is called the *Lah number*.

**10.1.4. Refinement and partial orders.** Let $X$ and $Y$ be partitions of $I$. We say that $Y$ *refines* $X$ if each block of $Y$ is contained in a block of $X$, or equivalently if each block of $X$ is a union of blocks of $Y$. In this case we write $X \leq Y$. This defines a partial order on the set of partitions of $I$ which is in fact a lattice. The top element is the partition into singletons and the bottom element is the partition whose only block is the whole set $I$.

*Warning.* Sometimes the opposite partial order on partitions is used in the literature.

Refinement is defined similarly for compositions of $I$ and for compositions of $n$. The bijection (10.1) defines an isomorphism between the poset of compositions of $n$ and the poset of subsets of $[n-1]$ (a Boolean poset).

We also define a partial order on the set of linear compositions of $I$ as follows. We view them as pairs consisting of a set composition and a finer linear order and declare $(F,C) \leq (G,D)$ if $C = D$ and $F \leq G$ ($G$ refines $F$). In the bar notation, this means that we go up in the partial order by turning some short bars into long bars.

We consider two partial orders on linear partitions. To this end, we make use of the notions of restriction, shuffle and concatenation of linear orders discussed in Examples 8.16 and 8.24.

Let $L$ and $M$ be two linear partitions of $I$. First, we write $L \leq' M$ if each ordered block of $M$ is a restriction of an ordered block of $L$, or equivalently if each ordered block of $L$ is a shuffle of ordered blocks of $M$. For instance,

$$\{l|a|k, s|h|m|i\} \leq' \{l|k, a, s|m, h|i\}.$$

Second, we write $L \leq M$ if the ordered blocks of $M$ are obtained by deconcatenating the ordered blocks of $L$, or equivalently, if each ordered block of $L$ is a concatenation of ordered blocks of $M$. For instance,

$$\{l|a|k, s|h|m|i\} \leq \{l|a, k, s|h, m|i\}.$$

Note that

$$L \leq M \implies L \leq' M.$$

**10.1.5. Type, support, and base.** The *type* of a composition $F$ of $I$ is the composition of $|I|$ whose parts are the sizes of the blocks of $F$. The *type* of a partition $X$ of $I$ is the partition of $|I|$ whose parts are the sizes of the blocks of $X$ (listed in decreasing order).

The *support* of a composition $F$ of $I$ is the partition $\mathrm{supp}(F)$ of $I$ obtained by forgetting the order among the blocks. The support of a composition of $n$ is the partition of $n$ obtained by reordering the parts in decreasing order.

The support and type maps commute with each other. This can be illustrated as follows.

$$\begin{tikzcd} la|ksh|mi \arrow[r, mapsto, "\mathrm{supp}"] \arrow[d, mapsto, "\mathrm{type}"'] & \{la, ksh, mi\} \arrow[d, mapsto, "\mathrm{type}"] \\ (2,3,2) \arrow[r, mapsto, "\mathrm{supp}"'] & (3,2,2). \end{tikzcd}$$

The *support* of a linear composition $(F, C)$ is the linear partition $\mathrm{supp}(F, C)$ obtained by forgetting the order among the blocks (but keeping the order within each block).

The *base* of a linear composition (partition) is the composition (partition) obtained by forgetting the orders within each block, or equivalently, by removing the short bars.

The support and base maps commute with each other. This can be illustrated as follows.

$$\begin{tikzcd} l|a\Big|k|s\Big|h|m|i \arrow[r, mapsto, "\mathrm{supp}"] \arrow[d, mapsto, "\mathrm{base}"'] & \{l|a, k|s, h|m|i\} \arrow[d, mapsto, "\mathrm{base}"] \\ la|ks|hmi \arrow[r, mapsto, "\mathrm{supp}"'] & \{la, ks, hmi\}. \end{tikzcd}$$

**10.1.6. Concatenation, restriction, shuffles and quasi-shuffles.** Throughout this section, we fix an ordered disjoint decomposition $I = S \sqcup T$ of a finite set $I$.

Given a composition $F$ of $I$, the *restriction* $F|_S$ is the composition of $S$ whose blocks are the nonempty intersections of the blocks of $F$ with $S$. If $F = F^1|\cdots|F^k$, we write

$$F|_S = (F^1 \cap S|\cdots|F^k \cap S)^\frown$$

where the hat indicates that empty intersections are removed from the list.

Given compositions $F = F^1|\cdots|F^k$ of $S$ and $G = G^1|\cdots|G^l$ of $T$, their *concatenation* is the composition $F \cdot G$ of $I$ defined by

$$F \cdot G := F^1|\cdots|F^k|G^1|\cdots|G^l.$$

A *quasi-shuffle* of $F$ and $G$ is a composition $H$ of $I$ such that $H|_S = F$ and $H|_T = G$. It follows that each block of $H$ is either a block of $F$, or a block of $G$, or a union of a block of $F$ and a block of $G$;

A *shuffle* of $F$ and $G$ is a quasi-shuffle $H$ such that each block of $H$ is either a block of $F$ or a block of $G$.

In other words, in a shuffle $H$ the blocks $F^i$ are listed in $H$ in the same order as in $F$, and similarly for the blocks of $G$. A quasi-shuffle is obtained from a shuffle by substituting any number of pairs of blocks $(F^i, G^j)$ for $F^i \sqcup G^j$, if they are adjacent in the shuffle.

For example,

$$\mathbf{g}|sh|i|\mathbf{au}|\mathbf{ri}|va \text{ is a shuffle of } sh|i|va \text{ and } \mathbf{g}|\mathbf{au}|\mathbf{ri},$$

and

$$v\mathbf{la}|i|sh\mathbf{ksh}|\mathbf{mi}|nu \text{ is a quasi-shuffle of } v|i|sh|nu \text{ and } \mathbf{la}|\mathbf{ksh}|\mathbf{mi}.$$

The notion of shuffle and quasi-shuffle can be extended to any finite number of set compositions.

Given a partition $X$ of $I$, the *restriction* $X|_S$ is defined in the same manner as for compositions.

Given partitions $X$ of $S$ and $Y$ of $T$, their *union* is the partition $X \sqcup Y$ of $I$ whose blocks are the blocks of $X$ and the blocks of $Y$. A *quasi-shuffle* of $X$ and $Y$ is any partition of $I$ whose restriction to $S$ is $X$ and whose restriction to $T$ is $Y$. For example,

$$\{sh, i, va\} \text{ is the union of } \{sh, i\} \text{ and } \{va\},$$

and

$$\{sh, i, va\},\ \{sh, iva\} \text{ and } \{shva, i\} \text{ are all their quasi-shuffles.}$$

Given a linear partition $L$ of $I$, the *restriction* $L|_S$ is the linear partition of $I$ whose blocks are the nonempty intersections of the blocks of $L$ with $S$, ordered as in $L$.

Given linear partitions $L$ of $S$ and $M$ of $T$, their *union* is the linear partition $L \sqcup M$ of $I$ whose ordered blocks are those of $L$ and those of $M$. A *quasi-shuffle* of $L$ and $M$ of $I$ is any linear partition of $I$ each of whose ordered blocks is either an ordered block of $L$, or one of $M$, or a concatenation of one of $L$ followed by one of $M$. For example, the quasi-shuffles of $\{v|i|s,h\}$ and $\{n|u\}$ are

$$\{v|i|s,h,n|u\}, \quad \{v|i|s,h|n|u\}, \quad \text{and} \quad \{v|i|s|n|u,h\}.$$

**10.1.7. Factorials and related numbers.** The *factorial* of a set partition $X$ is

$$X! := \prod_{S \in X} |S|!. \tag{10.2}$$

It counts the number of ways of endowing each block of $X$ with a linear order. The *cyclic factorial* of $X$ is

$$X\overset{\circ}{!} := \prod_{S \in X} (|S|-1)!.$$

It counts the number of ways of endowing each block of $X$ with a cyclic order. Note that

$$(X \sqcup Y)! = X!\,Y! \quad \text{and} \quad (X \sqcup Y)\overset{\circ}{!} = X\overset{\circ}{!}\,Y\overset{\circ}{!}. \tag{10.3}$$

The following relation between factorials and cyclic factorials is of importance.

$$\sum_{Y:\, X \leq Y} Y\overset{\circ}{!} = X!. \tag{10.4}$$

It may be derived as follows. Suppose $X$ has only one block $I$. Each permutation of $I$ determines a partition $Y$ of $I$ whose blocks are the cycles of the permutation. The left hand side counts the number of permutations of $I$ according to these cycle partitions, while the right hand side counts all permutations. The general case follows using (10.3).

The coefficients $X\overset{\circ}{!}$ appear in the work of Brown [70, Theorem 1] in the general setting of left regular bands; also see [12, Sections 2.5.5 and 2.6.2]. In these references, the notations $n_X$ and $c_X$ are used instead of $X\overset{\circ}{!}$ and $X!$.

Given set partitions $X$ and $Y$ with $Y$ refining $X$, let

$$(X:Y)! := \prod_{S \in X} (n_S)!, \tag{10.5}$$

where $n_S$ is the number of blocks of $Y$ that refine the block $S$ of $X$. Note that if $Y$ is the unique partition into singletons, then $X! = (X:Y)!$.

Let $F$ be any set composition with support $X$. Then

$$(X : Y)! = |\{G \mid F \leq G,\ \mathrm{supp}(G) = Y\}|. \tag{10.6}$$

The factorial of a set composition $F = F^1|\cdots|F^k$ is

$$F! := \prod_{i=1}^{k} |F^i|!. \tag{10.7}$$

This is the number of linear orders that refine $F$. The factorial of a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of an integer is

$$\alpha! := \prod_{i=1}^{k} \alpha_i!. \tag{10.8}$$

Note that

$$F! = (\mathrm{supp}\, F)! = (\mathrm{type}\, F)!.$$

## 10.2. Faces, chambers, flats and cones

In this section, we discuss the braid arrangement, along with the basic notions of faces, flats and cones which are attached to it. These notions are closely related to the notions of partitions and compositions discussed in Section 10.1. More details on the braid arrangement can be found in [45, 46, 47, 72].

**10.2.1. The braid arrangement.** The *braid arrangement* in Euclidean space $\mathbb{R}^n$ consists of the $\binom{n}{2}$ hyperplanes defined by

$$x_i = x_j,$$

where $1 \leq i < j \leq n$. The symmetric group $\mathrm{S}_n$ acts on this arrangement by permuting the coordinates.

One may replace the set $[n]$ by any finite set $I$. Let $\mathbb{R}^I$ be the vector space consisting of all functions from $I$ to $\mathbb{R}$. The braid arrangement in $\mathbb{R}^I$ consists of the hyperplanes $\mathrm{H}_{ij}$ defined by

$$x_i = x_j,$$

where $i \neq j$ range over the elements of $I$. Note that $\mathrm{H}_{ij} = \mathrm{H}_{ji}$. Let

$$\mathrm{Br}[I] := \{\mathrm{H}_{ij} \mid i, j \in I,\ i \neq j\}$$

denote the arrangement.

A bijection $I \cong J$ induces a linear isomorphism $\mathbb{R}^I \cong \mathbb{R}^J$ which sends $\mathrm{Br}[I]$ to $\mathrm{Br}[J]$. Thus, Br is a set species, and so will be each of the objects associated to it throughout this chapter.

**10.2.2. Faces and chambers.** For each pair $(i, j) \in I^2$ with $i \neq j$, the subset of $\mathbb{R}^I$ defined by

$$x_i \leq x_j$$

is a *half-space* of the braid arrangement. Its supporting hyperplane is $\mathrm{H}_{ij}$. Note that each hyperplane supports two half-spaces.

Two points $x, y \in \mathbb{R}^I$ *lie on opposite sides* of a hyperplane H if $x$ belongs to one half-space supported by H, $y$ belongs to the other half-space, and neither belongs to H. We say that two points are *separated* if they lie on opposite sides of at least one hyperplane $\mathrm{H} \in \mathrm{Br}[I]$.

A *face* of the braid arrangement $\mathrm{Br}[I]$ is a nonempty subset of $\mathbb{R}^I$ with the following two properties.

- If two points lie in the set, then they are not separated.
- If a point lies in the set, then any point that is not separated from it also lies in the set.

A face is defined by a system of equalities and inequalities which may be encoded by a composition of $I$: the equalities are used to define the blocks and the inequalities to order them. For example, for $I = \{a, b, c, d\}$,

$$x_a = x_c \leq x_b = x_d \qquad \longleftrightarrow \qquad ac|bd.$$

Thus, faces correspond to compositions of the set $I$.

Let $\Sigma[I]$ denote the set of faces of the arrangement $\mathrm{Br}[I]$. It is partially ordered by inclusion. The partial order on $\Sigma[I]$ corresponds to refinement of set compositions. The minimum element is the face

$$\{x \in \mathbb{R}^I \mid x_i = x_j \text{ for all } i, j \text{ in } I\}.$$

It corresponds to the composition with one block. The maximal faces are called *chambers*. They correspond to linear orders on $I$. For example,

$$x_a \leq x_c \leq x_b \leq x_d \qquad \longleftrightarrow \qquad a|c|b|d.$$

Let $\mathrm{L}[I]$ denote the set of chambers.

This defines the set species $\Sigma$ (of faces or set compositions) and $\mathrm{L}$ (of chambers or linear orders). The linearized species are denoted $\boldsymbol{\Sigma}$ and $\mathbf{L}$. The latter is the species of Example 8.3.

Since the braid arrangement is central (all hyperplanes pass through the origin), every face has an opposite face. In terms of set compositions, the opposite $\overline{F}$ of a face $F$ is obtained by reversing the order of the blocks: if $F = F^1|\cdots|F^l$, then

$$\overline{F} = F^l|\cdots|F^1.$$

The hyperplane $\mathrm{H}_{ij}$ is called a *wall* of a chamber $C$ if $i$ and $j$ are consecutive in the linear order $C$.

In particular, $\Sigma[n]$ and $\mathrm{L}[n]$ denote the sets of faces and chambers of the braid arrangement in $\mathbb{R}^n$. The action of $\mathrm{S}_n$ on $\Sigma[n]$ corresponds to its obvious action on compositions of $[n]$. This action is simply transitive on $\mathrm{L}[n]$. Hence, one may identify

$$\mathrm{S}_n \to \mathrm{L}[n] \qquad w \mapsto wC_{(n)}, \tag{10.9}$$

where $C_{(n)} := 1|\cdots|n$ is the canonical linear order on $[n]$. We refer to $C_{(n)}$ as the fundamental chamber.

Since there is no canonical order on $I$, for the arrangement $\mathrm{Br}[I]$ there is no canonical choice of fundamental chamber. On the other hand, let $n$ be the cardinality of the set $I$ and let $\mathrm{Bij}([n], I)$ be the set of bijections from $[n]$ to $I$. Then there is a bijection

$$\mathrm{Bij}([n], I) \to \mathrm{L}[I] \qquad w \mapsto wC_{(n)}, \tag{10.10}$$

where

$$C_{(n)} = 1|\cdots|n \qquad \text{and} \qquad wC_{(n)} = w(1)|w(2)|\cdots|w(n).$$

For $I = [n]$, this recovers (10.9).

**10.2.3. Flats.** A *flat* is a subspace obtained by intersecting some of the hyperplanes in the arrangement. Let $\Pi[I]$ denote the set of flats in the braid arrangement $\mathrm{Br}[I]$. It is partially ordered by inclusion. The poset of flats is a lattice.

Flats correspond to partitions of $I$. For example, for $I = \{k, r, i, s, h, n, a\}$,

$$(x_k = x_i) \cap (x_r = x_i) \cap (x_n = x_a) \qquad \longleftrightarrow \qquad \{kri, s, h, na\}.$$

We identify $\Pi[I]$ with the lattice of partitions of $I$.

This defines the set species $\Pi$ of flats, or equivalently, set partitions. The linearized species is denoted $\mathbf{\Pi}$. The partial order on flats given by inclusion corresponds to the partial order on set partitions given by refinement (Section 10.1.4).

Let $\mathrm{supp}\colon \Sigma[I] \to \Pi[I]$ be the map which sends a face to its linear span. Equivalently, $\mathrm{supp}(F)$ is the intersection of the hyperplanes containing the face $F$. In combinatorial terms, this coincides with the support map which sends a set composition to its underlying set partition (Section 10.1.5).

**10.2.4. Cones.** A *cone* of the braid arrangement $\mathrm{Br}[I]$ is defined to be an intersection of a subset of its half-spaces.

For example, for $I = \{a, b, c, d\}$,

$$\{x \in \mathbb{R}^I \mid x_a = x_c \leq x_b,\ x_d \leq x_b\}$$

is a cone. Note that a face of the arrangement is a cone. Similarly, a flat of the arrangement is also a cone.

A *top-dimensional cone* is a cone with a nonempty interior. In other words, it is a cone which contains a chamber. Note that a chamber of the arrangement is a top-dimensional cone and conversely a top-dimensional cone is the union of the chambers which belong to it. This defines the species of cones and the species of top-dimensional cones, both of whose $I$-components are posets under inclusion.

Observe that any flat in the braid arrangement inherits a hyperplane arrangement which is in fact isomorphic to a smaller braid arrangement. Now let $R$ be a cone. Define $X$ to be the flat obtained by intersecting all the hyperplanes which contain $R$. It follows that $R$ is a top-dimensional cone in the induced arrangement on $X$. Thus, every cone is a top-dimensional cone in some flat.

**10.2.5. The spherical representation.** Note that the intersection of all hyperplanes in the braid arrangement is the one-dimensional space where all coordinates are equal:

$$\bigcap_{i \neq j} \mathrm{H}_{ij} = \{x \in \mathbb{R}^I \mid x_i = x_j \text{ for all } i, j\}.$$

Let

$$\mathrm{H}_0 = \left\{ x \in \mathbb{R}^I \;\middle|\; \sum_{i \in I} x_i = 0 \right\}$$

be the orthogonal complement. We intersect all hyperplanes $\mathrm{H}_{ij}$ with $\mathrm{H}_0$ and no information is lost. Then we intersect with the unit sphere in $\mathrm{H}_0$ and we only lose the center of the arrangement. This is the spherical representation of the braid arrangement.

The procedure for $I = \{a, b, c\}$ is shown in Figure 10.1: $\mathrm{H}_0$ is shown in perspective as a horizontal plane, with the 3 vertical hyperplanes $\mathrm{H}_{ab}$, $\mathrm{H}_{bc}$ and $\mathrm{H}_{ac}$ cutting through it. The spherical representation is seen on the unit circle on the plane $\mathrm{H}_0$. It is shown in more detail in Figure 10.2.

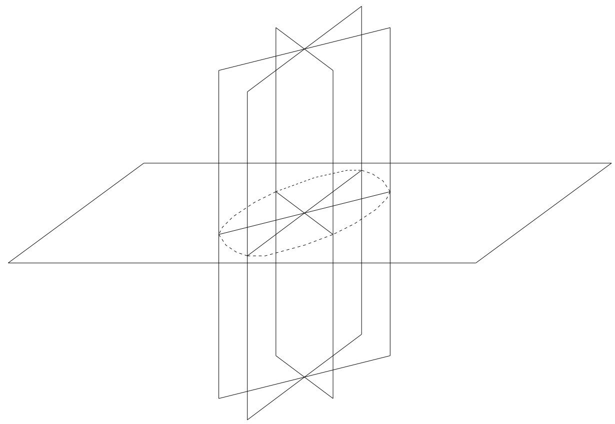


FIGURE 10.1. Euclidean and spherical representations.

## 10.3. The Coxeter complex of type $A$

In this section, we associate a simplicial complex to the braid arrangement. This is the Coxeter complex of type $A$. We discuss this along with explicit low dimensional examples.

**10.3.1. Simplicial complexes.** We begin with a quick review of simplicial complexes. More information can be found in [3, Appendix A.1] and [340].

Let $I$ be a finite set and $2^I$ the set of subsets of $I$ ordered by inclusion:

$$S \leq S' \iff S \subseteq S'.$$

This is the *Boolean poset*.

Let $V$ be a finite set. A *simplicial complex* with vertex set $V$ consists of a nonempty collection $k$ of subsets of $V$ with the following properties:

- for each $v \in V$, the singleton $\{v\}$ belongs to $k$;
- if $K \in k$ and $J \subseteq K$, then $J \in k$.

The collection of all subsets of $V$ is a simplicial complex, called the *simplex* with vertex set $V$ and denoted $\Delta_V$.

Let $k$ be a simplicial complex. The subsets of $V$ which belong to $k$ are its *faces*. Note that the empty set is a face of any simplicial complex. If $K$ is a face of $k$, then the collection of subfaces of $K$ forms a simplicial complex, equal to the simplex $\Delta_K$.

If $k$ is a simplicial complex, the collection $k$ is partially ordered by inclusion and satisfies properties (10.11a)–(10.11c) below. Conversely, any poset satisfying these properties is isomorphic to the poset of faces of a unique simplicial complex [3, Exercise A.3].

(10.11a) The poset $k$ has a minimum element.

(10.11b) For any $K \in k$, the subposet $\{J \in k \mid J \leq K\}$ is isomorphic to a Boolean poset.

(10.11c) If $J, K \in k$ have an upper bound, then they have a least upper bound.

The simplex $\Delta_V$ corresponds in this manner to the Boolean poset $2^V$.

Let $k$ be a simplicial complex and $K \in k$ a face. The *dimension* of $K$ is one less than its cardinality. In particular, the dimension of the empty face is $-1$. The *star* of $K$ consists of the faces of $k$ which contain $K$:

$$\mathrm{Star}_k(K) := \{J \in k \mid K \leq J\}.$$

It is a simplicial complex whose vertices are the faces of $k$ in which $K$ has codimension 1.

The complex $k$ is *pure* of dimension $d$ if all maximal faces, called *chambers*, have the same dimension $d$.

The Cartesian product $k_1 \times k_2$ of two simplicial complexes $k_1$ and $k_2$ is another simplicial complex, called the *join* of $k_1$ and $k_2$. The empty face gives rise to a canonical embedding of each factor in the join. For instance,

$$k_1 \hookrightarrow k_1 \times k_2, \qquad K \mapsto (K, \emptyset).$$

The vertex set of $k_1 \times k_2$ is the disjoint union of the vertex sets of $k_1$ and $k_2$.

A *balanced simplicial complex* is a pair $(k, \varphi)$ where $k$ is a simplicial complex and $\varphi\colon V \to [n]$ is a function that restricts to a bijection

$$C \xrightarrow{\cong} [n]$$

for each maximal face $C$ of $k$. This implies that $k$ is pure of dimension $n-1$.

Balanced complexes are called *colored complexes* in [3] and *labeled complexes* in [68]. If we think of $\varphi(K)$ as a color assigned to a vertex $K \in k$, then the condition on $\varphi$ implies that all vertices in a face receive different colors.

A *simplicial map* $f\colon k \to k'$ between simplicial complexes with vertex sets $V$ and $V'$ is a map $f\colon V \to V'$ such that $f(K)$ is a face of $k'$ for every face $K$ of $k$. The simplicial map is *nondegenerate* if it preserves face dimensions. A simplicial isomorphism is necessarily nondegenerate. A nondegenerate simplicial map $f\colon k \to k'$ restricts to an isomorphism $\Delta_K \to \Delta_{f(K)}$ for each face $K$ of $k$.

**10.3.2. The Coxeter complex.** Recall the poset of faces $\Sigma[I]$ associated to the braid arrangement in $\mathbb{R}^I$. One can easily check that it is the poset of faces of a simplicial complex. It admits the following (equivalent) descriptions:

- it is the reduced order complex of the Boolean poset $2^I$ (Example 13.21),
- it is the barycentric subdivision of the boundary of the simplex

$$\Big\{(x_i)_{i\in I} \in \mathbb{R}^I \;\Big|\; \sum_{i\in I} x_i = 1, x_i \geq 0\Big\},$$

- it is the triangulation of the unit sphere in the spherical representation of the braid arrangement.

From now on, we will identify $\Sigma[I]$ with this simplicial complex.

A significant property of $\Sigma[I]$ is that it is a Coxeter complex. The theory of Coxeter complexes was developed by Tits [360]. We recall some important features of these complexes: A Coxeter complex is balanced, gallery-connected, and it satisfies the gate property. The star of any face in a Coxeter complex is again a Coxeter complex. The join of two Coxeter complexes is again a Coxeter complex. Further, the set of faces of a Coxeter complex is a monoid; the product is constructed using the projection maps of Tits.

We will discuss some of these properties explicitly for the Coxeter complex of type $A$, namely $\Sigma[I]$. For this example, these properties can be checked directly.

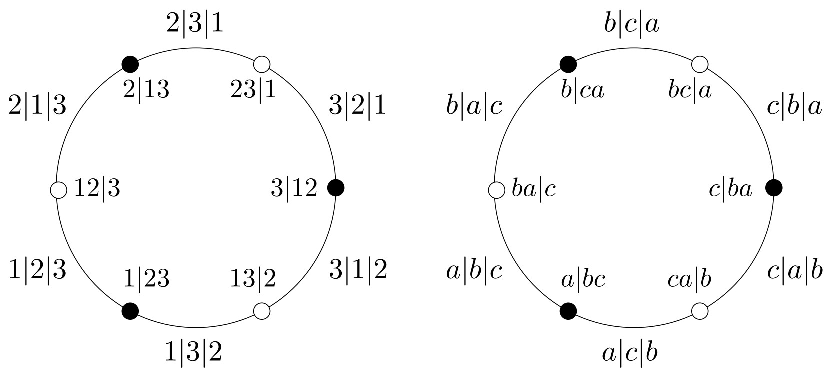



FIGURE 10.2. The simplicial complexes $\Sigma[3]$ and $\Sigma[\{a, b, c\}]$.

Hence familiarity with the general theory is not essential to follow the present discussion.

Let $n := |I|$. Note that the simplicial complex $\Sigma[I]$ is pure of dimension $n - 2$. Let $\Delta_{[n-1]}$ denote the set of compositions of $n$ (the simplex of dimension $n - 2$). Recall from Section 10.1.5 the type map

$$\Sigma[I] \to \Delta_{[n-1]} \tag{10.12}$$

which sends a composition $F$ of $I$ to the composition of $n$ whose parts are the sizes of the blocks of $F$. The type map is a nondegenerate simplicial map which turns $\Sigma[I]$ into a balanced complex (also see Proposition 13.18).

**10.3.3. Low dimensional examples.** Figure 10.2 shows the simplicial complexes $\Sigma[3]$ and $\Sigma[\{a, b, c\}]$. The circle is the same as the one shown in Figure 10.1. The vertices are of two types, shown in black and white. The set composition $abc$ (all elements in one block) indexes the center of the arrangement and does not show in the spherical representation.

The simplicial complex $\Sigma[\{a, b, c, d\}]$ is shown in Figure 10.3. It has been essentially reproduced from the paper of Brown, Billera and Diaconis [47]. This complex triangulates the sphere into twenty four triangles, eighteen of which can be seen (either partly or completely) in the figure. The edges and vertices have not been labeled for space constraints. Observe that the vertex in the center of the figure has label $abc|d$ and its star is isomorphic to the simplicial complex $\Sigma[\{a, b, c\}]$ shown on the right in Figure 10.2. The vertices are of three types. Those shown in black are of type $(1, 3)$, those in white are of type $(2, 2)$, and the vertex in the center is of type $(3, 1)$.

One can flatten the spherical representation so that all chambers except $d|c|b|a$ are visible. This is shown in Figure 10.4. The six hyperplanes can be seen in full as the six ovals.

## 10.4. Tits projection maps and the monoid of faces

There is an operation on the set of faces of the Coxeter complex which turns this set into a monoid. The operation is given by the projection maps of Tits. This section discusses these notions from a combinatorial perspective. The underlying geometry is discussed later in Section 10.5.

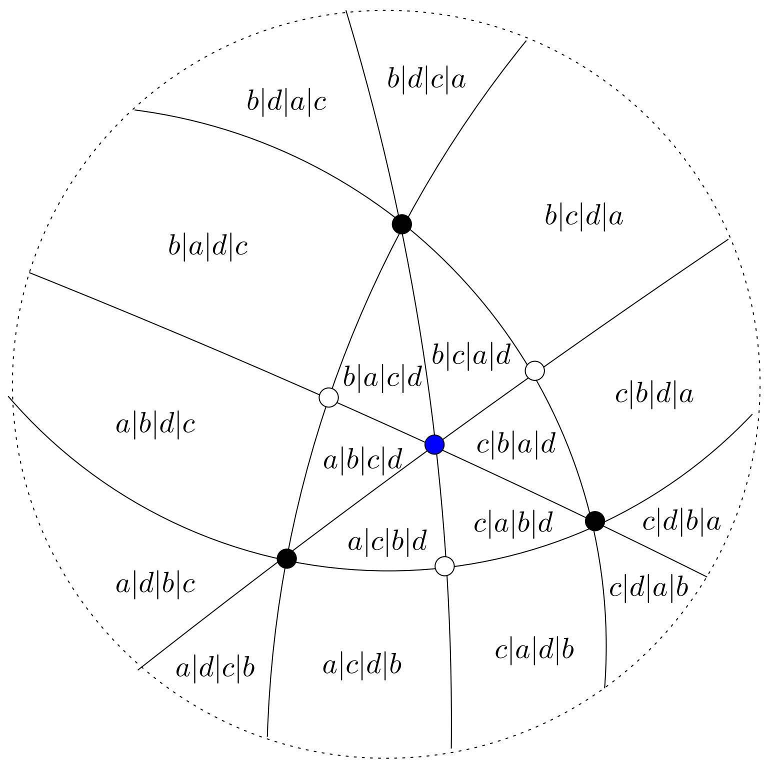



FIGURE 10.3. The simplicial complex $\Sigma[\{a, b, c, d\}]$.

**10.4.1. The monoid of faces.** The set $\Sigma[I]$ has the structure of a monoid. We view faces as set compositions of $I$ and multiply two such by intersecting their blocks and ordering them lexicographically. More precisely, if $F = F^1|\cdots|F^l$ and $G = G^1|\cdots|G^m$, then

$$FG := (F^1 \cap G^1|\cdots|F^1 \cap G^m|\cdots|F^l \cap G^1|\cdots|F^l \cap G^m)\widehat{\ },\tag{10.13}$$

where the hat indicates that any empty intersections should be deleted. For example,

$$(kri|shna)(s|khna|ri) = k|ri|s|hna.$$

It is clear that this product is associative. The set composition with one part serves as the unit. Thus, $\Sigma[I]$ is a monoid. It is not commutative. In fact,

$$FG = GF \iff F \text{ and } G \text{ are joinable,}\tag{10.14}$$

where joinable means that there is a face which contains both $F$ and $G$.

**Proposition 10.1.** *The product on $\Sigma[I]$ satisfies the following properties.*

(i) $F \leq FG$.
(ii) $F \leq G \iff FG = G$.
(iii) *If* $G \leq H$, *then* $FG \leq FH$.
(iv) *If* $C$ *is a chamber, then* $CF = C$ *and* $FC$ *is a chamber.*
(v) *If* $FG = K$ *and* $F \leq H \leq K$, *then* $HG = K$.
(vi) *If* $H_1F = K$ *and* $H_2F = K$, *then* $(H_1 \wedge H_2)F = K$.

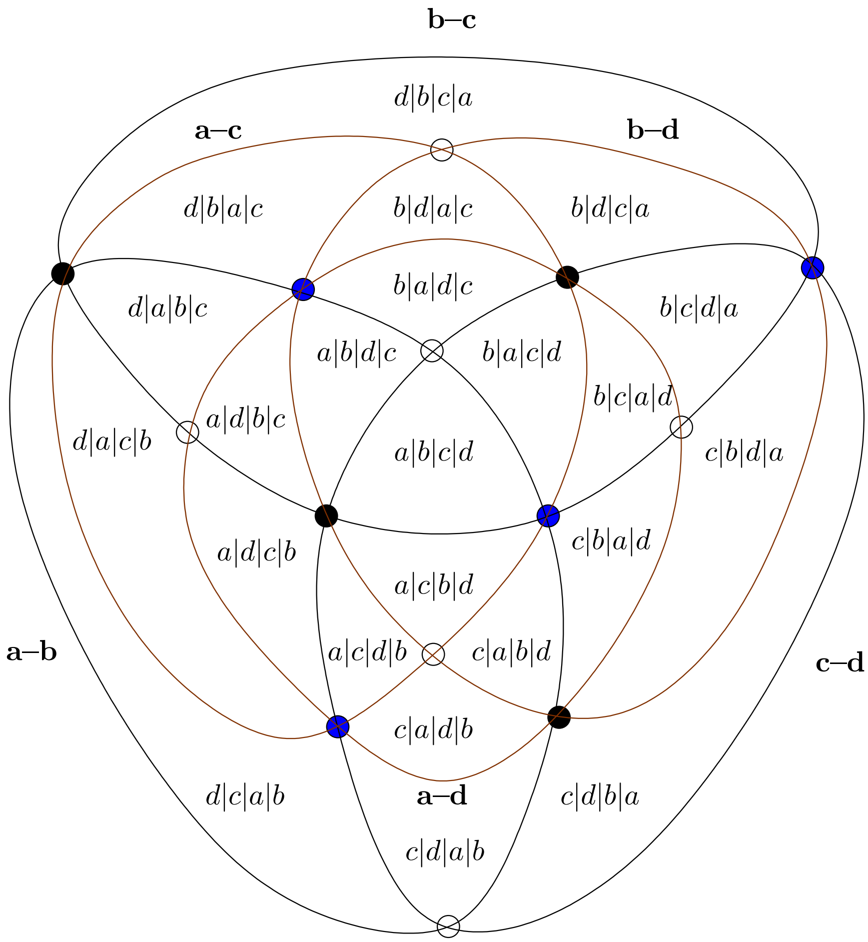



FIGURE 10.4. The flattened simplicial complex $\Sigma[\{a, b, c, d\}]$.

(vii) *If $E$ and $F$ are subfaces of a face, that is, if $E$ and $F$ have an upper bound, then $H(E \vee F) = HE \vee HF$ for any face $H$.*
(viii) $F\overline{F} = F$.
(ix) $FGF = FG$.
(x) *If $FPG = F\overline{P}G$, then $FP = F\overline{P} = F$.*
(xi) *If $F$ is a face and $D$ is a chamber, and $HF \neq D$ for any proper face $H$ of $D$, then $\overline{F} \leq D$.*

For any bijection $J \to I$, the corresponding map

$$\Sigma[J] \to \Sigma[I] \tag{10.15}$$

is both type and product preserving. In particular, the product of each $\Sigma[I]$ yields a morphism of set species

$$\Sigma \times \Sigma \to \Sigma, \tag{10.16}$$

where $\times$ denotes the Hadamard product on set species (8.36).

**Remark 10.2.** Property (ix) states that the monoid $\Sigma[I]$ is a left regular band (Section 8.7.7). Some of the properties listed above hold for all left regular bands.

**10.4.2. The lattice of flats as a quotient of the monoid of faces.** We have seen that the poset of flats $\Pi[I]$ is a lattice. We now view it as a commutative monoid with the product given by the join. The join $X \vee Y$ is the smallest common refinement of $X$ and $Y$. It is obtained by intersecting the parts of $X$ with the parts of $Y$ and deleting empty intersections. The similarity between the product in $\Sigma[I]$ and $\Pi[I]$ says that

$$\operatorname{supp}(FK) = \operatorname{supp}(F) \vee \operatorname{supp}(K). \tag{10.17}$$

Thus, the support map is a morphism of monoids. Using this fact, one may view $\Pi[I]$ as a left module over $\Sigma[I]$ via

$$K \cdot X := \operatorname{supp}(K) \vee X. \tag{10.18}$$

An alternative description of $\Pi[I]$ can be given as follows. Define an equivalence relation on $\Sigma[I]$:

$$F \sim G \iff FG = F \text{ and } GF = G. \tag{10.19}$$

It follows from (10.17) that

$$F \sim G \iff \operatorname{supp}(F) = \operatorname{supp}(G).$$

Thus, equivalence classes can be identified with flats and the canonical quotient map which sends a face to its equivalence class is the support map.

We reformulate the preceding discussion in combinatorial terms. Let $F$ and $G$ be two set compositions. Then (10.13) implies that

$$FG = F \text{ and } GF = G \iff F \text{ and } G \text{ consist of the same blocks.}$$

In other words, the equivalence class of $F$ consists of all its reorderings $G$. Thus, flats are identified with set partitions.

**10.4.3. Shuffles, quasi-shuffles, and the product of faces.** The monoid of faces is far from being a group. However, given faces $F$ and $H$ with $F \leq H$, there is always a face $G$ such that

$$FG = H.$$

In fact, we may just choose $G = H$. We now discuss all solutions $G$ to this equation, from a combinatorial perspective. A related point is addressed in Section 10.7.5.

We view faces as set compositions and make use of the operations of concatenation, shuffle and quasi-shuffle of Section 10.1.6. The statements below are direct consequences of (10.13).

Let $(F, H)$ be a pair of set compositions with $F \leq H$. Write $F = F^1|\cdots|F^i$. Since $H$ refines $F$, it is the concatenation of a composition of $F^1$, followed by a composition of $F^2$, and so on. We refer to these compositions as the blocks of $(F, H)$. For example, if

$$F = 135|24789|6 \quad \text{and} \quad H = 3|15|7|48|29|6,$$

then the blocks of $(F, H)$ are

$$3|15,\ 7|48|29 \text{ and } 6.$$

Note that $H$ is a linear order if and only if the blocks of $(F, H)$ are linear orders.

Now let $G$ be another set composition. Then,

$$FG = H \iff G \text{ is a quasi-shuffle of the blocks of } (F, H). \tag{10.20}$$

In addition,

$$FG = H \text{ and } GF = G \iff G \text{ is a shuffle of the blocks of } (F, H). \tag{10.21}$$

In particular, let $(F, D)$ be a linear set composition (so that the linear order $D$ refines $F$) and let $C$ be another linear order. Then,

$$FC = D \iff C \text{ is a shuffle of the blocks of } (F, D). \tag{10.22}$$

**10.4.4. Tits projection maps.** Let $F$ be a fixed face. The map given by left multiplication by $F$,

$$p_F \colon \Sigma[I] \to \mathrm{Star}_{\Sigma[I]}(F), \qquad G \mapsto FG \tag{10.23}$$

is called the *Tits projection* [360, Section 2.30]. Properties (i) and (ii) in Proposition 10.1 imply that the image of $p_F$ is the star of $F$ and that $p_F$ is idempotent. We say that $FG$ is the projection of $G$ on $F$.

## 10.5. The gallery metric and the gate property

In this section, we introduce the gallery metric on chambers. The Tits projection of a chamber onto a face is the closest chamber in the star of the face. Its existence is guaranteed by the gate property of the gallery metric. This is the geometric meaning of the product of faces of Section 10.4. In addition to reviewing these facts, we discuss a distance function on faces which generalizes the one on chambers.

**10.5.1. A distance function on chambers. The gallery metric.** We say two chambers are *adjacent* if they have a common codimension 1 face. A *gallery* is a sequence of chambers such that consecutive chambers are adjacent. Its *length* is one less than the number of chambers in the sequence. We have remarked earlier that $\Sigma[I]$ is a gallery-connected simplicial complex. This means that for any two chambers $C$ and $D$, there is a gallery from $C$ to $D$. We then define the *gallery distance* $\mathrm{dist}(C, D)$ to be the minimal length of a gallery connecting $C$ and $D$. Any gallery which achieves this minimum is called a *minimum gallery* from $C$ to $D$. This defines the *gallery metric* on $\mathrm{L}[I]$. It verifies the familiar properties of a metric:

$$\begin{aligned}
&\mathrm{dist}(C, D) \geq 0, \text{ with equality if and only if } C = D,\\
&\mathrm{dist}(C, D) = \mathrm{dist}(D, C),\\
&\mathrm{dist}(C, E) \leq \mathrm{dist}(C, D) + \mathrm{dist}(D, E),
\end{aligned}$$

with equality if and only if there is a minimum gallery from $C$ to $E$ which passes through the chamber $D$. We use the notation $C - D - E$ for such a minimum gallery.

The gallery metric is natural in $I$: For any bijection $\sigma \colon I \to J$,

$$\mathrm{dist}(C, D) = \mathrm{dist}(\sigma C, \sigma D). \tag{10.24}$$

Further, it is compatible with the opposite map:

$$\mathrm{dist}(C, D) = \mathrm{dist}(\overline{D}, \overline{C}). \tag{10.25}$$

If we view $\Sigma[I]$ as the set of faces of the braid arrangement, then $\mathrm{dist}(C, D)$ is the number of hyperplanes which separate $C$ and $D$. Let us make this more explicit. Write $C = C^1|\cdots|C^n$, where $n = |I|$. Define the inversion set of $(C, D)$ to be

$$\mathrm{Inv}(C, D) := \{(i, j) \in [n] \times [n] \mid i < j \text{ and } C^i \text{ appears after } C^j \text{ in } D\}.$$

Then

$$\mathrm{dist}(C, D) = |\,\mathrm{Inv}(C, D)|. \tag{10.26}$$

Let us now relate this to the inversion set (2.19) and the number of inversions (2.20) of an appropriate permutation. From (10.10), there are unique bijections $u$ and $v$ from $[n]$ to $I$ such that $C = uC_{(n)}$ and $D = vC_{(n)}$. Then

$$\mathrm{Inv}(C, D) = \mathrm{Inv}(uC_{(n)}, vC_{(n)}) = \mathrm{Inv}(C_{(n)}, u^{-1}vC_{(n)}) = \mathrm{Inv}(v^{-1}u).$$

Note that $w := u^{-1}v$ is a permutation. The second equality follows from naturality of the inversion set, and the last equality from the definitions (note that $w$ gets replaced by its inverse). It follows that

$$\mathrm{dist}(C, D) = \mathrm{inv}(v^{-1}u) = l(v^{-1}u). \tag{10.27}$$

Recall from Section 2.2.3 that $l(w)$ denotes the length of $w$, which coincides with the number of inversions of $w$. Note that (2.25) can be seen as a consequence of the symmetry of the distance function.

It is convenient to define, with notation as above,

$$d(C, D) := u^{-1}v. \tag{10.28}$$

This is known as the Weyl-valued distance between $C$ and $D$. It takes values in the symmetric group. In particular, for $I = [n]$, we obtain, for any permutation $\sigma$,

$$\sigma = d(C_{(n)}, \sigma C_{(n)}) \qquad \text{and hence} \qquad \mathrm{inv}(\sigma) = l(\sigma) = \mathrm{dist}(C_{(n)}, \sigma C_{(n)}). \tag{10.29}$$

It is clear that for any chambers $C$, $D$ and $E$,

$$d(C, E) = d(C, D)\, d(D, E). \tag{10.30}$$

**10.5.2. Gate property.** There is a geometric way of describing Tits projections, and hence the product of $\Sigma[I]$, which we discuss briefly. It relies on the fact that $\Sigma[I]$ has the gate property.

**Proposition 10.3 (Gate property).** *Let $F$ be a face and $D$ a chamber. Among the chambers containing $F$, there is a unique one that is closest to $D$ in the gallery metric. This unique chamber is $FD$.*

In other words, $FD$, which is the projection of $D$ on $F$, is the gate of the star of $F$ viewed from $D$. This is illustrated in Figure 10.5 which shows the relevant portion of a simplicial complex of dimension two. The big dot is a vertex named $F$, and both $D$ and $FD$ are chambers, which in dimension two are triangles.

The product of two arbitrary faces turns out to be

$$FG = \bigwedge FD,$$

where the meet is taken over all chambers $D$ which contain $G$.

The following is a consequence of Proposition 10.3.

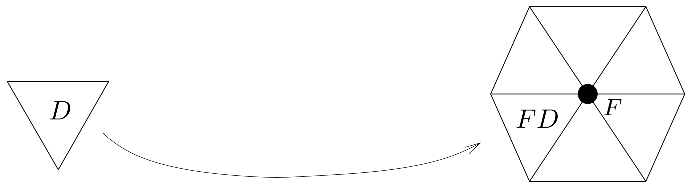



FIGURE 10.5. The projection map at work.

**Proposition 10.4.** *Let $C$ and $D$ be chambers and $F$ be a face of $C$. Then there exists a minimum gallery $C - FD - D$. In particular,*

$$\operatorname{dist}(C, D) = \operatorname{dist}(C, FD) + \operatorname{dist}(FD, D). \tag{10.31}$$

The gate property originated in the work of Tits [360, Section 3.19.6], and was abstracted later by Dress and Scharlau [320, 103]. It also appears in the work of Abels [2], Mühlherr [281] and Mahajan [253] (to name a few references). Some basic information on this property can be found in [12, Section 1.1.1]. The poset of faces of any real hyperplane arrangement satisfies this property. This fact can be used to define a semigroup structure on the set of faces of any real hyperplane arrangement [12, Equation (1.1)]. If the arrangement is central, the semigroup is in fact a monoid.

**10.5.3. A distance function on faces.** For a face $F$, let $\mathrm{L}_F$ denote the set of chambers containing $F$. It is straightforward [12, Lemma 2.2.1] to show that if faces $F$ and $G$ have the same support, then the projection

$$p_G\colon \mathrm{L}_F \to \mathrm{L}_G \qquad\qquad C \mapsto GC \tag{10.32}$$

is a bijection with inverse given by the projection $p_F\colon D \mapsto FD$.

Now let $F$ and $G$ be any two faces. Since $FG$ and $GF$ have the same support, the projection

$$p_{GF}\colon \mathrm{L}_{FG} \to \mathrm{L}_{GF}$$

is a bijection, with inverse $p_{FG}$. Further, if $C$ is any chamber containing $FG$, then by using the compatibility of the symmetric group action with the distance function and the projection map, we see that $\operatorname{dist}\bigl(C, p_{GF}(C)\bigr)$ is independent of the particular choice of $C$.

This observation allows us to define the distance between any two faces $F$ and $G$:

$$\operatorname{dist}(F, G) := \operatorname{dist}\bigl(C, p_{GF}(C)\bigr), \tag{10.33}$$

where $C$ is any chamber containing the face $FG$. Since $p_{GF}$ is a bijection with inverse $p_{FG}$, it follows that

$$\operatorname{dist}(F, G) = \operatorname{dist}(p_{FG}(D), D),$$

where $D$ is any chamber containing $GF$. This shows that the distance function is symmetric. It is also clear that

$$\operatorname{dist}(F, G) = \operatorname{dist}(FG, GF). \tag{10.34}$$

**Remark 10.5.** The above definition can in fact be made for the faces of any central hyperplane arrangement. The right-hand side of (10.33) is independent of the particular choice of $C$ and equals the number of hyperplanes which separate $F$ and $G$ (meaning that $F$ and $G$ lie on opposite sides of the hyperplane).

The distance function on faces does not define a metric. However, it does restrict to a metric on the set of faces with a fixed support. In particular,

$$\operatorname{dist}(F,G) = 0 \text{ and } \operatorname{supp} F = \operatorname{supp} G \iff F = G. \tag{10.35}$$

Since $FG$ and $GF$ have the same support, it follows from (10.34) and (10.14) that

$$\operatorname{dist}(F,G) = 0 \iff FG = GF \iff F \text{ and } G \text{ are joinable.} \tag{10.36}$$

In particular, the distance between a face and a subface is always 0. It follows that the triangle inequality fails as well; so the distance function on faces is not a pseudometric either.

Let us make the distance function more explicit. We first deal with the case of equal support. Let $F$ and $G$ be faces with the same support. Write $F = F^1|\cdots|F^k$. Then $G$ is a set composition obtained by permuting the $F^i$'s in some order. Define the inversion set of $(F,G)$ to be

$$\operatorname{Inv}(F,G) := \{(i,j) \in [k]\times[k] \mid i<j \text{ and } F^i \text{ appears after } F^j \text{ in } G\}.$$

Then

$$\operatorname{dist}(F,G) = \sum_{(i,j)\in\operatorname{Inv}(F,G)} |F^i|\,|F^j|. \tag{10.37}$$

Note that if $F$ and $G$ are both chambers, then $\operatorname{dist}(F,G) = |\operatorname{Inv}(F,G)|$ as noted in (10.26).

Now we go to the general case. Here, we have

$$\operatorname{dist}(F,G) = \sum_{\substack{i<k\\ j>l}} |F^i\cap G^j|\,|F^k\cap G^l|, \tag{10.38}$$

where $i$ and $k$ index the blocks of $F$ while $j$ and $l$ index the blocks of $G$.

## 10.6. Shuffle permutations

The set $\operatorname{Sh}(s,t)$ of $(s,t)$-shuffle permutations was defined in (2.21). In this section, we extend this notion to any composition, and then relate it to the gallery metric and Tits projection maps.

### 10.6.1. $T$-shuffle permutations and faces of type $T$.

Let $T = (t_1,\dots,t_k)$ be a composition of $n$. A permutation $\zeta \in \mathrm{S}_n$ is a *$T$-shuffle* if

$$\zeta(1) < \cdots < \zeta(t_1),\ \zeta(t_1{+}1) < \cdots < \zeta(t_1{+}t_2), \dots, \zeta(t_1{+}\cdots{+}t_{k-1}{+}1) < \cdots < \zeta(n).$$

We now discuss the geometric meaning of this notion. The definitions imply:

**Proposition 10.6.** *There is a canonical bijection between faces of type $T$ in the Coxeter complex $\Sigma[n]$ and $T$-shuffle permutations: For a face $F$ of type $T$, the corresponding $T$-shuffle permutation $\zeta$ is determined by*

$$FC_{(n)} = \zeta C_{(n)}. \tag{10.39}$$

*The left-hand side is the projection of $C_{(n)}$ on $F$, while the right-hand side is the action of $\zeta$ on $C_{(n)}$.*

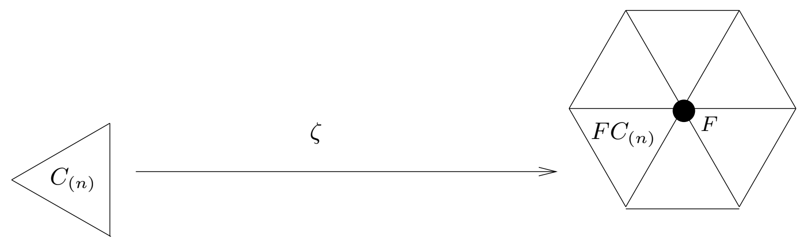



FIGURE 10.6. Faces of type $T$ correspond to $T$-shuffle permutations.

A more general result is given in [12, Lemma 5.3.1]. As a special case, we note that $\mathrm{Sh}\,(s,t)$ can be identified with the set of vertices of type $(s,t)$ in $\Sigma[n]$.

In view of Proposition 10.3, we note that under (10.9), $T$-shuffle permutations correspond to gates of the stars of faces of type $T$. This is illustrated in Figure 10.6. The black dot is a face $F$ of type $T$, and the six triangles around it are the chambers in its star. The $T$-shuffle permutation that corresponds to $F$ is $\zeta = d(C_{(n)}, FC_{(n)})$. It is shown as a vector pointing from $C_{(n)}$ to $FC_{(n)}$.

**10.6.2. Shuffles as coset representatives.** Recall from (2.22) that $(s,t)$-shuffle permutations are coset representatives for $\mathrm{S}_s \times \mathrm{S}_t$ as a subgroup of $\mathrm{S}_n$. We now explain the geometric meaning of this decomposition.

**Proposition 10.7.** *Let $G$ denote the face of $C_{(n)}$ of type $T$. Any chamber in the Coxeter complex $\Sigma[n]$ is uniquely determined by a $T$-shuffle permutation and a chamber in the star of $G$.*

PROOF. Let $C$ be any chamber. It has a unique face of type $T$; call it $F$. Let $\zeta$ be the corresponding $T$-shuffle permutation given by Proposition 10.6. Then the action of $\zeta$ maps the star of $G$ bijectively to the star of $F$. Thus, $C$ is uniquely determined by $\zeta$ and the chamber $\zeta^{-1}C$ which belongs to the star of $G$. $\square$

Let $G$ be a vertex. Then the chambers in the star of $G$ correspond under (10.9) precisely to those permutations which can be written in the form $\sigma \times \tau$ for $\sigma \in \mathrm{S}_s$ and $\tau \in \mathrm{S}_t$. The notation is as in (2.23). This observation along with Proposition 10.7 yields the decomposition (2.22).

## 10.7. The descent and global descent maps

In this section, we discuss the descent and global descent maps which associate a face to a pair of chambers, and further relate them to the descent and global descent maps on permutations.

The descent map on permutations is classical. The notion of global descents is closely related to that of *connected permutations*, which is also classical. The order properties of the global descent map on permutations were studied in [14]. Both descent and global descent maps on pairs of chambers were introduced in [12, Chapter 5] in the generality of finite Coxeter groups.

**10.7.1. Descents and global descents of permutations.** A permutation $w$ has a *descent* at position $p$ if $w(p) > w(p+1)$. Let $\mathrm{Des}(w)$ denote the set of descents of $w$. If $w$ is a permutation on $n$ letters, then $\mathrm{Des}(w)$ is a subset of $[n-1]$, or equivalently by (10.1), it is a composition of $n$. For example,

$$\mathrm{Des}(45132) = \{2, 4\} = (2, 2, 1).$$

A permutation $w$ has a *global descent* at position $p$ if for all $i \le p$ and $j \ge p+1$, we have $w(i) > w(j)$. Let $\mathrm{gDes}(w)$ denote the set of global descents of $w$. It is clear that $\mathrm{gDes}(w) \subseteq \mathrm{Des}(w)$, but these are not equal in general. For example,

$$\mathrm{gDes}(45132) = \{2\} = (2, 3).$$

We view Des and gDes as maps from $\mathrm{S}_n$ to $\Delta_{[n-1]}$, where the latter denotes the set of compositions of $n$.

We let $\mathrm{des}(w)$ and $\mathrm{gdes}(w)$ stand for the number of descents and global descents of a $w$. These are the cardinalities of $\mathrm{Des}(w)$ and $\mathrm{gDes}(w)$ respectively.

**10.7.2. Descents and global descents of pairs of chambers.** Let $\mathbb{L}[I]$ be the set whose elements are pairs of linear orders on $I$. For example,

$$(k|r|i|s|h|n|a, n|a|r|i|k|s|h)$$

is an element of $\mathbb{L}[\{k, r, i, s, h, n, a\}]$. This defines the set species $\mathbb{L}$. The linearized species is denoted $\boldsymbol{\mathbb{L}}$. We now proceed to define morphisms of species

$$\mathrm{Des}\colon \boldsymbol{\mathbb{L}} \to \Sigma \quad \text{and} \quad \mathrm{gDes}\colon \boldsymbol{\mathbb{L}} \to \Sigma.$$

We will refer to these as the *descent* and *global descent maps*.

Let $D$ be a linear order on $I$. A subset $S$ is called a *segment* of $D$ if all its elements appear contigously in $D$. For example,

$$\{k, s, h\} \text{ is a segment of } l|a|k|s|h|m|i.$$

Now let $C$ be another linear order. A segment of $D$ is compatible with respect to $C$ if the elements of that segment appear in the same order in $C$ and $D$. Partially order the set of compatible segments by inclusion. It is clear that the maximal compatible segments yield a partition of $I$.

**Definition 10.8.** Let $C$ and $D$ be two linear orders on $I$. Define $\mathrm{Des}(C, D)$ to be the face of $D$ whose blocks are the maximal compatible segments of $D$ with respect to $C$.

For example,

$$\mathrm{Des}(m|k|s|i|h|l|a, l|a|k|s|h|m|i) = la|ksh|mi.$$

In more geometric terms, $\mathrm{Des}(C, D)$ keeps track of those walls of $D$ which separate $C$ and $D$.

**Definition 10.9.** Let $C$ and $D$ be two linear orders on $I$. Define $\mathrm{gDes}(C, D)$ to be the maximal face $F$ of $D$ such that its opposite $\overline{F}$ is a face of $C$. In other words,

$$\mathrm{gDes}(C, D) = \overline{C} \wedge D.$$

For example,

$$\mathrm{gDes}(m|k|s|i|h|l|a, l|a|k|s|h|m|i) = la|kshmi.$$

**Remark 10.10.** Recall the descent cocycle from Section 9.7.2. The descent map on pairs of chambers is related to the descent cocycle as follows. Let $C$ be any chamber. Then the set $\mathrm{D}_{S,T}(C)$ defined in (9.40) consists of those walls of $C$ which separate $C$ and $KC$. Therefore, $\mathrm{d}_{S,T}(C)$ is the number of blocks of the face $\mathrm{Des}(KC, C)$, where $K = S|T$.

In particular, by letting $C = C_{(n)}$ and using (10.39), we see that $\mathrm{d}_{S,T}(C_{(n)})$ is the number of blocks of $\mathrm{Des}(\zeta C_{(n)}, C_{(n)})$. This yields (9.42).

**10.7.3. Relating the (global) descent maps.** The (global) descent maps on pairs of chambers and on permutations are related by the following commutative diagrams.

$$\begin{tikzcd} \mathbb{L}[n] \arrow[r, "\mathrm{Des}"] \arrow[d, "d"'] & \Sigma[n] \arrow[d, "\mathrm{type}"] \\ \mathrm{S}_n \arrow[r, "\mathrm{Des}"'] & \Delta_{[n-1]} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{L}[n] \arrow[r, "\mathrm{gDes}"] \arrow[d, "d"'] & \Sigma[n] \arrow[d, "\mathrm{type}"] \\ \mathrm{S}_n \arrow[r, "\mathrm{gDes}"'] & \Delta_{[n-1]} \end{tikzcd} \tag{10.40}$$

**10.7.4. The weak order on permutations.** Let $\mathrm{Inv}(\sigma)$ be the set of inversions of a permutation $\sigma$ as in (2.19). Given permutations $\sigma$ and $\tau$, let

$$\sigma \leq \tau \text{ if } \mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\tau).$$

This is the *weak left Bruhat order* on permutations. Equivalently, $\sigma \leq \tau$ if there is a minimum gallery $E - D - C$ such that $d(D, C) = u$ and $d(E, C) = v$. That is,

$$\sigma \leq \tau \iff \tau^{-1}C_{(n)} - \sigma^{-1}C_{(n)} - C_{(n)}.$$

The equivalence between the two definitions follows by noting that $\mathrm{Inv}(\sigma)$ can be identified with the set of hyperplanes which separate $C_{(n)}$ and $\sigma^{-1}C_{(n)}$ by letting the pair $(i, j)$ correspond to the hyperplane $x_i = x_j$.

Figure 10.7, which is taken from [14], shows the weak left Bruhat order on $\mathrm{S}_4$.

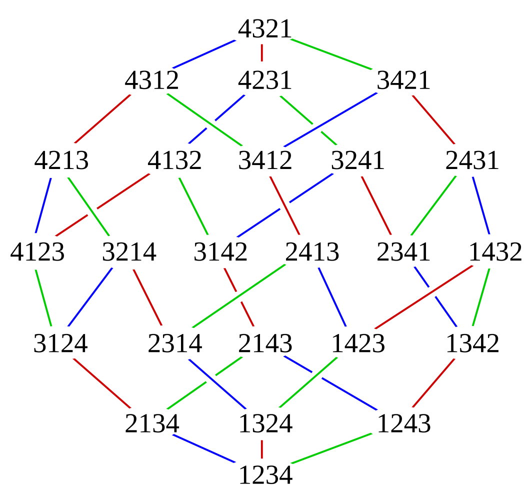



FIGURE 10.7. The weak left Bruhat order on $\mathrm{S}_4$.

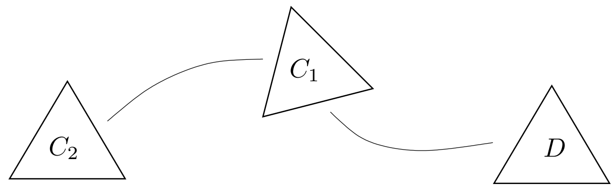



FIGURE 10.8. The partial order on the set of pairs of chambers.

We now define a partial order on the set of pairs of chambers:

$$(C_1, D_1) \le (C_2, D_2) \quad \text{if } D_1 = D_2 = D \text{ and } C_2 - C_1 - D, \tag{10.41}$$

where $C_2 - C_1 - D$ is a minimum gallery from $C_2$ to $D$ passing through $C_1$.

This partial order is illustrated in Figure 10.8 by a schematic two-dimensional picture. It stands for a minimal sequence of triangles starting with $C_2$, ending at $D$ and containing $C_1$ such that adjacent triangles share a common edge.

It is clear that the Weyl-distance map $d\colon \mathbb{L}[n] \to \mathrm{S}_n$ is order-preserving. Further, (10.40) may now be viewed as commutative diagrams of posets.

**10.7.5. Descents and the product of faces.** The discussion here complements the one in Section 10.4.3.

Let $H$ be a set composition and $C$ and $D$ two linear orders. It follows from the definition of the product of faces (10.13) that $HC = D$ if and only if the blocks of $H$ are compatible segments of $D$ with respect to $C$. In view of Definition 10.8, this may be expressed as follows:

$$HC = D \iff \mathrm{Des}(C, D) \le H \le D. \tag{10.42}$$

In other words, $\mathrm{Des}(C, D)$ is the smallest face $H$ of $D$ such that $HC = D$. This observation is due to Brown [70, Proposition 4], see also [12, Proposition 5.2.2].

We also note that

$$\mathrm{Des}(C, D) = D \iff \overline{C} = D.$$

To summarize:

**Proposition 10.11.** *Let $C$ and $D$ be chambers. Then the set of solutions to the equation $HC = D$ is a Boolean poset with minimal element $\mathrm{Des}(C, D)$ and maximal element $D$. The solution is unique precisely if $\overline{C} = D$.*

A more general result is given below.

**Proposition 10.12.** *Let $F$ and $G$ be any faces, and consider the equation $HF = G$. If $GF \neq G$, then it has no solutions. If $GF = G$, then the set of solutions is a Boolean poset with maximal element $G$. A solution exists and is unique if and only if $\overline{F} \le G$.*

PROOF. Suppose there is a $H$ such that $HF = G$. Multiplying by $F$ on the right we deduce $HF = GF$ (since $FF = F$), and hence $G = GF$. This proves the first claim.

For the second claim: Suppose $GF = G$. Let $A$ be the nonempty set of solutions to the equation $HF = G$. Using properties (v) and (vi) in Proposition 10.1, it follows that

- if $H \in A$ and $H \le L \le G$, then $L \in A$,

- if $H_1, H_2 \in A$, then $H_1 \wedge H_2 \in A$.

This shows that $A$ is a Boolean poset under containment of faces (refinement of compositions). The maximal element is clearly $G$.

For the third claim: Suppose $\overline{F} \leq G$. Then $G = G\overline{F}$. Multiplying by $F$ on the right and using property (viii), we deduce that $GF = G$. So $G$ is a solution. To show that it is unique, let $HF = G$. Multiplying by $\overline{F}$ on the left we deduce $\overline{F}H = G$. Since $\overline{F}$ and $H$ are joinable (both being faces of $G$), it follows from (10.14) that $HF = H\overline{F} = G$. By property (x), we conclude that $H = G$ proving uniqueness. Conversely, suppose a solution exists and is unique. So $GF = G$ and $HF \neq G$ for any proper face $H$ of $G$. One can then deduce from property (xi) that $\overline{F} \leq G$. $\square$

It is natural to ask whether the minimal element in the Boolean poset $A$ can be interpreted using descents, as is the case for chambers. In this regard, note the following. If $F$ and $G$ have the same support, say $X$, then the mimimal element of $A$ is precisely the face $\mathrm{Des}(F, G)$ obtained by applying Definition 10.8 to the complex $\Sigma[X]$. In the general case, the description of the mimimal element requires a more general notion of descents. We plan to explain this in a future work.

## 10.8. The action of faces on chambers and the descent algebra

Consider the species of faces $\Sigma$ and the species of pairs of chambers $\mathbb{L}$. Let $\boldsymbol{\Sigma}$ and $\mathbf{L\!L}$ denote their linearizations.

In this section, we equip $\mathbf{L\!L}$ with a product which is compatible with the product of $\boldsymbol{\Sigma}$ given by (10.13). Further, we show that by passing to invariants under the action of the symmetric group, the relation between $\boldsymbol{\Sigma}$ and $\mathbf{L\!L}$ recovers a well-known relation between Solomon's descent algebra and the group algebra of the symmetric group.

**10.8.1. Solomon's descent algebra.** Let $\mathrm{des}\colon \mathrm{S}_n \to \Delta_{[n-1]}$ be the descent map on permutations as defined in Section 10.7.1. Let $\Bbbk$ be a field and let $\Bbbk\mathrm{S}_n$ be the group algebra of $\mathrm{S}_n$ over $\Bbbk$. Solomon [333] showed that the subspace of $\Bbbk\mathrm{S}_n$ linearly spanned by the elements

$$d_T := \sum_{w\colon \mathrm{des}(w) \leq T} w, \tag{10.43}$$

as $T$ varies over subsets of $[n-1]$, is a subalgebra of $\Bbbk\mathrm{S}_n$. This subalgebra is known as the *descent algebra*. A geometric formulation of the descent algebra was given by Bidigare [45] and further clarified by Brown [70, Section 9.6]. An exposition is provided below. The relevant statement is given in Theorem 10.13.

**10.8.2. The action on chambers.** The set $\mathrm{L}[I]$ of chambers is a two-sided ideal of the monoid $\Sigma[I]$ of faces. This follows from property (iv) in Proposition 10.1. The right action is trivial, while the left action is given by the Tits projection

$$p_F\colon \mathrm{L}[I] \to \mathrm{L}[I], \quad D \mapsto FD.$$

Linearizing we obtain that $\mathbf{L}[I]$ is a left ideal in the algebra $\boldsymbol{\Sigma}[I]$. This gives rise to a morphism of algebras

$$\boldsymbol{\Sigma}[I] \to \mathrm{End}_{\mathsf{Vec}}(\mathbf{L}[I]) \qquad F \mapsto p_F,$$

one for each finite set $I$. Further, one can check that these morphisms are injective. These maps define an injective morphism of monoids

$$\boldsymbol{\Sigma} \to \mathcal{E}^{\times}(\mathbf{L}) \tag{10.44}$$

where $\mathcal{E}^{\times}$ is as in (8.80). Here the monoids are with respect to the Hadamard product on species. In other words, both $\boldsymbol{\Sigma}$ and $\mathcal{E}^{\times}(\mathbf{L})$ are species with values in the category of algebras.

We now point out the following canonical identifications.

$$\mathcal{E}^{\times}(\mathbf{L}) \cong \mathbf{L}^* \times \mathbf{L} \cong \mathbb{L}$$

The first one views a basis element $D^* \otimes C \in \mathbf{L}[I]^* \otimes \mathbf{L}[I]$ as the endomorphism

$$E \mapsto \begin{cases} C & \text{if } D = E, \\ 0 & \text{if not.} \end{cases}$$

The second one identifies $D^* \otimes C$ with the pair $(D, C) \in \mathrm{L}[I] \times \mathrm{L}[I]$.

Under these identifications, the product of the algebra $\mathbb{L}[I]$ is as follows.

$$(D_2, C_2)\,(D_1, C_1) \;=\; \begin{cases} (D_1, C_2) & \text{if } D_2 = C_1, \\ 0 & \text{otherwise.} \end{cases} \tag{10.45}$$

The morphism of monoids (10.44) takes the form

$$\boldsymbol{\Sigma} \to \mathbb{L}, \qquad F \mapsto \sum_{(D,C)\colon FD=C} (D, C). \tag{10.46}$$

**10.8.3. Invariant subalgebras.** Since $\boldsymbol{\Sigma}$ and $\mathbb{L}$ are species, the components $\boldsymbol{\Sigma}[n]$ and $\mathbb{L}[n]$ are $\mathrm{S}_n$-modules. Further, since these species are monoids with respect to the Hadamard product, the components are algebras whose products commute with the $\mathrm{S}_n$-action. This yields subalgebras of $\mathrm{S}_n$-invariants

$$(\boldsymbol{\Sigma}[n])^{\mathrm{S}_n} \hookrightarrow \boldsymbol{\Sigma}[n] \quad \text{and} \quad (\mathbb{L}[n])^{\mathrm{S}_n} \hookrightarrow \mathbb{L}[n].$$

We make these subalgebras explicit.

A basis for the subalgebra $(\boldsymbol{\Sigma}[n])^{\mathrm{S}_n}$ is given by

$$\sigma_T := \sum_{F\colon \mathrm{type}(F)=T} F, \tag{10.47}$$

as $T$ ranges over all subsets of $S$.

The subalgebra $(\mathbb{L}[n])^{\mathrm{S}_n}$ can be identified with the opposite of the group algebra as follows.

$$(\Bbbk\mathrm{S}_n)^{\mathrm{op}} \xrightarrow{\cong} (\mathbb{L}[n])^{\mathrm{S}_n}, \qquad w \mapsto \sum_{(D,C)\colon d(D,C)=w} (D, C).$$

The main assertion here is that this is a morphism of algebras. This is a consequence of (10.30) and (10.45).

Now consider the following commutative diagram of algebras.

$$\begin{tikzcd} \boldsymbol{\Sigma}[n] \arrow[r] & \mathbb{L}[n] \\ (\boldsymbol{\Sigma}[n])^{\mathrm{S}_n} \arrow[u] \arrow[r] & (\Bbbk\mathrm{S}_n)^{\mathrm{op}} \arrow[u] \end{tikzcd}$$

It follows from (10.42) and the first diagram in (10.40) that the bottom horizontal map sends $\sigma_T$ to $d_T$ as defined in (10.43) and (10.47). Hence the image of the bottom horizontal map is precisely the descent algebra. As a consequence:

**Theorem 10.13 (Bidigare).** *The descent algebra is isomorphic to* $\big((\mathbf{\Sigma}[n])^{\mathrm{S}_n}\big)^{\mathrm{op}}$.

## 10.9. Directed faces and directed flats

Recall that there are two fundamental objects associated to the Coxeter complex of type $A$, namely, faces and flats, and they are related by the support map. In this section, we show that there is a parallel theory for directed faces and directed flats. In combinatorial terms, this means that we replace set compositions (partitions) by linear set compositions (partitions).

We mainly follow the exposition in [12, Section 2.3], where this theory is explained in the generality of left regular bands. In that work, directed faces are called *pointed faces* and directed flats are called *lunes*.

**10.9.1. Directed faces and directed flats.** A *directed face* of the complex $\Sigma[I]$ is a pair $(G, D)$ where $G$ is a face and $D$ is a chamber containing $G$.

Directed faces of $\Sigma[I]$ are the same as linear compositions of $I$: The face $G$ is a composition of the set $I$ and the chamber $D$ determines a linear order on each block of $G$. Since $D$ refines $G$, the pair $(G, D)$ can be recovered from the linear set composition.

Directed faces may be visualized as in Figure 10.9. The pair $(G, D)$ tells us to stand at the face $G$ and look in the direction of the chamber $D$.

Let $\overrightarrow{\Sigma}[I]$ denote the set of directed faces. This defines the set species $\overrightarrow{\Sigma}$ of directed faces, or equivalently, of linear set compositions. The linearized species is denoted $\overrightarrow{\mathbf{\Sigma}}$.

Define an equivalence relation on $\overrightarrow{\Sigma}[I]$ as follows.

$$(G, D) \sim (F, C) \iff GF = G,\ GC = D,\ FG = F \text{ and } FD = C. \tag{10.48}$$

The equivalence classes are called *directed flats*.

It follows from (10.13) that $(G, D) \sim (F, C)$ if and only if the compositions $F$ and $G$ differ only in the ordering of the blocks, and the linear orders $C$ and $D$ agree on each of these blocks. Thus, directed flats are the same as linear partitions of $I$.

Let $\overrightarrow{\Pi}[I]$ denote the set of directed flats. This defines the set species $\overrightarrow{\Pi}$ of directed flats, or equivalently, of linear set partitions. The linearized species is denoted $\overrightarrow{\mathbf{\Pi}}$.

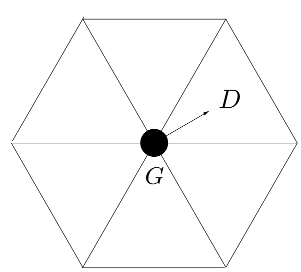



FIGURE 10.9. A directed face.

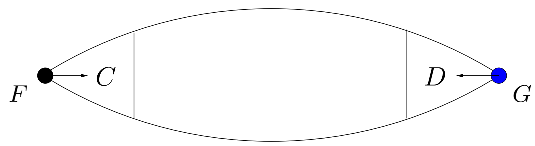



FIGURE 10.10. A schematic picture of a directed flat in dimension two.

**10.9.2. The base and support maps.** Let supp: $\overrightarrow{\Sigma}[I] \to \overrightarrow{\Pi}[I]$ be the canonical quotient map and base: $\overrightarrow{\Sigma}[I] \to \Sigma[I]$ the projection on the first coordinate. From (10.19) and (10.48) we have that

$$(G, D) \sim (F, C) \implies G \sim F.$$

It follows that there is an induced map $\overrightarrow{\Pi}[I] \to \Pi[I]$, also denoted base, fitting in the commutative diagram below.

$$\begin{tikzcd} \overrightarrow{\Sigma}[I] \arrow[r, "\text{base}"] \arrow[d, "\text{supp}"'] & \Sigma[I] \arrow[d, "\text{supp}"] \\ \overrightarrow{\Pi}[I] \arrow[r, "\text{base}"'] & \Pi[I] \end{tikzcd} \tag{10.49}$$

These maps coincide with the support and base maps defined in Section 10.1.5. In particular, if $\text{supp}(F, C) = L$, then the blocks of $L$ are the blocks of $F$ ordered according to $C$.

**10.9.3. Directed flats as top-dimensional cones.** A directed flat may be visualized as follows. Let $(G, D)$ be a directed face. Imagine all hyperplanes containing $G$ are opaque. Standing at $G$ and looking in the direction of $D$ one overlooks a portion of the ambient space. Two directed faces are equivalent under (10.48) if they overlook the same region. From this perspective, diagram (10.49) expresses the following fact: If two directed faces $(F, C)$ and $(G, D)$ overlook the same region, then the faces $F$ and $G$ have the same support.

This is illustrated in Figure 10.10. The directed faces $(F, C)$ and $(G, D)$ are equivalent: the oval in the figure is the region overlooked from either directed face. It is the intersection of the half-spaces (hemispheres) which contain $C$ and whose supporting hyperplane (great circle) contains $F$. Among these half-spaces, only those whose supporting hyperplanes are walls of $C$ are essential to determine the intersection. There are two of these in this case. The support of either $F$ or $G$ is the set $\{F, G\}$.

We now define the region overlooked from a directed face more precisely. To any directed face $(F, C)$, we associate a top-dimensional cone: intersect those half-spaces which contain $C$ and whose supporting hyperplane contains $F$. This cone is the region overlooked from $(F, C)$; we denote it by $\Psi(F, C)$.

We now describe the set of faces contained in this cone.

**Proposition 10.14.** *Let $(F, C)$ be a directed face. For any face $K$,*

$$K \subseteq \Psi(F, C) \iff FK \leq C.$$

*In particular, for any chamber* $D$,

$$D \subseteq \Psi(F,C) \iff FD = C.$$

PROOF. For simplicity we consider the second statement only. It follows from the definition of the cone that $D \subseteq \Psi(F,C)$ if and only if $C$ and $D$ lie on the same side of every hyperplane containing $F$. According to the description of chambers in Section 10.2.2, the latter is equivalent to the statement that if two elements $i$ and $j$ belong to the same block of $F$, then they appear in the same order in $C$ and $D$. In view of (10.13), this is in turn equivalent to $FD = C$. $\square$

**Proposition 10.15.** *Two directed faces yield the same cone if and only if they are equivalent under* (10.48). *Explicitly,*

$$(F,C) \sim (G,D) \iff \Psi(F,C) = \Psi(G,D).$$

In other words, the map $\Psi$ induces a bijection between the set of directed flats and the set of cones associated to directed faces. We continue to denote the induced map by $\Psi$. This bijection allows us to visualize directed flats as top-dimensional cones.

PROOF. Using symmetry, it is enough to show that

$$GF = G \text{ and } GC = D \iff \Psi(F,C) \subseteq \Psi(G,D). \tag{10.50}$$

Take a face $K$ in $\Psi(F,C)$. By Proposition 10.14, $FK \leq C$. If $GF = G$ and $GC = D$, then using property (iii) in Proposition 10.1 we have

$$FK \leq C \implies GFK \leq GC \implies GK \leq D.$$

This shows that $K$ is in $\Psi(G,D)$.

Conversely, assume $\Psi(F,C) \subseteq \Psi(G,D)$. Since $C \subseteq \Psi(F,C)$, then $C \subseteq \Psi(G,D)$, and by Proposition 10.14, we have $GC = D$. In addition, since both $F$ and $\overline{F}$ are in $\Psi(F,C)$, they are also in $\Psi(G,D)$, and $GF \leq D$ and $G\overline{F} \leq D$. Now from property (ii) in Proposition 10.1 we derive $GFD = G\overline{F}D = D$, and from property (x) it follows that $GF = G$. $\square$

Generalizations of Propositions 10.14 and 10.15 are given in [12, Lemmas 2.3.2 and 2.3.3].

Let $L$ be the support of the directed face $(F,C)$. Viewing $L$ as a linear set composition, the blocks of $L$ are the blocks of $(F,C)$ as in Section 10.4.3. It follows from Proposition 10.14 and (10.22) that

$$D \subseteq \Psi(L) \iff D \text{ is a shuffle of the blocks of } L.$$

Let us now look at a specific example to illustrate the preceding discussion. Two directed flats in the simplicial complex $\Sigma[\{a,b,c,d\}]$ are shown in Figure 10.11. The first directed flat is bounded by the hyperplanes $x_a = x_d$ and $x_c = x_d$. As a linear set partition, it is given by $\{b, a|d|c\}$. It is the cone associated to the directed face $(b|acd, b|a|d|c)$, or equivalently to $(acd|b, a|d|c|b)$. The four chambers that it contains are

$$b|a|d|c, \ a|b|d|c, \ a|d|b|c, \quad \text{and} \quad a|d|c|b.$$

These are precisely the shuffles of $b$ and $a|d|c$. The base of this directed flat is the set partition $\{b, adc\}$. This is the precisely the support of the vertices $b|acd$ and $acd|b$, which can be seen at the two corners of the directed flat.

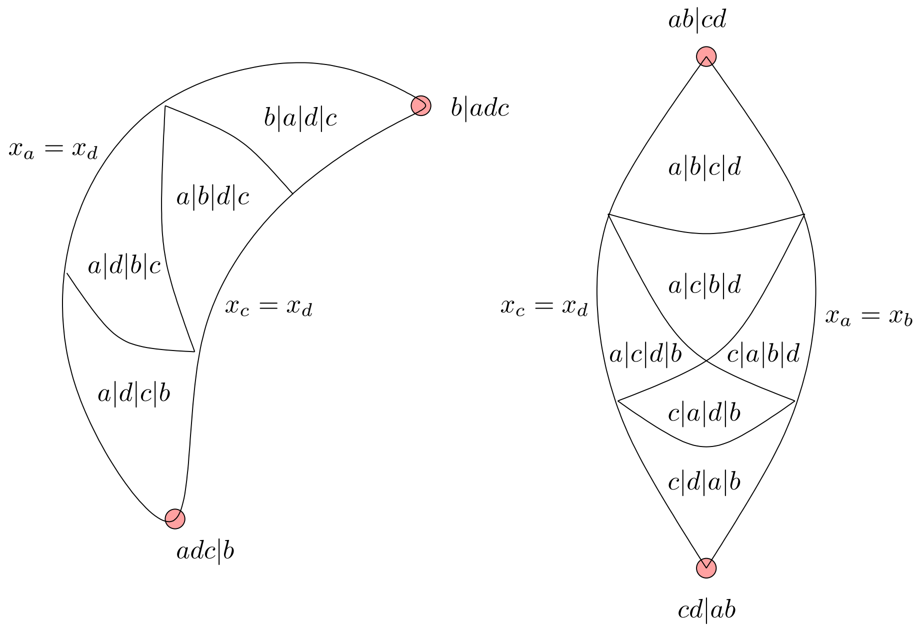



FIGURE 10.11. Two directed flats in the simplicial complex $\Sigma[\{a, b, c, d\}]$.

The second directed flat is bounded by the hyperplanes $x_a = x_b$ and $x_c = x_d$. As a linear set partition, it is given by $\{a|b, c|d\}$. It is the cone associated to the directed face $(ab|cd, a|b|c|d)$, or equivalently to $(cd|ab, c|d|a|b)$. The six chambers that it contains are

$$a|b|c|d,\ a|c|b|d,\ a|c|d|b,\ c|a|b|d,\ c|a|d|b, \quad \text{and} \quad c|d|a|b.$$

These are precisely the shuffles of $a|b$ and $c|d$. The base of this directed flat is the set partition $\{ab, cd\}$. This is the precisely the support of the vertices $ab|cd$ and $cd|ab$, which can be seen at the two corners of the directed flat.

These are two typical directed flats bounded by two hyperplanes, but general directed flats involve an arbitrary number of hyperplanes and range from the whole space and half-spaces at one end, to chambers at the other.

**10.9.4. Left modules over faces.** Given a face $K$ and a directed face $(G, D)$, define

$$K \cdot (G, D) := (KG, KD). \tag{10.51}$$

In view of properties (iii) and (iv) in Proposition 10.1, $(KG, KD)$ is a directed face. In this manner, the set of directed faces $\overrightarrow{\Sigma}[I]$ is a left module over the monoid of faces $\Sigma[I]$. Note that (10.48) can be rewritten as:

$$(G, D) \sim (F, C) \iff G \cdot (F, C) = (G, D) \text{ and } F \cdot (G, D) = (F, C).$$

Observe that for any face $K$,

$$(G, D) \sim (F, C) \implies K \cdot (G, D) \sim K \cdot (F, C).$$

It follows that $\overrightarrow{\Pi}[I]$ is also a left module over $\Sigma[I]$:

$$F \cdot M := \operatorname{supp}(FG, FD), \tag{10.52}$$

where $(G, D)$ is any directed face whose support is $M$. For example,

$$(lak|shmi) \cdot \{s|h|k, l|a, m|i\} = \{l|a, k, s|h, m|i\}.$$

By construction, the support map from directed faces to directed flats is a morphism of left modules.

**10.9.5. Partial orders.** Recall from Section 10.1.4 that we have a partial order on linear compositions, and two partial orders on linear partitions. We now phrase them in geometric terms.

The partial order on the set of directed faces is given by:

$$(F, C) \leq (G, D) \quad \text{if} \quad C = D \text{ and } F \leq G. \tag{10.53}$$

This is a disjoint union of Boolean posets, one for each chamber $C$.

We now discuss the two partial orders on the set of directed flats.

$$L \leq' M \quad \text{if} \quad H \cdot L = M \text{ for some face } H, \tag{10.54}$$

where $\cdot$ denotes the left module structure of directed flats (10.52).

Let $(F, C)$ and $(G, D)$ be directed faces. Each has an associated cone as in Section 10.9.3. We have:

$$\begin{aligned} \Psi(F, C) \subseteq \Psi(G, D) &\iff GF = G \text{ and } GC = D \\ &\iff G \cdot (F, C) = (G, D) \\ &\iff H \cdot (F, C) = (G, D) \text{ for some face } H \\ &\iff \operatorname{supp}(F, C) \leq' \operatorname{supp}(G, D). \end{aligned}$$

The first equivalence is (10.50). For the converse of the last implication, choose $H'$ as in (10.54), and let $H = GH'$. The remaining implications are straightforward.

**Proposition 10.16.** *Let $L$ and $M$ be directed flats. We have*

$$L \leq' M \iff \Psi(L) \subseteq \Psi(M).$$

Proof. This follows by applying the above to any directed faces $(F, C)$ and $(G, D)$ with supports $L$ and $M$ respectively. $\square$

Figure 10.4 shows that the cone associated to $\{a|d|b, c\}$ is contained in the cone associated to $\{a|b, c, d\}$. So

$$\{a|d|b, c\} \leq' \{a|b, c, d\}.$$

Let us now discuss the second partial order on directed flats.

$$L \leq M \quad \text{if} \quad H \cdot L = M, \ H \subseteq \Psi(L) \text{ for some face } H, \tag{10.55}$$

where $\Psi(L)$ is the cone defined in Section 10.9.3 and $\cdot$ denotes the left module structure of directed flats (10.52).

Observe that for any directed faces $(F, C)$ and $(G, D)$,

$$\begin{aligned} \operatorname{supp}(FG, C) = \operatorname{supp}(G, D) &\iff GF = G, \ GC = D, \ FD = C \\ &\iff GF = G, \ GC = D, \ FG \leq C \\ &\iff G \cdot (F, C) = (G, D), \ FG \leq C \\ &\iff H \cdot (F, C) = (G, D), \ FH \leq C \text{ for some face } H \\ &\implies \operatorname{supp}(F, C) \leq \operatorname{supp}(G, D). \end{aligned}$$

Note that the backward implication on the last line may fail. What one can say instead is the following: $\operatorname{supp}(F, C) \leq M$ if and only if there is a directed face $(G, D)$ with support $M$ such that $G \cdot (F, C) = (G, D)$ and $FG \leq C$ (or any of the above equivalent condition) holds. This further implies that $L \leq M$ if and only if there is a directed face with support $L$ which is less than a directed face with support $M$: Take directed faces $(F, C)$ and $(G, D)$ with supports $L$ and $M$ as above, and replace $(G, D)$ with $(FG, C)$.

**Proposition 10.17.** *Let $(F, C)$ and $(G, D)$ be directed faces such that $G \cdot (F, C) = (G, D)$ and $FG \leq C$. Then for any face $H$,*

$$H \cdot (F, C) = (G, D),\ FH \leq C \iff HF = G.$$

PROOF. The forward implication is clear. For the backward implication, we note that

$$HC = HFC = GC = D \qquad \text{and} \qquad FH \leq FG \leq C.$$

Thus, $HC = D$ and $FH \leq C$ as required. ☐

If $L \leq M$, then $L \leq' M$, and hence the cone associated to $L$ is contained in the cone associated to $M$. The converse, of course, is false since the two partial orders are distinct. For example,

$$\{a|d|b, c\} \leq' \{a|b, c, d\} \quad \text{but} \quad \{a|d|b, c\} \nleq \{a|b, c, d\}.$$

## 10.10. The dimonoid of directed faces

The monoid structure of the set of faces of the Coxeter complex has played a central role in the preceding sections. The set of faces $\Sigma[I]$ is a monoid and the set of flats $\Pi[I]$ is a quotient monoid under the support map (Section 10.4). It is natural to ask whether there is a similar structure on the set of directed faces $\overrightarrow{\Sigma}[I]$ and the set of directed flats $\overrightarrow{\Pi}[I]$. It turns out that $\overrightarrow{\Sigma}[I]$ is a bimodule over $\Sigma[I]$ and $\overrightarrow{\Pi}[I]$ is a bimodule over $\Pi[I]$. Moreover, there is a finer structure of *dimonoid* on each of these bimodules.

**10.10.1. Dimonoids.** We now take a small detour to the world of dimonoids. This notion was introduced by Loday [238, Section 1].

**Definition 10.18.** A dimonoid is a set $D$ equipped with two binary operations $\vdash$ and $\dashv$ such that:

(i) $x \dashv (y \dashv c) = (x \dashv y) \dashv z = x \dashv (y \vdash z)$,
(ii) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
(iii) $(x \dashv y) \vdash z = x \vdash (y \vdash z) = (x \vdash y) \vdash z$.

A bar-unit is an element $1 \in D$ such that

$$1 \vdash x = x = x \vdash 1$$

for every $x \in D$.

Bar-units need not be unique.

A monoid $M$ can be viewed as a dimonoid by setting

$$x \vdash y := xy =: x \dashv y.$$

More generally, suppose $B$ is a bimodule over a monoid $M$, and there is a map $\delta : B \to M$ a bimodules. Then [238, Example 2.2.d] $B$ can be turned into a dimonoid by setting

$$x \vdash y := \delta(x) \cdot y \quad \text{and} \quad x \dashv y := x \cdot \delta(y).$$

Every dimonoid arises in this manner from such a map $\delta$ [134, Proposition 1.6]. In fact, given a dimonoid $D$, let $M$ be the quotient by the dimonoid-ideal generated by the relations

$$x \vdash y \equiv x \dashv y.$$

Then $M$ is a dimonoid in which $\vdash = \dashv$, and so it is a monoid. Let $\delta : D \to M$ denote the quotient map. Then

$$\delta(x) \cdot y := x \vdash y \quad \text{and} \quad x \cdot \delta(y) := x \dashv y$$

yield a well-defined $M$-bimodule structure on $D$. The dimonoid associated to $\delta$ is the original one.

These constructions define a pair of adjoint functors between the category of dimonoids and a suitable category of maps $\delta$ as above. The functor from dimonoids is the left adjoint.

**10.10.2. The dimonoids of directed faces and of directed flats.** The set of directed faces $\overrightarrow{\Sigma}[I]$ is a bimodule over $\Sigma[I]$. The left and right module structures are:

$$F \cdot (G, D) := (FG, FD),$$
$$(F, C) \cdot G := (FG, FGC).$$

The left module structure is the same as in Section 10.9.4. For the right module structure, note that $(FG, FGC)$ is a directed face in view of properties (i) and (iii) in Proposition 10.1. In addition, property (ix) guarantees that the right structure is associative. The fact that the two structures commute follows.

Recall the base and support maps from Section 10.9.2. The map

$$\text{base}\colon \overrightarrow{\Sigma}[I] \to \Sigma[I]$$

(which simply projects on the first coordinate) is a map of $\Sigma[I]$-bimodules.

It then follows that $\overrightarrow{\Sigma}[I]$ is a dimonoid. The operations are

$$(F, C) \vdash (G, D) := F \cdot (G, D) = (FG, FD),$$
$$(F, C) \dashv (G, D) := (F, C) \cdot G = (FG, FGC).$$

Directed faces of the form $(\emptyset, C)$ are the bar-units of this dimonoid.

Similarly, the set of directed flats $\overrightarrow{\Pi}[I]$ is a bimodule over $\Pi[I]$:

$$X \cdot M := \text{supp}(KG, KD),$$
$$L \cdot X := \text{supp}(FK, FKC),$$

where $K$ is any face with support $X$, and $(F, C)$ and $(G, D)$ are directed faces with supports $L$ and $M$ respectively. The map

$$\text{base}\colon \overrightarrow{\Pi}[I] \to \Pi[I]$$

is a map of $\Pi[I]$-bimodules, and it follows that $\overrightarrow{\Pi}[I]$ is a dimonoid.

Thus the monoids $\Sigma[I]$ and $\Pi[I]$ are dimonoids as well. One can now check that:

**Proposition 10.19.** *Diagram* (10.49) *is a commutative diagram of dimonoids.*

We observe that since $\Sigma[I] \to \Pi[I]$ is a morphism of monoids, $\overrightarrow{\Pi}[I]$ is also a bimodule over $\Sigma[I]$. Explicitly,

$$\begin{aligned} K \cdot M &:= \operatorname{supp}(KG, KD), \\ L \cdot K &:= \operatorname{supp}(FK, FKC), \end{aligned}$$

where $(F, C)$ and $(G, D)$ are directed faces with supports $L$ and $M$ respectively.

As explained above, the dimonoids $\overrightarrow{\Sigma}[I]$ and $\overrightarrow{\Pi}[I]$ arise from the base maps $\overrightarrow{\Sigma}[I] \to \Sigma[I]$ and $\overrightarrow{\Pi}[I] \to \Pi[I]$ by means of the construction of Section 10.10.1. One can check that if one applies the left adjoint construction to these dimonoids one retrieves the monoids $\Sigma[I]$ and $\Pi[I]$ and the base maps.

**10.10.3. The Jacobson radical.** We now linearize the preceding discussion. The linearization of a dimonoid is a *dialgebra*. Proposition 10.19 yields the following commutative diagram of dialgebras.

$$\begin{tikzcd} \overrightarrow{\mathbf{\Sigma}}[I] \arrow[r, "\text{base}"] \arrow[d, "\text{supp}"'] & \mathbf{\Sigma}[I] \arrow[d, "\text{supp}"] \\ \overrightarrow{\mathbf{\Pi}}[I] \arrow[r, "\text{base}"'] & \mathbf{\Pi}[I] \end{tikzcd} \tag{10.56}$$

We discuss this diagram in more detail below. We pause to recall a basic fact. Let $A$ be an algebra and $J$ be a two-sided ideal. Then $A/J$ is an algebra and the quotient map $A \to A/J$ is a morphism of algebras. Now let $M$ be an $A$-bimodule. Then $M/JMJ$ is an $(A/J)$-bimodule and, in particular, an $A$-bimodule.

We proceed. Let $A$ be the algebra of faces $\mathbf{\Sigma}[I]$, and let $J$ be its Jacobson radical. Bidigare [45] showed that $J$ is precisely the kernel of its support map. This result was generalized to left regular bands by Brown [70], also see [12, Lemma 2.5.5]. Thus, $A/J$ is the algebra of flats $\mathbf{\Pi}[I]$, and the quotient map is the support map. Now let $M$ be the bimodule of directed faces $\overrightarrow{\mathbf{\Sigma}}[I]$. One can check that $JMJ = JM = MJ$ and that this subbimodule is the kernel of the support map from directed faces to directed flats. Thus, $M/JMJ$ is the space of directed flats $\overrightarrow{\mathbf{\Pi}}[I]$. It is a bimodule over $\mathbf{\Pi}[I]$.

## 10.11. The break and join maps

We mentioned earlier that Coxeter complexes are closed under the star and join operations. In this section, we explain how this property can be used to define break and join maps on the faces, flats, directed faces and directed flats of the Coxeter complex of type $A$. These maps along with Tits projection maps will play an important role in the construction of Hopf monoids in species (Chapter 12).

**10.11.1. The break and join maps for faces.** Let $K = S|T$ be a vertex of $\Sigma[I]$. A face $F$ which contains $K$ consists of a composition of $S$ followed by a composition of $T$. This yields a canonical identification

$$\operatorname{Star}(S|T) \cong \Sigma[S] \times \Sigma[T]$$

between the star of the vertex $S|T$ in $\Sigma[I]$ and the join of the complexes $\Sigma[S]$ and $\Sigma[T]$. We use

$$\mathrm{Star}(S|T) \underset{j_{S|T}}{\overset{b_{S|T}}{\rightleftarrows}} \Sigma[S] \times \Sigma[T] \tag{10.57}$$

to denote the inverse isomorphisms of simplicial complexes. We refer to $b_{S|T}$ and $j_{S|T}$ as the *break* and *join maps*, respectively. Explicitly, if $F = F^1|\cdots|F^i$ is a composition of $S$ and $G = G^1|\cdots|G^j$ is a composition of $T$, then

$$j_{S|T}(F, G) = F^1|\cdots|F^i|G^1|\cdots|G^j.$$

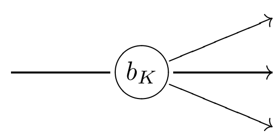
The star and join operations preserve Coxeter complexes; thus the break and join maps are simplicial isomorphisms between Coxeter complexes.

More generally, the break and join maps can be defined for any face: For $K = K^1|K^2|\cdots|K^j$, there are inverse isomorphisms of simplicial complexes

$$\mathrm{Star}(K^1|K^2|\cdots|K^j) \underset{j_K}{\overset{b_K}{\rightleftarrows}} \Sigma[K^1] \times \Sigma[K^2] \times \cdots \times \Sigma[K^j] \tag{10.58}$$

where $\mathrm{Star}(K)$ is the star of the face $K$ in $\Sigma[I]$.

The following is a useful way to picture the break map; the figure illustrates the case when $K$ has three parts. The disc at the center is an apparatus which takes one input larger than $K$ and produces three ordered outputs.




A similar picture can be drawn for the join map by reversing the arrows.

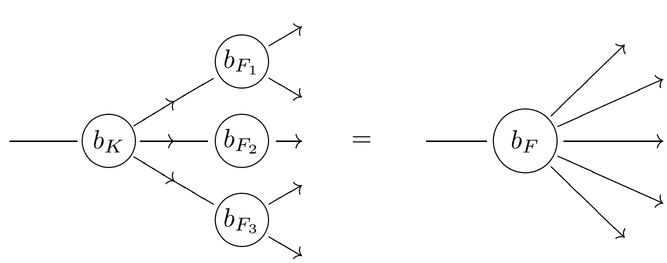
**10.11.2. Compatibilities.** The break and join maps are associative in the following sense: Let $K$ be a face of $F$, and let $b_K(F) = (F_1, \ldots, F_k)$. Then

$$b_F = (b_{F_1} \times b_{F_2} \times \cdots \times b_{F_k}) \circ b_K.$$

Equivalently, with the same setup,

$$j_F = j_K \circ (j_{F_1} \times j_{F_2} \times \cdots \times j_{F_k}).$$

An illustration for the associativity of the break map is provided below.




The break and join maps are compatible with the projection maps: For faces $F$ and $G$ which contain $K$,

$$b_K(FG) = b_K(F)b_K(G),$$

the product on the right being taken componentwise. Equivalently, for $K = K^1|K^2|\cdots|K^j$,

$$j_K(F_1,\dots,F_j)j_K(G_1,\dots,G_j) = j_K(F_1G_1,\dots,F_jG_j),$$

where $F_i$ and $G_i$ are compositions of $K^i$, as $i$ varies from 1 to $j$.

The break and join maps are compatible with the distance function on faces, and hence in particular with the gallery metric on chambers.

$$\operatorname{dist}\bigl(j_K(F_1,\dots,F_j), j_K(G_1,\dots,G_j)\bigr) = \sum_{i=1}^{j} \operatorname{dist}(F_i,G_i). \tag{10.59}$$

Equivalently, for faces $F$ and $G$ which contain $K$,

$$\operatorname{dist}(F,G) = \sum_{i=1}^{j} \operatorname{dist}(F_i,G_i), \tag{10.60}$$

where $b_K(F) = (F_1,\dots,F_j)$ and $b_K(G) = (G_1,\dots,G_j)$.

The compatibility of the distance function with projection maps (10.31), in conjunction with (10.60) yields the following important consequence.

Let $I = S \sqcup T$ be a decomposition, and let $K = S|T$. Further, let $C$ and $D$ be linear orders on $I$, $C_1$ and $D_1$ be linear orders on $S$, and $C_2$ and $D_2$ be linear orders on $T$, such that $b_K(KC) = (C_1,C_2)$ and $b_K(D) = (D_1,D_2)$. Then

$$\operatorname{dist}(C,D) = \operatorname{dist}(C_1,D_1) + \operatorname{dist}(C_2,D_2) + \operatorname{dist}(C,KC). \tag{10.61}$$

It is clear that this can be generalized by replacing the vertex $K$ by any face.

**10.11.3. Relation with shuffles and quasi-shuffles.** Recall the notions of shuffles and quasi-shuffles from Section 10.1.6. We now explain how they fit into the framework of break and join maps.

Let $K = S|T$ be a vertex of $\Sigma[I]$. Let $F$, $F_1$ and $F_2$ be faces of $\Sigma[I]$, $\Sigma[S]$ and $\Sigma[T]$ respectively. It follows from (10.20) that

$$\begin{aligned} b_K(KF) = (F_1,F_2) &\iff KF = j_K(F_1,F_2)\\ &\iff F \text{ is a quasi-shuffle of } F_1 \text{ and } F_2. \end{aligned} \tag{10.62}$$

For example, let $I = \{l,a,k,s,h,m,i\}$, $S = \{a,k,l\}$ and $T = \{h,i,m,s\}$. If the vertex $K$ is $alk|sihm$, then the set compositions

$$F = lsh|m|aki, \qquad F_1 = l|ak, \qquad \text{and} \qquad F_2 = sh|m|i$$

satisfy the conditions (10.62).

Going back to the general discussion, it follows from (10.21) that

$$\begin{aligned} b_K(KF) = (F_1,F_2) \text{ and } FK = F &\iff KF = j_K(F_1,F_2) \text{ and } FK = F\\ &\iff F \text{ is a shuffle of } F_1 \text{ and } F_2. \end{aligned} \tag{10.63}$$

With $I$, $S$, $T$ and $K$ as in the above example, the set compositions

$$F = l|sh|m|ak|i, \qquad F_1 = l|ak, \qquad \text{and} \qquad F_2 = sh|m|i.$$

satisfy conditions (10.63).

Let $C$, $C_1$ and $C_2$ are linear orders on $I$, $S$ and $T$ respectively and $K = S|T$. It follows from (10.22) that

$$\begin{aligned} b_K(KC) = (C_1,C_2) &\iff KC = j_K(C_1,C_2)\\ &\iff C \text{ is a shuffle of } C_1 \text{ and } C_2. \end{aligned} \tag{10.64}$$

With $I$, $S$, $T$ and $K$ as before, the linear orders

$$C = l|s|h|m|a|k|i, \qquad C_1 = l|a|k, \qquad \text{and} \qquad C_2 = s|h|m|i$$

satisfy conditions (10.64).

The above discussion can be generalized by replacing the vertex $K$ by any face.

**10.11.4. The break and join maps for flats.** We now discuss the analogues of the break and join maps for set partitions. For a partition $X$ of $I$, let $\mathrm{Star}(X)$ denote the star of $X$ in $\Pi[I]$. It consists of those partitions of $I$ which refine $X$. For $K = K^1|K^2|\cdots|K^j$, there are inverse isomorphisms

$$\mathrm{Star}\big(\mathrm{supp}(K)\big) \underset{j_K}{\overset{b_K}{\rightleftarrows}} \Pi[K^1] \times \Pi[K^2] \times \cdots \times \Pi[K^j]. \tag{10.65}$$

These are the break and join maps for flats. Note that $\mathrm{Star}\big(\mathrm{supp}(K)\big)$ consists precisely of those flats $X$ for which $K \cdot X = X$, with the module structure as in (10.18).

**10.11.5. The break and join maps for directed faces.** We now discuss the analogues of the break and join maps for linear set compositions.

For a set composition $K = K^1|K^2|\cdots|K^j$ of $I$, let

$$\mathrm{Star}_{\overrightarrow{\Sigma}[I]}(K)$$

denote the set of those linear set compositions $(G, D)$ for which $K \leq G$, or equivalently, $K \cdot (G, D) = (G, D)$, with the left module structure as in (10.51). In other words, it is the set of directed faces of the simplicial complex $\mathrm{Star}(K)$. Note that an element of this set is a linear set composition of $K^1$, followed by a linear set composition of $K^2$, and so on. This observation yields inverse bijections

$$\mathrm{Star}_{\overrightarrow{\Sigma}[I]}(K) \underset{j_K}{\overset{b_K}{\rightleftarrows}} \overrightarrow{\Sigma}[K^1] \times \overrightarrow{\Sigma}[K^2] \times \cdots \times \overrightarrow{\Sigma}[K^j]. \tag{10.66}$$

These are the break and join maps for directed faces.

Note that these break and join maps are defined by using the break and join maps for faces in both coordinates. For example, for a vertex $K$, if $b_K(F) = (F_1, F_2)$ and $b_K(D) = (D_1, D_2)$, then $b_K(F, D) = \big((F_1, D_1), (F_2, D_2)\big)$.

**10.11.6. The break and join maps for directed flats.** For a set composition $K = K^1|K^2|\cdots|K^j$ of $I$, let

$$\mathrm{Star}_{\overrightarrow{\Pi}[I]}(K)$$

denote the image of $\mathrm{Star}_{\overrightarrow{\Sigma}[I]}(K)$ under the support map. It consists of precisely those directed flats $M$ for which $K \cdot M = M$, with the left module structure as in (10.52). Alternatively, it is the set of directed flats of the simplicial complex $\mathrm{Star}(K)$. More explicitly, an element of this set is a disjoint union of a linear set partition of $K^1$, a linear set partition of $K^2$, and so on. This yields inverse bijections

$$\mathrm{Star}_{\overrightarrow{\Pi}[I]}(K) \underset{j_K}{\overset{b_K}{\rightleftarrows}} \overrightarrow{\Pi}[K^1] \times \overrightarrow{\Pi}[K^2] \times \cdots \times \overrightarrow{\Pi}[K^j]. \tag{10.67}$$

These are the break and join maps for directed flats.

## 10.12. The weighted distance function

In this section we discuss the weighted version of the distance function on chambers and faces (Section 10.5). These come in two flavors: additive and multiplicative. They depend on a matrix of size $r$ and a function $f\colon I \to [r]$. We denote the matrix by $A$ in the additive case, and by $Q$ in the multiplicative case. The two cases can be related by (2.33).

**10.12.1. The additive case.** Let $A$ be a fixed integer matrix of size $r$ and $f\colon I \to [r]$. To each half-space in the braid arrangement in $\mathbb{R}^I$, we assign a weight as follows:

$$w^A_f(x_j \le x_i) := a_{f(i)f(j)}. \tag{10.68}$$

Given chambers $C$ and $D$, define the *weighted additive distance* from $C$ to $D$ by

$$\operatorname{dist}^A_f(C,D) := \sum w^A_f(H), \tag{10.69}$$

where the sum is over all half-spaces $H$ which contain $C$ but do not contain $D$.

Explicitly, if $C = C^1|\cdots|C^n$, with $n = |I|$, then (10.26) generalizes as follows.

$$\operatorname{dist}^A_f(C,D) = \sum_{(i,j)\in \operatorname{Inv}(C,D)} a_{f(C^j)f(C^i)}. \tag{10.70}$$

Note that if all entries of $A$ are 1, then $\operatorname{dist}^A_f(C,D)$ is simply the gallery distance between $C$ and $D$.

Some basic properties of the weighted distance function are as follows. Let $C$, $D$ and $E$ be chambers in $\Sigma[I]$. Then

$$\operatorname{dist}^A_f(C,D) = \operatorname{dist}^{A^t}_f(D,C), \tag{10.71}$$

for any bijection $\sigma\colon I \to J$,

$$\operatorname{dist}^A_f(C,D) = \operatorname{dist}^A_{f\sigma^{-1}}(\sigma C,\sigma D), \tag{10.72}$$

if $C - D - E$ is a minimum gallery, then

$$\operatorname{dist}^A_f(C,E) = \operatorname{dist}^A_f(C,D) + \operatorname{dist}^A_f(D,E). \tag{10.73}$$

Further,

$$\operatorname{dist}^A_f(C,D) = \operatorname{dist}^A_f(\overline{D},\overline{C}). \tag{10.74}$$

Interesting special cases are when $A$ is symmetric or antisymmetric. We explain them briefly.

**Proposition 10.20.** *Let $A$ be a symmetric matrix with positive real entries. Then*

$$\begin{aligned}
\operatorname{dist}^A_f(C,C) &= 0,\\
\operatorname{dist}^A_f(C,D) &= \operatorname{dist}^A_f(D,C),\\
\operatorname{dist}^A_f(C,D) + \operatorname{dist}^A_f(D,E) &\ge \operatorname{dist}^A_f(C,E),
\end{aligned}$$

*with equality if $C - D - E$ is a minimum gallery.*

These are the familiar properties of "distance".

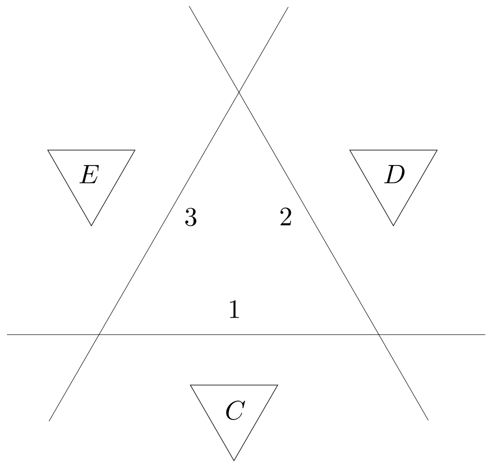



FIGURE 10.12. Additivity of antisymmetrically weighted distance.

**Proposition 10.21.** *Let $A$ be an antisymmetric matrix. Then*

$$\begin{aligned}\operatorname{dist}_f^A(C,C) &= 0,\\ \operatorname{dist}_f^A(C,D) + \operatorname{dist}_f^A(D,C) &= 0,\\ \operatorname{dist}_f^A(C,D) + \operatorname{dist}_f^A(D,E) &= \operatorname{dist}_f^A(C,E).\end{aligned}$$

These are the familiar properties of "displacement".

PROOF. We prove the last equality. For that, refer to Figure 10.12. We may assume that $C$, $D$ and $E$ are all distinct (the remaining cases are straightforward). Then there are three kinds of hyperplanes as shown in Figure 10.12 whose associated half-spaces may contribute to the weighted distances. The hyperplanes labeled 1 and 3 contribute once to both the left- and right-hand side via the half-space which contains $C$. The hyperplane labeled 2 does not contribute to the right-hand side and contributes twice to the left-hand side via the two half-spaces it supports. Since $A$ is antisymmetric, these contributions cancel. □

**10.12.2. The multiplicative case.** Let $Q$ be a matrix of size $r$ and $f\colon I \to [r]$ a function. Given chambers $C$ and $D$, define the *weighted multiplicative distance* from $C$ to $D$ by

$$\operatorname{dist}_f^Q(C,D) := \prod w_f^Q(H), \tag{10.75}$$

where $w_f^Q(H)$ is as in (10.68) and the product is over all half-spaces $H$ which contain $C$ but do not contain $D$. Clearly, if the matrices $Q$ and $A$ are related by (2.33), then

$$\operatorname{dist}_f^Q(C,D) = q^{\operatorname{dist}_f^A(C,D)}.$$

The multiplicative analogue of (10.70) is the following.

$$\operatorname{dist}_f^Q(C,D) = \prod_{(i,j)\in \operatorname{Inv}(C,D)} q_{f(C^j)f(C^i)}. \tag{10.76}$$

The multiplicative analogues of (10.71), (10.72) and (10.73) are as follows.

$$\operatorname{dist}^Q_f(C,D) = \operatorname{dist}^{Q^t}_f(D,C), \tag{10.77}$$

for any bijection $\sigma\colon I \to J$,

$$\operatorname{dist}^Q_f(C,D) = \operatorname{dist}^Q_{f\sigma^{-1}}(\sigma C, \sigma D), \tag{10.78}$$

if $C - D - E$ is a minimum gallery, then

$$\operatorname{dist}^Q_f(C,E) = \operatorname{dist}^Q_f(C,D)\operatorname{dist}^Q_f(D,E). \tag{10.79}$$

Further,

$$\operatorname{dist}^Q_f(C,D) = \operatorname{dist}^Q_f(\overline{D},\overline{C}). \tag{10.80}$$

The following is a multiplicative analogue of Proposition 10.21.

**Proposition 10.22.** *Let $Q$ be a log-antisymmetric matrix. Then*

$$\begin{aligned} \operatorname{dist}^Q_f(C,C) &= 1,\\ \operatorname{dist}^Q_f(C,D)\operatorname{dist}^Q_f(D,C) &= 1,\\ \operatorname{dist}^Q_f(C,D)\operatorname{dist}^Q_f(D,E) &= \operatorname{dist}^Q_f(C,E). \end{aligned}$$

**10.12.3. Compatibility with breaks, joins and projections.** We now discuss compatibilities of the weighted distance function with the break, join and projection maps. To start with, Proposition 10.4 in conjunction with (10.73) in the additive case and (10.79) in the multiplicative case yields:

**Proposition 10.23.** *Let $C$ and $D$ be chambers and $F$ be a face of $C$. Then there exists a minimum gallery $C - FD - D$. In particular,*

$$\begin{aligned} \operatorname{dist}^A_f(C,D) &= \operatorname{dist}^A_f(C,FD) + \operatorname{dist}^A_f(FD,D),\\ \operatorname{dist}^Q_f(C,D) &= \operatorname{dist}^Q_f(C,FD)\operatorname{dist}^Q_f(FD,D). \end{aligned} \tag{10.81}$$

The following are weighted analogues of (10.59) and (10.60) for the case of chambers. The notations are the same as before, so we do not repeat them here.

$$\begin{aligned} \operatorname{dist}^A_f\big(j_K(C_1,\dots,C_j), j_K(D_1,\dots,D_j)\big) &= \sum_{i=1}^{j} \operatorname{dist}^A_{f_i}(C_i,D_i),\\ \operatorname{dist}^Q_f\big(j_K(C_1,\dots,C_j), j_K(D_1,\dots,D_j)\big) &= \prod_{i=1}^{j} \operatorname{dist}^Q_{f_i}(C_i,D_i), \end{aligned} \tag{10.82}$$

where the $f_i$'s are appropriate restrictions of $f$. Equivalently, for chambers $C$ and $D$ which contain $K$,

$$\operatorname{dist}^A_f(C,D) = \sum_{i=1}^{j} \operatorname{dist}^A_{f_i}(C_i,D_i), \qquad \operatorname{dist}^Q_f(C,D) = \prod_{i=1}^{j} \operatorname{dist}^Q_{f_i}(C_i,D_i), \tag{10.83}$$

where $b_K(C) = (C_1,\dots,C_j)$ and $b_K(D) = (D_1,\dots,D_j)$.

By combining the above compatibilities, one obtains the following weighted analogue of (10.61).

Let $I = S \sqcup T$ be a decomposition, and let $K = S|T$. Let $f\colon I \to [r]$ and let $g$ and $h$ be the restrictions of $f$ to $S$ and $T$. Further, let $C$ and $D$ be linear orders on $I$, $C_1$ and $D_1$ be linear orders on $S$, and $C_2$ and $D_2$ be linear orders on $T$, such that $b_K(KC) = (C_1, C_2)$ and $b_K(D) = (D_1, D_2)$. Then

$$\begin{aligned} \operatorname{dist}_f^A(C, D) &= \operatorname{dist}_g^A(C_1, D_1) + \operatorname{dist}_h^A(C_2, D_2) + \operatorname{dist}_f^A(C, KC), \\ \operatorname{dist}_f^Q(C, D) &= \operatorname{dist}_g^Q(C_1, D_1) \operatorname{dist}_h^Q(C_2, D_2) \operatorname{dist}_f^Q(C, KC). \end{aligned} \tag{10.84}$$

**10.12.4. Integration over galleries.** We now view the weighted distance function on chambers as an integral. Since integration is traditionally defined using summations we formulate the discusssion for the weighted additive distance function. The same discussion can be carried out for the multiplicative case by replacing sums by products.

Let $\mathcal{G}(C, D)$ stand for the following gallery starting at $C$ and ending at $D$:

$$C = C_1 \xrightarrow{H_1} C_2 \xrightarrow{H_2} \cdots \xrightarrow{H_{n-1}} C_n = D.$$

The chambers $C_i$ and $C_{i+1}$ are distinct and adjacent, that is, they share a codimension 1 face and $H_i$ is the half-space containing $C_i$ whose supporting hyperplane supports the common codimension 1 face of $C_i$ and $C_{i+1}$. We view $\mathcal{G}(C, D)$ as an oriented path in the complex. Now define

$$\int_{\mathcal{G}(C,D)} (A, f) := \sum_{k=1}^{n-1} w_f^A(H_k) \tag{10.85}$$

with $w_f^A(H_k)$ as in (10.68). In other words, to integrate over a gallery we add the weights of all the half-spaces relevant to that gallery (in the above sense).

We now show that the weighted distance can be interpreted as an integral.

**Proposition 10.24.** *We have*

$$\operatorname{dist}_f^A(C, D) = \int_{\mathcal{G}(C,D)} (A, f)$$

*where $\mathcal{G}(C, D)$ is any minimum gallery from $C$ to $D$.*

Proof. The proof follows from the following chain of equalities.

$$\operatorname{dist}_f^A(C, D) = \sum_{i=1}^{n-1} \operatorname{dist}_f^A(C_i, C_{i+1}) = \sum_{i=1}^{n-1} w_f^A(H_i) = \int_{\mathcal{G}(C,D)} (A, f) \tag{10.86}$$

The first equality follows from (10.73), while the remaining two follow from the definitions. □

This result along with (10.71) implies that

$$\int_{\mathcal{G}} (A, f) = \int_{-\mathcal{G}} (A^t, f),$$

where $-\mathcal{G}$ denotes the gallery from $D$ to $C$ which traverses the chambers in the order opposite to that of $\mathcal{G}$.

We elaborate on the cases when $A$ is symmetric and antisymmetric.

If $A$ is symmetric, then one can assign a weight to each hyperplane using $f$ by:

$$w_f^A(x_i = x_j) := a_{f(i)f(j)} = a_{f(j)f(i)}.$$

It is clear that in this case the integral only depends on the path joining $C$ and $D$ and not on the orientation. That is, it does not matter whether one goes from $C$

to $D$ or from $D$ to $C$. Hence in this case, one may view (10.85) as an integral over an unoriented domain.

If $A$ is antisymmetric, then one can assign a weight to each hyperplane using $f$ but only up to a sign. In other words, the weights of the two half-spaces supported by a hyperplane differ by a sign. Hence in this case, one may view $(A, f)$ as a "1-form" on the complex and (10.85) as an integral over an oriented domain. Further:

**Proposition 10.25.** *If $A$ is antisymmetric, then*

$$\operatorname{dist}_f^A(C, D) = \int_{\mathcal{G}(C,D)} (A, f)$$

*where $\mathcal{G}(C, D)$ is any gallery from $C$ to $D$.*

PROOF. This follows from (10.86); the first equality in that chain now holds due to Proposition 10.21. □

The above result says that the 1-form $(A, f)$ is in fact "exact". By arbitrarily choosing the potential at a chamber, $(A, f)$ determines the potential at all other chambers and $\operatorname{dist}_f^A(C, D)$ is then the potential difference between $C$ and $D$.

**10.12.5. A weighted distance function on faces.** So far in this section, we have been discussing the gallery metric on chambers. Now we turn our attention to the weighted versions of the distance function on faces. To avoid repetition, we will freely use the setup of Section 10.5.3.

The weighted additive distance between faces $F$ and $G$ is defined as follows.

$$\operatorname{dist}_f^A(F, G) := \operatorname{dist}_f^A\big(C, p_{GF}(C)\big) = \operatorname{dist}_f^A(p_{FG}(D), D), \tag{10.87}$$

where $C$ is any chamber containing $FG$, and $D$ is any chamber containing $GF$. It is necessary to show (and one can show) that the definition is independent of the particular choice of $C$ or $D$. In fact, from (10.68) and (10.69), one can see that $\operatorname{dist}_f^A(F, G)$ is the sum of the weights of the half-spaces $H$ which contain $FG$ but do not contain $GF$. It follows that

$$\operatorname{dist}_f^A(F, G) = \operatorname{dist}_f^A(FG, GF). \tag{10.88}$$

More explicitly, if $F$ and $G$ have the same support, with $F = F^1|\cdots|F^k$, then (10.37) generalizes as follows.

$$\operatorname{dist}_f^A(F, G) = \sum_{(i,j)\in \operatorname{Inv}(F,G)} \sum_{\substack{s\in F^i \\ t\in F^j}} a_{f(t)f(s)}. \tag{10.89}$$

In the general case, (10.38) generalizes as follows.

$$\operatorname{dist}_f^A(F, G) = \sum_{\substack{i<k \\ j>l}} \sum_{\substack{s\in F^i\cap G^j \\ t\in F^k\cap G^l}} a_{f(t)f(s)}, \tag{10.90}$$

where $i$ and $k$ index the blocks of $F$ while $j$ and $l$ index the blocks of $G$.

The weighted multiplicative distance between faces $F$ and $G$ is defined similarly as follows.

$$\operatorname{dist}_f^Q(F, G) := \operatorname{dist}_f^Q\big(C, p_{GF}(C)\big) = \operatorname{dist}_f^Q(p_{FG}(D), D), \tag{10.91}$$

where $C$ is any chamber containing $FG$, and $D$ is any chamber containing $GF$.

The remaining discussion works in an analogous manner: (10.88) holds, and (10.89) and (10.90) hold with sums replaced by products as follows. For equal supports,

$$\operatorname{dist}_f^Q(F, G) = \prod_{(i,j)\in \operatorname{Inv}(F,G)} \prod_{\substack{s\in F^i \\ t\in F^j}} q_{f(t)f(s)}, \tag{10.92}$$

and in the general case,

$$\operatorname{dist}_f^Q(F, G) = \prod_{\substack{i<k \\ j>l}} \prod_{\substack{s\in F^i\cap G^j \\ t\in F^k\cap G^l}} q_{f(t)f(s)}. \tag{10.93}$$

## 10.13. The Schubert cocycle and the gallery metric

The Schubert statistic was introduced in Section 2.2. An equivalent formulation in terms of the Schubert cocycle was given in Section 9.7. We begin this section by relating them to the gallery metric by using projection maps.

A weighted version of the Schubert statistic was also discussed in Section 2.2. In this section, we give an equivalent formulation in terms of the weighted Schubert cocycle. We then relate these to the weighted gallery metric.

We conclude by introducing the Schubert cocycle on faces, along with its weighted version, and relating it to the distance function on faces. This generalizes the previous discussion.

**10.13.1. The Schubert cocycle and the gallery metric.** Let $I = S \sqcup T$ and let $C$ be a linear order on $I$. Also let $K = S|T$. Then it follows from the definitions that

$$\operatorname{sch}_{S,T}(C) = \operatorname{dist}(C, KC), \tag{10.94}$$

where the left-hand side is the Schubert cocycle (9.12). In particular, applying (9.13),

$$\operatorname{sch}_n(S) = \operatorname{dist}(C_{(n)}, KC_{(n)}), \tag{10.95}$$

where the left-hand side is the Schubert statistic (2.13).

We illustrate how properties of the Schubert statistic or cocycle can be established using this geometric interpretation. For example, for (2.15) or (9.15), we note that

$$\operatorname{dist}(KC, C) + \operatorname{dist}(C, \overline{K}C) = \operatorname{dist}(KC, \overline{K}C)$$

and the right-hand side, by (10.26), is equal to $st$, where $s = |S|$ and $t = |T|$.

Recall that $\omega_n$ is the permutation which sends $i$ to $n+1-i$ for each $i$. The following sequence of equalities establishes (2.16). The proof of (9.16) is contained in this argument.

$$\begin{aligned}
\operatorname{sch}_n\big(\omega_n(S)\big) &= \operatorname{dist}(C_{(n)}, \omega_n(K)C_{(n)}) \\
&= \operatorname{dist}(\omega_n(\overline{C}_{(n)}), \omega_n\big(K\overline{C}_{(n)}\big)) \\
&= \operatorname{dist}(\overline{C}_{(n)}, K\overline{C}_{(n)}) \\
&= \operatorname{dist}(C_{(n)}, \overline{K}C_{(n)}) \\
&= \operatorname{sch}_n(T)
\end{aligned}$$

The first and last equalities follow from (10.95) (note that $\overline{K} = T|S$). The second equality uses the fact that the projection map commutes with the group action, and that $\omega_n$ switches $C_{(n)}$ and its opposite $\overline{C}_{(n)}$. The third equality follows from (10.24), and the fourth follows from (10.25).

The cocycle condition (9.17) boils down to the following. For any decomposition $I = R \sqcup S \sqcup T$, and for any linear order $l$ on $I$,

$$\begin{aligned} \text{(10.96)} \quad \operatorname{dist}(l, (R|S \sqcup T)l) &+ \operatorname{dist}\big((R|S \sqcup T)l, (R|S|T)l\big) \\ &= \operatorname{dist}(l, (R \sqcup S|T)l) + \operatorname{dist}\big((R \sqcup S|T)l, (R|S|T)l\big). \end{aligned}$$

This identity can be proved as follows. The gate property implies that

$$l - (R|S \sqcup T)l - (R|S|T)l \qquad \text{and} \qquad l - (R \sqcup S|T)l - (R|S|T)l$$

are minimum galleries and hence both sides of the above identity equal $\operatorname{dist}(l, (R|S|T)l)$.

The multiplicative property of the cocycle (9.18) boils down to the following. Consider a pair of decompositions $I = S \sqcup T = S' \sqcup T'$ and let $A$, $B$, $C$, and $D$ be the resulting intersections, as in Lemma 8.7. Then

(10.97)
$$\operatorname{dist}(l \cdot m, (S'|T')l \cdot m) = \operatorname{dist}(l, (A|B)l) + \operatorname{dist}(m, (C|D)m) + \operatorname{dist}(B|C, C|B),$$

for any linear order $l$ on $S$, and linear order $m$ on $T$. This identity follows from the following sequence of equalities.

$$\begin{aligned} &\operatorname{dist}(l \cdot m, (S'|T')l \cdot m) \\ &\qquad = \operatorname{dist}(l \cdot m, (A|C|B|D)l \cdot m) \\ &\qquad = \operatorname{dist}(l \cdot m, (A|B|C|D)l \cdot m) + \operatorname{dist}\big((A|B|C|D)l \cdot m, (A|C|B|D)l \cdot m\big) \\ &\qquad = \operatorname{dist}(l \cdot m, (A|B|C|D)l \cdot m) + \operatorname{dist}(B|C, C|B) \\ &\qquad = \operatorname{dist}(l, (A|B)l) + \operatorname{dist}(m, (C|D)m) + \operatorname{dist}(B|C, C|B). \end{aligned}$$

The first equality follows by noting that

$$(S'|T')l \cdot m = (A|C|B|D)l \cdot m.$$

The gate property applied to the star of the face $A|B \sqcup C|D$ yields a minimum gallery

$$l \cdot m - (A|B|C|D)l \cdot m - (A|C|B|D)l \cdot m$$

which implies the second equality. The third equality uses the fact that to compute the distance between a face and its opposite it does not matter which chamber is used to do the computation. The last equality follows from the compatibility of the distance function with the join map (10.59).

The connection of the Schubert statistic with inversions (2.26) can be established as follows.

$$\text{(10.98)} \qquad \operatorname{sch}_n(S) = \operatorname{dist}(C_{(n)}, KC_{(n)}) = \operatorname{dist}(C_{(n)}, \zeta C_{(n)}) = l(\zeta) = \operatorname{inv}(\zeta).$$

The first equality is (10.95). The $(s,t)$-shuffle permutation $\zeta$ is defined using (10.39) from which the second equality follows. Explicitly, $\zeta$ is the unique permutation which sends $[s]$ to $S$ and $[s+1, s+t]$ to $T$ in an order-preserving manner. The third and fourth equalities follow from (10.29).

**10.13.2. The weighted Schubert cocycle and the gallery metric.** Let $A$ be a square matrix of size $r$. Let $l$ be a linear order on a finite set $I$, $S$ a subset of $I$, and $f\colon I \to [r]$ a function. The *weighted additive Schubert cocycle* is defined to be

$$\operatorname{sch}^A_{S,T,f}(l) := \sum_{(i,j)\in \operatorname{Sch}_{S,T}(l)} a_{f(i)f(j)} \tag{10.99}$$

where $\operatorname{Sch}_{S,T}(l)$ is as in (9.11). This is a reformulation of the weighted additive Schubert statistic (2.13). If $I = [n]$ and $C_{(n)}$ is the canonical linear order on $[n]$, then

$$\operatorname{Sch}_{S,T}(C_{(n)}) = \operatorname{Sch}_n(S) \quad \text{and} \quad \operatorname{sch}^A_{S,T,f}(C_{(n)}) = \operatorname{sch}^A_n(S,f). \tag{10.100}$$

Letting all the entries of $A$ to be 1 recovers the Schubert cocycle.

We now relate the weighted additive Schubert cocycle to the weighted distance function (10.69). Let $I = S \sqcup T$ and let $C$ be a linear order on $I$. Also let $K = S|T$. Then

$$\operatorname{sch}^A_{S,T,f}(C) = \operatorname{dist}^A_f(C, KC). \tag{10.101}$$

This generalizes (10.94).

Proof. Let us look at the right-hand side. Note that the half-space $x_j \leq x_i$ contains $C$ precisely if $j <_C i$, that is, if $i$ is greater than $j$ with respect to the linear order $C$ on $I$. In addition, the half-space $x_j \leq x_i$ does not contain $KC$ precisely if $i \in S$ and $j \in T$. Thus, the set of half-spaces which are used to define $\operatorname{dist}^A_f(C, KC)$ is in correspondence with the set $\operatorname{Sch}_{S,T}(C)$ which is used to define $\operatorname{sch}^A_{S,T,f}(C)$. One then checks that the corresponding weights match and the result follows. $\square$

We now go to the multiplicative case. Let $Q$ be a square matrix of size $r$. Let $l$ be a linear order on a finite set $I$, $S$ a subset of $I$, and $f\colon I \to [r]$ a function. The *weighted multiplicative Schubert cocycle* is

$$\operatorname{sch}^Q_{S,T,f}(l) := \prod_{(i,j)\in \operatorname{Sch}_{S,T}(l)} q_{f(i)f(j)} \tag{10.102}$$

where $\operatorname{Sch}_{S,T}(l)$ is as in (9.11). In terms of the weighted multiplicative distance function, this can be written as

$$\operatorname{sch}^Q_{S,T,f}(C) = \operatorname{dist}^Q_f(C, KC). \tag{10.103}$$

**10.13.3. The braid coefficients.** We now introduce the braid coefficients, which are closely related to those introduced in Section 2.2.7. The motivation for the terminology will become clear in Chapter 14 where we will use these coefficients to construct braidings on colored species.

Fix a decomposition $I = S \sqcup T$. Let $f\colon I \to [r]$ and let $g$ and $h$ be the restrictions of $f$ to $S$ and $T$ respectively. Define

$$\operatorname{brd}^A_{S,T,f} := \sum_{\substack{s\in S\\ t\in T}} a_{h(t)g(s)} \qquad \text{and} \qquad \operatorname{brd}^Q_{S,T,f} := \prod_{\substack{s\in S\\ t\in T}} q_{h(t)g(s)}. \tag{10.104}$$

We refer to these as the additive and multiplicative braid coefficients respectively. If $Q$ and $A$ are related by (2.33), then

$$\operatorname{brd}^Q_{S,T,f} = q^{\operatorname{brd}^A_{S,T,f}}.$$

If further, $r = 1$, $A = [1]$ and $Q = [q]$, then

$$\mathrm{brd}^A_{S,T,f} = |S|\,|T| \qquad \text{and} \qquad \mathrm{brd}^Q_{S,T,f} = q^{|S|\,|T|}.$$

It also follows that

$$\mathrm{brd}^A_{S,T,f} = \mathrm{brd}^{A^t}_{T,S,f} \qquad \text{and} \qquad \mathrm{brd}^Q_{S,T,f} = \mathrm{brd}^{Q^t}_{T,S,f}\,. \tag{10.105}$$

It is convenient to view the braid coefficients as the result of a multistep process: a step consists of an interchange of an element of $S$ and an element of $T$. To such a step, we associate a weight depending on the colors of the elements involved and then look up the corresponding entry in the matrix $A$ or $Q$. To get the braid coefficient, we add or multiply the weights of all possible interchanges, as may be the case. It follows that

$$\mathrm{brd}^A_{S,T,f} = \mathrm{brd}^A_{\mathsf{d}(g),\mathsf{d}(h)} \qquad \text{and} \qquad \mathrm{brd}^Q_{S,T,f} = \mathrm{brd}^Q_{\mathsf{d}(g),\mathsf{d}(h)} \tag{10.106}$$

where the right-hand sides are the braid coefficients of (2.36) and $\mathsf{d}(g)$ and $\mathsf{d}(h)$ are the multidegrees of the fibers of $g$ and $h$, as defined in (2.38).

We now provide a geometric interpretation for the braid coefficients in terms of the weighted distance function. First recall that $S|T$ and $T|S$ are opposite vertices in this complex; let us call them $K$ and $\overline{K}$ for simplicity. Then

$$\mathrm{brd}^A_{S,T,f} = \mathrm{dist}^A_f(KC, \overline{K}C) \qquad \text{and} \qquad \mathrm{brd}^Q_{S,T,f} = \mathrm{dist}^Q_f(KC, \overline{K}C), \tag{10.107}$$

for any chamber $C$, or equivalently,

$$\mathrm{brd}^A_{S,T,f} = \mathrm{dist}^A_f(K, \overline{K}) \qquad \text{and} \qquad \mathrm{brd}^Q_{S,T,f} = \mathrm{dist}^Q_f(K, \overline{K}).$$

Proof. The essential observation is that the set $S \times T$ which is used to define the braid coefficients (10.104) is in correspondence with the set of half-spaces which contain the vertex $K$ but do not contain $\overline{K}$, where $K = S|T$ via

$$(s,t) \longleftrightarrow x_s \geq x_t.$$

The result then follows from the definitions. $\square$

The multistep process described for the braid coefficients is equivalent to a choice of a path from $K$ to $\overline{K}$. More precisely, it is a choice of a minimum gallery from $KC$ and $\overline{K}C$, where $C$ is any chamber. Further, the weight associated to each step may be viewed as a weight associated to the corresponding half-space in the gallery. Thus, the braid coefficient can be viewed as an integral over a minimum gallery, see Proposition 10.24.

**10.13.4. Properties of the weighted Schubert cocycle.** We now record the weighted analogues of (9.14)–(9.18). They are reformulations of the properties of the weighted Schubert statistic (2.39)–(2.43). We also give the corresponding identities in terms of the gallery metric.

$$\begin{aligned} \mathrm{sch}^A_{I,\emptyset,f}(l) &= \mathrm{sch}^A_{\emptyset,I,f}(l) = 0, \\ \mathrm{sch}^Q_{I,\emptyset,f}(l) &= \mathrm{sch}^Q_{\emptyset,I,f}(l) = 1. \end{aligned} \tag{10.108}$$

$$\begin{aligned} \mathrm{sch}^{A^t}_{S,T,f}(l) + \mathrm{sch}^A_{T,S,f}(l) &= \mathrm{brd}^A_{S,T,f}, \\ \mathrm{sch}^{Q^t}_{S,T,f}(l)\, \mathrm{sch}^Q_{T,S,f}(l) &= \mathrm{brd}^Q_{S,T,f}\,. \end{aligned} \tag{10.109}$$

We see here an instance of how the braid coefficients relate to the Schubert cocyle. We will see another instance a little below. In geometric terms, the identities boil down to

$$\begin{aligned}\mathrm{dist}^{A^t}_f(C,KC)+\mathrm{dist}^{A}_f(C,\overline{K}C)&=\mathrm{dist}^{A}_f(KC,\overline{K}C),\\ \mathrm{dist}^{Q^t}_f(C,KC)\,\mathrm{dist}^{Q}_f(C,\overline{K}C)&=\mathrm{dist}^{Q}_f(KC,\overline{K}C),\end{aligned}\tag{10.110}$$

where $K=S|T$. To prove either of these, apply (10.73) or (10.79) to the minimum gallery

$$KC - C - \overline{K}C$$

and then use (10.71) or (10.77).

$$\begin{aligned}\mathrm{sch}^{A}_{S,T,f}(l)&=\mathrm{sch}^{A^t}_{T,S,f}(\bar{l}),\\ \mathrm{sch}^{Q}_{S,T,f}(l)&=\mathrm{sch}^{Q^t}_{T,S,f}(\bar{l}),\end{aligned}\tag{10.111}$$

where $\bar{l}$ is the linear order opposite to $l$. This follows from either (10.74) or (10.80).

For any decomposition $I=R\sqcup S\sqcup T$, and for any linear order $l$ on $I$, and $f\colon I\to[r]$,

$$\begin{aligned}\mathrm{sch}^{A}_{R,S\sqcup T,f}(l)+\mathrm{sch}^{A}_{S,T,f|_{S\sqcup T}}(l|_{S\sqcup T})&=\mathrm{sch}^{A}_{R\sqcup S,T,f}(l)+\mathrm{sch}^{A}_{R,S,f|_{R\sqcup S}}(l|_{R\sqcup S}),\\ \mathrm{sch}^{Q}_{R,S\sqcup T,f}(l)\,\mathrm{sch}^{Q}_{S,T,f|_{S\sqcup T}}(l|_{S\sqcup T})&=\mathrm{sch}^{Q}_{R\sqcup S,T,f}(l)\,\mathrm{sch}^{Q}_{R,S,f|_{R\sqcup S}}(l|_{R\sqcup S}).\end{aligned}\tag{10.112}$$

This is the cocycle condition. In geometric terms, it boils down to the following.

$$\begin{aligned}&\mathrm{dist}^{A}_f(l,(R|S\sqcup T)l)+\mathrm{dist}^{A}_f\big((R|S\sqcup T)l,(R|S|T)l\big)\\ &\qquad=\mathrm{dist}^{A}_f(l,(R\sqcup S|T)l)+\mathrm{dist}^{A}_f\big((R\sqcup S|T)l,(R|S|T)l\big),\\ &\mathrm{dist}^{Q}_f(l,(R|S\sqcup T)l)\,\mathrm{dist}^{Q}_f\big((R|S\sqcup T)l,(R|S|T)l\big)\\ &\qquad=\mathrm{dist}^{Q}_f(l,(R\sqcup S|T)l)\,\mathrm{dist}^{Q}_f\big((R\sqcup S|T)l,(R|S|T)l\big).\end{aligned}\tag{10.113}$$

This can be proved the same way as (10.96).

Consider a pair of decompositions $I=S\sqcup T=S'\sqcup T'$ and let $A$, $B$, $C$, and $D$ be the resulting intersections, as in Lemma 8.7. Also, let $f\colon I\to[r]$. Then, for any linear order $l$ on $S$, and linear order $m$ on $T$,

$$\begin{aligned}\mathrm{sch}^{A}_{S',T',f}(l\cdot m)&=\mathrm{sch}^{A}_{A,B,f|_S}(l)+\mathrm{sch}^{A}_{C,D,f|_T}(m)+\mathrm{brd}^{A}_{B,C,f|_{B\sqcup C}},\\ \mathrm{sch}^{Q}_{S',T',f}(l\cdot m)&=\mathrm{sch}^{Q}_{A,B,f|_S}(l)\,\mathrm{sch}^{Q}_{C,D,f|_T}(m)\,\mathrm{brd}^{Q}_{B,C,f|_{B\sqcup C}}.\end{aligned}\tag{10.114}$$

This is the multiplicative property of the weighted Schubert cocycle. In geometric terms,

$$\begin{aligned}\mathrm{dist}^{A}_f(l\cdot m,(S'|T')l\cdot m)&=\mathrm{dist}^{A}_{f|_S}(l,(A|B)l)+\mathrm{dist}^{A}_{f|_T}(m,(C|D)m)\\ &\qquad+\mathrm{dist}^{Q}_{f|_{B\sqcup C}}(B|C,C|B),\\ \mathrm{dist}^{Q}_f(l\cdot m,(S'|T')l\cdot m)&=\mathrm{dist}^{Q}_{f|_S}(l,(A|B)l)\,\mathrm{dist}^{Q}_{f|_T}(m,(C|D)m)\\ &\qquad\mathrm{dist}^{Q}_{f|_{B\sqcup C}}(B|C,C|B).\end{aligned}\tag{10.115}$$

This can be proved the same way as (10.97).

**10.13.5. The weighted inversion statistic and the gallery metric.** The weighted inversion statistic (2.44) and (2.45) is related to the weighted distance by:

$$\begin{aligned} \operatorname{inv}^A_f(\sigma^{-1}) &= \operatorname{dist}^A_f(C_{(n)}, \sigma C_{(n)}), \\ \operatorname{inv}^Q_f(\sigma^{-1}) &= \operatorname{dist}^Q_f(C_{(n)}, \sigma C_{(n)}). \end{aligned} \tag{10.116}$$

The connection of the weighted Schubert statistic and the weighted inversion statistic given in (2.46) can be established along the lines of (10.98): Use (10.101), (10.103) and (10.116).

The relation between the weighted inversion statistic of a permutation and its inverse given in (2.47) can be derived as follows.

$$\begin{aligned} \operatorname{inv}^Q_f(\sigma^{-1}) &= \operatorname{dist}^Q_f(C_{(n)}, \sigma C_{(n)}) \\ &= \operatorname{dist}^Q_{f\sigma}(\sigma^{-1} C_{(n)}, C_{(n)}) \\ &= \operatorname{dist}^{Q^t}_{f\sigma}(C_{(n)}, \sigma^{-1} C_{(n)}) \\ &= \operatorname{inv}^{Q^t}_{f\sigma}(\sigma). \end{aligned}$$

The equalities follow from (10.77), (10.78) and (10.116).

Let us now establish (2.49). In view of (10.116), this identity is equivalent to

$$\begin{aligned} &\operatorname{dist}^Q_f(C_{(n)}, \rho C_{(n)}) \\ &\qquad = \operatorname{dist}^Q_f(C_{(n)}, \zeta C_{(n)}) \operatorname{dist}^Q_{\overline{g}}(C_{(s)}, \sigma C_{(s)}) \operatorname{dist}^Q_{\overline{h}}(C_{(t)}, \tau C_{(t)}), \end{aligned} \tag{10.117}$$

where $\overline{g}$ and $\overline{h}$ are as in (2.48). To prove this, we apply (10.84): Put $C = C_{(n)}$ and $D = \rho C_{(n)}$. Now note from Proposition 10.6 that $\zeta C_{(n)} = K C_{(n)}$, where $K = S|T$. Further, using the naturality of the weighted distance (10.78), it follows that

$$\operatorname{dist}^Q_g(C_1, D_1) = \operatorname{dist}^Q_{\overline{g}}(C_{(s)}, \sigma C_{(s)}) \quad \text{and} \quad \operatorname{dist}^Q_h(C_2, D_2) = \operatorname{dist}^Q_{\overline{h}}(C_{(t)}, \tau C_{(t)}).$$

The required identity now follows.

**10.13.6. The Schubert cocycle on faces.** Given a set composition $H \in \Sigma[I]$ and a decomposition $I = S \sqcup T$, let

$$\operatorname{Sch}_{S,T}(H) := \{(i,j) \in S \times T \mid i > j \text{ according to } H\}, \tag{10.118}$$

where $i > j$ according to $H$ means that $i$ appears in a strictly later block of $H$ than $j$. Let

$$\operatorname{sch}_{S,T}(H) := |\operatorname{Sch}_{S,T}(H)|. \tag{10.119}$$

For instance, if

$$H = sh|iv|a, \quad S = \{i, s, a\}, \quad T = \{v, h\},$$

then

$$\operatorname{Sch}_{S,T}(H) = \{(i,h), (a,h), (a,v)\} \quad \text{and} \quad \operatorname{sch}_{S,T}(H) = 3.$$

Alternatively,

$$\operatorname{sch}_{S,T}(H) = \sum_{1 \le i < j \le k} |H^i \cap T|\, |H^j \cap S|,$$

where $H = H^1 | \cdots | H^k$.

We view $\operatorname{sch}_{S,T}$ as an integer-valued function on $\Sigma[I]$ and refer to the family of maps $\operatorname{sch}_{S,T}$ as the *Schubert cocycle on faces*. Its restriction to $\mathrm{L}[I]$ is the usual

Schubert cocycle. The Schubert cocycle on faces can be interpreted using the distance function on faces:

$$\mathrm{sch}_{S,T}(H) = \mathrm{dist}(H, K),$$

where $K$ is the vertex $S|T$. This follows from (10.38).

It satisfies the following generalizations of (9.14)–(9.18).

$$\mathrm{sch}_{I,\emptyset}(H) = \mathrm{sch}_{\emptyset,I}(H) = 0. \tag{10.120}$$

$$\mathrm{sch}_{S,T}(H) + \mathrm{sch}_{T,S}(H) + \sum_{i=1}^{k} |H^i \cap S|\,|H^i \cap T| = |S|\,|T|, \tag{10.121}$$

where $H = H^1|\cdots|H^k$.

$$\mathrm{sch}_{S,T}(H) = \mathrm{sch}_{T,S}(\overline{H}), \tag{10.122}$$

where $\overline{H} = H^k|\cdots|H^1$ denotes the face opposite to $H$.

For any decomposition $I = R \sqcup S \sqcup T$, and for any composition $H$ of $I$,

$$\mathrm{sch}_{R,S\sqcup T}(H) + \mathrm{sch}_{S,T}(H|_{S\sqcup T}) = \mathrm{sch}_{R\sqcup S,T}(H) + \mathrm{sch}_{R,S}(H|_{R\sqcup S}). \tag{10.123}$$

This is the cocycle condition.

Consider a pair of decompositions $I = S \sqcup T = S' \sqcup T'$ and let $A$, $B$, $C$, and $D$ be the resulting intersections, as in Lemma 8.7. Then

$$\mathrm{sch}_{S',T'}(F \cdot G) = \mathrm{sch}_{A,B}(F) + \mathrm{sch}_{C,D}(G) + |B||C|, \tag{10.124}$$

for any composition $F$ of $S$, and composition $G$ of $T$. Here $F \cdot G$ stands for the concatenation of $F$ and $G$. This is the multiplicative property of the cocycle.

We now briefly consider the weighted versions of the Schubert cocycle on faces. The setup is as for chambers. The *weighted additive Schubert cocycle on faces* is defined to be

$$\mathrm{sch}^{A}_{S,T,f}(H) := \sum_{(i,j)\in \mathrm{Sch}_{S,T}(H)} a_{f(i)f(j)}. \tag{10.125}$$

The *weighted multiplicative Schubert cocycle on faces* is

$$\mathrm{sch}^{Q}_{S,T,f}(H) := \prod_{(i,j)\in \mathrm{Sch}_{S,T}(H)} q_{f(i)f(j)}. \tag{10.126}$$

Alternatively,

$$\mathrm{sch}^{A}_{S,T,f}(H) = \sum_{1\leq i<j\leq k} \sum_{\substack{t\in F^i\cap T\\ s\in F^j\cap S}} a_{f(s)f(t)},$$

and

$$\mathrm{sch}^{Q}_{S,T,f}(H) = \prod_{1\leq i<j\leq k} \prod_{\substack{t\in F^i\cap T\\ s\in F^j\cap S}} q_{f(s)f(t)},$$

where $H = H^1|\cdots|H^k$. It follows from (10.90) and (10.93) that the relation to the weighted distance function on faces is given by

$$\mathrm{sch}^{A}_{S,T,f}(H) = \mathrm{dist}^{A}_{f}(H,K) \qquad \text{and} \qquad \mathrm{sch}^{Q}_{S,T,f}(H) = \mathrm{dist}^{Q}_{f}(H,K).$$

One can also write down weighted analogues of (10.120)–(10.124). The cocycle condition, for example, takes the following form.

For any decomposition $I = R \sqcup S \sqcup T$, and for any composition $H$ of $I$, and $f\colon I \to [r]$,

$$\begin{aligned}
&\mathrm{sch}^{A}_{R,S\sqcup T,f}(H) + \mathrm{sch}^{A}_{S,T,f|_{S\sqcup T}}(H|_{S\sqcup T}) \\
&\qquad\qquad = \mathrm{sch}^{A}_{R\sqcup S,T,f}(H) + \mathrm{sch}^{A}_{R,S,f|_{R\sqcup S}}(H|_{R\sqcup S}), \\
&\mathrm{sch}^{Q}_{R,S\sqcup T,f}(H)\,\mathrm{sch}^{Q}_{S,T,f|_{S\sqcup T}}(H|_{S\sqcup T}) \\
&\qquad\qquad = \mathrm{sch}^{Q}_{R\sqcup S,T,f}(H)\,\mathrm{sch}^{Q}_{R,S,f|_{R\sqcup S}}(H|_{R\sqcup S}).
\end{aligned} \tag{10.127}$$

This property may also be formulated in terms of the distance function; the details are omitted.

## 10.14. A bilinear form on chambers. Varchenko's result

In this section, we review some work of Varchenko [367]. It involves the factorization of a bilinear form on the set of chambers of a hyperplane arrangement. Special cases of relevance to this monograph are also discussed. They pertain to the braid arrangement.

**10.14.1. Weights on hyperplanes.** Fix a central hyperplane arrangement, and let $\mathrm{L}$ be its set of chambers. The bilinear form is defined on the linearization $\Bbbk\mathrm{L}$, where $\Bbbk$ is assumed to have characteristic zero. It is as follows. Assign a weight to each hyperplane in the arrangement, and let $\mathrm{wtdist}(C, D)$ be the product of the weights of the hyperplanes which separate $C$ and $D$. Define a symmetric bilinear form on $\Bbbk\mathrm{L}$:

$$\langle C, D\rangle := \mathrm{wtdist}(C, D). \tag{10.128}$$

The determinant of this bilinear form [367, Theorem (1.1)] or [366, Theorem 2.6.2] is given by:

$$\prod_{X} (1 - a(X)^2)^{l(X)}, \tag{10.129}$$

where the product is over all proper flats $X$ in the arrangement, $a(X)$ is the product of the weights of all hyperplanes that contain $X$, and $l(X)$ denotes the multiplicity of $X$ defined as follows. Pick any hyperplane $\mathrm{H}$ which contains $X$. Then $l(X)$ is half the number of chambers $C$ which have the property that $X$ is the support of $C \cap \mathrm{H}$.

Varchenko's proof of the factorization (10.129) is geometric in nature and similar to the spirit of the present chapter. It makes crucial use of directed flats which he calls cones. (In our terminology, directed flats are examples of top-dimensional cones.) The use of minimum galleries and the gate property is also evident in his proof.

**10.14.2. Weights on half-spaces.** It is useful to work in a slightly more general setup, where instead of hyperplanes, one assigns weights to half-spaces. Details follow. Assign a weight to each half-space in the arrangement, and let $\mathrm{wtdist}(C, D)$ be the product of the weights of the half-spaces which contain $C$ but do not contain $D$. Define a bilinear form on $\Bbbk\mathrm{L}$:

$$\langle C, D\rangle := \mathrm{wtdist}(C, D). \tag{10.130}$$

It is not symmetric in general. In fact, it is symmetric precisely if for each hyperplane, the weights of the two half-spaces it supports are equal. The determinant of this more general bilinear form is given by:

$$\prod_X (1 - b(X))^{l(X)}, \tag{10.131}$$

where the product is over all proper flats $X$ in the arrangement, $b(X)$ is the product of the weights of all half-spaces whose supporting hyperplane contains $X$, and $l(X)$ is as in (10.129).

If for each hyperplane, the weights of the two half-spaces it supports are equal, then $b(X) = a(X)^2$ and (10.131) reduces to (10.129). The factorization (10.131) can be established by generalizing Varchenko's proof.

The following is an immediate consequence of (10.131). To keep the exposition self-contained, we give a direct proof following Varchenko.

**Lemma 10.26.** *If $b(X) \neq 1$ for any proper flat $X$ in the arrangement, then the bilinear form (10.130) on $\Bbbk\mathrm{L}$ is nondegenerate.*

PROOF. Let $\gamma : \Bbbk\mathrm{L} \to \Bbbk\mathrm{L}^*$ be the map induced by the bilinear form. Explicitly,

$$\gamma(C) = \sum_D \operatorname{wtdist}(C, D)\, D^*.$$

Assume that $b(X) \neq 1$ for any proper flat $X$ in the arrangement. We want to show that $\gamma$ is an isomorphism, or equivalently that it is surjective.

For any directed face $(F, C)$, define

$$m(F, C) = \sum_{D \subseteq \Psi(F,C)} \operatorname{wtdist}(C, D)\, D^*,$$

where $\Psi(F, C)$ is the cone associated to $(F, C)$ as in Proposition 10.14.

We claim that $m(F, C)$ belongs to the image of $\gamma$. The proof proceeds by backward induction on the dimension of $F$. Note that $m(C, C) = \gamma(C)$, so the claim holds if $F$ has full dimension.

Let $F$ be any face and let $C$ and $D$ be chambers opposite to each other in $\operatorname{Star}(F)$. Then

$$\sum_{G : F \leq G \leq C} (-1)^{\deg(G)}\, m(G, C) = (-1)^{\deg(C)} \operatorname{wtdist}(C, D)\, m(F, D).$$

This is a consequence of inclusion-exclusion. Rearranging the terms,

$$m(F, C) - (-1)^{\deg(C) - \deg(F)} \operatorname{wtdist}(C, D)\, m(F, D) = \sum_{G : F \leq G \leq C, G \neq F} (-1)^{\deg(G) - \deg(F) + 1}\, m(G, C).$$

The right-hand side belongs to the image of $\gamma$ by the induction hypothesis. Interchanging the roles of $C$ and $D$, we note that

$$m(F, D) - (-1)^{\deg(D) - \deg(F)} \operatorname{wtdist}(D, C)\, m(F, C)$$

also belongs to the image of $\gamma$.

Let $X$ denote the support of $F$. Since by assumption $b(X) \neq 1$, we have

$$1 - \operatorname{wtdist}(C, D) \operatorname{wtdist}(D, C) = 1 - b(X) \neq 0.$$

It follows that both $m(F, C)$ and $m(F, D)$ belong to the image of $\gamma$. The claim follows.

To finish the proof, we note that for any chamber $C$, we have $m(\emptyset, C) = C^*$. By our claim, this is in the image of $\gamma$. Hence, $\gamma$ is surjective as required. $\square$

Let $w(H)$ denote the weight of the half-space $H$. View it as a variable. Note that for any proper flat $X$, $b(X)$ is a square free monomial in these variables. This yields the following.

**Lemma 10.27.** *Let $w(H)$ denote the weight of the half-space $H$. If no square free monomial in the $w(H)$'s equals 1, then the bilinear form (10.130) on $\Bbbk\mathrm{L}$ is nondegenerate.*

This result will be required in this monograph for questions related both to Hopf monoids in species and to Fock functors.

**10.14.3. Specialization: Equal weights.** Let $q \in \Bbbk$ be any scalar. Define a symmetric bilinear form on the space $\Bbbk\mathrm{L}$ indexed by chambers by:

$$\langle C, D\rangle := q^{\operatorname{dist}(C,D)}. \tag{10.132}$$

This is a special case of (10.128) in which weights of all hyperplanes are equal to $q$.

Note that a monomial in the hyperplane weights equals 1 if and only if $q$ is a root of unity. Hence, applying Lemma 10.27, we obtain:

**Lemma 10.28.** *If $q$ is not a root of unity, then the bilinear form (10.132) on $\Bbbk\mathrm{L}$ is nondegenerate.*

A weaker result with a direct proof is given below.

**Lemma 10.29.** *If $q$ is not an algebraic integer, then the bilinear form (10.132) on $\Bbbk\mathrm{L}$ is nondegenerate.*

PROOF. The determinant of the bilinear form on $\Bbbk\mathrm{L}$ is a polynomial in $q$ over $\mathbb{Z}$. The degree of this polynomial is the product of the number of chambers and the number of hyperplanes in the arrangement. Exactly one term in the determinant expansion gives the leading term: for each $C$, take inner product with $\overline{C}$. It follows that the coefficient of the leading term is either 1 or $-1$. Since $q$ is not an algebraic integer, the value of the polynomial will not be zero. $\square$

**Example 10.30.** The bilinear form (10.132) for the braid arrangement in $\mathbb{R}^n$ has been studied in several papers, including [96, 108, 159, 268, 380]. Its determinant is explicitly given by

$$\prod_{i=2}^{n} \Big(1 - q^{i(i-1)}\Big)^{\binom{n}{i}(i-2)!(n-i+1)!}. \tag{10.133}$$

This formula was first proved by Zagier [380, Theorem 2]. Zagier's formula (10.133) holds in the polynomial ring $\mathbb{Z}[q]$, and hence over a field of any characteristic. It is a specialization of Varchenko's formula (10.129). Details regarding this are given by Hanlon and Stanley [159]; additional information can be found in Krattenthaler's surveys [208, Theorem 55] and [209, Section 5.7]. Varchenko's bilinear form is also studied by Denham and Hanlon [97, 98, 96].

Let us explicitly look at the case $n = 2$. Then

$$\mathrm{L}[2] = \{1|2,\ 2|1\},$$

and the matrix of the bilinear form (10.132) on $\Bbbk\mathrm{L}$ is given by

$$\begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}.$$

Its determinant is $1 - q^2$; so the roots are 1 and $-1$. This agrees with (10.133).

**10.14.4. Specialization: Power weights.** Consider the more general case when all weights are a positive integral power of $q$. Again, applying Lemma 10.27, we obtain:

**Lemma 10.31.** *Let $q$ be a scalar which is not a root of unity. If all the hyperplane weights are a positive integral power of $q$, then the bilinear form* (10.128) *on $\Bbbk\mathrm{L}$ is nondegenerate.*

**Example 10.32.** Let $X$ be a partition of $I$. Let $\Sigma[X]$ be the poset of faces associated to the braid arrangement in $\mathbb{R}^X$. Similarly, let $\mathrm{L}[X]$ be the set of chambers. Observe that $\Sigma[X]$ can be identified with those compositions of $I$ whose support is less than $X$, and $\mathrm{L}[X]$ with those compositions of $I$ whose support is exactly $X$.

Now consider the following bilinear form on $\mathbf{L}[X]$.

$$\langle F, G\rangle := q^{\operatorname{dist}(F,G)},$$

where $\operatorname{dist}(F, G)$ is as given in (10.37). Note that $\operatorname{dist}(F, G)$ is not the gallery metric in $\Sigma[X]$, rather it is the distance function between faces in $\Sigma[I]$. Nevertheless, it can be interpreted as a weighted distance function between chambers of $\Sigma[X]$ as follows.

Let $X = \{X^1, \dots, X^k\}$. Then hyperplanes in $\Sigma[X]$ are given by $X^i = X^j$ where $i$ and $j$ vary between 1 and $k$. To each such hyperplane, we associate the weight

$$q^{|X^i|\,|X^j|}.$$

Then observe that $q^{\operatorname{dist}(F,G)}$ is the product of the weights of the hyperplanes in $\Sigma[X]$ which separate $F$ and $G$.

It follows from Lemma 10.31 that, for $q$ not a root of unity, the bilinear form on $\mathbf{L}[X]$ is nondegenerate.

**10.14.5. Specialization: Matrix weights.** We work with the braid arrangement. Let $Q$ be a square matrix of size $r$, and let $f\colon I \to [r]$. Define a bilinear form on $\Bbbk\mathrm{L}$:

$$\langle C, D\rangle := \operatorname{dist}^Q_f(C, D), \tag{10.134}$$

where the right-hand side is the weighted multiplicative distance (10.75). This bilinear form can be interpreted as follows. To the half-space $x_j \le x_i$, assign the weight $q_{f(i)f(j)}$. Then $\langle C, D\rangle$ is the product of the weights of the half-spaces which contain $C$ but do not contain $D$. Thus, (10.134) is a special case of (10.130). If all entries of $Q$ are equal to say $q$, then we recover (10.132).

Applying Lemma 10.27, we obtain:

**Lemma 10.33.** *If no monomial in the $q_{ij}$'s equals* 1*, then the bilinear form* (10.134) *on $\Bbbk\mathrm{L}$ is nondegenerate.*

**Example 10.34.** We work with the setup of Example 10.32 and generalize it as follows. Let $Q$ be a square matrix of size $r$, and let $f\colon I \to [r]$. To the hyperplane $X^j \leq X^i$, we associate the weight

$$\prod_{\substack{a \in X^i \\ b \in X^j}} q_{f(a)f(b)},$$

where $q_{ij}$ denotes the $ij$-th entry of $Q$. Note that the weight is a monomial in the $q_{ij}$'s. Now consider the bilinear form on $\mathbf{L}[X]$:

$$\langle F, G \rangle := \prod_{\mathrm{H}} w(H)$$

where the product is over all half-spaces $H$ which contain $F$ but do not contain $G$ and $w(H)$ denotes the weight of $H$. If all entries of $Q$ are equal to (say) $q$, then we recover the bilinear form of Example 10.32.

It follows from Lemma 10.33 that, if no monomial in the $q_{ij}$'s equals one, then the bilinear form on $\mathbf{L}[X]$ is nondegenerate.

## 10.15. Bilinear forms on directed faces and faces

In this section, we define bilinear forms on directed faces and faces, and study their nondegeneracy. This complements the discussion in Section 10.14. We continue to work in the setting of central hyperplane arrangements (though we are mainly interested in the braid arrangement).

Fix a central hyperplane arrangement, and let $\Sigma$ be its set of faces, and $\overrightarrow{\Sigma}$ be its set of directed faces. The bilinear forms are defined on the linearizations $\Bbbk\overrightarrow{\Sigma}$ and $\Bbbk\Sigma$, where $\Bbbk$ is assumed to have characteristic zero. A key role is played by the gallery metric and Tits projection maps. These can be defined for a central hyperplane arrangement in the same manner as for the braid arrangement (Section 10.5).

**10.15.1. A bilinear form on directed faces.** Define a symmetric bilinear form on the space $\Bbbk\overrightarrow{\Sigma}$ indexed by directed faces:

$$\langle (F, C), (G, D) \rangle := \begin{cases} q^{\operatorname{dist}(C,D)} & \text{if } GC = D \text{ and } FD = C, \\ 0 & \text{otherwise.} \end{cases} \tag{10.135}$$

**Lemma 10.35.** *If $q$ is not an algebraic integer, then the bilinear form on $\Bbbk\overrightarrow{\Sigma}$ is nondegenerate.*

Proof. We follow the pattern of the proof given for Lemma 10.29; the argument however is much more delicate. Firstly, $\overline{F}C$ belongs to the set $\Psi(F, C)$ (as in Proposition 10.14) and secondly, for any chamber $D$ in this set, there is a minimum gallery $C - D - \overline{F}C$ by [12, Fact 5.2.1]. Hence

$$\operatorname{dist}(C, D) \leq \operatorname{dist}(C, \overline{F}C).$$

It follows that

$$\langle (F, C), (G, D) \rangle = q^{\operatorname{dist}(C, \overline{F}C)} \qquad \text{if } D = \overline{F}C,\ \overline{F} \leq G \leq D.$$

This is illustrated in Figure 10.13.

Further, for any other directed face $(G, D)$, the bilinear form evaluates either to 0 or a strictly smaller power of $q$.

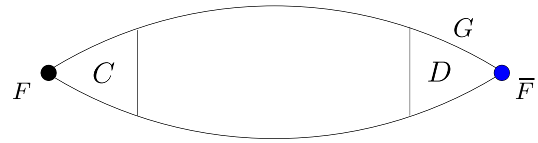



FIGURE 10.13. The bilinear form on directed faces.

By a backward induction on the dimension of $F$, it follows that the determinant of the bilinear form on $\Bbbk\overrightarrow{\Sigma}$ is a polynomial in $q$ of degree

$$\sum_{(F,C)\in\overrightarrow{\Sigma}} \operatorname{dist}(C, \overline{F}C)$$

and whose coefficient of the leading term is either 1 or $-1$. Exactly one term in the determinant expansion gives the leading term: for each $(F, C)$, take inner product with $(\overline{F}, \overline{F}C)$. Since $q$ is not an algebraic integer, the value of this polynomial will not be zero. □

**Example 10.36.** Let $\Sigma[2]$ be the Coxeter complex of the symmetric group $S_2$. Then

$$\overrightarrow{\Sigma}[2] = \{(12, 1|2),\ (1|2, 1|2),\ (12, 2|1),\ (2|1, 2|1)\}.$$

The matrix of the bilinear form on $\Bbbk\overrightarrow{\Sigma}[2]$, indexed in the above order, is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & q \\ 0 & 0 & 1 & 1 \\ 0 & q & 1 & 1 \end{pmatrix}.$$

Its determinant is $-q^2$. Observe directly that exactly one summand in the determinant expansion yields this term; the rest cancel out.

**10.15.2. A bilinear form on faces.** Define a symmetric bilinear form on the space $\Bbbk\Sigma$ indexed by faces as follows:

$$\langle F, G\rangle := \sum_{C:\, FG\leq C} q^{\operatorname{dist}(C, p_{GF}(C))}. \tag{10.136}$$

Recall that $p_{GF}(C)$ is the projection of the chamber $C$ on the face $GF$. It follows from (10.33) (also see Remark 10.5) that

$$\langle F, G\rangle = |\mathrm{L}_{FG}|\, q^{\operatorname{dist}(F,G)},$$

where $\mathrm{L}_{FG}$ is the set of chambers containing $FG$. In view of (10.7), we have

$$\langle F, G\rangle = (FG)!\, q^{\operatorname{dist}(F,G)}.$$

Note that this bilinear form is induced from the one on $\Bbbk\overrightarrow{\Sigma}$ by viewing $\Bbbk\Sigma$ as a subspace of $\Bbbk\overrightarrow{\Sigma}$ via the map $F \mapsto \sum(F, C)$, where the sum is over all chambers $C$ containing $F$. Explicitly,

$$\langle F, G\rangle = \sum_{C,D:\, F\leq C,\, G\leq D} \langle (F, C), (G, D)\rangle. \tag{10.137}$$

**Lemma 10.37.** *If $q$ is not an algebraic number, then the bilinear form on $\Bbbk\Sigma$ is nondegenerate.*

PROOF. The proof is similar to that of Lemma 10.35; we explain it briefly. For $F$ fixed and $G$ varying, the highest power of $q$ which appears in $\langle F, G\rangle$ is $\operatorname{dist}(F, \overline{F})$ and this happens precisely if $\overline{F} \leq G$. By a backward induction on the dimension of $F$, it then follows that the determinant of the bilinear form on $\Bbbk\Sigma$ is a polynomial in $q$ of degree

$$\sum_{G \in \Sigma} \operatorname{dist}(G, \overline{G}),$$

with integer coefficients, and whose leading coefficient is

$$\prod_{G} G!.$$

Exactly one term in the determinant expansion gives the leading term: for each $G$, take inner product with $\overline{G}$. Since $q$ does not satisfy any polynomial over $\mathbb{Q}$, the result follows. □

**Example 10.38.** Let $\Sigma[2]$ be the Coxeter complex of the symmetric group $\mathrm{S}_2$. Then

$$\Sigma[2] = \{12,\ 1|2,\ 2|1\}.$$

The matrix of the bilinear form on $\Bbbk\Sigma[2]$, indexed in the above order, is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & q \\ 1 & q & 1 \end{pmatrix}.$$

Its determinant is $-2q^2 + 2q$. Observe directly that exactly one summand in the determinant expansion contributes to the leading term $-2q^2$.

**Question 10.39.** For a central hyperplane arrangement, are there analogues of Varchenko's theorem for the bilinear forms on faces and directed faces? In other words, can one describe how the determinants of these bilinear forms factorize?

The roots of these determinants are important since for such values of $q$, one may then take the quotient by the radical of the form to construct new objects. The only case considered in the literature seems to be $q = 1$ and we discuss it briefly.

For $q = 1$, the radical of the bilinear form on $\Bbbk\Sigma$ is described in [12, Section 2.5.5]. The quotient by the radical is $\Bbbk\Pi$ where $\Pi$ is the lattice of flats. The situation appears to be much more complicated for $\Bbbk\overrightarrow{\Sigma}$. For $q = 1$, it is shown in the Coxeter case that the bilinear form on $\Bbbk\overrightarrow{\Sigma}$ is degenerate in general [12, Section 2.5.1]. In particular, for the braid arrangement, the form is degenerate for $n \geq 3$. However, a description of the radical is not known, even in the case of the braid arrangement.

**Remark 10.40.** The bilinear forms on faces and directed faces for $q = 1$ were defined in the wider context of left regular bands in [12, Section 2.5]. Left regular bands, however, may be too general for defining these bilinear forms for general $q$, since this requires the notion of a distance.

**10.15.3. A bilinear form on compositions and partitions.** Let $W$ be a Coxeter group and let $S$ be the set of generators in its standard presentation. For the symmetric group, $S$ can be taken to be the set of adjacent transpositions. More information about Coxeter presentations can be found in the references cited at the beginning of the chapter.

Let $\Sigma$ be the Coxeter complex associated to $W$ and $S$. The action of $W$ on $\Sigma$ is type-preserving and further, faces with the same type are in the same $W$-orbit. Since types of faces correspond to subsets of $S$, it follows that a basis for $(\Bbbk\Sigma)^W$, the space of $W$-invariants, is given by

$$\sigma_T := \sum_{F \colon \operatorname{type}(F)=T} F,$$

as $T$ ranges over all subsets of $S$. For the symmetric group, these are the elements considered in (10.47).

Since the bilinear form on faces (10.136) commutes with the $W$-action, one obtains a bilinear form on $(\Bbbk\Sigma)^W$. By Lemma 10.37, this form is nondegenerate if $q$ is not an algebraic integer. Explicitly, it is given by

$$\frac{1}{|W|}\langle \sigma_T, \sigma_U \rangle = \sum_{w \in W \colon \operatorname{Des}(w) \leq T,\, \operatorname{Des}(w^{-1}) \leq U} q^{l(w)},$$

where $\operatorname{Des}(w)$ stands for the descent set of $w$ (Section 10.7.1). The main steps in the calculation are indicated below.

$$\begin{aligned}
\langle \sigma_T, \sigma_U \rangle &= \sum_{\substack{F,G\colon \\ \operatorname{type}(F)=T,\, \operatorname{type}(G)=U}} \langle F, G \rangle \\
&= \sum_{\substack{(F,C),(G,D)\colon \\ \operatorname{type}(F)=T,\, \operatorname{type}(G)=U}} \langle (F,C), (G,D) \rangle \\
&= \sum_{\substack{(F,C),(G,D)\colon \\ \operatorname{type}(F)=T,\, \operatorname{type}(G)=U \\ FD=C,\, GC=D}} q^{\operatorname{dist}(C,D)} \\
&= \sum_{\substack{(C,D)\colon \\ \operatorname{Des}(C,D)\leq G,\, \operatorname{Des}(D.C)\leq F}} q^{\operatorname{dist}(C,D)} \\
&= |W| \sum_{\substack{w \in W\colon \\ \operatorname{Des}(w)\leq T,\, \operatorname{Des}(w^{-1})\leq U}} q^{l(w)}.
\end{aligned}$$

The first step is the definition, the second step uses (10.137), the third step uses definition (10.135), the fourth step uses (10.42) ($F$ and $G$ are dropped from the summation index since they are determined by $C$ and $D$), the last step replaces $(C, D)$ by $w = d(C, D)$ and uses the first diagram in (10.40) (which is valid for any finite Coxeter group).

We define the $H$ basis for the space of invariants by setting $\sigma_T = \sqrt{|W|}\, H_T$. This normalization implies that

$$\langle H_T, H_U \rangle = \sum_{\substack{w \in W\colon \\ \operatorname{Des}(w)\leq T,\, \operatorname{Des}(w^{-1})\leq U}} q^{l(w)}. \tag{10.138}$$

Now define the $K$ basis for the space of invariants by

$$H_T = \sum_{U \leq T} K_U.$$

A straightforward calculation shows that

$$\langle K_T, K_U \rangle = \sum_{\substack{w \in W:\\ \mathrm{Des}(w)=T,\, \mathrm{Des}(w^{-1})=U}} q^{l(w)}.$$

We note two special cases.

- For the Coxeter group of type $A$, the form (10.138) has been considered by Thibon and Ung [358, Formula (39)].
- The case of arbitrary finite Coxeter groups and $q = 1$ is discussed in detail in [12, Section 2.6]; some related references are [333, Theorem 3], [27], [139], [211, Corollary 3.11] and [70]. In this case, the bilinear form on $(\Bbbk\Sigma)^W$ is degenerate. The quotient by the radical is $(\Bbbk\Pi)^W$ where $\Pi$ is the lattice of flats.

Consider now the case of type $A$ and set $q = 1$. For each $n$, we obtain a bilinear form on the space spanned by all compositions of $n$, or equivalently, subsets of $[n-1]$ (10.1), given by

$$\langle K_T, K_U \rangle = |\{w \in \mathrm{S}_n : \mathrm{Des}(w) = T,\, \mathrm{Des}(w^{-1}) = U\}|. \tag{10.139}$$

This form is degenerate and the quotient by the radical is the space spanned by all partitions of $n$. The resulting nondegenerate form can be identified with the standard inner product of symmetric functions, which we recall has a basis indexed by partitions. Under this identification, the quotient map sends $H_T$ to $h_\lambda$, where $\lambda$ is the underlying partition of $T$ and $h$ denotes the basis of complete symmetric functions [252, Section I.2], [343, Section 7.5]. The *ribbon Schur functions* are the spanning set of the space of symmetric functions obtained as the image of the $K$ basis [359, Section 2.2]. The ribbon Schur functions are a special kind of skew Schur functions (for which the skew shape is a ribbon, also called rim-hook or border strip). Formula (10.139) recovers a result of Gessel [144, Theorem 5]; also see [343, Corollary 7.23.8] and [358, Formula (39)].

CHAPTER 11

# Universal Constructions of Hopf Monoids

Joyal described the free monoid and the free commutative monoid in $(\mathsf{Sp}, \cdot)$ on a species [181, Examples 42 and 43, Section 7]. In Section 11.2, we review the explicit construction of the free monoid and introduce the free Hopf monoid on a positive comonoid. The former is analogous to the tensor algebra of a vector space, the latter to the free Hopf algebra on a positively graded coalgebra; these constructions were reviewed in Sections 2.6.4 and 2.6.5. In Section 11.3 we present the corresponding constructions for free commutative monoids, analogous to the symmetric algebra of a vector space.

The dual results on cofree (and cofree cocommutative) comonoids are given in Sections 11.4 and 11.5. For finite-dimensional species, these results can be deduced from the preceding by duality, but we present direct proofs that do not require this assumption and allow us to be more concrete.

The notations for these universal objects for vector spaces and species are summarized in Table 11.1. In both cases, the universal objects are constructed as values of four functors: $\mathcal{T}$, $\mathcal{S}$, $\mathcal{T}^\vee$ and $\mathcal{S}^\vee$. For vector spaces, these yield the tensor, symmetric and shuffle algebras which were reviewed in Section 2.6.1. For species, the definitions of these functors involve the exponential species $\mathbf{E}$, the linear order species $\mathbf{L}$ and the substitution product $\circ$ on species. Note that $\mathcal{T}(\mathbf{q})$ and $\mathcal{T}^\vee(\mathbf{q})$ are identical as species; however they differ in the Hopf monoid structure. A similar remark applies to $\mathcal{S}(\mathbf{q})$ and $\mathcal{S}^\vee(\mathbf{q})$.

In Section 11.6, we relate the functors $\mathcal{T}$, $\mathcal{S}$, $\mathcal{T}^\vee$ and $\mathcal{S}^\vee$ via the norm and abelianization transformations. In Section 11.7, we consider deformations $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ which take values in the category of $q$-Hopf monoids. The parameter value $q = 1$ recovers $\mathcal{T}$ and $\mathcal{T}^\vee$. Similarly, we also consider signed analogues of $\mathcal{S}$ and $\mathcal{S}^\vee$. We denote them by $\Lambda$ and $\Lambda^\vee$. They take values in the category of $(-1)$-Hopf monoids. These objects are summarized in Table 11.2. The functors $\mathcal{T}_{-1}$, $\Lambda$, $\mathcal{T}_{-1}^\vee$ and $\Lambda^\vee$ relate via signed versions of the norm and abelianization transformations.

Table 11.1. Universal objects.

| Vector spaces | | Universal property | Species | |
|---|---|---|---|---|
| $\mathcal{T}(V)$ | tensor algebra | free | $\mathcal{T}(\mathbf{q}) = \mathbf{L} \circ \mathbf{q}$ | Section 11.2 |
| $\mathcal{T}^\vee(V)$ | shuffle algebra | cofree | $\mathcal{T}^\vee(\mathbf{q}) = \mathbf{L} \circ \mathbf{q}$ | Section 11.4 |
| $\mathcal{S}(V)$ | symmetric algebra | free commutative | $\mathcal{S}(\mathbf{q}) = \mathbf{E} \circ \mathbf{q}$ | Section 11.3 |
| $\mathcal{S}^\vee(V)$ | | cofree cocommutative | $\mathcal{S}^\vee(\mathbf{q}) = \mathbf{E} \circ \mathbf{q}$ | Section 11.5 |

TABLE 11.2. Universal objects. Deformed and signed versions.

| Vector spaces | | Species |
|---|---|---|
| $\mathcal{T}_q(V)$ | $q$-tensor algebra | $\mathcal{T}_q(\mathbf{q})$ |
| $\mathcal{T}_q^\vee(V)$ | $q$-shuffle algebra | $\mathcal{T}_q^\vee(\mathbf{q})$ |
| $\Lambda(V)$ | exterior algebra | $\Lambda(\mathbf{q})$ |
| $\Lambda^\vee(V)$ | | $\Lambda^\vee(\mathbf{q})$ |

In Section 11.8, we provide antipode formulas for $q$-Hopf monoids that arise as values of any of the above functors. In Section 11.9, we discuss related functors. These include the primitive element functor and the universal algebra functor.

The parameter value $q = 0$ is also of interest. This case is treated in Section 11.10. In particular, we provide a version for species of the rigidity result of Loday and Ronco given in Theorem 2.13.

## 11.1. The underlying species for the universal objects

Recall the exponential species $\mathbf{E}$ and the linear order species $\mathbf{L}$ of Example 8.3 and the substitution product $\circ$ on species (8.8). Let $\mathbf{q}$ be a positive species, that is, $\mathbf{q}[\emptyset] = 0$ (Section 8.9.2). As shown in Table 11.1, the underlying species for the universal objects are given by $\mathbf{L} \circ \mathbf{q}$ and $\mathbf{E} \circ \mathbf{q}$.

In this short section, we elaborate on these definitions and set up notations that will be used throughout the rest of this chapter.

**Notation 11.1.** Given a composition $F = F^1 | \cdots | F^k \vDash I$, we write

$$\mathbf{q}(F) := \mathbf{q}[F^1] \otimes \cdots \otimes \mathbf{q}[F^k].$$

Similarly, given a partition $X \vdash I$, we write

$$\mathbf{q}(X) := \bigotimes_{S \in X} \mathbf{q}[S].$$

These are the unbracketed and unordered tensor products of vector spaces, as in Example 1.30.

There are canonical identifications

$$\mathbf{q}(F) \otimes \mathbf{q}(G) \cong \mathbf{q}(F \cdot G), \tag{11.1}$$

$$\mathbf{q}(X) \otimes \mathbf{q}(Y) \cong \mathbf{q}(X \sqcup Y), \tag{11.2}$$

where $F \cdot G$ denotes concatenation of set compositions and $X \sqcup Y$ denotes disjoint union of set partitions (Section 10.1.6).

We claim that

$$(\mathbf{L} \circ \mathbf{q})[I] = \bigoplus_{F \vDash I} \mathbf{q}(F) = \bigoplus_{k \geq 0} \mathbf{q}^{\cdot k}[I], \tag{11.3}$$

$$(\mathbf{E} \circ \mathbf{q})[I] = \bigoplus_{X \vdash I} \mathbf{q}(X) = \bigoplus_{k \geq 0} (\mathbf{q}^{\cdot k}[I])_{\mathrm{S}_k} = \bigoplus_{k \geq 0} (\mathbf{q}^{\cdot k}[I])^{\mathrm{S}_k}. \tag{11.4}$$

We begin by explaining (11.3). According to (8.8), an $(\mathbf{L} \circ \mathbf{q})$-structure on $I$ consists of a partition $X$ of $I$ together with a linear order on $X$ and a $\mathbf{q}$-structure on each block of $X$. This is the same thing as a composition $F$ of $I$ together with a $\mathbf{q}$-structure on each block of $F$, which explains the first equality in (11.3). The second equality follows by repeated application of (8.6).

For (11.4), the first equality follows from (8.8), and the rest from Lemma B.18. We repeat the argument in the proof of that lemma in this special case. We have

$$\mathbf{q}^{\cdot k}[I] = \bigoplus_{F \vDash I} \mathbf{q}(F),$$

the sum being over all compositions $F$ of $I$ with $k$ blocks. The action of $\mathrm{S}_k$ on this set of compositions has no fixed points; hence, $\mathbf{q}^{\cdot k}[I]$ is free as $\mathrm{S}_k$-module. Therefore, invariants and coinvariants can be canonically identified (also see Lemma 2.20, (ii)). This proves the last equality. The second equality follows by noting that an orbit for the action of $\mathrm{S}_k$ on the above set of compositions can be identified with a partition $X$ of $I$.

## 11.2. The free monoid and the free Hopf monoid

In Section 2.6.5 we reviewed the construction of the free bialgebra on a coalgebra, including the case of noncounital and positively graded coalgebras. After reviewing Joyal's construction of the free monoid on a positive species, we describe in this section the free Hopf monoid on a positive comonoid.

### 11.2.1. The free monoid on a species.

**Definition 11.2.** Define a functor

$$\mathcal{T}\colon \mathsf{Sp}_+ \to \mathsf{Mon}(\mathsf{Sp}) \quad \text{by} \quad \mathcal{T}(\mathbf{q}) = \mathbf{L} \circ \mathbf{q}, \tag{11.5}$$

where $\mathbf{L}$ is the linear order species. To define the product on $\mathcal{T}(\mathbf{q})$ we make use of (11.3) and concatenation (11.1). Fix a decomposition $I = S \sqcup T$. The component $\mu_{S,T}$ of the product

$$\mathcal{T}(\mathbf{q})[S] \otimes \mathcal{T}(\mathbf{q})[T] \to \mathcal{T}(\mathbf{q})[I]$$

is the direct sum of all identity maps of the form

$$\mathbf{q}(F) \otimes \mathbf{q}(G) \xrightarrow{\mathrm{id}} \mathbf{q}(F \cdot G)$$

where $F \vDash S$ and $G \vDash T$. The unit is the identification $\Bbbk = \mathcal{T}(\mathbf{q})[\emptyset]$.

We call $\mathcal{T}$ the *free monoid functor*; the monoid $\mathcal{T}(\mathbf{q})$ is the free monoid on the species $\mathbf{q}$ in view of the result below. Let

$$(-)_+\colon \mathsf{Mon}(\mathsf{Sp}) \to \mathsf{Sp}_+$$

be the functor which sends a monoid $\mathbf{q}$ to the positive species $\mathbf{q}_+$ defined in (8.56).

**Theorem 11.3.** *The functor $\mathcal{T}$ is left adjoint to the $(-)_+$ functor. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Sp}_+}(\mathbf{q}, \mathbf{p}_+) \cong \mathrm{Hom}_{\mathsf{Mon}(\mathsf{Sp})}(\mathcal{T}(\mathbf{q}), \mathbf{p}) \tag{11.6}$$

*which are natural in $\mathbf{p}$ and $\mathbf{q}$.*

Proof. Note that (11.6) can be viewed as a composite of two adjunctions as below.

$$\begin{tikzcd} \mathsf{Sp}_+ \arrow[r, bend left, "inc"] & \mathsf{Sp} \arrow[l, bend left, "(-)_+"] \arrow[r, bend left, "\mathcal{T}"] & \mathsf{Mon}(\mathsf{Sp}). \arrow[l, bend left, "f\ell"] \end{tikzcd}$$

The first adjunction was discussed in (8.57) and the second adjunction follows by applying Proposition 6.69 to the category $(\mathsf{Sp}, \cdot)$. $\square$

The unit and counit of the adjunction $(\mathcal{T}, (-)_+)$ in (11.6) are maps

$$\eta(\mathbf{q})\colon \mathbf{q} \to \mathcal{T}(\mathbf{q})_+ \quad \text{and} \quad \xi(\mathbf{p})\colon \mathcal{T}(\mathbf{p}_+) \to \mathbf{p}, \tag{11.7}$$

which are morphisms of species and monoids respectively.

The map $\eta(\mathbf{q})$ is the natural inclusion of $\mathbf{q}$ in $\mathcal{T}(\mathbf{q})_+$, and the map $\xi(\mathbf{p})$ is the direct sum of the iterated products

$$(\mathbf{p}_+)^{\cdot n} \to \mathbf{p}$$

(for $n = 1$ this is the inclusion of $\mathbf{p}_+$ in $\mathbf{p}$ and for $n = 0$ this is the unit map of $\mathbf{p}$).

By general results on adjunctions [250, Theorem IV.1.2], the freeness of $\mathcal{T}(\mathbf{q})$ may be formulated in terms of a universal property.

**Theorem 11.4.** *Let $\mathbf{p}$ be a monoid, $\mathbf{q}$ a positive species, and $\zeta\colon \mathbf{q} \to \mathbf{p}_+$ a morphism of positive species. Then there exists a unique morphism of monoids $\hat{\zeta}\colon \mathcal{T}(\mathbf{q}) \to \mathbf{p}$ such that*

$$\begin{tikzcd} \mathcal{T}(\mathbf{q})_+ \arrow[rr, "\hat{\zeta}_+"] & & \mathbf{p}_+ \\ & \mathbf{q} \arrow[ul, "\eta(\mathbf{q})"] \arrow[ur, "\zeta"'] & \end{tikzcd} \tag{11.8}$$

*is a commutative diagram of positive species, where $\eta(\mathbf{q})$ is as in (11.7).*

It follows from the considerations preceding the proof of Theorem 11.3 that the morphism $\hat{\zeta}\colon \mathcal{T}(\mathbf{q}) \to \mathbf{p}$ has the following explicit form. For any composition $F = F^1|\cdots|F^k$ of a finite set $I$, let

$$\mu_{F^1,\ldots,F^k}\colon \mathbf{p}(F) = \mathbf{p}[F^1] \otimes \cdots \otimes \mathbf{p}[F^k] \to \mathbf{p}[I]$$

denote the corresponding component of the iterated product of $\mathbf{p}$. In addition, let $\zeta_{F^1,\ldots,F^k}\colon \mathbf{q}(F) \to \mathbf{p}(F)$ denote the map

$$\zeta_{F^1,\ldots,F^k} := \zeta_{F^1} \otimes \cdots \otimes \zeta_{F^k},$$

where $\zeta_I\colon \mathbf{p}[I] \to \mathbf{q}[I]$ denotes the $I$-component of $\zeta$. The $I$-component of the morphism $\hat{\zeta}$ is given by

$$\hat{\zeta}_I = \iota_\emptyset \quad \text{if } I = \emptyset, \qquad \hat{\zeta}_I = \sum_{F \vDash I} \mu_{F^1,\ldots,F^k} \zeta_{F^1,\ldots,F^k} \quad \text{otherwise,} \tag{11.9}$$

where $\iota_\emptyset$ is the nonzero component of the unit map of the monoid $\mathbf{p}$.

**Remark 11.5.** The adjunction in (11.6) can be viewed as a composite of adjunctions in another way as follows.

$$\begin{tikzcd} \mathsf{Sp}_+ \arrow[r, bend left, "\mathcal{T}"] & \mathsf{Mon}(\mathsf{Sp}^\circ) \arrow[l, bend left, "(-)_+"] \arrow[r, bend left, "inc"] & \mathsf{Mon}(\mathsf{Sp}) \arrow[l, bend left, "(-)^\circ"] \end{tikzcd}$$

The second adjunction was discussed in Section 8.9. The first one can be seen as a consequence of (11.6) and Proposition A.5 by noting: For a positive species $\mathbf{q}$, the species $\mathcal{T}(\mathbf{q})$ is connected, with $\mathcal{T}(\mathbf{q})[\emptyset]$ identified with $\Bbbk$ via the identity. Further, $\mathsf{Mon}(\mathsf{Sp}^\circ)$ is a full subcategory of $\mathsf{Mon}(\mathsf{Sp})$ as noted in Section 8.9.1.

**11.2.2. The free monoid as a Hopf monoid.** For a positive species $\mathbf{q}$, the monoid $\mathcal{T}(\mathbf{q})$ carries a canonical Hopf monoid structure. In order to define a coproduct on this species, fix a decomposition $I = S \sqcup T$. The component $\Delta_{S,T}$ of the coproduct

$$\mathcal{T}(\mathbf{q})[I] \to \mathcal{T}(\mathbf{q})[S] \otimes \mathcal{T}(\mathbf{q})[T]$$

is the direct sum of the following maps: for each composition $F \vDash I$ for which $S$ (and hence also $T$) is a union of blocks of $F$, there is one map

$$\mathbf{q}(F) \to \mathbf{q}(F|_S) \otimes \mathbf{q}(F|_T)$$

obtained by reordering the factors (with the restriction $F|_S$ as in Section 10.1.6). We refer to this map as *deshuffling*.

One verifies that, with this structure, $\mathcal{T}(\mathbf{q})$ is a cocommutative bimonoid. Since it is connected, it is also a Hopf monoid.

**11.2.3. A colax-colax adjunction.** Consider the adjunction

$$(11.10) \qquad \begin{tikzcd} \mathsf{Sp}_+ \arrow[r, bend left, "\mathcal{T}"] & \mathsf{Mon}(\mathsf{Sp}^\circ) \arrow[l, bend left, "(-)_+"] \end{tikzcd}$$

mentioned in Remark 11.5. Observe that both categories are symmetric monoidal categories, the latter with the Cauchy product and the former with the modified Cauchy product (8.55). It is then natural to wonder about the monoidal properties of this adjunction. In this regard, observe that (8.60) implies that the functor

$$(-)_+ \colon \mathsf{Mon}(\mathsf{Sp}^\circ, \cdot) \to (\mathsf{Sp}_+, \odot)$$

is braided strong. It is crucial here that we are working with monoids in connected species. This is why we prefer to work with (11.10) rather than with (11.6). Proposition 3.95 implies:

**Lemma 11.6.** *There is a unique braided colax structure on $\mathcal{T}$ such that the adjunction $(\mathcal{T}, (-)_+)$ in (11.10) is braided colax-colax.*

Using the descriptions of the unit and counit of the adjunction $(\mathcal{T}, (-)_+)$ given in (11.7), one can explicitly describe the colax structure on $\mathcal{T}$ as follows. Let maps

$$\psi_{\mathbf{p},\mathbf{q}} \colon \mathcal{T}(\mathbf{p} \odot \mathbf{q}) \to \mathcal{T}(\mathbf{p}) \cdot \mathcal{T}(\mathbf{q})$$

be as follows. Consider a summand of $\mathcal{T}(\mathbf{p} \odot \mathbf{q})$ as below.

$$(\mathbf{p} \odot \mathbf{q})^{\cdot k} = (\mathbf{p} \cdot \mathbf{q} + \mathbf{p} + \mathbf{q})^{\cdot k}$$

A factor in this $k$-fold product is a word in the letters $\mathbf{p}$ and $\mathbf{q}$ of length at most $2k$. We map from this word to a rearrangement of itself where the $\mathbf{p}$'s appear before the $\mathbf{q}$'s. The letters are rearranged by replacing occurrences of $\mathbf{q} \cdot \mathbf{p}$ for $\mathbf{p} \cdot \mathbf{q}$ using the braiding of $(\mathsf{Sp}, \cdot)$, that is, simply by reordering the tensor factors.

Further, define the map

$$\psi_0 \colon \mathcal{T}(\mathbf{0}) \to \mathbf{1}$$

to be the obvious isomorphism.

**11.2.4. A Hopf monoid starting from a positive comonoid.** Given a positive comonoid $\mathbf{q}$ (Definition 8.42), the monoid $\mathcal{T}(\mathbf{q})$ of Definition 11.2 carries a natural Hopf monoid structure as follows.

**Proposition 11.7.** *If $\mathbf{q}$ is a positive comonoid, then $\mathcal{T}(\mathbf{q})$ is a Hopf monoid.*

PROOF. By Lemma 11.6 the functor $(\mathcal{T}, \psi)$ is colax, so it preserves comonoids by Proposition 3.29. Hence if $\mathbf{q}$ is a comonoid in $(\mathsf{Sp}_+, \odot)$, that is, a positive comonoid, then $\mathcal{T}(\mathbf{q})$ is a comonoid in $\mathsf{Mon}(\mathsf{Sp}^{\circ})$, that is, a connected bimonoid. Since $\mathcal{T}(\mathbf{q})$ is connected, by Proposition 8.11 it follows that $\mathcal{T}(\mathbf{q})$ is a Hopf monoid. $\square$

The above construction defines a functor

$$\mathsf{Comon}(\mathsf{Sp}_+) \to \mathsf{Hopf}(\mathsf{Sp}), \tag{11.11}$$

which we again denote by $\mathcal{T}$.

The product on $\mathcal{T}(\mathbf{q})$ is given by concatenation, as described in Definition 11.2. It is easy to see that the coproduct admits the following description in terms of *dequasi-shuffles*. Fix a decomposition $I = S \sqcup T$. The component $\Delta_{S,T}$ of the coproduct

$$\mathcal{T}(\mathbf{q})[I] \to \mathcal{T}(\mathbf{q})[S] \otimes \mathcal{T}(\mathbf{q})[T]$$

is the direct sum of the following maps: for each composition $F = F^1|\cdots|F^k \vDash I$ there is a map

$$\mathbf{q}(F) \to \mathbf{q}(F|_S) \otimes \mathbf{q}(F|_T) \tag{11.12}$$

(with the restriction $F|_S$ as in Section 10.1.6). obtained by taking the tensor product of the maps below for $1 \leq i \leq k$ and then reordering the factors:

$$\mathbf{q}[F^i] \to \begin{cases} \mathbf{q}[F^i] & \text{if } F^i \cap S = \emptyset \text{ or } F^i \cap T = \emptyset, \\ \mathbf{q}[F^i \cap S] \otimes \mathbf{q}[F^i \cap T] & \text{otherwise.} \end{cases}$$

If $F^i \cap S = \emptyset$ or $F^i \cap T = \emptyset$, then we use the identity map; otherwise we use the appropriate component of the coproduct of $\mathbf{q}$.

We refer to the map (11.12) as *dequasi-shuffling*. Note that among all compositions $F \vDash I$, there are those for which $S$ and $T$ are unions of blocks of $F$. For such compositions we always have either $F^i \cap S = \emptyset$ or $F^i \cap T = \emptyset$, so in this case dequasi-shuffling simply involves reordering of the factors; it coincides with deshuffling.

**Remark 11.8.** The Hopf monoid described in Section 11.2.2 is a special case of the above construction as follows. Any positive species is a positive comonoid as follows: view it as a noncounital comonoid with the trivial coproduct (the zero map). We call it a *trivial positive comonoid*. In this case, the only dequasi-shuffles that contribute to the coproduct are the deshuffles. Thus, the above construction applied to a trivial positive comonoid yields the construction in Section 11.2.2.

**11.2.5. The free Hopf monoid on a positive comonoid.** We now show that the Hopf monoid $\mathcal{T}(\mathbf{q})$ is in fact the free Hopf monoid on the positive comonoid $\mathbf{q}$. This is essentially a consequence of the fact that a colax-colax adjunction induces an adjunction on the corresponding categories of comonoids. Details are as below.

There is a functor

$$(-)_+ \colon \mathsf{Hopf}(\mathsf{Sp}) \to \mathsf{Comon}(\mathsf{Sp}_+)$$

which sends a Hopf monoid $\mathbf{q}$ to the noncounital comonoid $\mathbf{q}_+$ (which is the same as a positive comonoid).

**Theorem 11.9.** *The functor*

$$\mathcal{T}\colon \mathsf{Comon}(\mathsf{Sp}_+) \to \mathsf{Hopf}(\mathsf{Sp})$$

*is left adjoint to the* $(-)_+$ *functor. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Comon}(\mathsf{Sp}_+)}(\mathbf{q}, \mathbf{p}_+) \cong \mathrm{Hom}_{\mathsf{Hopf}(\mathsf{Sp})}(\mathcal{T}(\mathbf{q}), \mathbf{p}),$$

*which are natural in* $\mathbf{p}$ *and* $\mathbf{q}$*.*

Proof. The above adjunction can be viewed as a composite of two adjunctions as below.

$$\begin{tikzcd} \mathsf{Comon}(\mathsf{Sp}_+) \arrow[r, bend left, "\mathcal{T}"] & \mathsf{Hopf}(\mathsf{Sp}^{\circ}) \arrow[l, bend left, "(-)_+"] \arrow[r, bend left, "inc"] & \mathsf{Hopf}(\mathsf{Sp}). \arrow[l, bend left, "(-)^{\circ}"] \end{tikzcd} \tag{11.13}$$

The second adjunction was discussed in Section 8.9. The first adjunction follows from Lemma 11.6 and Proposition 3.91. □

A restatement in terms of a universal property is given below.

**Theorem 11.10.** *Let* $\mathbf{p}$ *be a Hopf monoid,* $\mathbf{q}$ *a positive comonoid, and* $\zeta\colon \mathbf{q} \to \mathbf{p}_+$ *a morphism of positive comonoids. Then there exists a unique morphism of Hopf monoids* $\hat{\zeta}\colon \mathcal{T}(\mathbf{q}) \to \mathbf{p}$ *such that*

$$\begin{tikzcd} \mathcal{T}(\mathbf{q})_+ \arrow[rr, "\hat{\zeta}_+"] & & \mathbf{p}_+ \\ & \mathbf{q} \arrow[ul, "\eta(\mathbf{q})"] \arrow[ur, "\zeta"'] & \end{tikzcd} \tag{11.14}$$

*is a commutative diagram of positive comonoids, where* $\eta(\mathbf{q})$ *is as in* (11.7)*.*

Many familiar Hopf monoids and morphisms between Hopf monoids arise from the universal construction under discussion.

**Example 11.11.** View the positive species $\mathbf{X}$ in (8.3) as a trivial positive comonoid (as in Remark 11.8). Then

$$\mathcal{T}(\mathbf{X}) = \mathbf{L}$$

and one recovers the Hopf monoid structure on $\mathbf{L}$ defined in Example 8.16. In particular, $\mathbf{L}$ is the free monoid on one generator.

More generally, view the positive species $\mathbf{X}_V$ in (8.4) as a trivial positive comonoid. Then

$$\mathcal{T}(\mathbf{X}_V) = \mathbf{L} \times \mathbf{E}_V,$$

where $\mathbf{E}_V$ is the decorated exponential species of Example 8.18. If $V = \Bbbk$, then $\mathbf{E}_V$ is the exponential species which is the unit for the Hadamard product; thus one recovers the previous result in this case.

The Hopf monoids $\overrightarrow{\mathbf{\Sigma}} = \mathcal{T}(\mathbf{L}_+^*)$ and $\mathbf{\Sigma} = \mathcal{T}(\mathbf{E}_+^*)$ studied in Chapter 12 provide additional examples of free Hopf monoids; see Proposition 12.59.

## 11.3. The free commutative Hopf monoid

In this section, we outline a commutative version of the theory developed in Section 11.2. The role of the linear order species $\mathbf{L}$ is now played by the exponential species $\mathbf{E}$.

**11.3.1. The free commutative monoid.** Let $\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp})$ and $\mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp})$ be the categories of commutative monoids and commutative Hopf monoids in species.

**Definition 11.12.** Define a functor

$$\mathcal{S}\colon \mathsf{Sp}_+ \to \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}) \qquad \text{by} \qquad \mathcal{S}(\mathbf{q}) = \mathbf{E} \circ \mathbf{q}.$$

To define a product on $\mathcal{S}(\mathbf{q})$, we make use of (11.4) and union of partitions (11.2). The component $\mu_{S,T}$ of the product

$$\mathcal{S}(\mathbf{q})[S] \otimes \mathcal{S}(\mathbf{q})[T] \to \mathcal{S}(\mathbf{q})[I]$$

is the direct sum of the following maps

$$\mathbf{q}(X) \otimes \mathbf{q}(Y) \xrightarrow{\cong} \mathbf{q}(X \sqcup Y),$$

as $X$ and $Y$ run over all partitions of $S$ and $T$. The unit is the identification of $\Bbbk = \mathcal{S}(\mathbf{q})[\emptyset]$.

This turns $\mathcal{S}(\mathbf{q})$ into a commutative monoid in $(\mathsf{Sp}, \cdot)$. It is the free commutative monoid on the species $\mathbf{q}$ in the sense that

$$\begin{tikzcd} \mathsf{Sp}_+ \arrow[r, bend left, "\mathcal{S}"] & \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}) \arrow[l, bend left, "(-)_+"] \end{tikzcd}$$

is an adjunction, the functor $\mathcal{S}$ being the left adjoint to the functor $(-)_+$.

The restatement in terms of a universal property is given below.

**Theorem 11.13.** *Let $\mathbf{q}$ a positive species. Given a commutative monoid $\mathbf{p}$ and a morphism of positive species $\zeta\colon \mathbf{q} \to \mathbf{p}_+$, there exists a unique morphism of commutative monoids $\hat{\zeta}\colon \mathcal{S}(\mathbf{q}) \to \mathbf{p}$ such that the diagram*

$$\begin{tikzcd} \mathcal{S}(\mathbf{q})_+ \arrow[rr, "\hat{\zeta}_+"] & & \mathbf{p}_+ \\ & \mathbf{q} \arrow[ul, "\eta(\mathbf{q})"] \arrow[ur, "\zeta"'] & \end{tikzcd}$$

*commutes, where $\eta(\mathbf{q})$ is the natural inclusion of $\mathbf{q}$ in $\mathcal{S}(\mathbf{q})$.*

**11.3.2. The free commutative Hopf monoid on a positive comonoid.** Now suppose that $\mathbf{q}$ is a positive comonoid. A variant of the dequasi-shuffle coproduct of Section 11.2.5 then turns $\mathcal{S}(\mathbf{q})$ into a Hopf monoid. Fix a decomposition $I = S \sqcup T$. The component $\Delta_{S,T}$ of the coproduct

$$\mathcal{S}(\mathbf{q})[I] \to \mathcal{S}(\mathbf{q})[S] \otimes \mathcal{S}(\mathbf{q})[T]$$

is defined to be the direct sum of the following maps: for each partition $X \vdash I$ there is a map

$$\mathbf{q}(X) \to \mathbf{q}(X|_S) \otimes \mathbf{q}(X|_T) \tag{11.15}$$

(with the restriction $X|_S$ as in Section 10.1.6) obtained as the tensor product over the blocks $X^i$ of $X$ of the maps below:

$$\mathbf{q}[X^i] \to \begin{cases} \mathbf{q}[X^i] & \text{if } X^i \cap S = \emptyset \text{ or } X^i \cap T = \emptyset, \\ \mathbf{q}[X^i \cap S] \otimes \mathbf{q}[X^i \cap T] & \text{otherwise.} \end{cases}$$

If $X^i \cap S = \emptyset$ or $X^i \cap T = \emptyset$, then we use the identity map; otherwise we use the appropriate component of the coproduct of $\mathbf{q}$.

This turns $\mathcal{S}(\mathbf{q})$ into a connected commutative Hopf monoid. This leads to an adjunction

$$\begin{tikzcd} \mathsf{Comon}(\mathsf{Sp}_+) \arrow[r, bend left, "\mathcal{S}"] & \mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp}), \arrow[l, bend left, "(-)_+"] \end{tikzcd}$$

the functor $\mathcal{S}$ being the left adjoint to the functor $(-)_+$.

**Theorem 11.14.** *Let $\mathbf{q}$ be a positive comonoid. Given a commutative Hopf monoid $\mathbf{p}$ and a morphism of positive comonoids $\zeta\colon \mathbf{q} \to \mathbf{p}_+$, there exists a unique morphism of commutative Hopf monoids $\hat{\zeta}\colon \mathcal{S}(\mathbf{q}) \to \mathbf{p}$ such that the diagram*

$$\begin{tikzcd} \mathcal{S}(\mathbf{q})_+ \arrow[rr, "\hat{\zeta}_+"] & & \mathbf{p}_+ \\ & \mathbf{q} \arrow[ul, "\eta(\mathbf{q})"] \arrow[ur, "\zeta"'] & \end{tikzcd}$$

*commutes.*

**Example 11.15.** View the positive species $\mathbf{X}$ in (8.3) as a trivial positive comonoid (as in Remark 11.8). Then

$$\mathcal{S}(\mathbf{X}) = \mathbf{E}$$

and one recovers the Hopf monoid structure on $\mathbf{E}$ defined in Example 8.15. In particular, $\mathbf{E}$ is the free commutative monoid on one generator.

More generally, for the positive species $\mathbf{X}_V$ in (8.4),

$$\mathcal{S}(\mathbf{X}_V) = \mathbf{E}_V,$$

and one recovers the decorated exponential species of Example 8.18. In particular, $\mathbf{E}_V$ is the free commutative monoid on $\mathbf{X}_V$. This says that one should view this Hopf monoid as an analogue of the symmetric algebra of a vector space, rather than the tensor algebra.

The Hopf monoids $\overrightarrow{\mathbf{\Pi}} = \mathcal{S}(\mathbf{L}^*_+)$ and $\mathbf{\Pi} = \mathcal{S}(\mathbf{E}^*_+)$ discussed in Chapter 12 provide additional examples of free commutative monoids; see Proposition 12.59.

**Example 11.16.** Let $I$ be a set of cardinality $n$. The cyclic group of order $n$ acts on the set of linear orders on $I$. An orbit of this action is a *cycle* on $I$. Let $\mathbf{c}$ be the species of cycles and $\mathbf{b}$ the species of permutations. Thus, $\mathbf{c}[I]$ is the $\Bbbk$-span of the set of all cycles on $I$ and $\mathbf{b}[I]$ is the $\Bbbk$-span of the set of all bijections $I \to I$. By convention, $\mathbf{c}[\emptyset] = 0$. The species $\mathbf{b}$ should not be confused with the species $\mathbf{L}$ of linear orders; see [40, Section 1.2, p. 15]. Since a permutation is an unordered list of cycles, we have

$$\mathbf{b} = \mathbf{E} \circ \mathbf{c} = \mathcal{S}(\mathbf{c}).$$

Hence, if we view $\mathbf{c}$ as a trivial positive comonoid, we obtain a Hopf monoid structure on $\mathbf{b}$.

## 11.4. The cofree comonoid and the cofree Hopf monoid

In this section, we present results dual to those in Section 11.2. We choose to provide direct proofs instead of appealing to duality, which would require more restrictive hypotheses. We describe the cofree comonoid over a positive species and the cofree Hopf monoid over a positive monoid.

### 11.4.1. The cofree comonoid on a positive species.

**Definition 11.17.** Define a functor

$$\mathcal{T}^\vee\colon \mathsf{Sp}_+ \to \mathsf{Comon}(\mathsf{Sp}) \qquad \text{by} \qquad \mathcal{T}^\vee(\mathbf{q}) = \mathbf{L} \circ \mathbf{q}.$$

To define a coproduct on $\mathcal{T}^\vee(\mathbf{q})$, we make use of (11.3) and deconcatenation (11.1). Fix a decomposition $I = S \sqcup T$. The component $\Delta_{S,T}$ of the coproduct

$$\mathcal{T}^\vee(\mathbf{q})[I] \to \mathcal{T}^\vee(\mathbf{q})[S] \otimes \mathcal{T}^\vee(\mathbf{q})[T]$$

is the direct sum of the following maps: for each composition $F^1|\cdots|F^k \vDash I$ for which $S$ is the union of the first $i$ blocks (and hence $T$ is the union of the last $k-i$ blocks), take the identity map

$$\mathbf{q}(F^1|\cdots|F^k) \to \mathbf{q}(F^1|\cdots|F^i) \otimes \mathbf{q}(F^{i+1}|\cdots|F^k).$$

We note that if $I$ is empty, then $S$ and $T$ are necessarily empty and the component $\Delta_{\emptyset,\emptyset}$ is the canonical isomorphism $\Bbbk \to \Bbbk \otimes \Bbbk$.

The comonoid $\mathcal{T}^\vee(\mathbf{q})$ is the *cofree comonoid* on the species $\mathbf{q}$ in view of the result below. There is a functor

$$(-)_+\colon \mathsf{Comon}(\mathsf{Sp}) \to \mathsf{Sp}_+$$

which sends a comonoid $\mathbf{p}$ to the positive species $\mathbf{p}_+$.

**Theorem 11.18.** *The functor $\mathcal{T}^\vee$ is right adjoint to the $(-)_+$ functor. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Sp}_+}(\mathbf{p}_+, \mathbf{q}) \cong \mathrm{Hom}_{\mathsf{Comon}(\mathsf{Sp})}\big(\mathbf{p}, \mathcal{T}^\vee(\mathbf{q})\big),$$

*which are natural in $\mathbf{p}$ and $\mathbf{q}$.*

Proof. To prove the result, we will use the adjunction formulation given by (A.3). Accordingly, we first construct the counit and unit of the adjunction, namely,

$$\xi(\mathbf{q})\colon \mathcal{T}^\vee(\mathbf{q})_+ \to \mathbf{q} \qquad \text{and} \qquad \eta(\mathbf{p})\colon \mathbf{p} \to \mathcal{T}^\vee(\mathbf{p}_+). \tag{11.16}$$

Using (11.3), we may write

$$\xi(\mathbf{q})\colon \bigoplus_{n\geq 1} \mathbf{q}^{\cdot n} \to \mathbf{q} \qquad \text{and} \qquad \eta(\mathbf{p})\colon \mathbf{p} \to \bigoplus_{n \geq 0} \mathbf{p}_+^{\cdot n}.$$

We define $\xi(\mathbf{q})$ to be the projection that is the identity on $\mathbf{q}$ and annihilates $\mathbf{q}^{\cdot n}$ for $n \neq 1$. This is clearly a morphism of species and a natural transformation. In order to define $\eta(\mathbf{p})$, let $\Delta\colon \mathbf{p} \to \mathbf{p} \cdot \mathbf{p}$ be the coproduct of $\mathbf{p}$ and

$$\Delta^{(n)}\colon \mathbf{p} \to \mathbf{p}^{\cdot(n+1)} \quad \text{for } n \geq -1$$

be the iterated coproducts, where $\Delta^{(-1)}$, by usual convention, is the counit map $\mathbf{p} \to \mathbf{1}$, and $\Delta^{(0)}$ is the identity map. Similarly, let $\Delta_+\colon \mathbf{p}_+ \to \mathbf{p}_+ \cdot \mathbf{p}_+$ be the positive part of the coproduct (8.58) and

$$\Delta_+^{(n)}\colon \mathbf{p}_+ \to \mathbf{p}_+^{\cdot(n+1)} \quad \text{for } n \geq 0$$

be its iterates. Define

$$\begin{aligned}\eta(\mathbf{p}) &= \Delta^{(-1)} \oplus \bigoplus_{n\geq 0} (-)_+^{\cdot(n+1)} \circ \Delta^{(n)} \\ &= \Delta^{(-1)} \oplus \bigoplus_{n\geq 0} \Delta_+^{(n)} \circ (-)_+,\end{aligned}$$

where $(-)_+ \colon \mathbf{p} \to \mathbf{p}_+$ is the canonical projection. Since $(-)_+$ is a morphism of noncounital comonoids the second equality above is justified. In addition, note that each component of $\eta(\mathbf{p})$ involves a finite sum only. Indeed, in terms of (11.3), these are the maps

$$\eta(\mathbf{p})_I \colon \mathbf{p}[I] \to \bigoplus_{F\vDash I} \mathbf{p}(F)$$

obtained by adding the components (with nonempty parts) of the iterated coproduct of $\mathbf{p}$. To show that $\eta(\mathbf{p})$ is a morphism of comonoids, one needs to check that the following diagrams commute.

$$\begin{tikzcd} \mathbf{p}[I] \arrow[r, "\eta(\mathbf{p})_I"] \arrow[d, "\Delta_{S,T}"'] & \bigoplus_{F\vDash I} \mathbf{p}(F) \arrow[d, "\Delta_{S,T}"] \\ \mathbf{p}[S]\otimes\mathbf{p}[T] \arrow[r, "\eta(\mathbf{p})_S\otimes\eta(\mathbf{p})_T"'] & \bigoplus_{\substack{F_1\vDash S\\ F_2\vDash T}} \mathbf{p}(F_1)\otimes\mathbf{p}(F_2) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{p}[\emptyset] \arrow[rr, "\eta(\mathbf{p})_\emptyset"] \arrow[dr, "\epsilon_\emptyset"'] & & \Bbbk \arrow[dl, "\mathrm{id}"] \\ & \Bbbk & \end{tikzcd}$$

The first diagram commutes due to the coassociativity of the coproduct of $\mathbf{p}$, and the second diagram commutes by definition. Naturality in $\mathbf{p}$ of $\eta(\mathbf{p})$ holds because a morphism of comonoids commutes with all iterated coproducts.

We now check that the composites

$$\mathcal{T}^\vee(\mathbf{q}) \xrightarrow{\eta(\mathcal{T}^\vee(\mathbf{q}))} \mathcal{T}^\vee\big((\mathcal{T}^\vee(\mathbf{q})_+\big) \xrightarrow{\mathcal{T}^\vee(\xi(\mathbf{q}))} \mathcal{T}^\vee(\mathbf{q})$$
$$\mathbf{p}_+ \xrightarrow{(\eta(\mathbf{p}))_+} \mathcal{T}^\vee(\mathbf{p}_+)_+ \xrightarrow{\xi(\mathbf{p}_+)} \mathbf{p}_+$$

are the identity maps. The check for the bottom line is straightforward. The top line may be rewritten as below.

$$\bigoplus_{n\geq 0} \mathbf{q}^{\cdot n} \longrightarrow \bigoplus_{k\geq 0} \bigoplus_{n_1,\dots,n_k\geq 1} \mathbf{q}^{\cdot n_1}\cdot\mathbf{q}^{\cdot n_2}\cdot\dots\cdot\mathbf{q}^{\cdot n_k} \longrightarrow \bigoplus_{n\geq 0}\mathbf{q}^{\cdot n}$$

On the component with $n = 0$ and $k = 0$, the above reduces to $\mathbf{1} \cong \mathbf{1} \cong \mathbf{1}$. For a fixed $n \geq 1$, note that the first map takes $\mathbf{q}^{\cdot n}$ to the direct sum over all compositions $(n_1, \dots, n_k)$ of $n$. If we apply the second map to this sum, then, from the definition of $\xi$, only the summand corresponding to $n_1 = n_2 = \dots = n_k = 1$ survives in the image. So it maps onto $\mathbf{q}^{\cdot n}$ and the composite map is seen to be the identity. □

We formulate Theorem 11.18 in terms of a universal property.

**Theorem 11.19.** *Let* $\mathbf{p}$ *be a comonoid,* $\mathbf{q}$ *a positive species, and* $\zeta\colon \mathbf{p}_+ \to \mathbf{q}$ *a morphism of positive species. Then there exists a unique morphism of comonoids*

$\hat{\zeta}\colon \mathbf{p} \to \mathcal{T}^\vee(\mathbf{q})$ *such that*

$$\begin{tikzcd} \mathbf{p}_+ \arrow[rr, "\hat{\zeta}_+"] \arrow[dr, "\zeta"'] & & \mathcal{T}^\vee(\mathbf{q})_+ \arrow[dl, "\xi(\mathbf{q})"] \\ & \mathbf{q} & \end{tikzcd} \tag{11.17}$$

*is a commutative diagram of positive species with* $\xi(\mathbf{q})$ *as in* (11.16).

The morphism $\hat{\zeta}\colon \mathbf{p} \to \mathcal{T}^\vee(\mathbf{q})$ is given by the composite $\mathcal{T}^\vee(\zeta)\eta(\mathbf{p})$, with $\eta(\mathbf{p})$ as in (11.16). It has the following explicit form. For any composition $F = F^1|\cdots|F^k$ of a finite set $I$, let

$$\Delta_{F^1,\dots,F^k}\colon \mathbf{p}[I] \to \mathbf{p}(F)$$

denote the component of the iterated coproduct of $\mathbf{p}$ as in (8.26). In addition, let $\zeta_{F^1,\dots,F^k}\colon \mathbf{p}(F) \to \mathbf{q}(F)$ denote the map

$$\zeta_{F^1,\dots,F^k} := \zeta_{F^1} \otimes \cdots \otimes \zeta_{F^k},$$

where $\zeta_I\colon \mathbf{p}[I] \to \mathbf{q}[I]$ denotes the $I$-component of $\zeta$. The $I$-component of the morphism $\hat{\zeta}$ is given by

$$\hat{\zeta}_I = \epsilon_\emptyset \quad \text{if } I = \emptyset, \qquad \hat{\zeta}_I = \sum_{F \vDash I} \zeta_{F^1,\dots,F^k} \Delta_{F^1,\dots,F^k} \quad \text{otherwise,} \tag{11.18}$$

where $\epsilon_\emptyset$ is the nonzero component of the counit map of the comonoid $\mathbf{p}$.

**11.4.2. The cofree comonoid as a Hopf monoid.** Let $\mathbf{q}$ be a positive species and consider the comonoid $\mathcal{T}^\vee(\mathbf{q})$ of Definition 11.17. To define a product on $\mathcal{T}^\vee(\mathbf{q})$ we make use of (11.3) and shuffles (Section 10.1.6). For each pair of compositions $F \vDash S$, $G \vDash T$, and each shuffle $H$ of $F$ and $G$, there is a unique map

$$\mathbf{q}(F) \otimes \mathbf{q}(G) \to \mathbf{q}(H)$$

obtained by reordering the factors. We refer to this map as *shuffling*. The component $\mu_{S,T}$ of the product

$$\mathcal{T}^\vee(\mathbf{q})[S] \otimes \mathcal{T}^\vee(\mathbf{q})[T] \to \mathcal{T}^\vee(\mathbf{q})[I]$$

is the direct sum of the above maps over all such choices of $F$, $G$, and $H$.

One verifies that, with this structure, $\mathcal{T}^\vee(\mathbf{q})$ is a commutative bimonoid. Since it is connected, it is also a Hopf monoid.

**11.4.3. A Hopf monoid starting from a positive monoid.** We now generalize the above construction.

**Proposition 11.20.** *Let* $\mathbf{q}$ *be a positive monoid. There is a unique structure of a Hopf monoid on* $\mathcal{T}^\vee(\mathbf{q})$ *for which the map* $\xi(\mathbf{q})\colon \mathcal{T}^\vee(\mathbf{q})_+ \to \mathbf{q}$ *defined in* (11.16) *is a morphism of positive monoids.*

Proof. We apply the universal property of Theorem 11.19. Consider

$$\mathbf{p} = \mathcal{T}^\vee(\mathbf{q}) \cdot \mathcal{T}^\vee(\mathbf{q}).$$

Since $\mathcal{T}^\vee(\mathbf{q})$ is a comonoid, so is $\mathbf{p}$. According to (8.60),

$$\mathbf{p}_+ = \big(\mathcal{T}^\vee(\mathbf{q}) \cdot \mathcal{T}^\vee(\mathbf{q})\big)_+ = \mathcal{T}^\vee(\mathbf{q})_+ \odot \mathcal{T}^\vee(\mathbf{q})_+.$$

Let $\mu\colon \mathbf{q} \odot \mathbf{q} \to \mathbf{q}$ be the product of $\mathbf{q}$ viewed as a monoid in $(\mathsf{Sp}, \odot)$. Also, let $\zeta\colon \mathbf{p}_+ \to \mathbf{q}$ be the composite

$$\mathbf{p}_+ = \mathcal{T}^\vee(\mathbf{q})_+ \odot \mathcal{T}^\vee(\mathbf{q})_+ \xrightarrow{\xi(\mathbf{q})\odot\xi(\mathbf{q})} \mathbf{q} \odot \mathbf{q} \xrightarrow{\mu} \mathbf{q}.$$

Theorem 11.19 then yields a unique morphism of comonoids

$$\hat{\zeta}\colon \mathcal{T}^\vee(\mathbf{q}) \cdot \mathcal{T}^\vee(\mathbf{q}) \to \mathcal{T}^\vee(\mathbf{q})$$

such that the following diagram (an instance of (11.17)) commutes:

$$\begin{tikzcd}
\mathcal{T}^\vee(\mathbf{q})_+ \odot \mathcal{T}^\vee(\mathbf{q})_+ \arrow[r, "\hat{\zeta}_+"] \arrow[d, "\xi(\mathbf{q})\odot\xi(\mathbf{q})"'] & \mathcal{T}^\vee(\mathbf{q})_+ \arrow[d, "\xi(\mathbf{q})"] \\
\mathbf{q} \odot \mathbf{q} \arrow[r, "\mu"'] & \mathbf{q}.
\end{tikzcd} \tag{11.19}$$

From the associativity of $\mu$ it follows that both maps $\hat{\zeta}_+(\hat{\zeta}_+ \odot \mathrm{id})$ and $\hat{\zeta}_+(\mathrm{id} \odot \hat{\zeta}_+)$, which are morphisms of comonoids, fit in a commutative diagram of the form

$$\begin{tikzcd}
\mathcal{T}^\vee(\mathbf{q})_+ \odot \mathcal{T}^\vee(\mathbf{q})_+ \odot \mathcal{T}^\vee(\mathbf{q})_+ \arrow[r, dashed] \arrow[d, "\xi(\mathbf{q})\odot\xi(\mathbf{q})\odot\xi(\mathbf{q})"'] & \mathcal{T}^\vee(\mathbf{q})_+ \arrow[d, "\xi(\mathbf{q})"] \\
\mathbf{q} \odot \mathbf{q} \odot \mathbf{q} \arrow[r, "\mu^{(2)}"'] & \mathbf{q}.
\end{tikzcd}$$

By uniqueness, those two maps must agree, and therefore $\hat{\zeta}_+$ is an associative product on $\mathcal{T}^\vee(\mathbf{q})_+$. Similarly, from the unitality of $\mu$ one deduces the unitality of $\hat{\zeta}_+$. Thus, $\mathcal{T}^\vee(\mathbf{q})_+$ is a monoid in $(\mathsf{Sp}_+, \odot)$, that is, a positive monoid. Equivalently, according to Proposition 8.44, $\mathcal{T}^\vee(\mathbf{q})$ is a connected monoid. We claim that the product of this monoid is the map $\hat{\zeta}$: One only has to check that the $\emptyset$-component of $\hat{\zeta}$ is the identification $\Bbbk \otimes \Bbbk \to \Bbbk$. This is true since $\hat{\zeta}$ is a morphism of comonoids.

By construction $\hat{\zeta}$ is a morphism of comonoids, so $\mathcal{T}^\vee(\mathbf{q})$ is a connected bimonoid and hence a Hopf monoid. In addition, diagram (11.19) says that $\xi(\mathbf{q})$ is a morphism of positive monoids.

The uniqueness in Theorem 11.19 guarantees that such a Hopf monoid structure is unique. $\square$

The monoid structure of $\mathcal{T}^\vee(\mathbf{q})$ afforded by Proposition 11.20 can be made explicit. We make use of (11.3) and quasi-shuffles (Section 10.1.6). Fix a decomposition $I = S \sqcup T$, compositions $F \vDash S$, $G \vDash T$, and a quasi-shuffle $H$ of $F$ and $G$. Recall that $H$ shuffles the blocks of $F$ and $G$, and then merges a certain number of pairs of adjacent blocks, one block coming from $F$ and the other from $G$. Associated to this data there is a unique map

$$\mathbf{q}(F) \otimes \mathbf{q}(G) \to \mathbf{q}(H) \tag{11.20}$$

obtained by first reordering the factors according to the shuffle and then taking the tensor product of the maps below:

$$\mathbf{q}[H^k] \leftarrow \begin{cases} \mathbf{q}[H^k] & \text{if } H^k \text{ is a block of } F \text{ or of } G, \\ \mathbf{q}[F^i] \otimes \mathbf{q}[G^j] & \text{if } H^k = F^i \sqcup G^j. \end{cases}$$

If $H^k$ is a block of $F$ or of $G$ we use the identity map, and if $H^k = F^i \sqcup G^j$ we use the appropriate component of the product of $\mathbf{q}$.

We refer to the map $\mathbf{q}(F) \otimes \mathbf{q}(G) \to \mathbf{q}(H)$ as *quasi-shuffling*. The component $\mu_{S,T}$ of the product

$$\mathcal{T}^\vee(\mathbf{q})[S] \otimes \mathcal{T}^\vee(\mathbf{q})[T] \to \mathcal{T}^\vee(\mathbf{q})[I]$$

is the direct sum of quasi-shuffling maps over all such choices of $F$, $G$, and $H$.

Note that among all quasi-shuffles $H$ of $F$ and $G$ there are those that are shuffles, that is, that do not involve any merging of blocks. For such $H$, quasi-shuffling simply involves reordering of the factors, so it coincides with shuffling.

**Remark 11.21.** The Hopf monoid described in Section 11.4.2 is a special case of the above construction as follows. Any positive species is a positive monoid as follows: view it as a nonunital monoid with the trivial product (the zero map). We call it a *trivial positive monoid*. In this case, the only quasi-shuffles that contribute to the product are the shuffles. Thus the above construction applied to a trivial positive monoid yields the construction in Section 11.4.2.

**11.4.4. The cofree Hopf monoid on a positive monoid.** In Section 11.4.3, we constructed a Hopf monoid starting with a positive monoid. More formally, we have a functor $\mathcal{T}^\vee\colon \mathsf{Mon}(\mathsf{Sp}_+) \to \mathsf{Hopf}(\mathsf{Sp})$. There is also a functor which goes in the other direction which sends a Hopf monoid $\mathbf{p}$ to the nonunital monoid $\mathbf{p}_+$ (which is the same as a positive monoid). The Hopf monoid $\mathcal{T}^\vee(\mathbf{q})$ is the cofree Hopf monoid on a positive monoid $\mathbf{q}$ in view of the following result.

**Theorem 11.22.** *The functor*

$$\mathcal{T}^\vee\colon \mathsf{Mon}(\mathsf{Sp}_+) \to \mathsf{Hopf}(\mathsf{Sp})$$

*is right adjoint to the $(-)_+$ functor. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Mon}(\mathsf{Sp}_+)}(\mathbf{p}_+, \mathbf{q}) \cong \mathrm{Hom}_{\mathsf{Hopf}(\mathsf{Sp})}\big(\mathbf{p}, \mathcal{T}^\vee(\mathbf{q})\big),$$

*which are natural in $\mathbf{p}$ and $\mathbf{q}$.*

We will prove the following equivalent formulation in terms of a universal property.

**Theorem 11.23.** *Let $\mathbf{p}$ be a Hopf monoid, $\mathbf{q}$ a positive monoid, and $\zeta\colon \mathbf{p}_+ \to \mathbf{q}$ a morphism of positive monoids. Then there exists a unique morphism of Hopf monoids $\hat{\zeta}\colon \mathbf{p} \to \mathcal{T}^\vee(\mathbf{q})$ such that*

(11.21)
$$\begin{tikzcd} \mathbf{p}_+ \arrow[rr, "\hat{\zeta}_+"] \arrow[dr, "\zeta"'] & & \mathcal{T}^\vee(\mathbf{q})_+ \arrow[dl, "\xi(\mathbf{q})"] \\ & \mathbf{q} & \end{tikzcd}$$

*is a commutative diagram of positive monoids, with $\xi(\mathbf{q})$ as in (11.16).*

Proof. Note that $\mathsf{Hopf}(\mathsf{Sp}^\circ)$ is a full subcategory of $\mathsf{Hopf}(\mathsf{Sp})$, so in view of Proposition A.5, we may assume that $\mathbf{p}$ is connected. Theorem 11.19 yields a unique morphism of comonoids $\hat{\zeta}$ for which (11.21) commutes. Thus, it suffices to show that if $\zeta$ is a morphism of positive monoids, then $\hat{\zeta}$ is a morphism of monoids. In turn, in view of the equivalences (8.61), it is enough to show that $\hat{\zeta}_+$ is a morphism of positive monoids.

Consider the following diagram:

$$\begin{tikzcd}
\mathbf{p}_+ \odot \mathbf{p}_+ \arrow[r, "\hat{\zeta}_+ \odot \hat{\zeta}_+"] \arrow[d, "\mu"'] \arrow[dr, "\zeta\odot\zeta"'] & \mathcal{T}^\vee(\mathbf{q})_+ \odot \mathcal{T}^\vee(\mathbf{q})_+ \arrow[r, "\mu"] \arrow[d, "\xi(\mathbf{q})\odot\xi(\mathbf{q})"] & \mathcal{T}^\vee(\mathbf{q})_+ \arrow[dd, "\xi(\mathbf{q})"] \\
\mathbf{p}_+ \arrow[d, "\hat{\zeta}_+"'] \arrow[drr, "\zeta"'] & \mathbf{q} \odot \mathbf{q} \arrow[dr, "\mu"] & \\
\mathcal{T}^\vee(\mathbf{q})_+ \arrow[rr, "\xi(\mathbf{q})"'] & & \mathbf{q}
\end{tikzcd}$$

in which we have written $\mu$ for all the monoid structures. The squares commute because $\zeta$ and $\xi(\mathbf{q})$ are morphisms of positive monoids (the former by hypothesis, the latter by Proposition 11.20). The triangles commute because of diagram (11.21). Therefore the outer diagram commutes. Since the maps $\mu(\hat{\zeta}_+ \odot \hat{\zeta}_+)$ and $\hat{\zeta}_+\mu$ are morphisms of positive comonoids, they must be equal by the uniqueness in Theorem 11.19. This proves that $\hat{\zeta}_+$ is a morphism of monoids. $\square$

**Example 11.24.** View $\mathbf{X}$ as a trivial positive monoid (as in Remark 11.21). Then

$$\mathcal{T}^\vee(\mathbf{X}) = \mathbf{L}^*,$$

the dual of the Hopf monoid $\mathbf{L}$ discussed in Example 8.24. This can be derived directly, or from Example 11.11 by noting that $\mathcal{T}^\vee$ is the contragredient of $\mathcal{T}$, and $\mathbf{X}$ is self-dual. More generally,

$$\mathcal{T}^\vee(\mathbf{X}_V) = \mathcal{T}(\mathbf{X}_{V^*})^* = (\mathbf{L} \times \mathbf{E}_{V^*})^* = \mathbf{L}^* \times \mathbf{E}_V.$$

In Section 12.8, we give two examples of Hopf monoids of the form $\mathcal{T}^\vee(\mathbf{q})$ and we discuss instances of the above universal property for them; see Proposition 12.58 and Examples 12.61 and 12.62.

## 11.5. The cofree cocommutative Hopf monoid

We provide a brief outline of a cocommutative version of the theory developed in Section 11.4. The constructions are dual to those in Section 11.3.

**11.5.1. The cofree cocommutative comonoid.** Let ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{Sp})$ be the category of cocommutative comonoids in species and ${}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp})$ that of cocommutative Hopf monoids.

**Definition 11.25.** Define a functor

$$\mathcal{S}^\vee \colon \mathsf{Sp}_+ \to {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{Sp}) \qquad \text{by} \qquad \mathcal{S}^\vee(\mathbf{q}) = \mathbf{E} \circ \mathbf{q}.$$

To define a comonoid structure on $\mathcal{S}^\vee(\mathbf{q})$, we make use of (11.3) and (11.2) as follows. Fix a decomposition $I = S \sqcup T$. Given a partition $X \vdash I$, for which $S$ (and hence $T$) is a union of blocks of $X$, consider the identity map

$$\mathbf{q}(X) \to \mathbf{q}(X|_S) \otimes \mathbf{q}(X|_T).$$

The component $\Delta_{S,T}$ of the coproduct

$$\mathcal{S}^\vee(\mathbf{q})[I] \to \mathcal{S}^\vee(\mathbf{q})[S] \otimes \mathcal{S}^\vee(\mathbf{q})[T]$$

is the direct sum of these maps over all such partitions $X \vdash I$.

This turns $\mathcal{S}^\vee(\mathbf{q})$ into a connected cocommutative comonoid in $(\mathsf{Sp}, \cdot)$. It is the cofree cocommutative comonoid on the species $\mathbf{q}$ in the sense that

$$\begin{tikzcd} {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{Sp}) \arrow[r, bend left, "(-)_+"] & \mathsf{Sp}_+ \arrow[l, bend left, "\mathcal{S}^\vee"] \end{tikzcd}$$

is an adjunction, the functor $\mathcal{S}^\vee$ being the right adjoint to the functor $(-)_+$. The restatement in terms of a universal property is given below.

**Theorem 11.26.** *Let $\mathbf{p}$ be a cocommutative comonoid, $\mathbf{q}$ a positive species, and $\zeta\colon \mathbf{p}_+ \to \mathbf{q}$ a morphism of positive species. Then there exists a unique morphism of cocommutative comonoids $\hat{\zeta}\colon \mathbf{p} \to \mathcal{S}^\vee(\mathbf{q})$ such that*

$$\begin{tikzcd} \mathbf{p}_+ \arrow[rr, "\hat{\zeta}_+"] \arrow[dr, "\zeta"'] & & \mathcal{S}^\vee(\mathbf{q})_+ \arrow[dl, "\xi(\mathbf{q})"] \\ & \mathbf{q} & \end{tikzcd}$$

*is a commutative diagram of positive species, where $\xi(\mathbf{q})$ is the natural projection similar to (11.16).*

**11.5.2. The cofree cocommutative Hopf monoid on a positive monoid.** Now suppose that $\mathbf{q}$ is a positive monoid. A variant of the quasi-shuffle product of Section 11.4.3 turns $\mathcal{S}^\vee(\mathbf{q})$ into a connected cocommutative Hopf monoid. This leads to an adjunction

$$\begin{tikzcd} {}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}) \arrow[r, bend left, "(-)_+"] & \mathsf{Mon}(\mathsf{Sp}_+), \arrow[l, bend left, "\mathcal{S}^\vee"] \end{tikzcd}$$

the functor $\mathcal{S}^\vee$ being the right adjoint to the functor $(-)_+$. Thus, $\mathcal{S}^\vee(\mathbf{q})$ is the cofree cocommutative Hopf monoid on the positive monoid $\mathbf{q}$. In terms of a universal property, we have:

**Theorem 11.27.** *Let $\mathbf{p}$ be a cocommutative Hopf monoid, $\mathbf{q}$ a positive monoid, and $\zeta\colon \mathbf{p}_+ \to \mathbf{q}$ a morphism of positive monoids. Then there exists a unique morphism of cocommutative Hopf monoids $\hat{\zeta}\colon \mathbf{p} \to \mathcal{S}^\vee(\mathbf{q})$ such that*

$$\begin{tikzcd} \mathbf{p}_+ \arrow[rr, "\hat{\zeta}_+"] \arrow[dr, "\zeta"'] & & \mathcal{S}^\vee(\mathbf{q})_+ \arrow[dl, "\xi(\mathbf{q})"] \\ & \mathbf{q} & \end{tikzcd} \tag{11.22}$$

*is a commutative diagram of positive monoids, where $\xi(\mathbf{q})$ is the natural projection similar to (11.16).*

**Example 11.28.** View $\mathbf{X}$ as a trivial positive monoid. Then

$$\mathcal{S}^\vee(\mathbf{X}) = \mathbf{E}.$$

In particular, $\mathbf{E}$ is the cofree cocommutative comonoid on one generator. More generally,

$$\mathcal{S}^\vee(\mathbf{X}_V) = \mathbf{E}_V.$$

The Hopf monoids $\overrightarrow{\mathbf{\Pi}}^* = \mathcal{S}^\vee(\mathbf{L}_+)$ and $\mathbf{\Pi}^* = \mathcal{S}^\vee(\mathbf{E}_+)$ discussed in Chapter 12 provide additional examples; see Proposition 12.58.

## 11.6. The norm transformation and the abelianization

In the preceding sections, we discussed the functors $\mathcal{T}$, $\mathcal{T}^\vee$, $\mathcal{S}$ and $\mathcal{S}^\vee$ in detail. The goal of this section is to relate them. The discussion parallels the one for graded vector spaces in Section 2.6.1.

**11.6.1. The contragredient of the free monoid functor.** The duality functor $(-)^*$ is a contravariant bistrong monoidal functor on the category of finite-dimensional species (Definition 8.2). Therefore, it takes monoids to comonoids and viceversa, and Hopf monoids to Hopf monoids. For the present discussion, we restrict to finite-dimensional species.

Consider the following two compositions of functors

$$\mathsf{Sp}_+ \xrightarrow{\quad(-)^*\quad} \mathsf{Sp}_+ \xrightarrow{\quad\mathcal{T}\quad} \mathsf{Mon}(\mathsf{Sp}) \xrightarrow{(-)^*} \mathsf{Comon}(\mathsf{Sp})$$
$$\mathsf{Mon}(\mathsf{Sp}_+) \xrightarrow{(-)^*} \mathsf{Comon}(\mathsf{Sp}_+) \xrightarrow{\mathcal{T}} \mathsf{Hopf}(\mathsf{Sp}) \xrightarrow{(-)^*} \mathsf{Hopf}(\mathsf{Sp})$$

where the middle functors $\mathcal{T}$ are as in (11.5) and (11.11). This defines two functors, namely

$$\mathsf{Sp}_+ \to \mathsf{Comon}(\mathsf{Sp}) \qquad \text{and} \qquad \mathsf{Mon}(\mathsf{Sp}_+) \to \mathsf{Hopf}(\mathsf{Sp}).$$

This is an instance of the contragredient construction (Section 3.10). The same construction can be carried out for the functor $\mathcal{S}$.

**Theorem 11.29.** *For finite-dimensional positive species (comonoids), the functors $\mathcal{T}^\vee$ and $\mathcal{S}^\vee$ are the contragredients of $\mathcal{T}$ and $\mathcal{S}$ respectively. In particular, for any (finite-dimensional) positive comonoid $\mathbf{q}$, there is a canonical isomorphism of Hopf monoids*

$$\mathcal{T}(\mathbf{q})^* \cong \mathcal{T}^\vee(\mathbf{q}^*) \qquad \textit{and} \qquad \mathcal{S}(\mathbf{q})^* \cong \mathcal{S}^\vee(\mathbf{q}^*).$$

Proof. The proof for $\mathcal{T}$ proceeds as follows. The image of an object $\mathbf{q}$ under the contragredient of $\mathcal{T}$ is

$$(\mathbf{L} \circ \mathbf{q}^*)^* \cong \mathbf{L}^* \circ (\mathbf{q}^*)^* \cong \mathbf{L}^* \circ \mathbf{q} \cong \mathbf{L} \circ \mathbf{q}.$$

The first isomorphism follows from (8.8); it relies on the fact that $\mathbf{q}$ is positive. Since $\mathbf{L}$ is linearized, it is canonically isomorphic to its dual. This explains the last isomorphism. Thus, as species, $\mathcal{T}^\vee$ agrees with contragredient of $\mathcal{T}$. One checks that the concatenation product dualizes to the deconcenation coproduct, while the coproduct defined using dequasi-shuffling dualizes to the product defined using shuffling.

The argument for $\mathcal{S}$ is similar and omitted. $\square$

This result can be used to deduce results for $\mathcal{T}^\vee$ or $\mathcal{S}^\vee$ using those for $\mathcal{T}$ or $\mathcal{S}$ and viceversa.

**11.6.2. The abelianization.** Let $\mathbf{q}$ be a positive comonoid and consider the free Hopf monoid $\mathcal{T}(\mathbf{q})$ and the free commutative Hopf monoid $\mathcal{S}(\mathbf{q})$ on $\mathbf{q}$. By freeness of the former (Theorem 11.10), there is a unique morphism of Hopf monoids

$$\pi_{\mathbf{q}} \colon \mathcal{T}(\mathbf{q}) \to \mathcal{S}(\mathbf{q})$$

such that

$$\begin{tikzcd}
\mathcal{T}(\mathbf{q})_+ \arrow[rr, "(\pi_{\mathbf{q}})_+"] & & \mathcal{S}(\mathbf{q})_+ \\
& \mathbf{q} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes, where the universal arrows are unlabeled. We refer to the map $\pi_{\mathbf{q}}$ as the *abelianization*. It defines a natural transformation $\pi \colon \mathcal{T} \Rightarrow \mathcal{S}$.

In terms of the decompositions (11.3), $\pi_{\mathbf{q}}$ admits the following description. For each composition $F \vDash I$, let $\operatorname{supp}(F) \vdash I$ be the underlying partition (Section 10.1.5). Then $\pi_{\mathbf{q}}$ consists of the canonical isomorphisms

$$\mathbf{q}(F) = \mathbf{q}[F^1] \otimes \cdots \otimes \mathbf{q}[F^k] \xrightarrow{\cong} \bigotimes_{F^i \in \operatorname{supp}(F)} \mathbf{q}[F^i] = \mathbf{q}\big(\operatorname{supp}(F)\big) \tag{11.23}$$

between the unbracketed and the unordered tensor products (the map $\pi$ is not an isomorphism since a given partition underlies many compositions).

There is a natural transformation $\pi^\vee \colon \mathcal{S}^\vee \Rightarrow \mathcal{T}^\vee$: For each partition $X$ of $I$, the map

$$\mathbf{q}(X) \to \bigoplus_{F : \operatorname{supp}(F) = X} \mathbf{q}(F) \tag{11.24}$$

identifies $\mathbf{q}(X)$ and $\mathbf{q}(F)$, for each $F$ in the direct sum. If one restricts to finite-dimensional species, then the above transformation is the contragredient (Section 3.10) of the abelianization.

The simplest instance of the abelianization map occurs when $\mathbf{q} = \mathbf{X}$. In this case, $\pi_{\mathbf{X}}$ coincides with the morphism $\pi \colon \mathbf{L} \to \mathbf{E}$ given in (8.31). The contragredient applied to the species $\mathbf{X}$ yields the dual morphism $\pi^* \colon \mathbf{E}^* \to \mathbf{L}^*$ of (8.33).

**11.6.3. The norm transformation.** We now relate the functors $\mathcal{T}$ and $\mathcal{T}^\vee$. We first note that these functors are defined on different categories: $\mathcal{T}$ is defined on positive comonoids while $\mathcal{T}^\vee$ is defined on positive monoids. So in order to relate these functors, we consider

$$\mathcal{T}, \mathcal{T}^\vee \colon \mathsf{Sp}_+ \to \mathsf{Hopf}(\mathsf{Sp})$$

by viewing a positive species as a positive (co)monoid in the trivial way. The norm transformation

$$\kappa \colon \mathcal{T} \Rightarrow \mathcal{T}^\vee$$

is defined as follows. Fix a composition $F$ of $I$. Let $G$ be any composition with the same support as $F$. Consider the map

$$\mathbf{q}(F) \to \mathbf{q}(G)$$

which reorders the tensor factors. By summing over all such $G$, we obtain

$$\mathbf{q}(F) \to \bigoplus_G \mathbf{q}(G). \tag{11.25}$$

This is the norm transformation. One can check that it is a natural tranformation, that is, $\kappa(\mathbf{q})\colon \mathcal{T}(\mathbf{q}) \to \mathcal{T}^\vee(\mathbf{q})$ is a morphism of Hopf monoids.

Suppose now that the species $\mathbf{q}$ is finite-dimensional. Observe that $\kappa$ is related to its contragredient (3.47) as follows.

$$\begin{tikzcd} \mathcal{T}^\vee(\mathbf{q})^* \arrow[r, "(\kappa_{\mathbf{q}})^*"] \arrow[d, equal] & \mathcal{T}(\mathbf{q})^* \arrow[d, equal] \\ \mathcal{T}(\mathbf{q}^*) \arrow[r, "\kappa_{\mathbf{q}^*}"'] & \mathcal{T}^\vee(\mathbf{q}^*) \end{tikzcd}$$

This means that the norm transformation is self-dual (Definition 3.108).

**11.6.4. Self-duality of $\mathcal{S}$.** If we view a positive species $\mathbf{q}$ as a trivial positive monoid and as a trivial positive comonoid, then

$$\mathcal{S}(\mathbf{q}) = \mathcal{S}^\vee(\mathbf{q})$$

as Hopf monoids. This follows directly from the definitions. We explain the comonoid part.

Fix a decomposition $I = S \sqcup T$. Among all partitions $X \vdash I$, there are those for which $S$ and $T$ are unions of blocks. For any block $X^i$ of such a partition $X$ we have either $X^i \cap S = \emptyset$ or $X^i \cap T = \emptyset$, so in this case the map (11.15) is the identity. If $\mathbf{q}$ is a trivial positive comonoid, then the only partitions that contribute to the coproduct of $\mathcal{S}(\mathbf{q})$ are those of this kind. In this case, the coproduct on $\mathcal{S}(\mathbf{q})$ coincides with that of $\mathcal{S}^\vee(\mathbf{q})$ described in Section 11.5.

This shows that

$$\mathcal{S}\colon \mathsf{Sp}_+ \to \mathsf{Hopf}(\mathsf{Sp})$$

is self-dual in the sense of Definition 3.105 (over a field of any characteristic).

Over a field of characteristic 0, any finite-dimensional $\mathrm{S}_n$-module is isomorphic to its dual (noncanonically). Therefore, in this case, any finite-dimensional positive species $\mathbf{q}$ is isomorphic to $\mathbf{q}^*$, and hence self-dual. Since self-dual functors preserve self-dual objects (Proposition 3.107), it follows that for any finite-dimensional positive species $\mathbf{q}$ in characteristic 0, the Hopf monoid $\mathcal{S}(\mathbf{q})$ is self-dual (though noncanonically). If, in addition, $\mathbf{q}$ is linearized, then the self-duality is canonical (send a basis element to its dual basis element).

An instance is provided by $\mathbf{E}_V = \mathcal{S}(\mathbf{X}_V)$; see the discussion in Example 8.23.

**11.6.5. The image of the norm.** The norm transformation is far from being an isomorphism. In fact, it factors through the abelianization to yield the following self-dual commutative diagram.

$$\begin{tikzcd} \mathcal{T} \arrow[r, Rightarrow, "\kappa"] \arrow[d, Rightarrow, "\pi"'] & \mathcal{T}^\vee \\ \mathcal{S} \arrow[r, Rightarrow, "\mathrm{id}"'] & \mathcal{S}^\vee \arrow[u, Rightarrow, "\pi^\vee"'] \end{tikzcd} \tag{11.26}$$

This is the species analogue of (2.66). The vertical transformations are the abelianization and its contragredient. The transformation $\mathcal{S} \Rightarrow \mathcal{S}^\vee$ is the identity. The commutativity of (11.26) follows from (11.23), (11.24) and (11.25). Thus, $\mathcal{S}$ (or $\mathcal{S}^\vee$) is the image of the norm transformation. The self-duality of $\mathcal{S}$ also follows from the discussion in Section 3.11.6.

**Example 11.30.** The simplest instance of the norm transformation is the map

$$\kappa(\mathbf{X})\colon \mathbf{L} \to \mathbf{L}^* \qquad\qquad l' \mapsto \sum l^*,$$

where the sum is over all linear orders $l$ on $I$. More generally, applying (11.26) to the positive species $\mathbf{X}$, one obtains the commutative diagram (8.34) of Hopf monoids.

## 11.7. The deformed free and cofree Hopf monoids

In this section, we briefly sketch a deformation theory for the preceding sections. This includes the construction of the functors $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ which are one-parameter deformations of $\mathcal{T}$ and $\mathcal{T}^\vee$ (with $q = 1$ recovering the latter), the construction of the functors $\Lambda$ and $\Lambda^\vee$ which are signed analogues of $\mathcal{S}$ and $\mathcal{S}^\vee$, and the construction of the norm transformation $\kappa_q$ (which recovers $\kappa$ for $q = 1$).

These constructions make use of some of the notions introduced in Chapter 10, namely, the distance function on faces (Section 10.5.3) and the Schubert cocycle on faces (Section 10.13.6). In addition, we employ a result of Varchenko concerning a bilinear form on chambers studied in Section 10.15 to deduce the nondegeneracy of the $q$-norm in the generic case.

The case $q = 0$ exhibits particular features and is treated separately in Section 11.10.

**11.7.1. The functor $\mathcal{T}_q$.** Recall that $q$-$\mathsf{Hopf}(\mathsf{Sp})$ is the category of $q$-Hopf monoids in species (Section 9.1.2). Define a functor

$$\mathcal{T}_q\colon \mathsf{Comon}(\mathsf{Sp}_+) \to q\text{-}\mathsf{Hopf}(\mathsf{Sp})$$

by

$$\mathcal{T}_q(\mathbf{q}) := \mathbf{L} \circ \mathbf{q},$$

where $\mathbf{q}$ is a positive comonoid. In other words, $\mathcal{T}_q(\mathbf{q})$ and $\mathcal{T}(\mathbf{q})$ have the same underlying species. The product is the same as before, namely, concatenation. The component $\Delta_{S,T}$ of the coproduct is the direct sum over all compositions $H \vDash I$ of the map

$$\mathbf{q}(H) \to \mathbf{q}(H|_S) \otimes \mathbf{q}(H|_T)$$

which is defined as the map in (11.12) multiplied by the coefficient

$$q^{\mathrm{sch}_{S,T}(H)}, \tag{11.27}$$

where $\mathrm{sch}_{S,T}(H)$ is the Schubert cocycle on faces (10.119).

The functor $\mathcal{T}_q$ satisfies a universal property, namely, $\mathcal{T}_q(\mathbf{q})$ is the free $q$-Hopf monoid on the positive comonoid $\mathbf{q}$. This provides a $q$-analogue to Theorems 11.9 and 11.10.

Let $\mathbf{q}$ be a linearized positive comonoid. In this case, the $q$-bimonoid $\mathcal{T}_q(\mathbf{q})$ can be viewed as a deformation of $\mathcal{T}(\mathbf{q})$ by a normal multiplicative 2-cocycle of twist 1 (Section 9.6.4). This cocycle is defined by

$$\gamma_{S,T}(x) := \mathrm{sch}_{S,T}(H), \tag{11.28}$$

where $H$ is the composition of $I$ such that $x \in \mathbf{q}(H)$.

**Example 11.31.** View $\mathbf{X}$ as a trivial positive comonoid. Then

$$\mathcal{T}_q(\mathbf{X}) = \mathbf{L}_q$$

and one recovers the Hopf monoid structure on $\mathbf{L}_q$ given in Definition 9.13. Thus, $\mathbf{L}_q$ is the free $q$-Hopf monoid on one generator. Note that in this example, $\mathbf{X}(H)$ is nonzero only if $H$ is a linear order, say $C$. Further, for a linear order, $\mathbf{X}(C)$ is one-dimensional and spanned by $C$. The cocycle (11.28) then specializes to

$$\gamma_{S,T}(C) = \mathrm{sch}_{S,T}(C),$$

which is the Schubert cocycle on linear orders (9.12). This is consistent with our earlier observation that $\mathbf{L}_q$ is the deformation of $\mathbf{L}$ by the Schubert cocycle.

More generally,

$$\mathcal{T}_q(\mathbf{X}_V) = \mathbf{L}_q \times \mathbf{E}_V.$$

The $q$-Hopf monoids $\mathbf{\Sigma}_q = \mathcal{T}_q(\mathbf{E}_+^*)$ and $\overrightarrow{\mathbf{\Sigma}}_q = \mathcal{T}_q(\mathbf{L}_+^*)$ studied in Chapter 12 provide additional examples; see Proposition 12.63. These are based on set compositions and linear set compositions. In these examples, $\mathbf{q}(H)$ is always nonzero; hence the Schubert cocycle on faces is fully visible in the definitions of these $q$-Hopf monoids.

### 11.7.2. The functor $\mathcal{T}_q^\vee$.

We now define the functor

$$\mathcal{T}_q^\vee : \mathsf{Mon}(\mathsf{Sp}_+) \to q\text{-}\mathsf{Hopf}(\mathsf{Sp})$$

as the contragredient to $\mathcal{T}_q$. Explicitly, let $\mathbf{q}$ be a positive monoid. Define

$$\mathcal{T}_q^\vee(\mathbf{q}) := \mathbf{L} \circ \mathbf{q}.$$

The coproduct is deconcatenation as in the undeformed case. We now explain the product. Fix a decomposition $I = S \sqcup T$. Let $F \vDash S$ and $G \vDash T$ be compositions, and let $H$ be a quasi-shuffle of $F$ and $G$ (Section 10.1.6). Now consider the map

$$\mathbf{q}(F) \otimes \mathbf{q}(G) \to \mathbf{q}(H)$$

as in (11.20) multiplied by the coefficient (11.27). In terms of the blocks of $F$ and $G$, this coefficient is given by

$$q^{\sum |F^i|\,|G^j|}$$

where the sum is over $i$ and $j$ such that the block $F^i$ appears after $G^j$ in the quasi-shuffle $H$. The component $\mu_{S,T}$ of the product is given by summing over all choices of $F$, $G$ and $H$.

The functor $\mathcal{T}_q^\vee$ satisfies a universal property, namely, $\mathcal{T}_q^\vee(\mathbf{q})$ is the cofree $q$-Hopf monoid on the positive monoid $\mathbf{q}$. This provides a $q$-analogue to Theorems 11.22 and 11.23.

**Example 11.32.** View $\mathbf{X}$ as a trivial positive monoid. Then

$$\mathcal{T}_q^\vee(\mathbf{X}) = \mathbf{L}_q^*,$$

the dual to $\mathbf{L}_q$. Thus, $\mathbf{L}_q^*$ is the cofree $q$-Hopf monoid on one generator. More generally,

$$\mathcal{T}_q^\vee(\mathbf{X}_V) = \mathbf{L}_q^* \times \mathbf{E}_V.$$

The $q$-Hopf monoids $\mathbf{\Sigma}_q^* = \mathcal{T}_q^\vee(\mathbf{E}_+)$ and $\overrightarrow{\mathbf{\Sigma}}_q^* = \mathcal{T}_q^\vee(\mathbf{L}_+)$ studied in Chapter 12 provide additional examples; see Proposition 12.63. These are based on set compositions and linear set compositions.

**11.7.3. The functors $\mathbf{\Lambda}$ and $\mathbf{\Lambda}^\vee$.** We now turn our attention to the functors

$$\Lambda\colon \mathsf{Comon}(\mathsf{Sp}_+) \to (-1)\text{-}\mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp})$$

and

$$\Lambda^\vee\colon \mathsf{Mon}(\mathsf{Sp}_+) \to (-1)\text{-}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}).$$

These are signed analogues of $\mathcal{S}$ and $\mathcal{S}^\vee$.

Let $\mathbf{q}$ be a positive comonoid. Define

$$\Lambda(\mathbf{q})[I] := \bigoplus_{X\vdash I} \mathbf{q}(X)\otimes \mathrm{Det}(\Bbbk X) = \bigoplus_{k\geq 0} (\mathbf{q}^{\cdot k}[I])_{\mathrm{S}_k},$$

where the $\mathrm{S}_k$-coinvariants are taken with respect to the action induced by the braiding $\beta_{-1}$. The notation $\mathrm{Det}(\Bbbk X)$ generalizes the notation of (9.9): It denotes the one-dimensional subspace of the free graded commutative algebra on $\Bbbk X$ spanned by

$$X^1 \wedge \cdots \wedge X^k$$

where $X = \{X^1, \dots, X^k\}$. Here we are using $\wedge$ to denote the product of the algebra. The grading on $\Bbbk X$ is defined by setting the degree of $X^i$ to be $|X^i|$. For example, for $k = 3$,

$$X^1 \wedge X^2 \wedge X^3 = (-1)^{|X^2|\,|X^3|} X^1 \wedge X^3 \wedge X^2$$

spans $\mathrm{Det}(\Bbbk X)$. Note that if any block of $X$, say $X^1$, has even degree, then $X^1 \wedge X^1$ is not zero, and the free graded commutative algebra on $\Bbbk X$ is not finite-dimensional. If $X$ is the partition of $I$ into singletons, then $\mathrm{Det}(\Bbbk X)$ coincides with $\mathrm{Det}(\Bbbk I)$ as in (9.9).

**Remark 11.33.** Let $X = \{X^1, \dots, X^k\}$ be a set partition. For any species $\mathbf{q}$, view $\mathbf{q}[X^i]$ as a graded vector space concentrated in degree $|X^i|$. Then

$$\mathbf{q}(X)\otimes \mathrm{Det}(\Bbbk X)$$

is the same as the unordered tensor product of $\mathbf{q}[X^1],\dots,\ \mathbf{q}[X^k]$ in the category $(\mathsf{gVec}, \otimes, \beta_{-1})$. If one takes the unordered tensor product with respect to the braiding $\beta$, then one obtains $\mathbf{q}(X)$ as before.

We now turn $\Lambda(\mathbf{q})$ into a $(-1)$-Hopf monoid. The component $\mu_{S,T}$ of the product is given by tensoring the map (11.2) with the map

$$\begin{gathered}\mathrm{Det}(\Bbbk X)\otimes(\Bbbk Y) \to \mathrm{Det}\big(\Bbbk(X \sqcup Y)\big)\\ (X^1\wedge\cdots\wedge X^k)\otimes(Y^1\wedge\cdots\wedge Y^l) \mapsto X^1\wedge\cdots\wedge X^k\wedge Y^1\wedge\cdots\wedge Y^l,\end{gathered}$$

and then summing over all $X \vdash S$ and $Y \vdash T$.

The coproduct is given by tensoring the map

$$\mathbf{q}(X) \to \mathbf{q}(X|_S)\otimes \mathbf{q}(X|_T)$$

as in (11.15) with the map

$$\mathrm{Det}(\Bbbk X) \to \mathrm{Det}(\Bbbk X|_S)\otimes \mathrm{Det}(\Bbbk X|_T)$$

given as follows.

$$\begin{aligned} X^1\wedge\cdots\wedge X^k \mapsto (-1)^{\mathrm{sch}_{S,T}(X^1|\cdots|X^k)} &(X^1\cap S)\wedge\cdots\wedge(X^k\cap S)\\ &\otimes (X^1\cap T)\wedge\cdots\wedge(X^k\cap T),\end{aligned}$$

where $\mathrm{sch}_{S,T}(X^1|\cdots|X^k)$ is the Schubert cocycle on faces (10.119). The sign in front makes this map well-defined. This map can also be viewed as a composite as follows.

Let $Y := X \vee \{S,T\}$. Let $X^1|\cdots|X^k$ be a set composition with support $X$. This is equivalent to choosing a linear order on the blocks of $X$. Then the products (10.13)

$$(X^1|\cdots|X^k)(S|T) \qquad \text{and} \qquad (S|T)(X^1|\cdots|X^k)$$

are two set compositions with support $Y$. They define two different linear orders on the blocks of $Y$, and will be implicitly used in the definitions below. First define the map

$$\mathrm{Det}(\Bbbk X) \to \mathrm{Det}(\Bbbk Y)$$

by

$$X^1 \wedge \cdots \wedge X^k \mapsto (X^1 \cap S) \wedge (X^1 \cap T) \wedge \cdots \wedge (X^k \cap S) \wedge (X^k \cap T).$$

Now define the map

$$\mathrm{Det}(\Bbbk Y) \to \mathrm{Det}(\Bbbk X|_S) \otimes \mathrm{Det}(\Bbbk X|_T)$$

by

$$\begin{aligned}(X^1 \cap S) \wedge \cdots \wedge (X^k \cap S) \wedge (X^1 \cap T) \wedge \cdots \wedge (X^k \cap T)& \\ \mapsto (X^1 \cap S) \wedge \cdots \wedge (X^k \cap S) \otimes (X^1 \cap T) \wedge \cdots \wedge (X^k \cap T)&.\end{aligned}$$

We did not see the Schubert cocycle on faces explicitly in this description. It is hidden in the fact that the two maps above used different basis elements for $\mathrm{Det}^Q_f(\Bbbk Y)$.

The functor $\Lambda^\vee$ is the contragredient of $\Lambda$. An explicit description is omitted. The functors $\Lambda$ and $\Lambda^\vee$ satisfy the following universal properties: $\Lambda(\mathbf{q})$ is the free commutative $(-1)$-Hopf monoid on the positive comonoid $\mathbf{q}$, while $\Lambda^\vee(\mathbf{q})$ is the cofree cocommutative $(-1)$-Hopf monoid on the positive monoid $\mathbf{q}$. These provide signed analogues to Theorems 11.14 and 11.27.

**Example 11.34.** View $\mathbf{X}$ as a trivial positive (co)monoid. Then

$$\Lambda(\mathbf{X}) = \Lambda^\vee(\mathbf{X}) = \mathbf{E}^-,$$

the signed exponential species of Section 9.3. In other words, $\mathbf{E}^-$ is the free commutative and cofree cocommutative $(-1)$-Hopf monoid on one generator.

More generally,

$$\Lambda(\mathbf{X}_V) = \Lambda^\vee(\mathbf{X}_V) = \mathbf{E}^-_V,$$

the signed partner of the decorated exponential species. Explicitly,

$$\mathbf{E}^-_V[I] = V^{\otimes I} \otimes \mathrm{Det}(\Bbbk I).$$

The right-hand side is the unordered tensor product of $V$ over $I$ with respect to the braiding $\beta_{-1}$ (viewing $V$ as a graded vector space in degree 1).

Evaluating $\Lambda^\vee$ on $\mathbf{E}_+$ and $\mathbf{L}_+$, and $\Lambda$ on $\mathbf{E}^*_+$ and $\mathbf{L}^*_+$, yields signed analogues of the Hopf monoids of set partitions and linear set partitions. We do not consider the signed analogues any further in this monograph. The unsigned objects are studied in Chapter 12.

**11.7.4. The signed abelianization.** There is a signed analogue of the abelianization denoted

$$\pi_{-1}\colon \mathcal{T}_{-1} \Rightarrow \Lambda.$$

We refer to it as the *signed abelianization*. It is defined by summing the maps

$$\begin{aligned} \mathbf{q}(F) = \mathbf{q}[F^1] \otimes \cdots \otimes \mathbf{q}[F^k] &\xrightarrow{\cong} \mathbf{q}(X) \otimes \mathrm{Det}(\Bbbk X) \\ x &\longmapsto \pi(x) \otimes (F^1 \wedge \cdots \wedge F^k), \end{aligned} \tag{11.29}$$

where $X = \mathrm{supp}(F)$ and $\pi$ is as in (11.23).

There is a natural transformation

$$\pi_{-1}^{\vee}\colon \Lambda^{\vee} \Rightarrow \mathcal{T}_{-1}^{\vee}.$$

For each partition $X$ of $I$, the map

$$\mathbf{q}(X) \otimes \mathrm{Det}(\Bbbk X) \to \bigoplus_{F:\mathrm{supp}(F)=X} \mathbf{q}(F)$$

identifies the left hand side with $\mathbf{q}(F)$, using the inverse of (11.29), for each $F$ in the direct sum. If one restricts to finite-dimensional species, then the above transformation is the contragredient of the signed abelianization.

**11.7.5. The norm transformation.** We now relate the functors $\mathcal{T}_q$ and $\mathcal{T}_q^{\vee}$ by a $q$-analogue of the norm transformation. First, we view

$$\mathcal{T}_q, \mathcal{T}_q^{\vee}\colon \mathsf{Sp}_+ \to q\text{-}\mathsf{Hopf}(\mathsf{Sp})$$

by viewing a positive species as a positive (co)monoid in the trivial way. The $q$-norm transformation

$$\kappa_q\colon \mathcal{T}_q \Rightarrow \mathcal{T}_q^{\vee}$$

is defined as follows. Fix a composition $F$ of $I$. Let $G$ be any composition with the same support as $F$. Consider the map

$$\mathbf{q}(F) \to \mathbf{q}(G)$$

which reorders the tensor factors and multiplies by the coefficient

$$q^{\mathrm{dist}(F,G)}, \quad \text{or equivalently,} \quad q^{\sum_{(i,j)\in \mathrm{Inv}(F,G)} |F^i|\,|F^j|},$$

with notation as in (10.37). By summing over all such $G$, we obtain

$$\mathbf{q}(F) \to \bigoplus_{G} \mathbf{q}(G).$$

This defines a map

$$\kappa_q(\mathbf{q})\colon \mathcal{T}_q(\mathbf{q}) \to \mathcal{T}_q^{\vee}(\mathbf{q})$$

which is a natural morphism of $q$-Hopf monoids. This is the $q$-norm transformation. It continues to be self-dual.

**11.7.6. The image of the norm.** The norm transformation is an isomorphism in the generic case. More precisely:

**Theorem 11.35.** *If $q$ is not a root of unity, then $\kappa_q$ is an isomorphism.*

PROOF. Fix a partition $X$ of $I$. It is enough to show that each of the restrictions

$$\bigoplus_{F:\mathrm{supp}(F)=X} \mathbf{q}(F) \to \bigoplus_{F:\mathrm{supp}(F)=X} \mathbf{q}(F) \tag{11.30}$$

of $\kappa_q$ is an isomorphism.

For this, let us first do the case $\mathbf{q} = \mathbf{E}_+$. Then (11.30) can be viewed as a map

$$\mathbf{L}[X] \to \mathbf{L}[X]$$

where we recall that $\mathbf{L}[X]$ is the span of linear orders on $X$. Further, this map arises from the bilinear form

$$\langle F, G\rangle = q^{\mathrm{dist}(F,G)},$$

where $F$ and $G$ are linear orders on $X$ (which is the same as compositions with underlying partition $X$). We showed in Example 10.32 that, for $q$ not a root of unity, the above bilinear form is nondegenerate; hence $\kappa_q$ is an isomorphism.

Now let us go to the general case. Pick a basis for $\mathbf{q}[X^i]$, for each block $X^i$ of $X$. This determines a basis for $\mathbf{q}(F)$ for each $F$ with support $X$. Further, there is a canonical bijection between the basis of $\mathbf{q}(F)$ and the basis of $\mathbf{q}(G)$. Then (11.30) is a direct sum of maps of the form

$$\bigoplus_{F:\mathrm{supp}(F)=X} \Bbbk e_F \to \bigoplus_{F:\mathrm{supp}(F)=X} \Bbbk e_F$$

where for each $F$, $e_F$ is a chosen basis element of $\mathbf{q}(F)$ such that the choices for varying $F$ correspond to one another under the bijections mentioned above. The above map is an isomorphism by the previous case, and the result follows. $\square$

For $q = 1$, the situation is as in (11.26), while for $q = -1$, it is as follows.

$$\begin{tikzcd} \mathcal{T}_{-1} \arrow[r, Rightarrow, "\kappa_{-1}"] \arrow[d, Rightarrow, "\pi_{-1}"'] & \mathcal{T}_{-1}^{\vee} \\ \Lambda \arrow[r, Rightarrow, "\mathrm{id}"'] & \Lambda^{\vee} \arrow[u, Rightarrow, "\pi_{-1}^{\vee}"'] \end{tikzcd} \tag{11.31}$$

This is the species analogue of (2.68). The vertical maps are the signed abelianization and its contragredient. Thus, $\Lambda$ (or $\Lambda^{\vee}$) is the image of the signed norm transformation. The facts in Section 11.6.4 hold for $\Lambda$ as well. In particular,

$$\Lambda\colon \mathsf{Sp}_+ \to (-1)\text{-}\mathsf{Hopf}(\mathsf{Sp})$$

is canonically self-dual (over any characteristic). This also follows from the discussion in Section 3.11.6.

**Example 11.36.** The simplest instance of the $q$-norm tranformation is its value on the positive species $\mathbf{X}$. It follows from Theorem 11.35 that if $q$ is not a root of unity, then

$$\kappa_q(\mathbf{X})\colon \mathbf{L}_q \to \mathbf{L}_q^*$$

is an isomorphism. In particular, $\mathbf{L}_q$ is self-dual if $q$ is not a root of unity. This result is explained in more detail in Proposition 12.6.

Now let us consider the case $q = -1$. Applying diagram (11.31) to the positive species $\mathbf{X}$, one obtains the commutative diagram (9.21) of $(-1)$-Hopf monoids.

## 11.8. Antipode formulas

In this section, we give antipode formulas for the Hopf monoids which arise as the values of the functors $\mathcal{T}_q$, $\mathcal{S}$ and $\Lambda$ on some positive comonoid $\mathbf{q}$. We also write down similar formulas for the functors $\mathcal{T}_q^\vee$, $\mathcal{S}^\vee$ and $\Lambda^\vee$. If $\mathbf{q}$ is finite-dimensional, then one can pass from one situation to the other by applying the contragredient. However, the formulas hold without this assumption.

**11.8.1. The antipode for $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$.** Let $F$ and $G$ be compositions of $I$ and let $F \leq G$. Let $k$ be the number of blocks of $F$ and write

$$b_F(G) = (G^1, \dots, G^k), \tag{11.32}$$

where $b_F$ is the break map (10.58). Explicitly, $G^i$ consists of those blocks of $G$ which refine the $i$-th block of $F$, $i = 1, \dots, k$. For example,

$$F = lak|sh|mi, \qquad G = la|k|s|h|m|i, \qquad b_F(G) = (la|k, s|h, m|i).$$

We pause to review a notation. For any face $H = H^1|\cdots|H^k$, let

$$\mu_H := \mu_{H^1,\dots,H^k} \qquad \text{and} \qquad \Delta_H := \Delta_{H^1,\dots,H^k} \tag{11.33}$$

where the right-hand sides are the components of the iterated product (8.25) and coproduct (8.26). If $H$ has only one part, then $\mu_H$ and $\Delta_H$ are the identity. We proceed.

Let $\mathbf{q}$ be a positive comonoid. Define, for $F \leq G$,

$$\Delta_{G/F} \colon \mathbf{q}(F) \to \mathbf{q}(G) \qquad \text{by} \qquad \Delta_{G/F} := \Delta_{G^1} \otimes \cdots \otimes \Delta_{G^k},$$

where the $G^i$'s are as in (11.32) and the $\Delta_{G^i}$'s are as in (11.33). In other words, starting with an unbracketed tensor product over the blocks of $F$, we apply the appropriate component of the iterated coproduct of $\mathbf{q}$ to each block to obtain an unbracketed tensor product over the blocks of $G$. For example,

$$F = lak|sh|mi, \qquad G = la|k|s|h|m|i, \qquad \Delta_{G/F} = \Delta_{la,k} \otimes \Delta_{s,h} \otimes \Delta_{m,i}.$$

Dually, let $\mathbf{q}$ be a positive monoid. For $F \leq G$, define

$$\mu_{F\backslash G} \colon \mathbf{q}(G) \to \mathbf{q}(F) \qquad \text{by} \qquad \mu_{F\backslash G} := \mu_{G^1} \otimes \cdots \otimes \mu_{G^k},$$

where the $G^i$'s are as in (11.32) and the $\mu_{G^i}$'s are as in (11.33).

For faces $F$ and $G$ having the same support, let

$$\beta_{G,F} \colon \mathbf{q}(F) \to \mathbf{q}(G)$$

be the map which reorders the tensor factors. If $F = S|T$ and $G = T|S$, then $\beta_{G,F} = \beta_{S,T}$, the component of the braiding (8.13).

Before stating the antipode formulas, we prove a preliminary result.

**Lemma 11.37.** *Let $F$ and $G$ be any faces. Then*

$$\sum_{H:\, HF=G} (-1)^{\deg(H)} = \begin{cases} (-1)^{\deg(G)} & \text{if } \overline{F} \leq G, \\ 0 & \text{otherwise,} \end{cases}$$

*where $HF$ is the product of faces (10.13) and $\deg(H)$ denotes the number of blocks in $H$.*

Proof. We apply Proposition 10.12. Suppose $GF \neq G$. Then two things happen. The index set of the sum in the left hand side is empty, and we do not have $\bar{F} \leq G$. So the result holds in this case (both sides being 0). Now supppose $GF = G$. Then

$$A_{(F,G)} = \{H \mid HF = G\}$$

is a Boolean poset under containment of faces, and it is a singleton precisely when $\overline{F} \leq G$. So the result holds by inclusion-exclusion. □

**Theorem 11.38.** *Let $F$ be a composition of $I$ and let $\mathbf{q}$ be a positive comonoid. The antipode $\mathrm{s}\colon \mathcal{T}_q(\mathbf{q}) \to \mathcal{T}_q(\mathbf{q})$ on the $F$-component is given by*

$$\begin{aligned} \mathbf{q}(F) &\to \bigoplus_{G:G\models I} \mathbf{q}(G) \\ \mathrm{s}_I(x) &= q^{\operatorname{dist}(F,\overline{F})} \sum_{G:\overline{F}\leq G} (-1)^{\deg(G)} \Delta_{G/\overline{F}} \beta_{\overline{F},F}(x), \end{aligned}$$

*where $\deg(G)$ is the number of blocks in $G$, $\overline{F}$ denotes the opposite of $F$, and $\operatorname{dist}(F,G)$ is as in (10.33).*

Proof. Using the definition of the product and coproduct for $\mathcal{T}_q(\mathbf{q})$, we see that the iterated coproduct followed by the iterated product, $\mu_H \Delta_H$, sends the component $\mathbf{q}(F)$ to $\mathbf{q}(HF)$. More precisely, it is the composite

$$\mathbf{q}(F) \xrightarrow{\Delta_{FH/F}} \mathbf{q}(FH) \xrightarrow{\beta_{HF,FH}} \mathbf{q}(HF),$$

multiplied by the coefficient $q^{\operatorname{dist}(F,H)}$.

Applying Takeuchi's formula (8.27), the result follows from the calculation below. Let $x$ belong to the $F$-component. Then

$$\begin{aligned} \mathrm{s}_I(x) &= \sum_H (-1)^{\deg(H)} q^{\operatorname{dist}(F,H)} \beta_{HF,FH} \Delta_{FH/F}(x) \\ &= \sum_G \Big( \sum_{H:HF=G} (-1)^{\deg(H)} \Big) q^{\operatorname{dist}(F,G)} \beta_{G,FG} \Delta_{FG/F}(x) \\ &= \sum_{G:\overline{F}\leq G} (-1)^{\deg(G)} q^{\operatorname{dist}(F,G)} \beta_{G,FG} \Delta_{FG/F}(x) \\ &= q^{\operatorname{dist}(F,\overline{F})} \sum_{G:\overline{F}\leq G} (-1)^{\deg(G)}\ \Delta_{G/\overline{F}} \beta_{\overline{F},F}(x). \end{aligned}$$

The main step where the cancellations occur is the third equality. It follows from Lemma 11.37. The last equality follows from naturality of the braiding, and the fact that $\operatorname{dist}(F,G) = \operatorname{dist}(F,\overline{F})$ if $\overline{F} \leq G$. □

Special cases of the above result along with examples are discussed in detail in Chapter 12; see Proposition 12.3 and Theorems 12.24 and 12.36.

By dualizing Theorem 11.38 or by proceeding directly, one obtains:

**Theorem 11.39.** *Let $G$ be a composition of $I$ and let $\mathbf{q}$ be a positive monoid. The antipode* $\mathrm{S}\colon \mathcal{T}_q^\vee(\mathbf{q}) \to \mathcal{T}_q^\vee(\mathbf{q})$ *on the $G$-component is given by*

$$\mathbf{q}(G) \to \bigoplus_{F : F \vDash I} \mathbf{q}(F)$$
$$\mathrm{S}_I(x) = (-1)^{\deg(G)} \sum_{F : F \leq \overline{G}} q^{\mathrm{dist}(\overline{F}, F)}\, \beta_{F,\overline{F}} \mu_{\overline{F} \backslash G}(x),$$

*where the notation is as in Theorem 11.38.*

Special cases of the above result are discussed in Proposition 12.5 and Theorems 12.21 and 12.34.

**11.8.2. The antipode for $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}^\vee$.** Let $X$ and $Y$ be partitions of $I$ and let $X \leq Y$. Fix a face $F$ with support $X$. Write

$$b_F(Y) = (Y^1, \dots, Y^k), \tag{11.34}$$

where $b_F$ is the break map for set partitions (10.65). Explicitly, $Y^i$ consists of those blocks of $Y$ which refine the $i$-th block of $F$, $i = 1, \dots, k$. For example,

$$F = lak|sh|mi, \qquad Y = \{la, k, s, h, m, i\}, \qquad b_F(Y) = (\{la, k\}, \{s, h\}, \{m, i\}).$$

Let $\mathbf{q}$ be a positive comonoid and let $X \leq Y$. Fix $F$ to be any set composition with support $X$. Define $\Delta_{Y/X}$ by the commutativity of the following diagram.

$$\begin{tikzcd}
\mathbf{q}(F) \arrow[rr, "\bigoplus_{G:F\leq G,\mathrm{supp}(G)=Y} \Delta_{G/F}"] \arrow[d, "\pi_{\mathbf{q}}"'] & & \displaystyle\bigoplus_{\substack{G:F\leq G \\ \mathrm{supp}(G)=Y}} \mathbf{q}(G) \arrow[d, "\pi_{\mathbf{q}}"] \\
\mathbf{q}(X) \arrow[rr, dashed, "\Delta_{Y/X}"'] & & \mathbf{q}(Y)
\end{tikzcd} \tag{11.35}$$

The vertical maps are the abelianizations.

Now assume for the moment that $\mathbf{q}$ is cocommutative. In this case, the map $\Delta_{Y/X}$ can be understood more directly as follows. First note that if $F$ and $G$ have the same support, then $\Delta_F = \beta_{F,G}\Delta_G$. Hence, for any partition $X$ of $I$, by fixing a composition $F$ with support $X$, one obtains a map $\Delta_X$ by the commutativity of the diagram

$$\begin{tikzcd}
 & \mathbf{q}(F) \arrow[d, "\pi_{\mathbf{q}}"] \\
\mathbf{q}[I] \arrow[ur, "\Delta_F"] \arrow[r, dashed, "\Delta_X"'] & \mathbf{q}(X).
\end{tikzcd} \tag{11.36}$$

For example,

$$\Delta_{\{la,ksh,mi\}} := \pi_{\mathbf{q}} \Delta_{la,ksh,mi} = \pi_{\mathbf{q}} \Delta_{ksh,mi,la}.$$

It now follows that

$$\Delta_{Y/X} = (X : Y)! \Big( \bigotimes_i \Delta_{Y^i} \Big),$$

where the $Y^i$ are as defined in (11.34) and

$$(X:Y)! = |\{G \mid F \leq G,\ \operatorname{supp}(G) = Y\}|$$

are the coefficients in (10.5).

**Theorem 11.40.** *Let $X$ be a partition of $I$ and let $\mathbf{q}$ be a positive comonoid. The antipode $\mathrm{S}\colon \mathcal{S}(\mathbf{q}) \to \mathcal{S}(\mathbf{q})$ on the $X$-component is given by*

$$\begin{aligned}\mathbf{q}(X) &\to \bigoplus_{Y:Y\vdash I} \mathbf{q}(Y)\\ \mathrm{S}_I(x) &= \sum_{Y:X\leq Y} (-1)^{\deg(Y)}\, \Delta_{Y/X}(x),\end{aligned}$$

*where $\deg(Y)$ is the number of blocks in $Y$.*

The result follows by applying the abelianization to the antipode formula of Theorem 11.38. The simplest example that illustrates this theorem is the antipode of the exponential species (Example 8.15). Further examples are given in Chapter 12; see Theorems 12.47 and 12.56.

Let $\mathbf{q}$ be a positive monoid and let $X \leq Y$. Define $\mu_{X\backslash Y}$ by the commutativity of the following diagram.

(11.37)
$$\begin{tikzcd}
\displaystyle\bigoplus_{G:\operatorname{supp}(G)=Y} \mathbf{q}(G) \arrow[r, "\bigoplus \mu_{F\backslash G}"] & \displaystyle\bigoplus_{F:\operatorname{supp}(F)=X} \mathbf{q}(F) \\
\mathbf{q}(Y) \arrow[u, "\pi_{\mathbf{q}}^{\vee}"] \arrow[r, dashed, "\mu_{X\backslash Y}"'] & \mathbf{q}(X) \arrow[u, "\pi_{\mathbf{q}}^{\vee}"']
\end{tikzcd}$$

The vertical maps are the contragredient of the abelianization. The top horizontal map above is obtained by summing $\mu_{F\backslash G}$ over all $F$ with support $X$, and $G$ with support $Y$ such that $F \leq G$. Note that there exist $G$ for which there is no corresponding $F$; these components map to zero.

If $\mathbf{q}$ is commutative, then

$$\mu_{X\backslash Y} = (X:Y)!\Big(\bigotimes_i \mu_{Y^i}\Big),$$

where the $Y^i$ are as in (11.34), and $\mu_{Y^i}$ are defined by the dual of (11.36), and $(X:Y)!$ is as in (10.5).

By either dualizing Theorem 11.40, or by using the contragredient of the abelianization along with Theorem 11.39, we obtain:

**Theorem 11.41.** *Let $Y$ be a partition of $I$ and let $\mathbf{q}$ be a positive monoid. The antipode $\mathrm{S}\colon \mathcal{S}^{\vee}(\mathbf{q}) \to \mathcal{S}^{\vee}(\mathbf{q})$ on the $Y$-component is given by*

$$\begin{aligned}\mathbf{q}(Y) &\to \bigoplus_{X:X\vdash I} \mathbf{q}(X)\\ \mathrm{S}_I(x) &= (-1)^{\deg(Y)} \sum_{X:X\leq Y} \mu_{X\backslash Y}(x).\end{aligned}$$

The simplest example that illustrates this theorem is again the antipode of the exponential species (Example 8.15). Further examples are given in Theorems 12.44 and 12.51.

**11.8.3. The antipode for $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}^\vee$.** We now consider the signed case. The general setup is the same as in the unsigned case. Define $\Delta^-_{Y/X}$ by the commutativity of the following diagram.

$$\begin{tikzcd}
\mathbf{q}(F) \arrow[rr, "\bigoplus_{G:F\leq G,\operatorname{supp}(G)=Y} \Delta_{G/F}"] \arrow[d, "(\pi_{-1})_{\mathbf{q}}"'] & & \displaystyle\bigoplus_{\substack{G:F\leq G\\ \operatorname{supp}(G)=Y}} \mathbf{q}(G) \arrow[d, "(\pi_{-1})_{\mathbf{q}}"] \\
\mathbf{q}(X)\otimes \operatorname{Det}(\Bbbk X) \arrow[rr, dashed, "\Delta^-_{Y/X}"'] & & \mathbf{q}(Y)\otimes \operatorname{Det}(\Bbbk Y)
\end{tikzcd}$$

This is similar to (11.35); the difference is that the vertical maps are the signed abelianizations.

Now assume for the moment that $\mathbf{q}$ is graded cocommutative. In this case, for any partition $X$ of $I$, by fixing a composition $F$ with support $X$, one obtains a map $\Delta^-_X$ by the commutativity of the diagram

$$\begin{tikzcd}
 & \mathbf{q}(F) \arrow[d, "(\pi_{-1})_{\mathbf{q}}"] \\
\mathbf{q}[I] \arrow[ru, "\Delta_F"] \arrow[r, dashed, "\Delta^-_X"'] & \mathbf{q}(X)\otimes \operatorname{Det}(\Bbbk X).
\end{tikzcd}$$

For example,

$$\Delta^-_{\{la,ksh,mi\}} := (\pi_{-1})_{\mathbf{q}}\Delta_{la,ksh,mi} = (\pi_{-1})_{\mathbf{q}}\Delta_{ksh,mi,la}.$$

It follows that

$$\Delta^-_{Y/X} = (X:Y)!\Big(\bigotimes_i \Delta^-_{Y^i}\Big).$$

**Theorem 11.42.** *Let $X$ be a partition of $I$ and let $\mathbf{q}$ be a positive comonoid. The antipode $\mathrm{s}\colon \Lambda(\mathbf{q}) \to \Lambda(\mathbf{q})$ on the $X$-component is given by*

$$\mathbf{q}(X) \to \bigoplus_{Y:Y\vdash I} \mathbf{q}(Y)$$
$$\mathrm{s}_I(x) = \sum_{Y:X\leq Y} (-1)^{\deg(Y)}\, \Delta^-_{Y/X}(x).$$

The result follows by applying the signed abelianization to the antipode formula of Theorem 11.38. The antipode of the signed exponential species (Section 9.3) illustrates this theorem.

Let us now briefly consider the dual situation. Define $\mu^-_{X\backslash Y}$ along the lines of (11.37) by using the contragredients of the signed abelianizations for the vertical maps.

**Theorem 11.43.** *Let $Y$ be a partition of $I$ and let $\mathbf{q}$ be a positive monoid. The antipode* $\mathrm{s}\colon \Lambda^\vee(\mathbf{q}) \to \Lambda^\vee(\mathbf{q})$ *on the $Y$-component is given by*

$$\mathbf{q}(Y) \to \bigoplus_{X : X \vdash I} \mathbf{q}(X)$$
$$\mathrm{s}_I(x) = (-1)^{\deg(Y)} \sum_{X : X \leq Y} \mu^-_{X \setminus Y}(x).$$

## 11.9. Primitive elements and related functors

Primitive elements were discussed in Section 8.10. In this section, we consider the functor which associates the species of primitive elements to a Hopf monoid. It is right adjoint to the free monoid functor of Section 11.2.

The species of primitive elements of a Hopf monoid $\mathbf{h}$ is a Lie submonoid of $\mathbf{h}$, when this is viewed as a Lie monoid under the commutator bracket. In this manner, primitive elements give rise to another functor, this time from Hopf monoids to Lie monoids. Its left adjoint yields the universal enveloping monoid of a Hopf monoid. We review these notions briefly.

We also analyze the primitive elements of the universal Hopf monoids of earlier sections.

**11.9.1. Lie monoids in species.** We recall that Lie monoids can be defined in any linear symmetric monoidal category (possibly without a unit) (Definition 1.25). Let $\mathsf{Lie}(\mathsf{Sp})$ and $\mathsf{Lie}(\mathsf{Sp}_+)$ be the categories of Lie monoids in $(\mathsf{Sp}, \cdot)$ and $(\mathsf{Sp}_+, \cdot)$ respectively. The latter is a full subcategory of the former. A Lie monoid in $(\mathsf{Sp}, \cdot)$ is equivalent to a *twisted Lie algebra* in the sense of Barratt [33, Definition 4].

An important example of a Lie monoid in species is the **Lie** operad. It is discussed in Appendix B, see Example B.5. Recall that as a species, $\mathbf{Lie}[I]$ is spanned by bracket sequences on $I$. A *bracket sequence* on $I$ is a way to parenthesize the elements of $I$, each one appearing exactly once. For example, $[[x\ y]\,[w\ z]]$ is a bracket sequence on the set $\{x, y, z, w\}$. The space $\mathbf{Lie}[I]$ is the span of all bracket sequences of $I$ subject to the relations generated by antisymmetry and the Jacobi identity. For example, $\mathbf{Lie}[\{x, y\}]$ is the span of $[x\ y]$ and $[y\ x]$ subject to the relation

$$[x\ y] = -[y\ x];$$

hence it is one-dimensional. Also $\mathbf{Lie}[\{*\}] = \Bbbk$.

We now turn **Lie** into a Lie monoid. Fix a decomposition $I = S \sqcup T$. The component $\mu_{S,T}$ of the product sends a bracket sequence $\alpha$ on the set $S$ and $\beta$ on the set $T$ to the bracket sequence $[\alpha\,\beta]$ on $I$. This structure in terms of twisted Lie algebras is discussed in [11, Sections 5.1 and 5.3].

Define a functor

$$\mathcal{L}ie\colon \mathsf{Sp}_+ \to \mathsf{Lie}(\mathsf{Sp}_+) \qquad \text{by} \qquad \mathcal{L}ie(\mathbf{q}) = \mathbf{Lie} \circ \mathbf{q}.$$

The component $\mu_{S,T}$ of the product of $\mathbf{Lie} \circ \mathbf{q}$ is defined by: Let $X$ and $Y$ be partitions of $S$ and $T$ respectively. Then

$$\Big(\mathbf{Lie}[X] \otimes \bigotimes_{X^i \in X} \mathbf{q}[X^i]\Big) \otimes \Big(\mathbf{Lie}[Y] \otimes \bigotimes_{Y^j \in Y} \mathbf{q}[Y^j]\Big) \longrightarrow \mathbf{Lie}[X \sqcup Y] \otimes \Big(\bigotimes_{I' \in X \sqcup Y} \mathbf{q}[I']\Big)$$

is defined using the component $\mu_{X,Y}$ of the product of **Lie**.

The Lie monoid $\mathbf{Lie} \circ \mathbf{q}$ is the free Lie monoid on the species $\mathbf{q}$, that is, there is an adjunction

$$\begin{tikzcd} \mathsf{Sp}_+ \arrow[r, bend left, "\mathcal{L}ie"] & \mathsf{Lie}(\mathsf{Sp}_+). \arrow[l, bend left, "f\ell"] \end{tikzcd} \tag{11.38}$$

In particular, **Lie** is the free Lie monoid on **X**. In this sense, **Lie** plays the same role for Lie monoids as **L** plays for monoids and **E** plays for commutative monoids. The functor $\mathcal{L}ie$ is the Lie analogue of the functors $\mathcal{T}$ and $\mathcal{S}$.

### 11.9.2. The primitive element functor.

Recall that for a Hopf algebra $H$, its space of primitive elements is defined by

$$\mathcal{P}(H) := \{x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1\},$$

where 1 is the unit element of $H$. Further, if one regards $H$ as a Lie algebra via the commutator, then the space of primitive elements is a Lie subalgebra of $H$.

We now discuss the analogue of this construction for species. Let $\mathbf{h}$ be a Hopf monoid. Its species of primitive elements, denoted $\mathcal{P}(\mathbf{h})$, is given by

$$\mathcal{P}(\mathbf{h})[I] := \{x \in \mathbf{h}[I] \mid \Delta(x) = 1 \otimes x + x \otimes 1\},$$

where we consider 1 as an element of $\mathbf{h}[\emptyset]$ via the nonzero component $\iota_\emptyset$ of the unit map of $\mathbf{h}$.

If one regards $\mathbf{h}$ as a Lie monoid via the commutator as in Proposition 1.26, then it follows that $\mathcal{P}(\mathbf{h})$ is a Lie submonoid of $\mathbf{h}$. This defines a functor

$$\mathcal{P}\colon \mathsf{Hopf}(\mathsf{Sp}) \to \mathsf{Lie}(\mathsf{Sp}). \tag{11.39}$$

We refer to it as the *primitive element functor*. Note that $\mathcal{P}(\mathbf{h})[\emptyset] = \mathcal{P}(\mathbf{h}[\emptyset])$, where the latter is the space of primitive elements of the Hopf algebra $\mathbf{h}[\emptyset]$. Further, if $\mathbf{h}$ is connected, then

$$\mathcal{P}(\mathbf{h})[\emptyset] = 0 \quad \text{and} \quad \mathcal{P}(\mathbf{h})[I] = \ker \Delta_+[I],$$

where $\Delta_+$ is as defined in (8.58). This agrees with the definition of primitive elements given in Section 8.10. We obtain a functor

$$\mathcal{P}\colon \mathsf{Hopf}(\mathsf{Sp}^\circ) \to \mathsf{Lie}(\mathsf{Sp}_+).$$

**Example 11.44.** We record the species of primitive elements for the Hopf monoids considered in Examples 8.15, 8.16, 8.22 and 8.24. We have

$$\mathcal{P}(\mathbf{E}) = \mathcal{P}(\mathbf{E}^*) = \mathcal{P}(\mathbf{L}^*) = \mathbf{X},$$

where $\mathbf{X}$ is the species in (8.3). Since these Hopf monoids are commutative, the Lie bracket is trivial.

We have that $\mathcal{P}(\mathbf{L})$ is the operad **Lie**. This fact appears in work of Fresse [137, Proposition 1.2.16] and of Patras and Reutenauer [291, Proposition 17]. We deduce this from a more general result below, namely $\mathcal{P}(\mathbf{L} \circ \mathbf{q}) = \mathbf{Lie} \circ \mathbf{q}$ (Corollary 11.46). A generalization of this fact appears in recent work of Livernet and Patras [234, Theorem 3.2.1]. Since $\mathbf{Lie} = \mathcal{P}(\mathbf{L})$, the Lie operad is a Lie submonoid of $\mathbf{L}$. This fact was obtained by other means in [11, Section 5.3].

**11.9.3. Adjoints to the primitive element functor.** We turn to various adjoint functors to the primitive element functor. The results that follow are analogues of well known results for primitive elements of connected Hopf algebras.

Let ${}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}})$ be the category of cocommutative Hopf monoids in $(\mathsf{Sp}^{\mathsf{o}}, \cdot)$. Similarly, let ${}^{\mathsf{co}}\mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp}^{\mathsf{o}}))$ be the category of commutative and cocommutative Hopf monoids in $(\mathsf{Sp}^{\mathsf{o}}, \cdot)$. Note that they are full subcategories of $\mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}})$.

Recall from Section 11.2.2 that the free monoid $\mathcal{T}(\mathbf{q})$ carries the structure of a Hopf monoid. The same is true of the free commutative monoid $\mathcal{S}(\mathbf{q})$. This defines functors

$$\mathcal{T} \colon \mathsf{Sp}_+ \to {}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}}) \qquad \text{and} \qquad \mathcal{S} \colon \mathsf{Sp}_+ \to {}^{\mathsf{co}}\mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp}^{\mathsf{o}}).$$

We now show that these are the left adjoints to the primitive element functor (defined with the appropriate source category).

**Proposition 11.45.** *There are adjunctions*

$$\mathsf{Sp}_+ \underset{\mathcal{P}}{\overset{\mathcal{T}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}}) \qquad \textit{and} \qquad \mathsf{Sp}_+ \underset{\mathcal{P}}{\overset{\mathcal{S}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp}^{\mathsf{o}}),$$

*the functors above the arrows being left adjoints to the functors below the arrows.*

PROOF. We explain the first adjunction, the second being similar. We claim the following adjunctions:

$$\mathsf{Sp}_+ \underset{\mathcal{P}}{\overset{inc}{\rightleftarrows}} \mathsf{Comon}(\mathsf{Sp}_+) \underset{(-)_+}{\overset{\mathcal{T}}{\rightleftarrows}} \mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}}).$$

The first adjunction is as in Section 8.10. The second adjunction was proved in (11.13). The composite of the two implies the first adjunction in the proposition by noting that ${}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}})$ is a full subcategory of $\mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}})$ and using Proposition A.5. $\square$

The first adjunction in Proposition 11.45 can be viewed as the following composite.

$$\mathsf{Sp}_+ \underset{f\ell}{\overset{\mathcal{L}ie}{\rightleftarrows}} \mathsf{Lie}(\mathsf{Sp}_+) \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}^{\mathsf{o}})$$

The Hopf monoid $\mathcal{U}(\mathbf{g})$ is the universal enveloping monoid of the Lie monoid $\mathbf{g}$; see [182, 346] for a definition. The first adjunction was discussed in (11.38). The adjointness between $\mathcal{U}$ and $\mathcal{P}$ is part of the Cartier–Milnor–Moore theorem for species and can be found in Stover [346, Proposition 7.10 and Theorem 8.4]. It says that $\mathcal{U}$ and $\mathcal{P}$ define an adjoint equivalence, independent of the field characteristic. In particular, for any Lie monoid $\mathbf{g}$ and any connected cocommutative Hopf monoid $\mathbf{h}$, we have

$$\mathcal{P}\big(\mathcal{U}(\mathbf{g})\big) = \mathbf{g} \qquad \text{and} \qquad \mathcal{U}\big(\mathcal{P}(\mathbf{h})\big) = \mathbf{h}. \tag{11.40}$$

**Corollary 11.46.** *For a positive species* $\mathbf{q}$, *we have*

$$\mathcal{P}\big(\mathcal{S}(\mathbf{q})\big) = \mathbf{q} \qquad \textit{and} \qquad \mathcal{P}\big(\mathcal{T}(\mathbf{q})\big) = \mathcal{L}ie(\mathbf{q}) = \mathbf{Lie} \circ \mathbf{q},$$

*where* $\mathbf{Lie}$ *is the Lie operad. In particular,* $\mathcal{P}(\mathbf{L}) = \mathbf{Lie}$.

PROOF. Viewing $\mathbf{q}$ as a trivial Lie monoid, we have $\mathcal{S}(\mathbf{q}) = \mathcal{U}(\mathbf{q})$ as Hopf monoids. Applying (11.40), the first assertion follows.

From the above discussion, we know that both $\mathcal{T}$ and $\mathcal{U} \circ \mathcal{L}ie$ are left adjoints to the primitive element functor from connected cocommutative Hopf monoids to positive species. By uniqueness of adjoints, we have $\mathcal{T}(\mathbf{q}) = \mathcal{U}\big(\mathcal{L}ie(\mathbf{q})\big)$. Applying (11.40), the second assertion follows. $\square$

There is also an analogue of the Poincaré-Birkhoff-Witt theorem for species which can be found in Joyal [182, Section 4.2, Theorem 2] and Stover [346, Theorem 11.3]. It says that $\mathcal{U}(\mathbf{g}) = \mathcal{S}(\mathbf{g})$ as comonoids for any Lie monoid $\mathbf{g}$.

**Remark 11.47.** The classical context for Lie theory is the category of vector spaces. In this context the Poincaré–Birkhoff–Witt and Cartier–Milnor–Moore theorems can be found in [300, Appendix B, Theorems 2.3 and 4.5] or [274, Theorems 5.15 and 5.18]. Usually the field characteristic is assumed to be zero in these theorems; however no such assumption is necessary if one works in the category of species.

Fresse [135, Section 4] develops Lie theory in the symmetric monoidal category of right modules over an operad $\mathbf{p}$. The monoidal structure is given by the Cauchy product. For $\mathbf{p} = \mathbf{X}$, this category coincides with that of species. Fresse discusses the Poincaré–Birkhoff–Witt theorem [135, Theorem 4.1.5] and the Cartier–Milnor–Moore theorem [135, Theorem 4.1.6] in this more general context. He also discusses the cofree cocommutative comonoid in the category of right modules over an operad $\mathbf{p}$ [135, Section 3.2]. This generalizes the functor $\mathcal{S}^\vee$ of Definition 11.25.

**11.9.4. Coradical filtrations.** We now explicitly describe the species of primitive elements $\mathcal{P}(\mathbf{h})$ and the coradical filtration $\mathcal{P}^{(k)}(\mathbf{h})$ (Section 8.10) for the connected Hopf monoids $\mathbf{h}$ which arise as values of the functors $\mathcal{T}^\vee$, $\mathcal{S}^\vee$ and $\Lambda^\vee$. Since this notion pertains only to the comonoid structure, the parameter $q$ does not play any part in this discussion. Thus the situation for $\mathcal{T}_q^\vee$ is identical to that for $\mathcal{T}^\vee$. Recall that these functors are evaluated on a positive monoid $\mathbf{q}$. The monoid structure of $\mathbf{q}$ plays no role for the same reason.

Let $I$ be a nonempty set. We have

$$\mathcal{P}^{(k)}\big(\mathcal{T}^\vee(\mathbf{q})\big)[I] = \bigoplus \mathbf{q}(F),$$

where the sum is over all compositions $F \vDash I$ into at most $k$ blocks. This is a direct consequence of Definition 11.17 of the deconcatenation coproduct. In particular, the primitive element species is precisely $\mathbf{q}$.

Similarly,

$$\mathcal{P}^{(k)}\big(\mathcal{S}^\vee(\mathbf{q})\big)[I] = \bigoplus \mathbf{q}(X),$$

where the sum is over all partitions $X \vdash I$ into at most $k$ blocks. In particular, the primitive element species is again $\mathbf{q}$. The same description works for $\Lambda^\vee(\mathbf{q})$ with $\mathbf{q}(X) \otimes \mathrm{Det}(\Bbbk X)$ instead of $\mathbf{q}(X)$.

**Example 11.48.** Recall from Example 11.24 that $\mathbf{L}^*$ is the value of $\mathcal{T}^\vee$ on the species $\mathbf{X}$ characteristic of singletons. It follows that

$$\mathcal{P}^{(k)}(\mathbf{L}^*)[I] = \begin{cases} \mathbf{L}^*[I] & \text{if } |I| \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

for any nonempty set $I$. In particular, the primitive element species of $\mathbf{L}^*$ is $\mathbf{X}$.

Recall from Example 11.28 that $\mathbf{E}$ is the value of $\mathcal{S}^\vee$ on the species $\mathbf{X}$. It follows that

$$\mathcal{P}^{(k)}(\mathbf{E})[I] = \begin{cases} \mathbf{E}[I] & \text{if } |I| \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

for any nonempty set $I$. In particular, the primitive element species of $\mathbf{E}$ is $\mathbf{X}$.

The Hopf monoids $\mathbf{\Sigma}^*$ and $\overrightarrow{\mathbf{\Sigma}}^*$ also arise as values of the functor $\mathcal{T}^\vee$. Their coradical filtrations are described in Sections 12.4.1 and 12.5.2.

## 11.10. The free and cofree 0-Hopf monoids

Recall the constructions of universal $q$-Hopf monoids given in Section 11.7. Along with parameter values $q = \pm 1$, the case $q = 0$ is also of considerable interest. In this section, we look at this case in more detail. The relevant functors are

$$\mathcal{T}_0 \colon \mathsf{Comon}(\mathsf{Sp}_+) \to 0\text{-}\mathsf{Hopf}(\mathsf{Sp}) \quad \text{and} \quad \mathcal{T}_0^\vee \colon \mathsf{Mon}(\mathsf{Sp}_+) \to 0\text{-}\mathsf{Hopf}(\mathsf{Sp}).$$

We describe these functors explicitly and also provide antipode formulas following the results in Section 11.8.

The functors $\mathcal{T}_0$ and $\mathcal{T}_0^\vee$ agree on positive species (equipped with the trivial coproduct and products). This feature is unique to the parameter value 0. Further, it turns out that any connected 0-bimonoid is necessarily of the form $\mathcal{T}_0(\mathbf{q})$ for some positive species $\mathbf{q}$ (Theorem 11.49).

**11.10.1. The free 0-Hopf monoid on a positive comonoid.** Let $\mathbf{q}$ be a positive comonoid. The 0-Hopf monoid $\mathcal{T}_0(\mathbf{q})$ has the same underlying species and the same product as the Hopf monoid $\mathcal{T}(\mathbf{q})$ (concatenation). We now discuss the coproduct, by setting $q = 0$ in the description of Section 11.7.1. Fix a decomposition $I = S \sqcup T$. Let $H \vDash I$ be such that no element of $T$ appears in an earlier block of $H$ than an element of $S$. Equivalently, there are compositions $F \vDash S$ and $G \vDash T$ such that $H$ is either $F \cdot G$ or $F \smile G$. The former is the concatenation of $F$ and $G$ and the latter is the unique quasishuffle of $F$ and $G$ in which the last block of $F$ is merged with the first block of $G$. Now consider the map

$$\mathbf{q}(H) \to \mathbf{q}(H|_S) \otimes \mathbf{q}(H|_T)$$

as in (11.12). The component $\Delta_{S,T}$ of the coproduct of $\mathcal{T}_0(\mathbf{q})$ is the direct sum of these maps over all $H \vDash I$ of the above form (in which case $H|_S = F$ and $H|_T = G$). Note that the coproduct of $\mathbf{q}$ is involved only when $H = F \smile G$.

The antipode formula of Theorem 11.38 takes the following form for $\mathcal{T}_0$.

$$(\mathrm{s}_0)_I(x) = \begin{cases} \sum\limits_{G} (-1)^{\deg(G)} \, \Delta_G(x) & \text{if } x \in \mathbf{q}[I], \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the antipode is 0 on $\mathbf{q}(F)$ if $F$ has more than one block.

For $\mathbf{q} = \mathbf{X}$, this formula specializes to the antipode formula for $\mathbf{L}_0$ given in (9.19).

**11.10.2. The cofree 0-Hopf monoid on a positive monoid.** Let $\mathbf{q}$ be a positive monoid. The 0-Hopf monoid $\mathcal{T}_0^\vee(\mathbf{q})$ has the same underlying species and the same coproduct as the Hopf monoid $\mathcal{T}^\vee(\mathbf{q})$ (deconcatenation). We now discuss the product, by setting $q = 0$ in the description of Section 11.7.2. Fix a decomposition $I = S \sqcup T$. Let $F \vDash S$ and $G \vDash T$ be compositions, and let $H$ be either $F \cdot G$ or

$F \smile G$, as above. In either case, $H$ is a quasi-shuffle of $F$ and $G$. Now consider the map

$$\mathbf{q}(F) \otimes \mathbf{q}(G) \to \mathbf{q}(H)$$

as in (11.20). The component $\mu_{S,T}$ of the product is given by summing over all such choices of $F$, $G$ and $H$. Note that the product of $\mathbf{q}$ is involved only when $H = F \smile G$.

The antipode formula of Theorem 11.39 takes the following form for $\mathcal{T}_0^\vee$.

$$(\mathrm{s}_0)_I(x) = (-1)^{\deg(G)} \mu_G(x) \quad \text{for } x \in \mathbf{q}(G).$$

Note that the image always lies in $\mathbf{q}[I]$ irrespective of $G$.

**11.10.3. The free and cofree 0-Hopf monoid on a positive species.** Let us now restrict the functors $\mathcal{T}_0$ and $\mathcal{T}_0^\vee$ to trivial positive comonoids and monoids respectively:

$$\mathcal{T}_0, \mathcal{T}_0^\vee \colon \mathsf{Sp}_+ \to 0\text{-}\mathsf{Hopf}(\mathsf{Sp}).$$

In this case, the functors $\mathcal{T}_0$ and $\mathcal{T}_0^\vee$ coincide. Indeed, comparing the descriptions of these functors given above, we see that for any positive species $\mathbf{q}$, the 0-Hopf monoid structure of

$$\mathcal{T}_0(\mathbf{q}) = \mathcal{T}_0^\vee(\mathbf{q})$$

is in both cases given by concatenation and deconcatenation. Moreover, the norm $\kappa_0$ (Section 11.7.5) is the identity. This follows from (10.35). Hence, on finite-dimensional species, $\mathcal{T}_0$ is a self-dual functor.

It turns out that any connected 0-bimonoid is of this form. First, note that

$$\mathcal{P}\big(\mathcal{T}_0(\mathbf{q})\big) = \mathbf{q}. \tag{11.41}$$

This follows from the results in Section 11.9.4, taking into account that

$$\mathcal{T}_0(\mathbf{q}) = \mathcal{T}_0^\vee(\mathbf{q}) = \mathcal{T}^\vee(\mathbf{q})$$

(the latter as comonoids only). The precise result is:

**Theorem 11.49.** *Let $\mathbf{h}$ be a connected 0-bimonoid. There is a canonical isomorphism of 0-Hopf monoids*

$$\mathbf{h} \cong \mathcal{T}_0\big(\mathcal{P}(\mathbf{h})\big).$$

PROOF. The universal property of $\mathcal{T}_0$ gives a morphism of 0-Hopf monoids

$$\varphi : \mathcal{T}_0\big(\mathcal{P}(\mathbf{h})\big) \to \mathbf{h}$$

extending the inclusion $\mathcal{P}(\mathbf{h}) \hookrightarrow \mathbf{h}$. Explicitly,

$$\varphi = \mu^{(k-1)} \text{ on } \mathcal{P}(\mathbf{h})^{\cdot k} \text{ for each } k \geq 1.$$

By construction, and in view of (11.41), $\varphi$ is injective on primitives. Therefore, it is injective everywhere, by Proposition 8.46.

We now show it is surjective, by induction. Take $z \in \mathbf{h}$. By (8.63), there exists $k \geq 1$ such that $z \in \mathcal{P}^{(k)}(\mathbf{h})$. If $k = 1$, then $z = \varphi(z)$. Suppose $k \geq 2$. By (8.64), we have

$$\Delta_+^{(k-1)}(z) \in \mathcal{P}(\mathbf{h})^{\cdot k}$$

and therefore

$$\varphi\big(\Delta_+^{(k-1)}(z)\big) = \mu^{(k-1)} \Delta_+^{(k-1)}(z).$$

On the other hand, from Lemma 9.4 we have

$$\Delta_+^{(k-1)} \mu^{(k-1)} \Delta_+^{(k-1)}(z) = \Delta_+^{(k-1)}(z).$$

Hence,

$$z - \mu^{(k-1)}\Delta_+^{(k-1)}(z) \in \ker(\Delta_+^{(k-1)}) = \mathcal{P}^{(k-1)}(\mathbf{h}).$$

By induction hypothesis, this element belongs to the image of $\varphi$. We conclude that

$$\begin{aligned} z &= \mu^{(k-1)}\Delta_+^{(k-1)}(z) + \Big(z - \mu^{(k-1)}\Delta_+^{(k-1)}(z)\Big) \\ &= \varphi\big(\Delta_+^{(k-1)}(z)\big) + \Big(z - \mu^{(k-1)}\Delta_+^{(k-1)}(z)\Big) \end{aligned}$$

is also in the image of $\varphi$. □

The preceding is the version for species of the result of Loday and Ronco given in Theorem 2.13. The proof above is essentially the same as theirs.

CHAPTER 12

# Hopf Monoids from Geometry

The goal of this chapter is to construct Hopf monoids in species motivated by the geometry of the Coxeter complex of type $A$. Since this complex can be described in purely combinatorial terms, it follows that the Hopf monoids also admit simple combinatorial descriptions. Throughout this chapter, we assume that the characteristic of the base field $\Bbbk$ is zero.

The Hopf monoids we consider in this chapter are summarized in Table 12.1. We begin by discussing the underlying species. The species $\mathbf{E}$ and $\mathbf{L}$ are the exponential and linear order species respectively. The remaining species in Table 12.1 have all been considered in Chapter 10: the ones pertaining to faces and flats in Section 10.2, the ones pertaining to directed faces and directed flats in Section 10.9, and the one pertaining to pair of chambers in Section 10.7. These geometric objects can be interpreted combinatorially in terms of partitions and compositions; the relevant terminology was given in Section 10.1. All species are connected and linearized and hence naturally isomorphic to their duals. The letters $H$, $K$, and so on denote the different bases that we consider on these species.

Each of these species can be turned into a Hopf monoid, the dual species being given the dual structure. The functors $\mathcal{T}$ and $\mathcal{S}$, and their contragredients $\mathcal{T}^\vee$ and $\mathcal{S}^\vee$ were constructed in Chapter 11. The functor $\mathcal{E}^\times$ is the internal endomorphism functor for the Hadamard product on species (8.80). The Hopf monoids under consideration can all be obtained as values of these functors as indicated in Table 12.1. The Hopf monoids $\mathbf{L}$ and $\mathbf{E}$ were described explicitly in Examples 8.15

TABLE 12.1. Hopf monoids.

| Hopf monoid | | | Indexing sets | | Dual Hopf monoid | | |
|---|---|---|---|---|---|---|---|
| $\mathbf{L}$ | $\mathcal{T}(\mathbf{X}^*)$ | | linear orders | chambers | $\mathbf{L}^*$ | $\mathcal{T}^\vee(\mathbf{X})$ | |
| $\mathbf{\Sigma}$ | $\mathcal{T}(\mathbf{E}^*_+)$ | $H$ | set compositions | faces | $\mathbf{\Sigma}^*$ | $\mathcal{T}^\vee(\mathbf{E}_+)$ | $M$ |
| $\overrightarrow{\mathbf{\Sigma}}$ | $\mathcal{T}(\mathbf{L}^*_+)$ | $H, K$ | linear set compositions | directed faces | $\overrightarrow{\mathbf{\Sigma}}^*$ | $\mathcal{T}^\vee(\mathbf{L}_+)$ | $M, F$ |
| $\mathbb{L}$ | $\mathcal{E}^\times(\mathbf{L})$ | $H, K$ | pairs of linear orders | pairs of chambers | $\mathbb{L}^*$ | $\mathcal{E}^\times(\mathbf{L}^*)$ | $M, F$ |
| $\mathbf{E}$ | $\mathcal{S}(\mathbf{X}^*)$ | | one-block partition | ambient space | $\mathbf{E}^*$ | $\mathcal{S}^\vee(\mathbf{X})$ | |
| $\mathbf{\Pi}$ | $\mathcal{S}(\mathbf{E}^*_+)$ | $h, q$ | set partitions | flats | $\mathbf{\Pi}^*$ | $\mathcal{S}^\vee(\mathbf{E}_+)$ | $m, p$ |
| $\overrightarrow{\mathbf{\Pi}}$ | $\mathcal{S}(\mathbf{L}^*_+)$ | $h$ | linear set partitions | directed flats | $\overrightarrow{\mathbf{\Pi}}^*$ | $\mathcal{S}^\vee(\mathbf{L}_+)$ | $m$ |

and 8.16, and their duals were described in Examples 8.22 and 8.24. The remaining Hopf monoids will be defined explicitly and studied in detail in this chapter. The components of the product and coproduct of each of these Hopf monoids can be understood via three geometric constructions: the projection, break and join maps. These notions were explained in Sections 10.4, 10.5 and 10.11.

Some of these Hopf monoids can be deformed to $q$-Hopf monoids. In geometric terms, the deformation is done using the gallery metric, while in combinatorial terms, it is done using the Schubert cocycle. Whenever such a deformation is possible, we directly study the $q$ case. This avoids repetition and one can easily recover the undeformed case by setting $q = 1$.

We begin this chapter with a discussion of the various bases (Section 12.1). In Section 12.2, we revisit the $q$-Hopf monoid of linear orders $\mathbf{L}_q$ and its dual $\mathbf{L}_q^*$, and explain this example in geometric terms. Subsequent Sections 12.3–12.7 discuss more advanced examples: $\mathbb{L}$, $\mathbf{\Sigma}$, $\overrightarrow{\mathbf{\Sigma}}$, $\mathbf{\Pi}$ and $\overrightarrow{\mathbf{\Pi}}$, and their duals. We provide explicit antipode formulas in all cases. In Section 12.8, we explain the inter-relationships between these Hopf monoids (see diagrams (12.14) and (12.20)) and provide details on their universal properties.

## 12.1. Bases

It is useful to define more than one linear basis for many of the Hopf monoids we consider. These are summarized in Table 12.1. Since the underlying species are linearized, each Hopf monoid comes equipped with a canonical basis. The second basis is then defined in terms of this one.

**Notation 12.1.** In this context, the letters $H$ and $M$ denote a pair of dual basis. Thus, if a Hopf monoid $\mathbf{H}$ has a linear basis $\{H_x\}$, where $x$ runs over a certain set, then $\{M_x\}$ denotes the dual basis of the Hopf monoid $\mathbf{H}^*$:

$$\langle H_x,\, M_y\rangle = \delta(x, y). \tag{12.1}$$

Besides $(H, M)$, other pairs of dual bases that we employ are $(K, F)$, $(h, m)$, and $(p, q)$:

$$\langle K_x,\, F_y\rangle = \langle h_x,\, m_y\rangle = \langle p_x,\, q_y\rangle = \delta(x, y). \tag{12.2}$$

The letters used for each Hopf monoid are indicated in Table 12.1. For the basis of the Hopf monoids $\mathbf{E}$ and $\mathbf{L}$ we do not employ any letters. This choice of notation is the same as that in [12, Table 5.2].

**12.1.1. Pairs of chambers.** We begin with the species of pairs of linear orders $\mathbb{L}^*$. Let $F$ denote its canonical basis. We now proceed to define another basis on it, called the $M$ basis. Recall from (10.41) the partial order on the set of pairs of chambers:

$$(C_1, D_1) \leq (C_2, D_2) \quad \text{if } D_1 = D_2 = D \text{ and } C_2 - C_1 - D,$$

where $C_2 - C_1 - D$ stands for a minimum gallery from $C_2$ to $D$ passing through $C_1$. The $F$ and $M$ bases are related to each other via the above partial order as follows.

$$F_{(E,D)} = \sum_{C:(E,D)\leq(C,D)} M_{(C,D)} \tag{12.3}$$

In other words, an element in the $F$ basis is obtained by summing those elements in the $M$ basis which are larger than itself in the partial order. This is the definition of the $M$ basis. It can be made more explicit by inverting the above equation.

By dualizing, we obtain two bases on $\mathbb{L}$: the dual to $F$ is denoted $K$, and the dual to $M$ is denoted $H$. This is in agreement with Notation 12.1.

**12.1.2. Directed faces.** We now consider the species of directed faces $\overrightarrow{\mathbf{\Sigma}}^*$. Let $F$ denote its canonical basis. We proceed to define another basis on it, called the $M$ basis. Recall from (10.53) the partial order on the set of directed faces:

$$(F, C) \leq (G, D) \iff C = D \text{ and } F \leq G,$$

where $F \leq G$ denotes inclusion of faces. The $F$ and $M$ bases are related to each other as follows.

$$F_{(G,D)} = \sum_{H:G \leq H \leq D} M_{(H,D)}. \tag{12.4}$$

This is in analogy with what we did for pairs of chambers, namely, an element in the $F$ basis is obtained by summing those elements in the $M$ basis which are larger than itself in the partial order.

By dualizing, we obtain two bases on $\overrightarrow{\mathbf{\Sigma}}$: the dual to $F$ is denoted $K$, and the dual to $M$ is denoted $H$. The relationship between the two bases of $\overrightarrow{\mathbf{\Sigma}}$ is

$$H_{(H,C)} = \sum_{F:F \leq H} K_{(F,C)}.$$

**12.1.3. Flats.** We now consider the species of flats $\mathbf{\Pi}^*$. The situation here is somewhat different. Let $m$ denote its canonical basis. Define the $p$ basis by

$$p_Y = \sum_{X:X \leq Y} m_X, \tag{12.5}$$

where $\leq$ denotes the partial order on set partitions given by refinement (Section 10.2.3).

By dualizing, we obtain two bases on $\mathbf{\Pi}$: the dual to $m$ is denoted $h$, and the dual to $p$ is denoted $q$, in agreement with Notation 12.1. The bases of $\mathbf{\Pi}$ are related by

$$h_X = \sum_{Y:X \leq Y} q_Y. \tag{12.6}$$

The motivation for the $p$ and $q$ bases is as follows. View $\Pi[I]$ as a commutative monoid with product given by the join as in Section 10.4.2. Then the monoid algebra $\Bbbk\Pi[I]$ is semisimple and the elements of the $q$ basis are the primitive idempotents. This result is due to Solomon [332], see also Greene [153], Stanley [341, Section 3.9] and [12, Lemma 5.6.3].

## 12.2. The $q$-Hopf monoid of chambers

Recall the $q$-Hopf monoids $\mathbf{L}_q$ and $\mathbf{L}_q^*$ of linear orders from Section 9.5. In this section, we describe these objects in geometric terms. The starting point is the observation that linear orders on $I$ correspond to chambers in the Coxeter complex $\Sigma[I]$. In this setting, it is convenient to describe the components of the product and coproduct if the decomposition is into nonempty subsets. The descriptions involve the break and join maps, namely, $b_K$ and $j_K$ (10.57), and the notion of gallery distance $\operatorname{dist}(C, D)$ between chambers $C$ and $D$ (Section 10.5.1).

We also show that $\mathbf{L}_q$ is self-dual for generic values of $q$. This will make use of the bilinear form on chambers studied in Section 10.14.

### 12.2.1. A geometric description.

**Proposition 12.2.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by*

$$\begin{aligned} \mathbf{L}_q[I] &\to \mathbf{L}_q[S] \otimes \mathbf{L}_q[T] \\ C &\mapsto q^{\operatorname{dist}(C,KC)}\, C_1 \otimes C_2, \end{aligned}$$

*where $K$ is the vertex $S|T$ in $\Sigma[I]$, and the chambers $C_1$ and $C_2$ are defined by $b_K(KC) = (C_1, C_2)$.*

*The product is given by*

$$\begin{aligned} \mathbf{L}_q[S] \otimes \mathbf{L}_q[T] &\to \mathbf{L}_q[I] \\ C_1 \otimes C_2 &\mapsto j_K(C_1, C_2), \end{aligned}$$

*where $K$ is the vertex $S|T$ of $\Sigma[I]$.*

Proof. We need to show that this is equivalent to Definition 9.13. The equivalence between the product formulas is immediate. To see the equivalence between the coproduct formulas, the key observation is that

$$b_K(KC) = (C_1, C_2) \iff C_1 = C|_S \text{ and } C_2 = C|_T.$$

This follows from (10.64). The power of $q$ works out correctly because of the relation between the gallery metric and the Schubert statistic (10.94). □

Let $I = S_1 \sqcup \cdots \sqcup S_k$ be an ordered decomposition of $I$ into nonempty subsets $S_i$. Let $H = S_1|\cdots|S_k$ be the face corresponding to this ordered decomposition. Then the component $\Delta_{S_1,\dots,S_k}$ of the iterated coproduct is given by

$$\begin{aligned} \mathbf{L}_q[I] &\to \mathbf{L}_q[S_1] \otimes \cdots \otimes \mathbf{L}_q[S_k] \\ C &\mapsto q^{\operatorname{dist}(C,HC)}\, C_1 \otimes \cdots \otimes C_k, \end{aligned}$$

where $C_1, \dots, C_k$ are defined by $b_H(HC) = (C_1, \dots, C_k)$. Similarly, the component $\mu_{S_1,\dots,S_k}$ of the iterated product is given by

$$\begin{aligned} \mathbf{L}_q[S_1] \otimes \cdots \otimes \mathbf{L}_q[S_k] &\to \mathbf{L}_q[I] \\ C_1 \otimes \cdots \otimes C_k &\mapsto j_H(C_1, \dots, C_k). \end{aligned}$$

These formulas follow from the various compatibilities of the break and join map discussed in Section 10.11.2.

**Proposition 12.3.** *The antipode of $\mathrm{S}\colon \mathbf{L}_q \to \mathbf{L}_q$ is given by*

$$\begin{aligned} &\mathbf{L}_q[I] \to \mathbf{L}_q[I] \\ &\mathrm{S}_I(C) = (-1)^{\deg(C)} q^{\operatorname{dist}(C,\overline{C})}\, \overline{C}. \end{aligned}$$

This is a restatement of Proposition 9.14 in geometric terms: note that for any chamber $C$ in $\Sigma[I]$,

$$\deg(C) \;=\; |I|, \qquad \operatorname{dist}(C, \overline{C}) \;=\; \binom{|I|}{2}, \tag{12.7}$$

and if $C$ is indexed by the linear order $l$, then the opposite chamber $\overline{C}$ is indexed by the reversal $\bar{l}$. We repeat the proof to clarify the geometric ideas that go into the computation.

Proof. By the above formulas for the components of the iterated product and coproduct, the fact that break and join are inverses, and Takeuchi's formula (8.27), we obtain:

$$\begin{aligned} \mathrm{s}_I(C) &= \sum_H (-1)^{\deg(H)} q^{\operatorname{dist}(C,HC)}\, HC \\ &= \sum_D \Big( \sum_{H:HC=D} (-1)^{\deg(H)} \Big) q^{\operatorname{dist}(C,D)}\, D. \end{aligned}$$

It follows from Proposition 10.11 that the sum in parenthesis above is over a Boolean poset with minimal element $\operatorname{Des}(C,D)$ and maximal element $D$. The sum will be zero unless $\operatorname{Des}(C,D) = D$; this happens precisely when $D = \overline{C}$. $\square$

By dualizing the formulas, we obtain the following descriptions for $\mathbf{L}_q^*$.

**Proposition 12.4.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by*

$$\begin{aligned} &\mathbf{L}_q^*[I] \to \mathbf{L}_q^*[S] \otimes \mathbf{L}_q^*[T] \\ &D^* \mapsto \begin{cases} D_1^* \otimes D_2^* & \text{if } K = S|T \text{ is a vertex of } D, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

*where $D_1$ and $D_2$ are defined by $b_K(D) = (D_1, D_2)$.*

*The product is given by*

$$\begin{aligned} &\mathbf{L}_q^*[S] \otimes \mathbf{L}_q^*[T] \to \mathbf{L}_q^*[I] \\ &D_1^* \otimes D_2^* \mapsto \sum_{D:KD=j_K(D_1,D_2)} q^{\operatorname{dist}(KD,D)}\, D^*. \end{aligned}$$

*The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum.*

**Proposition 12.5.** *The antipode of $\mathrm{s}\colon \mathbf{L}_q^* \to \mathbf{L}_q^*$ is given by*

$$\begin{aligned} &\mathbf{L}_q^*[I] \to \mathbf{L}_q^*[I] \\ &\mathrm{s}_I(D^*) = (-1)^{\deg(D)} q^{\operatorname{dist}(D,\overline{D})}\, \overline{D}^*. \end{aligned}$$

Recall that the above parameters depend only on $|I|$ (12.7).

**12.2.2. Self-duality.** Recall that the Hopf monoid $\mathbf{L}$ is cocommutative but not commutative. Hence it cannot be self-dual. However, it turns out that in the generic case, $\mathbf{L}_q$ is self-dual. More precisely:

**Proposition 12.6.** *The map*

$$\mathbf{L}_q \to \mathbf{L}_q^* \qquad C \mapsto \sum_D q^{\operatorname{dist}(C,D)}\, D^*$$

*is a morphism of $q$-Hopf monoids. Further, if $q$ is not a root of unity, then it is an isomorphism, and in this case, the Hopf monoid $\mathbf{L}_q$ is self-dual.*

For $q = 0$, the $q$-coefficient is nonzero only if $C = D$. So, in this case, the map sends a chamber to itself (viewed in the dual) and is clearly an isomorphism.

PROOF. It can be directly checked that the above map is a morphism of monoids and comonoids. Both checks make use of (10.61). It then follows that the above map is a morphism of $q$-Hopf monoids. The next observation is that the above map is induced from the bilinear form on chambers given in (10.132). It is shown in Lemma 10.28 that if $q$ is not a root of unity, then this bilinear form is nondegenerate. The result follows. □

The above result is an instance of a much more general result given in Theorem 11.35; see the discussion in Example 11.36.

## 12.3. The $q$-Hopf monoid of pairs of chambers

In this section, we study the Hopf monoid $\mathbf{L}^*$ based on pairs of chambers in detail. This Hopf monoid is self-dual, free and cofree. We first describe $\mathbf{L}^*$ in the fundamental or $F$ basis in both combinatorial and geometric terms, and then do the same for the $M$ basis. The advantage of the $M$ basis it that it allows us to describe explicitly the coradical filtration, and hence the primitive elements of $\mathbf{L}^*$ and conclude that $\mathbf{L}^*$ is cofree. We also describe the antipode in both the $F$ and the $M$ basis.

The Hopf monoid $\mathbf{L}^*$ admits a one-parameter deformation to a $q$-Hopf monoid $\mathbf{L}^*_q$ meaning that $\mathbf{L}^* = \mathbf{L}^*_1$. To avoid repetition, we present everything directly for $\mathbf{L}^*_q$. On a first reading one may restrict attention to Hopf monoids and replace all occurrences of $q$ for 1.

In the geometric setting, it is convenient to describe the components of the product and coproduct if the decomposition is into nonempty subsets. The descriptions involve the break and join maps (10.57).

### 12.3.1. The $F$ and $K$ bases.

**Definition 12.7.** Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by

$$\begin{gathered}\mathbf{L}^*_q[I] \to \mathbf{L}^*_q[S] \otimes \mathbf{L}^*_q[T]\\ F_{(C,D)} \mapsto \begin{cases} F_{(C_1,D_1)} \otimes F_{(C_2,D_2)} & \text{if } K = S|T \text{ is a vertex of } D,\\ 0 & \text{otherwise,}\end{cases}\end{gathered}$$

where $C_1$, $C_2$, $D_1$ and $D_2$ are defined by $b_K(D) = (D_1, D_2)$ and $b_K(KC) = (C_1, C_2)$.

The product is given by

$$\begin{gathered}\mathbf{L}^*_q[S] \otimes \mathbf{L}^*_q[T] \to \mathbf{L}^*_q[I]\\ F_{(C_1,D_1)} \otimes F_{(C_2,D_2)} \mapsto \sum_{D:\, KD = j_K(D_1,D_2)} q^{\operatorname{dist}(KD,D)} F_{(j_K(C_1,C_2),D)}.\end{gathered}$$

The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum.

By using the definitions of the projection, break and join maps, one obtains the following combinatorial descriptions for the coproduct and product. We use the notations of Examples 8.16 and 8.24.

**Proposition 12.8.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by*

$$\mathbb{L}_q^*[I] \to \mathbb{L}_q^*[S] \otimes \mathbb{L}_q^*[T]$$
$$F_{(C,D)} \mapsto \begin{cases} F_{(C|_S,D|_S)} \otimes F_{(C|_T,D|_T)} & \text{if } S \text{ is an initial segment of } D, \\ 0 & \text{otherwise.} \end{cases}$$

*The product is given by*

$$\mathbb{L}_q^*[S] \otimes \mathbb{L}_q^*[T] \to \mathbb{L}_q^*[I]$$
$$F_{(C_1,D_1)} \otimes F_{(C_2,D_2)} \mapsto \sum_{D:D \text{ a shuffle of } D_1 \text{ and } D_2} q^{\mathrm{sch}_{S,T}(D)} F_{(C_1 \cdot C_2, D)},$$

*where $\mathrm{sch}_{S,T}(D)$ is the Schubert cocycle (9.12).*

For example,

$$F_{(t|a|s|i,s|i|t|a)} \mapsto 1 \otimes F_{(t|a|s|i,s|i|t|a)} + F_{(s,s)} \otimes F_{(t|a|i,i|t|a)}$$
$$+ F_{(s|i,s|i)} \otimes F_{(t|a,t|a)} + F_{(t|s|i,s|i|t)} \otimes F_{(a,a)} + F_{(t|a|s|i,s|i|t|a)} \otimes 1.$$
$$F_{(s|i,s|i)} \otimes F_{(t|a,a|t)} \mapsto F_{(s|i|t|a,s|i|a|t)} + q\, F_{(s|i|t|a,s|a|i|t)} + q^2\, F_{(s|i|t|a,a|s|i|t)}$$
$$+ q^2\, F_{(s|i|t|a,s|a|t|i)} + q^3\, F_{(s|i|t|a,a|s|t|i)} + q^4\, F_{(s|i|t|a,a|t|s|i)}.$$

Let $\mathbb{L}_q$ denote the Hopf dual of $\mathbb{L}_q^*$. Recall that $K$ denotes the basis of $\mathbb{L}_q$ dual to the $F$ basis of $\mathbb{L}_q^*$ (12.2). The product and coproduct on the $K$ basis are as follows.

**Proposition 12.9.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The product is given by*

$$\mathbb{L}_q[S] \otimes \mathbb{L}_q[T] \to \mathbb{L}_q[I]$$
$$K_{(D_1,C_1)} \otimes K_{(D_2,C_2)} \mapsto \sum_{D:KD = j_K(D_1,D_2)} K_{(D, j_K(C_1,C_2))}.$$

*The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum.*

*The coproduct is given by*

$$\mathbb{L}_q[I] \to \mathbb{L}_q[S] \otimes \mathbb{L}_q[T]$$
$$K_{(D,C)} \mapsto \begin{cases} q^{\mathrm{dist}(C,KC)} K_{(D_1,C_1)} \otimes K_{(D_2,C_2)} & \text{if } K = S|T \text{ is a vertex of } D, \\ 0 & \text{otherwise,} \end{cases}$$

*where $C_1, C_2, D_1$ and $D_2$ are defined by $\mathrm{b}_K(D) = (D_1, D_2)$ and $\mathrm{b}_K(KC) = (C_1, C_2)$.*

**12.3.2. Hadamard products and self-duality.** Recall that the Hadamard product of two bimonoids is again a bimonoid. It follows from the definitions that

$$\begin{array}{ccc} \mathbb{L}_q^* \xrightarrow{\cong} \mathbf{L} \times \mathbf{L}_q^* & \quad\text{and}\quad & \mathbb{L}_q \xrightarrow{\cong} \mathbf{L}^* \times \mathbf{L}_q \\ F_{(C,D)} \mapsto (C, D^*) & & K_{(D,C)} \mapsto (D^*, C) \end{array} \tag{12.8}$$

as $q$-Hopf monoids. One may also deduce one fact from the other by Corollary 8.62.

Applying the signature functor yields:

$$(\mathbb{L}_q)^- \cong \mathbb{L}_{-q} \qquad \text{and} \qquad (\mathbb{L}_q^*)^- \cong \mathbb{L}_{-q}^*.$$

In other words, the signed partners of $\mathbb{L}_q$ and $\mathbb{L}_q^*$ are $\mathbb{L}_{-q}$ and $\mathbb{L}_{-q}^*$ respectively. This follows from (12.8) and the corresponding results for the $q$-Hopf monoids of linear orders (9.20).

In view of (12.8), one may be tempted to consider $\mathbf{L}_p \times \mathbf{L}_q^*$ as a two parameter deformation of $\mathbb{L}^*$. However, as the results below show, this generalization is superficial; there is really only one parameter involved.

**Proposition 12.10.** *Let $q$ be a nonzero scalar, and $p$ be any scalar. The map*

$$\mathbf{L}_q \times \mathbf{L}_p^* \to \mathbf{L} \times \mathbf{L}_{pq}^* \qquad (C, D^*) \to q^{\operatorname{dist}(C,D)}\,(C, D^*)$$

*is an isomorphism of $pq$-Hopf monoids. In particular, as $0$-Hopf monoids, for $q \neq 0$,*

$$\mathbf{L}_q \times \mathbf{L}_0^* \cong \mathbf{L} \times \mathbf{L}_0^*,$$

*and hence, by duality,*

$$\mathbf{L}_0 \times \mathbf{L}_q^* \cong \mathbf{L}_0 \times \mathbf{L}^*.$$

Proof. The product and coproduct are preserved because of (10.61). $\square$

**Corollary 12.11.** *Let $p, q, p'$ and $q'$ be nonzero scalars such that $pq = p'q'$. Then*

$$\mathbf{L}_q \times \mathbf{L}_p^* \cong \mathbf{L}_{q'} \times \mathbf{L}_{p'}^*$$

*as $pq$-Hopf monoids.*

The above results show that by working with $\mathbb{L}_q^*$ instead of $\mathbf{L}_q \times \mathbf{L}_p^*$, the only object that gets left out is $\mathbf{L}_0 \times \mathbf{L}_0^*$.

Define a map $s_q \colon \mathbb{L}_q \to \mathbb{L}_q^*$ by

$$K_{(D,C)} \mapsto q^{\operatorname{dist}(C,D)}\, F_{(C,D)}. \tag{12.9}$$

We refer to $s_q$ as the *switch map* on the $q$-Hopf monoid of pairs of linear orders. Setting $p = 1$ in Proposition 12.10 and using (12.8) we deduce:

**Proposition 12.12.** *For $q \neq 0$, the switch map $s_q$ is an isomorphism of $q$-Hopf monoids. Hence, $\mathbb{L}_q$ is self-dual if $q \neq 0$.*

The self-duality of $\mathbb{L}$ (which is the case $q = 1$) follows more simply from Corollary 8.61. It is also true that the 0-Hopf monoid $\mathbb{L}_0$ is self-dual. This point is addressed in Section 12.3.6.

### 12.3.3. The $M$ and $H$ bases.

Recall that the $M$ and $F$ bases are related by (12.3). The coproduct and product on the $M$ basis are given by the following formulas.

**Theorem 12.13.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by*

$$\begin{gathered}\mathbb{L}_q^*[I] \to \mathbb{L}_q^*[S] \otimes \mathbb{L}_q^*[T] \\ M_{(C,D)} \mapsto \begin{cases} M_{(C_1,D_1)} \otimes M_{(C_2,D_2)} & \text{if } K = S|T \text{ is a vertex of } D \text{ and } \overline{C}, \\ 0 & \text{otherwise,}\end{cases}\end{gathered}$$

*where $\overline{C}$ denotes the chamber opposite to $C$, and $C_1, C_2, D_1$ and $D_2$ are defined by $b_K(D) = (D_1, D_2)$ and $b_K(KC) = (C_1, C_2)$, or equivalently, $b_{\overline{K}}(C) = (C_2, C_1)$.*

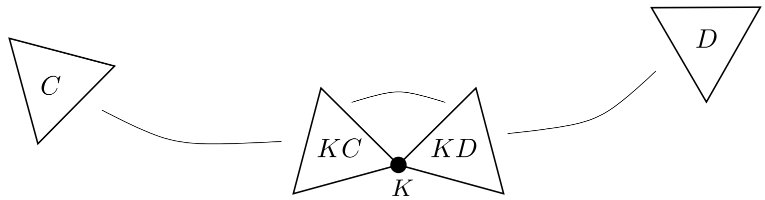



FIGURE 12.1. A mimimal gallery from $C$ to $D$ via the vertex $K$.

*The product is given by*

$$\begin{gathered}\mathbb{L}_q^*[S] \otimes \mathbb{L}_q^*[T] \to \mathbb{L}_q^*[I] \\ M_{(C_1,D_1)} \otimes M_{(C_2,D_2)} \mapsto \sum_{\substack{(C,D): \\ KD=j_K(D_1,D_2) \\ KC=j_K(C_1,C_2) \\ C-KC-KD-D}} q^{\operatorname{dist}(KD,D)}\, M_{(C,D)}.\end{gathered}$$

*The vertex $K = S|T$ is fixed in the above sum.*

The proof is similar to those of [12, Theorems 7.3.1 and 7.3.4].

The condition $C - KC - KD - D$ says that there is a mimimal gallery from $C$ to $D$, which passes through $KC$ and $KD$. This condition is illustrated in Figure 12.1, using the notation of Figure 10.8. Roughly, it means that the vertex $K$ lies in between the chambers $C$ and $D$.

We now write down the formulas on the $M$ basis in combinatorial terms.

**Theorem 12.14.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by*

$$\begin{gathered}\mathbb{L}_q^*[I] \to \mathbb{L}_q^*[S] \otimes \mathbb{L}_q^*[T] \\ M_{(C,D)} \mapsto \begin{cases} M_{(C|_S, D|_S)} \otimes M_{(C|_T, D|_T)} & \text{if } S \text{ is an initial segment of } D \text{ and} \\ & \text{a final segment of } C, \\ 0 & \text{otherwise.}\end{cases}\end{gathered}$$

*The product is given by*

$$\begin{gathered}\mathbb{L}_q^*[S] \otimes \mathbb{L}_q^*[T] \to \mathbb{L}_q^*[I] \\ M_{(C_1,D_1)} \otimes M_{(C_2,D_2)} \mapsto \sum q^{\operatorname{sch}_{S,T}(D)}\, M_{(C,D)},\end{gathered}$$

*where the sum is over all $(C, D)$ such that $C$ is a shuffle of $C_1$ and $C_2$, $D$ is a shuffle of $D_1$ and $D_2$, and an element of $T$ cannot precede an element of $S$ in both $C$ and $D$.*

For example,

$$M_{(t|a|s|i,s|i|t|a)}) \mapsto 1 \otimes M_{(t|a|s|i,s|i|t|a)} + M_{(s|i,s|i)} \otimes M_{(t|a,t|a)} + M_{(t|a|s|i,s|i|t|a)} \otimes 1.$$

$$\begin{aligned} M_{(t|a,a|t)} \otimes M_{(i,i)} \mapsto {} & M_{(t|a|i,a|t|i)} + M_{(t|i|a,a|t|i)} + M_{(i|t|a,a|t|i)} \\ & + q\, M_{(t|a|i,a|i|t)} + q\, M_{(t|i|a,a|i|t)} + q^2\, M_{(t|a|i,i|a|t)}.\end{aligned}$$

Recall that $H$ denotes the basis of $\mathbb{L}_q$ dual to the $M$ basis of $\mathbb{L}_q^*$ (12.1). Dualizing the product and coproduct in Theorem 12.13, one obtains:

**Theorem 12.15.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The product is given by*

$$\begin{gathered}\mathbb{L}_q[S] \otimes \mathbb{L}_q[T] \to \mathbb{L}_q[I] \\ H_{(D_1,C_1)} \otimes H_{(D_2,C_2)} \mapsto H_{\left(j_{\overline{K}}(D_2,D_1),\, j_K(C_1,C_2)\right)},\end{gathered}$$

*where $K = S|T$ and $\overline{K} = T|S$.*

*The coproduct is given by*

$$\begin{aligned}&\mathbb{L}_q[I] \to \mathbb{L}_q[S] \otimes \mathbb{L}_q[T] \\ &H_{(D,C)} \mapsto \begin{cases} q^{\operatorname{dist}(C,KC)}\, H_{(D_1,C_1)} \otimes H_{(D_2,C_2)} & \text{if } D - KD - KC - C, \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

*where the chambers $C_1, C_2, D_1$ and $D_2$ are defined by $b_K(KD) = (D_1, D_2)$ and $b_K(KC) = (C_1, C_2)$.*

**12.3.4. Interchanging the coordinates on the $M$ and $H$ bases.** Assume now that $q \neq 0$. Consider the map

$$t_q \colon \mathbb{L}_q^* \to \mathbb{L}_{q^{-1}}^* \qquad M_{(C,D)} \mapsto q^{-\operatorname{dist}(C,D)} M_{(D,C)}. \tag{12.10}$$

The dual map is given by

$$t_q^* \colon \mathbb{L}_{q^{-1}} \to \mathbb{L}_q \qquad H_{(D,C)} \mapsto q^{-\operatorname{dist}(C,D)}\, H_{(C,D)}.$$

These maps are not to be confused with the switch map $s_q$ (12.9). In fact, we show below that $t_q$ reverses coproducts, while $s_q$ preserves them.

Recall that for any Hopf monoid $H$, one can define Hopf monoids ${}^{\mathrm{op}}H$ and $H^{\mathrm{cop}}$ by twisting the product and coproduct on $H$ by the braiding, see Proposition 1.21.

**Corollary 12.16.** *The map $t_q \colon \mathbb{L}_q^* \to (\mathbb{L}_{q^{-1}}^*)^{\mathrm{cop}}$ is an isomorphism of $q$-Hopf monoids. Dually, the map $t_q^* \colon {}^{\mathrm{op}}(\mathbb{L}_{q^{-1}}) \to \mathbb{L}_q$ is an isomorphism of $q$-Hopf monoids.*

Proof. This follows directly from the symmetry of the two coordinates in the $M$ basis formulas in Theorem 12.13. The powers of $q$ are controlled by formula (10.61) and the interaction of the distance metric with the braiding. □

**12.3.5. The coradical filtration.** We now explicitly describe the species of primitive elements and more generally the species $\mathcal{P}^{(k)}(\mathbb{L}_q^*)$ (Section 11.9.4). This notion pertains only to the comonoid structure and hence the parameter $q$ does not play any part in this discussion.

Given the factors $M_{(C_1,D_1)}$ and $M_{(C_2,D_2)}$ of the component $\Delta_{S,T}$ of the coproduct on the $M$ basis, one can uniquely recover $M_{(C,D)}$ by

$$D = j_K(D_1, D_2) \qquad \text{and} \qquad C = \overline{K} j_K(C_1, C_2)$$

where $K = S|T$. Equivalently,

$$D = D_1 \cdot D_2 \qquad \text{and} \qquad C = C_2 \cdot C_1,$$

where $\cdot$ stands for concatenation of linear orders. This shows that

$$\{M_{(C,D)} \mid \dim(D \wedge \overline{C}) \leq k-2\}$$

is a linear basis of the species $\mathcal{P}^{(k)}(\mathbb{L}_q^*)$. The condition on $(C, D)$ can also be phrased in terms of global descents (Definition 10.9). The case $k = 1$ yields the primitive element species;

$$\{M_{(C,D)} \mid D \wedge \overline{C} = \emptyset\} \tag{12.11}$$

is a linear basis. The dimensions of the first 3 components of $\mathcal{P}(\mathbb{L}_q^*)$ are 1, 2, and 18.

**12.3.6. 0-Hopf monoids.** We now briefly discuss the connected 0-Hopf monoids $\mathbb{L}_0^*$ and $\mathbb{L}_0$. In this discussion, $K$ denotes the vertex $S|T$. Recall from (12.8) that

$$\mathbb{L}_0^* \xrightarrow{\cong} \mathbf{L} \times \mathbf{L}_0^* \quad \text{and} \quad \mathbb{L}_0 \xrightarrow{\cong} \mathbf{L}^* \times \mathbf{L}_0.$$

The coproduct of $\mathbb{L}_0^*$ is as for $\mathbb{L}_q^*$ (since the coproduct of $\mathbb{L}_q^*$ does not depend on $q$). The product is given by

$$\begin{aligned} \mathbb{L}_0^*[S] \otimes \mathbb{L}_0^*[T] &\to \mathbb{L}_0^*[I] \\ F_{(C_1,D_1)} \otimes F_{(C_2,D_2)} &\mapsto F_{(j_K(C_1,C_2), j_K(D_1,D_2))} \\ M_{(C_1,D_1)} \otimes M_{(C_2,D_2)} &\mapsto \sum_{C: KC = j_K(C_1,C_2)} M_{(C, j_K(D_1,D_2))}. \end{aligned}$$

Dually, the product of $\mathbb{L}_0$ is the same as for $\mathbb{L}_q$. The coproduct is given by

$$\begin{aligned} &\mathbb{L}_0[I] \to \mathbb{L}_0[S] \otimes \mathbb{L}_0[T] \\ &K_{(D,C)} \mapsto \begin{cases} K_{(D_1,C_1)} \otimes K_{(D_2,C_2)} & \text{if } K \text{ is a vertex of } D \text{ and } C, \\ 0 & \text{otherwise,} \end{cases} \\ &H_{(D,C)} \mapsto \begin{cases} H_{(D_1,C_1)} \otimes H_{(D_2,C_2)} & \text{if } K \text{ is a vertex of } C, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where the chambers $C_1, C_2, D_1$ and $D_2$ are defined by $b_K(KD) = (D_1, D_2)$ and $b_K(C) = (C_1, C_2)$.

It follows from these formulas that the map

$$\mathbb{L}_0 \to \mathbb{L}_0^*, \quad K_{(D,C)} \mapsto M_{(\overline{C},D)}$$

is an isomorphism of comonoids, and the map

$$\mathbb{L}_0 \to \mathbb{L}_0^*, \quad K_{(D,C)} \mapsto M_{(D,C)}$$

is an isomorphism of monoids. However, neither map is an isomorphism of Hopf monoids. It turns out that these 0-Hopf monoids are nevertheless isomorphic, as we now explain.

Recall that connected 0-Hopf monoids are determined by the subspecies of primitive elements (Theorem 11.49). It is clear that

$$\{K_{(D,C)} \mid D \wedge C = \emptyset\} \tag{12.12}$$

is a linear basis for the primitive element species of $\mathbb{L}_0$. The coproduct of $\mathbb{L}_q^*$ does not depend on $q$, so (12.11) tells us that

$$\{M_{(C,D)} \mid D \wedge \overline{C} = \emptyset\}$$

is a linear basis for the primitive element species of $\mathbb{L}_0^*$. It follows that there is an isomorphism of species

$$\mathcal{P}(\mathbb{L}_0) \to \mathcal{P}(\mathbb{L}_0^*) \qquad K_{(D,C)} \mapsto M_{(\overline{C},D)}.$$

In view of Theorem 11.49, we can now deduce that the 0-Hopf monoids $\mathbb{L}_0$ and $\mathbb{L}_0^*$ are isomorphic, and hence self-dual. This complements the result of Proposition 12.12.

We briefly consider the 0-Hopf monoid $\mathbf{L}_0 \times \mathbf{L}_0^*$. (Recall from Section 12.3.2 that this object is not of the form $\mathbb{L}_q$ or $\mathbb{L}_q^*$ for any $q$.) The definition implies that the product is concatenation in both coordinates and the coproduct is deconcatenation in both coordinates. In particular, the primitive element species has a linear basis as in (12.12). Again applying Theorem 11.49, we conclude that $\mathbf{L}_0 \times \mathbf{L}_0^*$ is isomorphic to $\mathbb{L}_0$ and $\mathbb{L}_0^*$.

**12.3.7. The antipode on the *F* and *M* basis.** Let $\deg(F)$ be the number of blocks in the set composition $F$. In other words, we have $\deg(F) = \dim(F) + 2$.

**Theorem 12.17.** *The antipode of* $\mathrm{s}\colon \mathbb{L}_q^* \to \mathbb{L}_q^*$ *on the* $F$ *basis is given by*

$$\mathbb{L}_q^*[I] \to \mathbb{L}_q^*[I]$$
$$\mathrm{s}_I(F_{(C,D)}) = \sum (-1)^{\deg(C' \wedge D)} q^{\mathrm{dist}(D,D')} F_{(C',D')},$$

*where the sum is over the pairs* $(C', D')$ *such that there is a unique face* $H$ *with* $HC = C'$ *and* $HD' = D$. *(In this case, the face* $H$ *must equal* $C' \wedge D$*).*

Figure 12.2 illustrates this situation. The nice symmetry in the description of the face $H$ given above, which is manifest in the figure, underlies the fact that, up to switching the coordinates, the dual of the above antipode formula is itself. In this figure, the two instances of $\overline{H}$ represent the same face.

Proof. We will use Takeuchi's formula (8.27) to prove the result. As a first step, we compute $\mu_{S,T}\Delta_{S,T}$ using the geometric Definition 12.7. Since the break and join maps are inverses, we have,

$$\mu_{S,T}\big(\Delta_{S,T}(F_{(C,D)})\big) = \sum_{(C',D')} q^{\mathrm{dist}(D,D')} F_{(C',D')},$$

where the sum is over all $(C', D')$ which for $K = S|T$ satisfy $KC = C'$ and $KD' = D$. The same formula holds for all the iterated composites, namely for

$$\mu_{S_1,\dots,S_k}\Delta_{S_1,\dots,S_k},$$

the sum is over all $(C', D')$ which for $H = S_1|\cdots|S_k$ satisfy $HC = C'$ and $HD' = D$. Therefore by Takeuchi's formula, we obtain:

$$\mathrm{s}_I(F_{(C,D)}) = \sum_{(C',D')} \Bigg( \sum_{H \in A_{((C,D),(C',D'))}} (-1)^{\deg(H)} \Bigg) q^{\mathrm{dist}(D,D')} F_{(C',D')}, \tag{12.13}$$

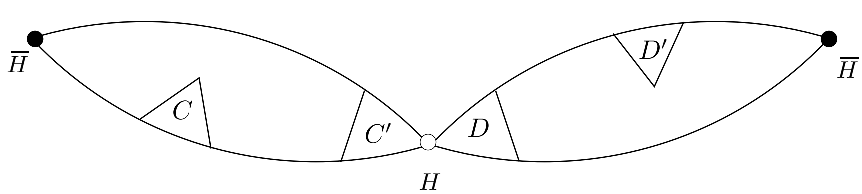



Figure 12.2. The antipode on the $F$ basis of $\mathbb{L}_q^*$.

where

$$A_{((C,D),(C',D'))} = \{H \mid HC = C',\ HD' = D\}.$$

We make two simple observations about this set, which we abbreviate to $A$ from now on. These follow from properties (v) and (vi) of the projection product in Proposition 10.1.

- If $H \in A$ and $H \leq G \leq C' \wedge D$, then $G \in A$.
- If $H_1, H_2 \in A$, then $H_1 \wedge H_2 \in A$.

In particular, if $A$ is nonempty, then $C' \wedge D \in A$. This shows that the set $A$ is a Boolean poset. So the alternating sum inside the parenthesis above will be zero unless $A$ consists of the singleton element $C' \wedge D$. This finishes the proof. □

The antipode formula given by (12.13) is also useful for working out specific examples and can be rewritten as follows.

$$\mathrm{s}_I(F_{(C,D)}) = \sum_{H:H\leq D}\ \sum_{D':HD'=D} (-1)^{\deg(H)} q^{\mathrm{dist}(D,D')}\, F_{(HC,D')}.$$

For example, using this formula,

$$\begin{aligned}
\mathrm{s}(F_{(u|m|a,m|a|u)}) = -F_{(u|m|a,m|a|u)}& \\
&+ F_{(m|a|u,m|a|u)} + q\, F_{(m|a|u,m|u|a)} + q^2\, F_{(m|a|u,u|m|a)} \\
&+ F_{(m|u|a,m|a|u)} + q\, F_{(m|u|a,a|m|u)} + q^2\, F_{(m|u|a,a|u|m)} \\
&- F_{(m|a|u,m|a|u)} - q\, F_{(m|a|u,m|u|a)} - q^2\, F_{(m|a|u,u|m|a)} \\
&- q\, F_{(m|a|u,a|m|u)} - q^2\, F_{(m|a|u,a|u|m)} - q^3\, F_{(m|a|u,u|a|m)}.
\end{aligned}$$

The above proof showed that terms which repeat must necessarily cancel. Thus we have

$$\begin{aligned}
\mathrm{s}(F_{(u|m|a,m|a|u)}) = -F_{(u|m|a,m|a|u)} + F_{(m|u|a,m|a|u)} + qF_{(m|u|a,a|m|u)} + q^2F_{(m|u|a,a|u|m)}& \\
- qF_{(m|a|u,a|m|u)} - q^2\, F_{(m|a|u,a|u|m)} - q^3\, F_{(m|a|u,u|a|m)}&.
\end{aligned}$$

This is the formula provided by Theorem 12.17, where all cancellations have been accounted for. The coefficient of any term, up to a power of $q$, is either 0, 1 or $-1$.

**Theorem 12.18.** *The antipode of* $\mathrm{s}\colon \mathbf{L}^*_q \to \mathbf{L}^*_q$ *on the* $M$ *basis is given by*

$$\begin{gathered}
\mathbf{L}^*_q[I] \to \mathbf{L}^*_q[I] \\
\mathrm{s}_I(M_{(C,D)}) = (-1)^{\deg(\overline{C}\wedge D)} \sum q^{\mathrm{dist}(D,D')}\, M_{(C',D')},
\end{gathered}$$

*where the sum is over all pairs* $(C', D')$ *such that* $C' - D - D'$ *and such that there is a unique face* $H$ *with* $\overline{H}C' = C$ *and* $HD' = D$. *(In this case, the face* $H$ *must equal* $\overline{C} \wedge D$ *itself).*

Figure 12.3 illustrates this situation. It does not show the symmetry of Figure 12.2, because the relation between the $M$ and its dual $H$ basis is more complicated than simply switching the coordinates. We point out that $HC$ is not $\overline{C}$. In fact, these two chambers are opposite to each other in the star of $H$.

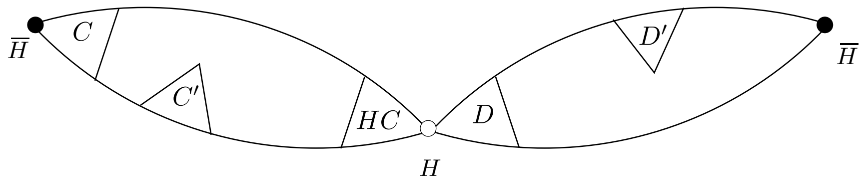



FIGURE 12.3. The antipode on the $M$ basis of $\mathbf{L}^*_q$.

PROOF. The proof is along the lines of the proof of Theorem 12.17 and identical in complexity; hence we will be brief. By the geometric formulations for the product and coproduct given by Theorem 12.13, and Takeuchi's formula (8.27), we obtain:

$$\mathrm{s}_I(M_{(C,D)}) = \sum_{(C',D'):C'-D-D'} \Bigg( \sum_{H \in B_{((C,D),(C',D'))}} (-1)^{\deg(H)} \Bigg) q^{\operatorname{dist}(D,D')}\, M_{(C',D')},$$

where

$$B_{((C,D),(C',D'))} = \{H \mid H \leq \overline{C} \wedge D,\ \overline{H}C' = C,\ HD' = D\}.$$

We make two simple observations about this set, which we abbreviate to $B$ from now on. These follow from properties (v) and (vi) of the projection product in Proposition 10.1.

- If $H \in B$ and $H \leq G \leq \overline{C} \wedge D$, then $G \in B$.
- If $H_1, H_2 \in B$, then $H_1 \wedge H_2 \in B$.

In particular if $B$ is nonempty, then $\overline{C} \wedge D \in B$. This shows that the set $B$ is a Boolean poset. So the alternating sum inside the parenthesis above will be zero unless $B$ consists of the singleton element $\overline{C} \wedge D$. Since the term $\overline{C} \wedge D$ does not depend on $(C', D')$, we can pull the sign out of the sum. This finishes the proof. $\square$

For example,

$$\begin{aligned}\mathrm{s}(M_{(r|i|h|a,h|a|r|i)}) &= M_{(r|h|i|a,h|a|r|i)} + M_{(h|r|i|a,h|a|r|i)} + M_{(r|h|a|i,h|a|r|i)} \\ &+ M_{(h|r|a|i,h|a|r|i)} + M_{(h|a|r|i,h|a|r|i)} + q\, M_{(h|a|r|i,h|r|a|i)} + q^2\, M_{(h|a|r|i,r|h|a|i)} \\ &\qquad + q^2\, M_{(h|a|r|i,h|r|i|a)} + q^3\, M_{(h|a|r|i,r|h|i|a)} + q^4\, M_{(h|a|r|i,r|i|h|a)}.\end{aligned}$$

In this case, $H = ha|ri$, $C = r|i|h|a$ and $D = h|a|r|i$; so $HC = D$ (this will not happen in general). Hence to satisfy $C' - D - D'$, either $C' = D$ or $D' = D$. This explains why either the first or the second coordinate of each term on the right is $h|a|r|i$.

Antipode formulas on the $K$ and $H$ bases of $\mathbf{L}_q$ can be derived by duality (12.2).

## 12.4. The $q$-Hopf monoids of faces

In this section, we study the Hopf monoids $\mathbf{\Sigma}^*$ and $\mathbf{\Sigma}$ which are both based on set compositions (or faces) and dual to each other. We write $M$ for the basis of $\mathbf{\Sigma}^*$ and $H$ for the dual basis of $\mathbf{\Sigma}$, in agreement with Notation 12.1.

These Hopf monoids admit one-parameter deformations to $q$-Hopf monoids which we denote by $\mathbf{\Sigma}^*_q$ and $\mathbf{\Sigma}_q$. This means that $\mathbf{\Sigma}^* = \mathbf{\Sigma}^*_1$ and $\mathbf{\Sigma} = \mathbf{\Sigma}_1$. The deformed objects $\mathbf{\Sigma}^*_q$ and $\mathbf{\Sigma}_q$ are dual $q$-Hopf monoids. It is convenient to present the theory directly for these $q$-Hopf monoids.

There are a number of operations on set compositions such as concatenation, restriction, shuffle, and quasi-shuffle (Section 10.1.6) which are relevant to this section. On the geometric side, we make use of the break and join maps (10.57).

### 12.4.1. The $M$ basis.

**Definition 12.19.** Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by

$$\begin{aligned} \mathbf{\Sigma}_q^*[I] &\to \mathbf{\Sigma}_q^*[S] \otimes \mathbf{\Sigma}_q^*[T] \\ M_G &\mapsto \begin{cases} M_{G_1} \otimes M_{G_2} & \text{if } K = S|T \text{ is a vertex of } G, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $b_K(G) = (G_1, G_2)$.

The product is given by

$$\begin{aligned} \mathbf{\Sigma}_q^*[S] \otimes \mathbf{\Sigma}_q^*[T] &\to \mathbf{\Sigma}_q^*[I] \\ M_{G_1} \otimes M_{G_2} &\mapsto \sum_{G:KG=j_K(G_1,G_2)} q^{\operatorname{dist}(K,G)} M_G, \end{aligned}$$

where the vertex $K = S|T$ is fixed, and $\operatorname{dist}(K, G)$ is as in (10.33).

We now give a combinatorial formulation of the above definitions. Let $G$ be a composition of $I$. The subset $S$ is called an initial segment of $G$ if it is the union of the first few blocks of $G$.

**Proposition 12.20.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by deconcatenation, namely*

$$\begin{aligned} \mathbf{\Sigma}_q^*[I] &\to \mathbf{\Sigma}_q^*[S] \otimes \mathbf{\Sigma}_q^*[T] \\ M_G &\mapsto \begin{cases} M_{G|_S} \otimes M_{G|_T} & \textit{if } S \textit{ is an initial segment of } G, \\ 0 & \textit{otherwise,} \end{cases} \end{aligned}$$

*with the restriction $G|_S$ as in Section 10.1.6.*

*The product is given by quasi-shuffling, namely*

$$\begin{aligned} \mathbf{\Sigma}_q^*[S] \otimes \mathbf{\Sigma}_q^*[T] &\to \mathbf{\Sigma}_q^*[I] \\ M_{G_1} \otimes M_{G_2} &\mapsto \sum_{G:G \textit{ a quasi-shuffle of } G_1 \textit{ and } G_2} q^{\operatorname{sch}_{S,T}(G)} M_G, \end{aligned}$$

*where $\operatorname{sch}_{S,T}(G)$ is the Schubert cocycle on faces (10.119).*

For example,

$$\begin{aligned} M_{vi|sh|nu} &\mapsto 1 \otimes M_{vi|sh|nu} + M_{vi} \otimes M_{sh|nu} + M_{vi|sh} \otimes M_{nu} + M_{vi|sh|nu} \otimes 1 \\ M_{la|ksh} \otimes M_{mi} &\mapsto M_{la|ksh|mi} + q^6 M_{la|mi|ksh} + q^{10} M_{mi|la|ksh} \\ &\qquad + M_{la|kshmi} + q^6 M_{lami|ksh}. \end{aligned}$$

The coradical filtration of $\mathbf{\Sigma}_q^*$ can be readily described using the coproduct formula as follows. This discussion does not depend on the parameter $q$ and is a special case of that in Section 11.9.4.

Given the factors $M_{G_1}$ and $M_{G_2}$ of the component $\Delta_{S,T}$ of the coproduct, one can uniquely recover $M_G$ by

$$G = j_K(G_1, G_2), \quad \text{or equivalently,} \quad G = G_1 \cdot G_2,$$

where $K = S|T$ and $\cdot$ denotes concatenation of faces. This shows that $\mathcal{P}^{(k)}(\mathbf{\Sigma}_q^*)$ is the species spanned by

$$\{M_G \mid \dim(G) \leq k-2\}.$$

The case $k = 1$ yields the primitive element species. It is spanned by the empty faces, or equivalently, the one-block compositions; so it is one-dimensional in each component. In other words, the primitive element species is $\mathbf{E}_+$.

**Theorem 12.21.** *The antipode* $\mathrm{s}\colon \mathbf{\Sigma}_q^* \to \mathbf{\Sigma}_q^*$ *is given by*

$$\mathbf{\Sigma}_q^*[I] \to \mathbf{\Sigma}_q^*[I]$$
$$\mathrm{s}_I(M_G) = (-1)^{\deg(G)} \sum_{F:F\leq \overline{G}} q^{\operatorname{dist}(\overline{F},F)}\, M_F,$$

*where* $\deg(G)$ *is the number of blocks in* $G$, *and* $\overline{G}$ *denotes the opposite of* $G$, *and* $\operatorname{dist}(G,F)$ *is as in* (10.33).

The above result is an instance of a much more general antipode formula given in Theorem 11.39: set $\mathbf{q} = \mathbf{E}_+$. The proof is briefly indicated below. It follows the proof pattern of the antipode formulas for $\mathbf{L}_q^*$ given in Section 12.3.7.

Proof. By the geometric Definition 12.19 for the product and coproduct and Takeuchi's formula (8.27), we obtain:

$$\mathrm{s}_I(M_G) = \sum_F \Bigl(\sum_{H:HF=G} (-1)^{\deg(H)}\Bigr) q^{\operatorname{dist}(G,F)}\, M_F.$$

The result now follows by applying Lemma 11.37 and noting that $F \leq \overline{G}$ implies $\operatorname{dist}(G,F) = \operatorname{dist}(\overline{F},F)$. $\square$

For example,

$$\mathrm{s}\bigl(M_{mi|ksh|la}\bigr) = -M_{lakshmi} - q^{10}\, M_{laksh|mi} - q^{10}\, M_{la|kshmi} - q^{16}\, M_{la|ksh|mi}.$$

**12.4.2. The $H$ basis.** Recall that $H$ denotes the basis of $\mathbf{\Sigma}_q$ dual to the $M$ basis of $\mathbf{\Sigma}_q^*$ (12.1). We now state the formulas for the coproduct, product and antipode on the $H$ basis of $\mathbf{\Sigma}_q$ obtained by duality from the formulas on the $M$ basis.

**Proposition 12.22.** *Fix a decomposition* $I = S \sqcup T$ *into nonempty subsets. The coproduct is given by*

$$\mathbf{\Sigma}_q[I] \to \mathbf{\Sigma}_q[S] \otimes \mathbf{\Sigma}_q[T]$$
$$H_F \mapsto q^{\operatorname{dist}(F,K)}\, H_{F_1} \otimes H_{F_2},$$

*where* $K$ *is the vertex* $S|T$, *and* $F_1$ *and* $F_2$ *are defined by* $b_K(KF) = (F_1, F_2)$, *and* $\operatorname{dist}(K,F)$ *is as in* (10.33).

*The product is given by*

$$\mathbf{\Sigma}_q[S] \otimes \mathbf{\Sigma}_q[T] \to \mathbf{\Sigma}_q[I]$$
$$H_{F_1} \otimes H_{F_2} \mapsto H_{j_K(F_1,F_2)}$$

*where the vertex* $K$ *is defined to be* $S|T$.

We now formulate this in combinatorial terms.

**Proposition 12.23.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by dequasi-shuffling; namely,*

$$\begin{aligned} \mathbf{\Sigma}_q[I] &\to \mathbf{\Sigma}_q[S] \otimes \mathbf{\Sigma}_q[T] \\ H_F &\mapsto q^{\mathrm{sch}_{S,T}(F)}\, H_{F|_S} \otimes H_{F|_T} \end{aligned}$$

*where the restriction $F|_S$ is as in Section 10.1.6 and $\mathrm{sch}_{S,T}(F)$ is the Schubert cocycle on faces (10.119).*

*The product is given by concatenation, namely*

$$\begin{aligned} \mathbf{\Sigma}_q[S] \otimes \mathbf{\Sigma}_q[T] &\to \mathbf{\Sigma}_q[I] \\ H_{F_1} \otimes H_{F_2} &\mapsto H_{F_1 \cdot F_2}. \end{aligned}$$

For example,

$$H_{la|ksh} \otimes H_{mi} \mapsto H_{la|ksh|mi}$$

$$\begin{aligned} H_{u|ma} \mapsto 1 \otimes H_{u|ma} + H_u \otimes H_{ma} + q\, H_m \otimes H_{u|a} + q\, H_a \otimes H_{u|m} \\ + H_{u|m} \otimes H_a + H_{u|a} \otimes H_m + q^2\, H_{ma} \otimes H_u + H_{u|ma} \otimes 1. \end{aligned}$$

**Theorem 12.24.** *The antipode $\mathrm{s}\colon \mathbf{\Sigma}_q \to \mathbf{\Sigma}_q$ is given by*

$$\begin{aligned} \mathbf{\Sigma}_q[I] &\to \mathbf{\Sigma}_q[I] \\ \mathrm{s}_I(H_F) &= q^{\mathrm{dist}(F,\overline{F})} \sum_{G:\, \overline{F} \leq G} (-1)^{\deg(G)}\, H_G. \end{aligned}$$

For example,

$$\mathrm{s}\big(H_{ma|u}\big) = q^2\, H_{u|ma} - q^2\, H_{u|m|a} - q^2\, H_{u|a|m}.$$

The above result is an instance of a much more general antipode formula given in Theorem 11.38: set $\mathbf{q} = \mathbf{E}^*_+$. It is instructive to compare the proof of that theorem with the proof that we gave for Theorem 12.21 (which is the dual case).

**12.4.3. Deformation via the Schubert cocycle on faces.** Consider the Hopf monoid $\mathbf{\Sigma}$; this is the case $q = 1$ in the preceding discussion. The components of the product and coproduct preserve the $H$-basis elements of $\mathbf{\Sigma}$. So $\mathbf{\Sigma}$ is a linearized bimonoid in the sense of Section 8.7.3.

**Proposition 12.25.** *The Schubert cocycle on faces defines a normal 2-cocycle on the linearized comonoid $\mathbf{\Sigma}$. Moreover, this cocycle is multiplicative of twist 1.*

This is a generalization of Proposition 9.24; the proof is along similar lines: Normality (9.24) follows from (10.120), and multiplicativity (9.33) (with $m = 1$) follows from (10.124). It follows that $\mathbf{\Sigma}_q$ is a bimonoid deformation of $\mathbf{\Sigma}$ in the sense of Proposition 9.21.

**12.4.4. Self-duality.**

**Proposition 12.26.** *For $q$ not an algebraic number, the Hopf monoid $\mathbf{\Sigma}_q$ is self-dual. The isomorphism to its dual is given by*

$$\mathbf{\Sigma}_q \to \mathbf{\Sigma}_q^* \qquad H_F \mapsto \sum_G (FG)!\, q^{\mathrm{dist}(F,G)}\, M_G,$$

*where $(FG)!$ is as in (10.7) and $\mathrm{dist}(F,G)$ is as in (10.33).*

PROOF. It can be directly checked that the above map is a morphism of monoids and comonoids. Hence it is a morphism of $q$-Hopf monoids. We next note that the bilinear form on faces which induces the above map coincides with the bilinear form in Lemma 10.37. The result follows. $\square$

The signed partner of $\mathbf{\Sigma}_q$ is not $\mathbf{\Sigma}_{-q}$. Similarly, the signed partner of $\mathbf{\Sigma}^*_q$ is not $\mathbf{\Sigma}^*_{-q}$. In fact, $\mathbf{\Sigma}_{-1}$ and $\mathbf{\Sigma}^-$ are not even isomorphic as species (this does not depend on $q$). For example, $\mathbf{\Sigma}_{-1}[2]$ is three-dimensional and contains 2 trivial representations and one sign representation of $\mathrm{S}_2$; so clearly, it cannot be isomorphic to $\mathbf{\Sigma}^-[2]$, which displays the opposite feature.

**12.4.5. 0-Hopf monoids.** We now briefly discuss the connected 0-Hopf monoids $\mathbf{\Sigma}^*_0$ and $\mathbf{\Sigma}_0$.

The coproduct of $\mathbf{\Sigma}^*_0$ is the same as given in Definition 12.19 (it does not depend on $q$). The product is given by

$$\begin{aligned}
\mathbf{\Sigma}^*_0[S] \otimes \mathbf{\Sigma}^*_0[T] &\to \mathbf{\Sigma}^*_0[I] \\
M_{G_1} \otimes M_{G_2} &\mapsto \sum_{G: KG = j_K(G_1, G_2) = GK} M_G,
\end{aligned}$$

where the vertex $K = S|T$ is fixed. This follows from (10.36). Explicitly, the sum consists of exactly two summands: $G$ is either the concatenation of $G_1$ and $G_2$, or it is the concatenation with the last block of $G_1$ merged with the first block of $G_2$.

For example,

$$M_{la|ksh} \otimes M_{mi} \mapsto M_{la|ksh|mi} + M_{la|kshmi}.$$

This can also be seen by setting $q = 0$ in our product example for $\mathbf{\Sigma}^*_q$.

The antipode is given by

$$\begin{aligned}
\mathbf{\Sigma}^*_0[I] &\to \mathbf{\Sigma}^*_0[I] \\
(\mathrm{s}_0)_I(M_G) &= (-1)^{\deg(G)}\, M_{\emptyset},
\end{aligned}$$

where $\emptyset$ is the empty face of the complex, or equivalently, the composition of $I$ with one block.

We now consider the dual situation. The product of $\mathbf{\Sigma}_0$ is the same as given in Proposition 12.22 (it does not depend on $q$). The coproduct is given by

$$\begin{aligned}
\mathbf{\Sigma}_0[I] &\to \mathbf{\Sigma}_0[S] \otimes \mathbf{\Sigma}_0[T] \\
H_F &\mapsto \begin{cases} H_{F_1} \otimes H_{F_2} & \text{if } FK = KF, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where $K$ is the vertex $S|T$, and $F_1$ and $F_2$ are defined by $b_K(KF) = (F_1, F_2)$. Explicitly, the coproduct is nonzero only if either $F$ is a concatenation of a composition of $S$ with a composition of $T$, or a concatenation in which the last block of the first composition is merged with the first block of the second composition.

The antipode is given by

$$\begin{aligned}
\mathbf{\Sigma}_0[I] &\to \mathbf{\Sigma}_0[I] \\
(\mathrm{s}_0)_I(H_F) &= \begin{cases} \sum_G (-1)^{\deg(G)}\, H_G & \text{if } F = \emptyset, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For example,

$$\begin{aligned}\mathrm{s}_0(H_{uma}) = -H_{uma} + H_{u|ma} + H_{m|au} + H_{a|um} + H_{um|a} + H_{ua|m} + H_{ma|u} \\ - H_{u|m|a} - H_{u|a|m} - H_{m|u|a} - H_{a|u|m} - H_{m|a|u} - H_{a|m|u}.\end{aligned}$$

We now address the issue of self-duality. Recall that $\mathcal{P}^{(k)}(\mathbf{\Sigma}_0^*)$ is the species spanned by

$$\{M_G \mid \deg(G) \leq k\}.$$

For the dual Hopf monoid, one can show that $\mathcal{P}^{(k)}(\mathbf{\Sigma}_0)$ is the species spanned by

$$\Big\{ \sum_{G:F\leq G} (-1)^{\deg(G)} H_G \mid \deg(F) \leq k \Big\}.$$

In particular, by letting $k = 1$, it follows that for each $I$, the $I$-component of the primitive element species is one-dimensional with basis element

$$\sum_{G\vDash I} (-1)^{\deg(G)} H_G.$$

As a consequence,

$$\mathcal{P}(\mathbf{\Sigma}_0^*) \cong \mathcal{P}(\mathbf{\Sigma}_0) \cong \mathbf{E}_+.$$

Hence by Theorem 11.49, the 0-Hopf monoids $\mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_0^*$ are isomorphic, and hence self-dual. This complements the result of Proposition 12.26.

**Question 12.27.** Describe in explicit terms the coradical filtration and, in particular, the primitive element species of $\mathbf{\Sigma}_q$. The answer for the case $q = 0$ is given above.

## 12.5. The $q$-Hopf monoids of directed faces

In this section, we study the Hopf monoids $\overrightarrow{\mathbf{\Sigma}}^*$ and $\overrightarrow{\mathbf{\Sigma}}$ which are both based on linear set compositions (or directed faces) and dual to each other. These admit one-parameter deformations to $q$-Hopf monoids which we denote by $\overrightarrow{\mathbf{\Sigma}}_q^*$ and $\overrightarrow{\mathbf{\Sigma}}_q$. This means that $\overrightarrow{\mathbf{\Sigma}}^* = \overrightarrow{\mathbf{\Sigma}}_1^*$ and $\overrightarrow{\mathbf{\Sigma}} = \overrightarrow{\mathbf{\Sigma}}_1$. The deformed objects $\overrightarrow{\mathbf{\Sigma}}_q^*$ and $\overrightarrow{\mathbf{\Sigma}}_q$ are dual $q$-Hopf monoids. As done in earlier sections, we present the theory directly for these $q$-Hopf monoids. The projection, break and join maps continue to play a crucial role in describing the structure of the Hopf monoids. The bases for $\overrightarrow{\mathbf{\Sigma}}^*$ are $F$ and $M$ and the bases for $\overrightarrow{\mathbf{\Sigma}}$ are $H$ and $K$.

### 12.5.1. The $F$ basis.

**Definition 12.28.** Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by

$$\begin{aligned}&\overrightarrow{\mathbf{\Sigma}}_q^*[I] \to \overrightarrow{\mathbf{\Sigma}}_q^*[S] \otimes \overrightarrow{\mathbf{\Sigma}}_q^*[T]\\ &F_{(G,D)} \mapsto \begin{cases} F_{(G_1,D_1)} \otimes F_{(G_2,D_2)} & \text{if } K = S|T \text{ is a vertex of } D,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where the directed faces $(G_1, D_1)$ and $(G_2, D_2)$ are defined by $b_K(KG) = (G_1, G_2)$ and $b_K(D) = (D_1, D_2)$.

The product is given by

$$\overrightarrow{\mathbf{\Sigma}}^*_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}^*_q[T] \to \overrightarrow{\mathbf{\Sigma}}^*_q[I]$$
$$F_{(G_1,D_1)} \otimes F_{(G_2,D_2)} \mapsto \sum_{D:KD=j_K(D_1,D_2)} q^{\mathrm{dist}(K,G)}\, F_{(G,D)},$$

where the vertex $K = S|T$ is fixed, $\mathrm{dist}(K,G)$ is as in (10.33), and $G$ is the smallest face of $D$ such that

$$G\, j_K(\overline{G}_1, \overline{G}_2) \leq D \quad \text{or equivalently} \quad j_K(G_1,G_2) \leq KG \text{ and } GK \leq D.$$

The equivalence mentioned above can be seen as a consequence of the following result by letting $H := j_K(G_1,G_2)$ and $H' := j_K(\overline{G}_1,\overline{G}_2)$.

**Proposition 12.29.** *Let $K \leq H \leq KD$ and let $H'$ be the face opposite to $H$ in* $\mathrm{Star}(K)$*. Then for any face $G$ of $D$,*

$$GH' \leq D \iff H \leq KG \text{ and } GK \leq D.$$

*If either of these equivalent statements holds, then*

$$GH = GH' = GK.$$

Proof. We provide an outline. Let us begin with the forward implication. Suppose $GH' \leq D$. Then $GK \leq D$. Hence $GK \leq GH \leq GKD = D$, and so $GH \leq D$. Since $H$ and $H'$ are opposite in $\mathrm{Star}(K)$, it follows that $GH$ and $GH'$ are opposite in $\mathrm{Star}(GK)$. But they are both faces of $D$, so $GH = GH' = GK$. Now $H$ and $KG$ are faces of $KD$ and $KGH = KG$. Hence $H \leq KG$ as required.

For the backward implication, suppose $H \leq KG$ and $GK \leq D$. Then since $H$ and $H'$ have the same support, we have $GKH' = GK$ which implies $GH' \leq D$. $\square$

We now give a combinatorial formulation of Definition 12.28.

**Proposition 12.30.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by*

$$\overrightarrow{\mathbf{\Sigma}}^*_q[I] \to \overrightarrow{\mathbf{\Sigma}}^*_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}^*_q[T]$$
$$F_{(G,D)} \mapsto \begin{cases} F_{(G|_S,D|_S)} \otimes F_{(G|_T,D|_T)} & \text{if } S \text{ is an initial segment of } D, \\ 0 & \text{otherwise,} \end{cases}$$

*with the restriction $G|_S$ as in Section 10.1.6.*

*The product is given by*

$$\overrightarrow{\mathbf{\Sigma}}^*_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}^*_q[T] \to \overrightarrow{\mathbf{\Sigma}}^*_q[I]$$
$$F_{(G_1,D_1)} \otimes F_{(G_2,D_2)} \mapsto \sum_{D:D \text{ a shuffle of } D_1 \text{ and } D_2} q^{\mathrm{sch}_{S,T}(D)}\, F_{(G,D)},$$

*where $\mathrm{sch}_{S,T}(D)$ is the Schubert cocycle (9.12) and $G$ is the face of $D$ determined as follows. Let $D = D^1|\cdots|D^n$. Then $D^i$ and $D^{i+1}$ occur in different blocks of $G$ if and only if*

- *$D^i$ occurs in $G_2$ and $D^{i+1}$ occurs in $G_1$, or*
- *$D^i$ and $D^{i+1}$ occur in different blocks of $G_1$, or*
- *$D^i$ and $D^{i+1}$ occur in different blocks of $G_2$.*

In the situation of the above proposition, $\mathrm{sch}_{S,T}(D) = \mathrm{sch}_{S,T}(G)$, the latter being the Schubert cocycle on faces (9.12).

An example for the coproduct and product is given below.

$$\begin{aligned}F_{(sh|iv|a,s|h|i|v|a)} \mapsto 1 \otimes F_{(sh|iv|a,s|h|i|v|a)} + F_{(s,s)} \otimes F_{(h|iv|a,h|i|v|a)} \\ + F_{(sh,s|h)} \otimes F_{(iv|a,i|v|a)} + F_{(sh|i,s|h|i)} \otimes F_{(v|a,v|a)} \\ + F_{(sh|iv,s|h|i|v)} \otimes F_{(a,a)} + F_{(sh|iv|a,s|h|i|v|a)} \otimes 1.\end{aligned}$$

$$\begin{aligned}F_{(ha,h|a)} \otimes F_{(r|i,r|i)} \mapsto F_{(har|i,h|a|r|i)} + q\, F_{(hr|ai,h|r|a|i)} + q^2\, F_{(r|hai,r|h|a|i)} \\ + q^2\, F_{(hr|i|a,h|r|i|a)} + q^3\, F_{(r|hi|a,r|h|i|a)} + q^4\, F_{(r|i|ha,r|i|h|a)}.\end{aligned}$$

We provide a formula for the antipode on the $F$ basis.

**Theorem 12.31.** *The antipode* $\mathrm{s} : \overrightarrow{\mathbf{\Sigma}}^*_q \to \overrightarrow{\mathbf{\Sigma}}^*_q$ *is given by*

$$\begin{aligned}\overrightarrow{\mathbf{\Sigma}}^*_q[I] &\to \overrightarrow{\mathbf{\Sigma}}^*_q[I] \\ \mathrm{s}_I(F_{(G,D)}) &= \sum_{\substack{(F,H):\\ F\leq H\leq D,\, G\vee F=D}} (-1)^{\deg(D)-\deg(H)+\deg(F)}\; q^{\mathrm{dist}(\overline{F},F)}\, F_{(\overline{F}H,\overline{F}D)},\end{aligned}$$

*where* $\deg(G)$ *is the number of blocks in* $G$, $\overline{G}$ *denotes the opposite of* $G$, *and* $\mathrm{dist}(F,G)$ *is as in* (10.33).

The proof is omitted. As an example,

$$\mathrm{s}\left(F_{(ha|ri,h|a|r|i)}\right) = q^5\, F_{(i|ra|h,i|a|r|h)} - q^5\, F_{(i|a|r|h,i|a|r|h)} + q^6\, F_{(i|r|a|h,h|a|r|i)}.$$

**12.5.2. The $M$ basis.** Recall that the $M$ and $F$ bases are related by (12.4). The coproduct and product on the $M$ basis are given by the following formulas.

**Theorem 12.32.** *Fix a decomposition* $I = S \sqcup T$ *into nonempty subsets. The coproduct is given by*

$$\begin{aligned}\overrightarrow{\mathbf{\Sigma}}^*_q[I] &\to \overrightarrow{\mathbf{\Sigma}}^*_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}^*_q[T] \\ M_{(G,D)} &\mapsto \begin{cases} M_{(G_1,D_1)} \otimes M_{(G_2,D_2)} & \text{if } K = S|T \text{ is a vertex of } G,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

*where* $G_1, G_2, D_1$ *and* $D_2$ *are defined by* $b_K(G) = (G_1, G_2)$ *and* $b_K(D) = (D_1, D_2)$.

*The product is given by*

$$\begin{aligned}\overrightarrow{\mathbf{\Sigma}}^*_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}^*_q[T] &\to \overrightarrow{\mathbf{\Sigma}}^*_q[I] \\ M_{(G_1,D_1)} \otimes M_{(G_2,D_2)} &\mapsto \sum_{G:KG=j_K(G_1,G_2)} q^{\mathrm{dist}(K,G)}\, M_{(G,Gj_K(D_1,D_2))},\end{aligned}$$

*where the vertex* $K = S|T$ *is fixed, and* $\mathrm{dist}(K,G)$ *is as in* (10.33).

The proof is similar to those of [12, Theorems 8.2.1 and 8.2.3]; we omit it. We now give a combinatorial formulation of the product and coproduct.

**Theorem 12.33.** *Fix a decomposition* $I = S \sqcup T$. *The coproduct is given by*

$$\begin{aligned}\overrightarrow{\mathbf{\Sigma}}^*_q[I] &\to \overrightarrow{\mathbf{\Sigma}}^*_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}^*_q[T] \\ M_{(G,D)} &\mapsto \begin{cases} M_{(G|_S,D|_S)} \otimes M_{(G|_T,D|_T)} & \text{if } S \text{ is an initial segment of } G,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

*with the restriction* $G|_S$ *as in Section* 10.1.6.

*The product is given by*

$$\begin{aligned} \overrightarrow{\mathbf{\Sigma}}^*_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}^*_q[T] &\to \overrightarrow{\mathbf{\Sigma}}^*_q[I] \\ M_{(G_1,D_1)} \otimes M_{(G_2,D_2)} &\mapsto \sum_{G:G \text{ a quasi-shuffle of } G_1 \text{ and } G_2} q^{\mathrm{sch}_{S,T}(D)}\, M_{(G,D)}, \end{aligned}$$

*where* $\mathrm{sch}_{S,T}(D)$ *is the Schubert cocycle* (9.12) *and* $D$ *is the unique shuffle of* $D_1$ *and* $D_2$, *that refines* $G$ *and keeps the blocks of* $D_1$ *before* $D_2$, *whenever there is such a choice.*

In the situation of the above theorem, $\mathrm{sch}_{S,T}(D) = \mathrm{sch}_{S,T}(G)$, the latter being the Schubert cocycle on faces (9.12).

An example for the coproduct and product is given below.

$$\begin{aligned} M_{(sh|iv|a,s|h|i|v|a)} \mapsto 1 \otimes M_{(sh|iv|a,s|h|i|v|a)} + M_{(sh,s|h)} \otimes M_{(iv|a,i|v|a)} \\ + M_{(sh|iv,s|h|i|v)} \otimes M_{(a,a)} + M_{(sh|iv|a,s|h|i|v|a)} \otimes 1. \end{aligned}$$

$$\begin{aligned} M_{(ha,h|a)} \otimes M_{(r|i,r|i)} \mapsto M_{(ha|r|i,h|a|r|i)} + M_{(har|i,h|a|r|i)} + q^2\, M_{(r|ha|i,r|h|a|i)} \\ + q^2\, M_{(r|hai,r|h|a|i)} + q^4\, M_{(r|i|ha,r|i|h|a)}. \end{aligned}$$

The coradical filtration of $\overrightarrow{\mathbf{\Sigma}}^*_q$ can be readily described using the coproduct formula as follows. This discussion does not depend on the parameter $q$ and is a special case of that in Section 11.9.4.

Given the factors $M_{(G_1,D_1)}$ and $M_{(G_2,D_2)}$ of the component $\Delta_{S,T}$ of the coproduct, one can uniquely recover $M_{(G,D)}$ by

$$G = j_K(G_1, G_2) \qquad \text{and} \qquad D = j_K(D_1, D_2)$$

where $K = S|T$. This shows that $\mathcal{P}^{(k)}(\overrightarrow{\mathbf{\Sigma}}^*_q)$ is the species spanned by

$$\{M_{(G,D)} \mid \dim(G) \le k-2\}.$$

The case $k = 1$ yields the primitive element species. It is spanned by the elements $M_{(\emptyset,D)}$.

**Theorem 12.34.** *The antipode* $\mathrm{s}\colon \overrightarrow{\mathbf{\Sigma}}^*_q \to \overrightarrow{\mathbf{\Sigma}}^*_q$ *is given by*

$$\begin{gathered} \overrightarrow{\mathbf{\Sigma}}^*_q[I] \to \overrightarrow{\mathbf{\Sigma}}^*_q[I] \\ \mathrm{s}_I(M_{(G,D)}) = (-1)^{\deg(G)} \sum_{F:F \le \overline{G}} q^{\mathrm{dist}(\overline{F},F)}\, M_{(F,FD)}, \end{gathered}$$

*where* $\deg(G)$ *is the number of blocks in* $G$, *and* $\overline{G}$ *denotes the opposite of* $G$, *and* $\mathrm{dist}(F,G)$ *is as in* (10.33).

The proof is similar to that of the antipode formula for $\mathbf{\Sigma}^*_q$ given in Theorem 12.21, and hence omitted. As an example,

$$\begin{aligned} \mathrm{s}\big(M_{m|i|k|s|h|l|a}\big) = -M_{m|i|k|s|h|l|a} - q^{10}\, M_{k|s|h|l|a|m|i} \\ - q^{10}\, M_{l|a|m|i|k|s|h} - q^{16}\, M_{l|a|k|s|h|m|i}. \end{aligned}$$

**12.5.3. The $H$ basis.** We now describe the coproduct, product and antipode of $\overrightarrow{\mathbf{\Sigma}}_q$ on the $H$ basis by dualizing the formulas on the $M$ basis.

**Theorem 12.35.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by*

$$\overrightarrow{\mathbf{\Sigma}}_q[I] \to \overrightarrow{\mathbf{\Sigma}}_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}_q[T]$$

$$H_{(F,C)} \mapsto \begin{cases} q^{\operatorname{dist}(F,K)}\, H_{(F_1,C_1)} \otimes H_{(F_2,C_2)} & \text{if } K = S|T \text{ satisfies } FK \leq C, \\ 0 & \text{otherwise,} \end{cases}$$

*where $\mathrm{b}_K(KF) = (F_1, F_2)$ and $\mathrm{b}_K(KC) = (C_1, C_2)$.*

*The product is given by*

$$\overrightarrow{\mathbf{\Sigma}}_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}_q[T] \to \overrightarrow{\mathbf{\Sigma}}_q[I]$$
$$H_{(F_1,C_1)} \otimes H_{(F_2,C_2)} \mapsto H_{(j_K(F_1,F_2),\, j_K(C_1,C_2))},$$

*where the vertex $K$ is defined to be $S|T$.*

For example,

$$H_{u|m|a} \mapsto 1 \otimes H_{u|m|a} + q^2\, H_a \otimes H_{u|m} + H_u \otimes H_{m|a}$$
$$+ q\, H_{u|a} \otimes H_m + H_{u|m} \otimes H_a + H_{u|m|a} \otimes 1.$$

$$H_{l|a|k} \otimes H_{s|h|m|i} \mapsto H_{l|a|k|s|h|m|i}.$$

Note that:

- only the vertices which belong to the cone $\Psi(F, C)$ associated to the directed face $(F, C)$ as in Proposition 10.14 contribute to the coproduct,
- in the above definitions, we are implicitly using the break and join maps on directed faces (10.66),
- for the coproduct, we are implicitly using the left module structure of directed faces over the algebra of faces (10.51).

**Theorem 12.36.** *The antipode $\mathrm{s}\colon \overrightarrow{\mathbf{\Sigma}}_q \to \overrightarrow{\mathbf{\Sigma}}_q$ is given by*

$$\overrightarrow{\mathbf{\Sigma}}_q[I] \to \overrightarrow{\mathbf{\Sigma}}_q[I]$$
$$\mathrm{s}_I(H_{(F,C)}) = q^{\operatorname{dist}(F,\overline{F})} \sum_{G:\, \overline{F} \leq G,\, FG \leq C} (-1)^{\deg(G)}\, H_{(G,GC)}.$$

For example,

$$\mathrm{s}(H_{m|a|u}) = q^2\, H_{u|m|a} - q^2\, H_{u|m|a}.$$

The above result is an instance of a much more general antipode formula given in Theorem 11.38: set $\mathbf{q} = \mathbf{L}_+^*$.

**12.5.4. The $K$ basis.** The descriptions for the structure maps of $\overrightarrow{\mathbf{\Sigma}}_q$ on the $K$ basis follow from those for $\overrightarrow{\mathbf{\Sigma}}_q^*$ on the $F$ basis by duality. The formula for the product is worth-stating in combinatorial terms.

**Theorem 12.37.** *The product is given by*

$$\overrightarrow{\mathbf{\Sigma}}_q[S] \otimes \overrightarrow{\mathbf{\Sigma}}_q[T] \to \overrightarrow{\mathbf{\Sigma}}_q[I]$$
$$K_{(F_1,C_1)} \otimes K_{(F_2,C_2)} \mapsto K_{(F_1 \cdot F_2,\, C_1 \cdot C_2)} + K_{(F_1 \smile F_2,\, C_1 \cdot C_2)},$$

*where $F_1 \cdot F_2$ is the concatenation of the set compositions $F_1$ and $F_2$, and $F_1 \smile F_2$ is the unique quasishuffle of $F_1$ and $F_2$ in which the last block of $F_1$ is merged with the first block of $F_2$.*

### 12.5.5. Self-duality.

**Proposition 12.38.** *For $q$ not an algebraic integer, the Hopf monoid $\overrightarrow{\mathbf{\Sigma}}_q$ is self-dual. The isomorphism to its dual is given by*

$$\overrightarrow{\mathbf{\Sigma}}_q \to \overrightarrow{\mathbf{\Sigma}}_q^* \qquad\qquad H_{(F,C)} \mapsto \sum_{(G,D)\colon FD=C,\, GC=D} q^{\operatorname{dist}(C,D)}\, M_{(G,D)}.$$

Proof. It can be directly checked that the above map is a morphism of monoids and comonoids. Hence it is a morphism of $q$-Hopf monoids. We next note that the bilinear form on directed faces which induces the above map coincides with the bilinear form in Lemma 10.35. The result follows. $\square$

Applying the signature functor yields:

$$(\overrightarrow{\mathbf{\Sigma}}_q)^- \cong \overrightarrow{\mathbf{\Sigma}}_{-q} \qquad \text{and} \qquad (\overrightarrow{\mathbf{\Sigma}}_q^*)^- \cong \overrightarrow{\mathbf{\Sigma}}_{-q}^*.$$

In other words, $\overrightarrow{\mathbf{\Sigma}}_q$ and $\overrightarrow{\mathbf{\Sigma}}_{-q}$ are signed partners, and $\overrightarrow{\mathbf{\Sigma}}_q^*$ and $\overrightarrow{\mathbf{\Sigma}}_{-q}^*$ are signed partners.

**12.5.6. 0-Hopf monoids.** We now briefly discuss the connected 0-Hopf monoids $\overrightarrow{\mathbf{\Sigma}}_0^*$ and $\overrightarrow{\mathbf{\Sigma}}_0$ on the $M$ and $H$ basis.

The coproduct of $\overrightarrow{\mathbf{\Sigma}}_0^*$ is the same as given in Theorem 12.32 (it does not depend on $q$). The product is given by

$$\begin{gathered}\overrightarrow{\mathbf{\Sigma}}_0^*[S] \otimes \overrightarrow{\mathbf{\Sigma}}_0^*[T] \to \overrightarrow{\mathbf{\Sigma}}_0^*[I] \\ M_{(G_1,D_1)} \otimes M_{(G_2,D_2)} \mapsto \sum_{G: KG=j_K(G_1,G_2)=GK} M_{(G,Gj_K(D_1,D_2))},\end{gathered}$$

where the vertex $K = S|T$ is fixed. This follows from (10.36). Explicitly, as observed for $\mathbf{\Sigma}_0^*$, the sum consists of exactly two summands.

For example,

$$M_{(la|ksh,l|a|k|s|h)} \otimes M_{(mi,m|i)} \mapsto M_{(la|ksh|mi,l|a|k|s|h|m|i)} + M_{(la|kshmi,l|a|k|s|h|m|i)}.$$

The antipode is given by

$$\begin{gathered}\overrightarrow{\mathbf{\Sigma}}_0^*[I] \to \overrightarrow{\mathbf{\Sigma}}_0^*[I] \\ (\mathrm{s}_0)_I(M_{(G,D)}) = (-1)^{\deg(G)} M_{(\emptyset,D)},\end{gathered}$$

where $\emptyset$ is the empty face of the complex, or equivalently, the composition of $I$ with one block.

We now consider the dual situation. The product of $\overrightarrow{\mathbf{\Sigma}}_0$ is the same as given in Theorem 12.35 (it does not depend on $q$). The coproduct is given by

$$\begin{gathered}\overrightarrow{\mathbf{\Sigma}}_0[I] \to \overrightarrow{\mathbf{\Sigma}}_0[S] \otimes \overrightarrow{\mathbf{\Sigma}}_0[T] \\ H_{(F,C)} \mapsto \begin{cases} H_{(F_1,C_1)} \otimes H_{(F_2,C_2)} & \text{if } K = S|T \text{ satisfies } K \leq C, \\ 0 & \text{otherwise,}\end{cases}\end{gathered}$$

where $b_K(KF) = (F_1, F_2)$, $b_K(C) = (C_1, C_2)$. Explicitly, the coproduct is nonzero only if $C$ is a concatenation of an order on $S$ with an order on $T$.

For example,

$$H_{s|h|i\big|v|a} \mapsto 1 \otimes H_{s|h|i\big|v|a} + H_s \otimes H_{h|i\big|v|a} + H_{s|h} \otimes H_{i\big|v|a} \\ + H_{s|h|i} \otimes H_{v|a} + H_{s|h|i\big|v} \otimes H_a + H_{s|h|i\big|v|a} \otimes 1.$$

The antipode is given by

$$\overrightarrow{\mathbf{\Sigma}}_0[I] \to \overrightarrow{\mathbf{\Sigma}}_0[I]$$
$$(\mathrm{s}_0)_I(H_{(F,C)}) = \begin{cases} \sum_{G:G\leq C} (-1)^{\deg(G)} H_{(G,C)} & \text{if } F = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$\mathrm{s}_0(H_{u|m|a}) = -H_{u|m|a} + H_{u\big|m|a} + H_{u|m\big|a} - H_{u\big|m\big|a}.$$

We now address the issue of self-duality. Recall that $\mathcal{P}^{(k)}(\overrightarrow{\mathbf{\Sigma}}^*_0)$ is the species spanned by

$$\{M_{(G,D)} \mid \deg(G) \leq k\}.$$

For the dual Hopf monoid, one can show that $\mathcal{P}^{(k)}(\overrightarrow{\mathbf{\Sigma}}_0)$ is the species spanned by

$$\Big\{ \sum_{G:F\leq G\leq C} (-1)^{\deg(G)} H_{(G,C)} \mid F \leq C \text{ and } \deg(F) \leq k \Big\}.$$

In particular, by letting $k = 1$, it follows that for each $I$, the $I$-component of the primitive element species has basis

$$\Big\{ \sum_{G:G\leq C} (-1)^{\deg(G)} H_{(G,C)} \Big\},$$

one element for each linear order $C$ on $I$. As a consequence,

$$\mathcal{P}(\overrightarrow{\mathbf{\Sigma}}^*_0) \cong \mathcal{P}(\overrightarrow{\mathbf{\Sigma}}_0) \cong \mathbf{L}_+.$$

Hence by Theorem 11.49, the 0-Hopf monoids $\overrightarrow{\mathbf{\Sigma}}_0$ and $\overrightarrow{\mathbf{\Sigma}}^*_0$ are isomorphic, and hence self-dual. This complements the result of Proposition 12.38.

**Question 12.39.** Describe in explicit terms the coradical filtration and, in particular, the primitive element species of $\overrightarrow{\mathbf{\Sigma}}_q$. The answer for the case $q = 0$ is given above.

**12.5.7. An aside on lattice congruences.** Going over the results and proofs of the preceding sections reveals a striking parallel, particularly between those for the Hopf monoid $\mathbb{L}$ of pairs of chambers (Section 12.3) and those for the Hopf monoid $\overrightarrow{\mathbf{\Sigma}}$ of directed faces (in this section).

In fact, it is possible to view both $\overrightarrow{\mathbf{\Sigma}}$ and $\mathbb{L}$ as special cases of a very general construction of Hopf monoids. This is based on the notion of *lattice congruence* [88, 150, 314]. We plan to pursue this theme in a future work. Ideas of Nathan Reading play a key role in this. In a series of papers [303, 304, 306], Reading has studied lattice congruences on the poset of regions of a hyperplane arrangement, and in [305] he has applied the case of the braid arrangement to the construction of Hopf algebras. His methods can be adapted to yield a construction of Hopf monoids of which $\overrightarrow{\mathbf{\Sigma}}$ and $\mathbb{L}$ are extreme examples.

## 12.6. The Hopf monoids of flats

In this section, we study the Hopf monoids $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ which are both based on set partitions (or flats) and dual to each other. Each one has two bases; we use $h$ and $q$ for the former and $m$ and $p$ for the latter, in agreement with Notation (12.1). Duality is as indicated in (12.2). It turns out that these Hopf monoids are isomorphic, and hence self-dual (Proposition 12.48). This may be regarded as a commutative and cocommutative version of the situation for the Hopf monoids of pairs of chambers.

The discussion below will make use of the break and join maps for flats (10.65) and the module structure of flats over faces (10.18).

### 12.6.1. The $m$ and $p$ bases.

We define the product and coproduct of $\mathbf{\Pi}^*$ on the $m$ basis and then describe it on the $p$ basis. Recall that these bases are related by (12.5).

**Definition 12.40.** Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by

$$\mathbf{\Pi}^*[I] \to \mathbf{\Pi}^*[S] \otimes \mathbf{\Pi}^*[T]$$
$$m_X \mapsto \begin{cases} m_{X_1} \otimes m_{X_2} & \text{if } K = S|T \text{ satisfies } K \cdot X = X, \\ 0 & \text{otherwise,} \end{cases}$$

where $X_1$ and $X_2$ are defined by $b_K(X) = (X_1, X_2)$.

The product is given by

$$\mathbf{\Pi}^*[S] \otimes \mathbf{\Pi}^*[T] \to \mathbf{\Pi}^*[I]$$
$$m_{X_1} \otimes m_{X_2} \mapsto \sum_{X : K \cdot X = j_K(X_1, X_2)} m_X.$$

The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum.

**Proposition 12.41.** *The coproduct of $\mathbf{\Pi}^*$ is identical on the $m$ and $p$ bases. The product on the $p$ basis is given by*

$$\mathbf{\Pi}^*[S] \otimes \mathbf{\Pi}^*[T] \to \mathbf{\Pi}^*[I]$$
$$p_{X_1} \otimes p_{X_2} \mapsto p_{j_K(X_1, X_2)}.$$

The proof is omitted.

The product and coproduct can be described combinatorially using the notions of restriction and quasi-shuffle of set partitions (Section 10.1.6). They are as follows.

**Proposition 12.42.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by*

$$\mathbf{\Pi}^*[I] \to \mathbf{\Pi}^*[S] \otimes \mathbf{\Pi}^*[T]$$
$$m_X \mapsto \begin{cases} m_{X|_S} \otimes m_{X|_T} & \textit{if } S \textit{ is the union of some blocks of } X, \\ 0 & \textit{otherwise,} \end{cases}$$

*where $X|_S$ is the restriction of $X$ to $S$. The coproduct on the $p$ basis is given by the same formula.*

*The product is given by*

$$\begin{aligned}
\mathbf{\Pi}^*[S] \otimes \mathbf{\Pi}^*[T] &\to \mathbf{\Pi}^*[I] \\
m_{X_1} \otimes m_{X_2} &\mapsto \sum_{X:X \text{ a quasi-shuffle of } X_1 \text{ and } X_2} m_X \\
p_{X_1} \otimes p_{X_2} &\mapsto p_{X_1 \sqcup X_2}.
\end{aligned}$$

For example,

$$m_{\{um,a\}} \mapsto 1 \otimes m_{\{um,a\}} + m_{\{um\}} \otimes m_{\{a\}} + m_{\{a\}} \otimes m_{\{um\}} + m_{\{um,a\}} \otimes 1.$$

Since the coproduct is identical on the $m$ and $p$ basis, the same example can be given on the $p$ basis. For the product,

$$\begin{aligned}
m_{\{sh,i\}} \otimes m_{\{va\}} &\mapsto m_{\{sh,i,va\}} + m_{\{sh,iva\}} + m_{\{shva,i\}} \\
p_{\{sh,i\}} \otimes p_{\{va\}} &\mapsto p_{\{sh,i,va\}}.
\end{aligned}$$

The coradical filtration of $\mathbf{\Pi}^*$ can be readily described using the coproduct formula: $\mathcal{P}^{(k)}(\mathbf{\Pi}^*)$ is the species spanned by

$$\{m_X \mid X \text{ has } k \text{ blocks}\}.$$

The case $k = 1$ yields the primitive element species. It is spanned by the one-block partitions; so it is one-dimensional in each component. In other words, the primitive element species is $\mathbf{E}_+$.

We now turn our attention to the antipode formula for $\mathbf{\Pi}^*$. We begin with a preliminary lemma.

**Lemma 12.43.** *Let $X$ and $Y$ be set partitions with $X \leq Y$. Then*

$$\sum_{H:H\cdot X=Y} (-1)^{\deg(H)} = (-1)^{\deg(Y)} (X:Y)!,$$

*where* $\deg(Y)$ *is the number of blocks in* $Y$ *and* $(X:Y)!$ *is given by* (10.5).

Proof. Fix a face $F$ whose support is $X$. Then we claim

$$\sum_{\substack{H:\\ H\cdot X=Y}} (-1)^{\deg(H)} = \sum_{\substack{G:\\ \operatorname{supp}(G)=Y}} \sum_{\substack{H:\\ HF=G}} (-1)^{\deg(H)} = \sum_{\substack{G:\\ \operatorname{supp}(G)=Y\\ \overline{F}\leq G}} (-1)^{\deg(G)}.$$

The first equality is clear. The second equality follows from Lemma 11.37. The result now follows from (10.6). ∎

**Theorem 12.44.** *The antipode* $\mathrm{S}\colon \mathbf{\Pi}^* \to \mathbf{\Pi}^*$ *is given by*

$$\begin{aligned}
\mathbf{\Pi}^*[I] &\to \mathbf{\Pi}^*[I] \\
\mathrm{S}_I(m_Y) &= (-1)^{\deg(Y)} \sum_{X:X\leq Y} (X:Y)!\, m_X, \\
\mathrm{S}_I(p_Y) &= (-1)^{\deg(Y)}\, p_Y.
\end{aligned}$$

This is an instance of a much more general antipode formula given in Theorem 11.41: set $\mathbf{q} = \mathbf{E}_+$. A direct proof is given below.

PROOF. For deriving the antipode formula on the $p$ basis, we begin by noting that for a singleton set $Y$, the element $p_Y$ is primitive and hence $\mathrm{S}_I(p_Y) = -p_Y$. The fact that $\mathrm{S}$ is an antimorphism of monoids then gives the general formula.

For deriving the antipode formula on the $m$ basis, we employ Takeuchi's formula (8.27). As a first step, we compute $\mu_{S,T}\Delta_{S,T}$ using the geometric definitions. Since the break and join maps are inverses, we have,

$$\mu_{S,T}\big(\Delta_{S,T}(m_Y)\big) = \sum_X m_X,$$

where the sum is over all $X$ which for $K = S|T$ satisfy $K \cdot X = Y$. The same formula holds for all the iterated composites, namely for

$$\mu_{S_1,\dots,S_k}\Delta_{S_1,\dots,S_k},$$

the sum is over all $X$ which for $H = S_1|\cdots|S_k$ satisfy $H \cdot X = Y$. Therefore by Takeuchi's formula, we obtain:

$$\mathrm{S}_I(m_Y) = \sum_X \Big( \sum_{H:H\cdot X = Y} (-1)^{\deg(H)} \Big) m_X.$$

The result now follows from Lemma 12.43. □

For example,

$$\begin{aligned}
\mathrm{S}\big(p_{\{sh,i,va\}}\big) &= -p_{\{sh,i,va\}}\\
\mathrm{S}\big(m_{\{sh,i,va\}}\big) &= -m_{\{sh,i,va\}} - 2\,m_{\{shi,va\}} - 2\,m_{\{sh,iva\}} - 2\,m_{\{shva,i\}} - 6\,m_{\{shiva\}}.
\end{aligned}$$

**12.6.2. The $h$ basis.** We now describe the product, coproduct and antipode on the $h$ basis of $\mathbf{\Pi}$ by dualizing the formulas in the $m$ basis.

**Proposition 12.45.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by*

$$\begin{aligned}
\mathbf{\Pi}[I] &\to \mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T]\\
h_Y &\mapsto h_{Y_1} \otimes h_{Y_2}
\end{aligned}$$

*where $Y_1$ and $Y_2$ are defined by $b_K(K \cdot Y) = (Y_1, Y_2)$.*

*The product is given by*

$$\begin{aligned}
\mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T] &\to \mathbf{\Pi}[I]\\
h_{Y_1} \otimes h_{Y_2} &\mapsto h_{j_K(Y_1,Y_2)}.
\end{aligned}$$

*The vertex $K = S|T \in \Sigma[I]$.*

**Proposition 12.46.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by*

$$\begin{aligned}
\mathbf{\Pi}[I] &\to \mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T]\\
h_Y &\mapsto h_{Y|_S} \otimes h_{Y|_T},
\end{aligned}$$

*with the restriction $Y|_S$ as in Section 10.1.6.*

*The product is given by*

$$\begin{aligned}
\mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T] &\to \mathbf{\Pi}[I]\\
h_{Y_1} \otimes h_{Y_2} &\mapsto h_{Y_1 \sqcup Y_2}.
\end{aligned}$$

For example,

$$h_{\{lak,s\}} \otimes h_{\{h,mi\}} \mapsto h_{\{lak,s,h,mi\}}.$$

$$\begin{aligned} h_{\{um,a\}} \mapsto 1 \otimes h_{\{um,a\}} + h_{\{u\}} \otimes h_{\{m,a\}} + h_{\{m\}} \otimes h_{\{u,a\}} + h_{\{u,a\}} \otimes h_{\{m\}} \\ + h_{\{m,a\}} \otimes h_{\{u\}} + h_{\{a\}} \otimes h_{\{um\}} + h_{\{um\}} \otimes h_{\{a\}} + h_{\{um,a\}} \otimes 1. \end{aligned}$$

**Theorem 12.47.** *The antipode* $\mathrm{S}\colon \mathbf{\Pi} \to \mathbf{\Pi}$ *is given by*

$$\begin{aligned} &\mathbf{\Pi}[I] \to \mathbf{\Pi}[I] \\ &\mathrm{S}_I(h_X) = \sum_{Y: X \leq Y} (-1)^{\deg(Y)}\, (X:Y)!\, h_Y, \end{aligned}$$

*where* $\deg(Y)$ *is the number of blocks in* $Y$*.*

For example,

$$\mathrm{S}\big(h_{\{sh,i,va\}}\big) = -h_{\{sh,i,va\}} + 2\, h_{\{s,h,i,va\}} + 2\, h_{\{sh,i,v,a\}} - 4\, h_{\{s,h,i,v,a\}}.$$

The above result is an instance of a much more general antipode formula given in Theorem 11.40: set $\mathbf{q} = \mathbf{E}_+^*$.

**12.6.3. Self-duality.** Observe by dualizing that the formulas on the $q$ basis of $\mathbf{\Pi}$ are the same as the formulas on the $p$ basis of $\mathbf{\Pi}^*$. This shows that the Hopf monoids $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ are isomorphic by means of the map

$$q_X \mapsto p_X.$$

Thus, $\mathbf{\Pi}$ is a self-dual Hopf monoid. There is another isomorphism between $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ which we consider next. Recall the numbers $X!$ and $X¡$ associated to a set partition $X$ from Section 10.1.7.

**Proposition 12.48.** *The Hopf monoid* $\mathbf{\Pi}$ *is commutative, cocommutative, and self-dual. Explicitly, the map* $\psi\colon \mathbf{\Pi} \to \mathbf{\Pi}^*$ *defined by*

$$h_Y \mapsto \sum_X (X \vee Y)!\, m_X$$

*is an isomorphism of Hopf monoids, where* $X \vee Y$ *is the smallest refinement of* $X$ *and* $Y$*. In addition,* $\psi$ *maps*

$$q_X \mapsto X¡\, p_X.$$

PROOF. Define $\psi$ using the expression on the $q$ and $p$ bases. Formula (10.3) and the fact that the product and coproduct on the $q$ and $p$ bases are given by identical formulas imply that $\psi$ is an isomorphism of Hopf monoids. The expression for $\psi$ on the $h$ and $m$ bases follows from (12.5), (12.6) and (10.4). □

The isomorphism $\psi$ is part of a large diagram (12.14) which we discuss in Section 12.8.

## 12.7. The Hopf monoids of directed flats

In this section, we study the Hopf monoids $\overrightarrow{\mathbf{\Pi}}$ and $\overrightarrow{\mathbf{\Pi}}^*$ which are both based on linear set partitions (or directed flats) and dual to each other. We use $h$ for the basis of the former and $m$ for the dual basis of the latter, in agreement with Notation 12.1. The Hopf monoid $\overrightarrow{\mathbf{\Pi}}$ is commutative but not cocommutative. Dually, $\overrightarrow{\mathbf{\Pi}}^*$ is not commutative but cocommutative. It follows that these Hopf monoids cannot be isomorphic.

The discussion below makes use of the break and join maps for directed flats defined in (10.67), the left module structure of directed flats over faces (10.52), and the description of the cone $\Psi(L)$ associated to a directed flat $L$ given in Section 10.9.3.

**12.7.1. The $\boldsymbol{m}$ basis.** We describe the product, coproduct and antipode of $\overrightarrow{\mathbf{\Pi}}^*$ on the $m$ basis.

**Definition 12.49.** Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by

$$\begin{aligned} \overrightarrow{\mathbf{\Pi}}^*[I] &\to \overrightarrow{\mathbf{\Pi}}^*[S] \otimes \overrightarrow{\mathbf{\Pi}}^*[T] \\ m_L &\mapsto \begin{cases} m_{L_1} \otimes m_{L_2} & \text{if } K = S|T \text{ satisfies } K \cdot L = L, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $L_1$ and $L_2$ are defined by $b_K(L) = (L_1, L_2)$.

The product is given by

$$\begin{aligned} \overrightarrow{\mathbf{\Pi}}^*[S] \otimes \overrightarrow{\mathbf{\Pi}}^*[T] &\to \overrightarrow{\mathbf{\Pi}}^*[I] \\ m_{L_1} \otimes m_{L_2} &\mapsto \sum_{L:K \cdot L = j_K(L_1, L_2),\, K \subseteq \Psi(L)} m_L. \end{aligned}$$

The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum. The condition $K \subseteq \Psi(L)$ means that $K$ belongs to the cone $\Psi(L)$ associated to $L$ as in Section 10.9.3.

The product and coproduct can be described combinatorially using the notions of restriction and quasi-shuffle of linear set partitions (Section 10.1.6). They are as follows.

**Proposition 12.50.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by*

$$\begin{aligned} \overrightarrow{\mathbf{\Pi}}^*[I] &\to \overrightarrow{\mathbf{\Pi}}^*[S] \otimes \overrightarrow{\mathbf{\Pi}}^*[T] \\ m_L &\mapsto \begin{cases} m_{L|_S} \otimes m_{L|_T} & \textit{if } S \textit{ is the union of some blocks of } L, \\ 0 & \textit{otherwise,} \end{cases} \end{aligned}$$

*where $L|_S$ is the restriction of $L$ to $S$.*

*The product is given by*

$$\begin{aligned} \overrightarrow{\mathbf{\Pi}}^*[S] \otimes \overrightarrow{\mathbf{\Pi}}^*[T] &\to \overrightarrow{\mathbf{\Pi}}^*[I] \\ m_{L_1} \otimes m_{L_2} &\mapsto \sum_{L:L \textit{ a quasi-shuffle of } L_1 \textit{ and } L_2} m_L. \end{aligned}$$

For example,

$$\begin{aligned} m_{\{s|h|i,v|a\}} \mapsto 1 \otimes m_{\{s|h|i,v|a\}} &+ m_{\{s|h|i\}} \otimes m_{\{v|a\}} \\ &+ m_{\{v|a\}} \otimes m_{\{s|h|i\}} + m_{\{s|h|i,v|a\}} \otimes 1. \\ m_{\{v|i|s,h\}} \otimes m_{\{n|u\}} \mapsto m_{\{v|i|s,h,n|u\}} &+ m_{\{v|i|s,h|n|u\}} + m_{\{v|i|s|n|u,h\}}. \end{aligned}$$

The coradical filtration of $\overrightarrow{\mathbf{\Pi}}^*$ can be readily described using the coproduct formula: $\mathcal{P}^{(k)}(\overrightarrow{\mathbf{\Pi}}^*)$ is the species spanned by

$$\{m_L \mid L \text{ has } k \text{ blocks}\}.$$

The case $k = 1$ yields the primitive element species. It is spanned by the one-block linear partitions. These are same as linear orders; so the primitive element species is $\mathbf{L}_+$.

**Theorem 12.51.** *The antipode* $\mathrm{s}\colon \overrightarrow{\mathbf{\Pi}}^* \to \overrightarrow{\mathbf{\Pi}}^*$ *is given by*

$$\overrightarrow{\mathbf{\Pi}}^*[I] \to \overrightarrow{\mathbf{\Pi}}^*[I]$$
$$\mathrm{s}_I(m_M) = (-1)^{\deg(M)} \sum_{L:L\leq M} m_L,$$

*where* $\deg(M)$ *is the number of blocks of the partition underlying* $M$, *and* $\leq$ *is the partial order on linear set partitions* (10.55).

For example,

$$\begin{aligned}\mathrm{s}\big(m_{\{s|h,i,v|a\}}\big) = -m_{\{s|h,i,v|a\}} - m_{\{s|h|i,v|a\}} - m_{\{i|s|h,v|a\}} - m_{\{s|h,i|v|a\}} - m_{\{s|h,v|a|i\}} \\ - m_{\{s|h|v|a,i\}} - m_{\{v|a|s|h,i\}} - m_{\{s|h|i|v|a\}} - m_{\{i|s|h|v|a\}} \\ - m_{\{s|h|v|a|i\}} - m_{\{i|v|a|s|h\}} - m_{\{v|a|s|h|i\}} - m_{\{v|a|i|s|h\}}.\end{aligned}$$

The above result is an instance of a much more general antipode formula given in Theorem 11.41: set $\mathbf{q} = \mathbf{L}_+$. More directly, it follows from Takeuchi's formula (8.27) and Lemma 12.52 below. The details are similar to the proof of Theorem 12.44.

**Lemma 12.52.** *Let* $L$ *and* $M$ *be linear set partitions with* $L \leq M$. *Then*

$$\sum_{H:H\cdot L=M,\, H\subseteq\Psi(L)} (-1)^{\deg(H)} = (-1)^{\deg(M)}.$$

Proof. Fix a directed face $(F, C)$ with support $L$. Now we claim

$$\begin{aligned}\sum_{\substack{H:\\ H\cdot L=M\\ H\subseteq\Psi(L)}} (-1)^{\deg(H)} &= \sum_{\substack{(G,D):\\ \mathrm{supp}(G,D)=M}} \; \sum_{\substack{H:\\ H\cdot(F,C)=(G,D)\\ FH\leq C}} (-1)^{\deg(H)} \\ &= \sum_{\substack{(G,D):\\ \mathrm{supp}(G,D)=M\\ G\cdot(F,C)=(G,D)\\ FG\leq C}} \; \sum_{\substack{H:\\ HF=G}} (-1)^{\deg(H)} \\ &= \sum_{\substack{(G,D):\\ \mathrm{supp}(G,D)=M\\ G\cdot(F,C)=(G,D)\\ FG\leq C,\ \overline{F}\leq G}} (-1)^{\deg(G)}.\end{aligned}$$

The first equality follows from the definitions. The second equality follows from Proposition 10.17 (and the discussion preceding it). The third equality follows from Lemma 11.37.

To finish the proof, we note that the conditions in the last summation uniquely determine $(G, D)$: $D = \overline{F}C$ and $G$ is the unique face between $\overline{F}$ and $\overline{F}C$ such that the support of $(G, D)$ is $M$. $\square$

**Remark 12.53.** In contrast to Theorem 12.51, the antipode formula for $\mathbf{\Pi}^*$ (Theorem 12.44) involves the nontrivial coefficients $(X : Y)!$. Analogous to the $(X : Y)!$, one may define coefficients $(L:M)!$ for $L \leq M$ as follows. Fix a directed face $(F,C)$ with support $L$. Let $(L:M)!$ be the cardinality of the set

$$\{(G,D) \mid (F,C) \leq (G,D),\ \mathrm{supp}(G,D) = M\}.$$

This is analogous to expression (10.6) for the coefficients $(X:Y)!$. The partial order on directed faces forces $D = C$ and then $G$ is uniquely determined. Thus, $(L:M)! = 1$ always, and so we do not see any coefficients other than 1 in the antipode formula of Theorem 12.51.

**12.7.2. The *h* basis.** We now describe the product, coproduct and antipode on the $h$ basis of $\overrightarrow{\mathbf{\Pi}}$ by dualizing the formulas in the $m$ basis.

**Proposition 12.54.** *Fix a decomposition $I = S \sqcup T$ into nonempty subsets. The coproduct is given by*

$$\overrightarrow{\mathbf{\Pi}}[I] \to \overrightarrow{\mathbf{\Pi}}[S] \otimes \overrightarrow{\mathbf{\Pi}}[T]$$
$$h_L \mapsto \begin{cases} h_{L_1} \otimes h_{L_2} & \text{if } K = S|T \text{ satisfies } K \subseteq \Psi(L), \\ 0 & \text{otherwise,} \end{cases}$$

*where $L_1$ and $L_2$ are defined by $b_K(K \cdot L) = (L_1, L_2)$.*

*The product is given by*

$$\overrightarrow{\mathbf{\Pi}}[S] \otimes \overrightarrow{\mathbf{\Pi}}[T] \to \overrightarrow{\mathbf{\Pi}}[I]$$
$$h_{L_1} \otimes h_{L_2} \mapsto h_{j_K(L_1,L_2)}.$$

*The vertex $K = S|T \in \Sigma[I]$.*

**Proposition 12.55.** *Fix a decomposition $I = S \sqcup T$. The coproduct is given by*

$$\overrightarrow{\mathbf{\Pi}}[I] \to \overrightarrow{\mathbf{\Pi}}[S] \otimes \overrightarrow{\mathbf{\Pi}}[T]$$
$$h_L \mapsto \begin{cases} h_{L|_S} \otimes h_{L|_T} & \text{if } S < T \text{ in } L, \\ 0 & \text{otherwise,} \end{cases}$$

*where $S < T$ in $L$ means that in each block $l^i$ of $L$, elements of $l^i \cap S$ precede the elements of $l^i \cap T$ according to the linear order on $l^i$.*

*The product is given by*

$$\overrightarrow{\mathbf{\Pi}}[S] \otimes \overrightarrow{\mathbf{\Pi}}[T] \to \overrightarrow{\mathbf{\Pi}}[S \sqcup T]$$
$$h_{L_1} \otimes h_{L_2} \mapsto h_{L_1 \sqcup L_2}.$$

For example,

$$h_{\{u|m,a\}} \mapsto 1 \otimes h_{\{u|m,a\}} + h_{\{a\}} \otimes h_{\{u|m\}} + h_{\{u|m\}} \otimes h_{\{a\}} \\ + h_{\{u\}} \otimes h_{\{m,a\}} + h_{\{m,a\}} \otimes h_{\{u\}} + h_{\{u|m,a\}} \otimes 1.$$
$$h_{\{l|a|k,s\}} \otimes h_{\{h,m|i\}} \mapsto h_{\{l|a|k,s,h,m|i\}}.$$

**Theorem 12.56.** *The antipode $\mathrm{s}\colon \overrightarrow{\mathbf{\Pi}} \to \overrightarrow{\mathbf{\Pi}}$ is given by*

$$\overrightarrow{\mathbf{\Pi}}[I] \to \overrightarrow{\mathbf{\Pi}}[I]$$
$$\mathrm{s}_I(h_L) = \sum_{M : L \leq M} (-1)^{\deg(M)}\, h_M,$$

*where $\deg(M)$ is the number of blocks of the partition underlying $M$.*

For example,

$$\mathrm{s}\big(h_{\{s|h,i,v|a\}}\big) = -h_{\{s|h,i,v|a\}} + h_{\{s,h,i,v|a\}} + h_{\{s|h,i,v,a\}} - h_{\{s,h,i,v,a\}}.$$

The above result is an instance of a much more general antipode formula given in Theorem 11.40: set $\mathbf{q} = \mathbf{L}_+^*$.

## 12.8. Relating the Hopf monoids

In this section, we relate the various Hopf monoids that have been studied in the preceding sections. The main result is as follows.

**Theorem 12.57.** *The following is a commutative diagram of Hopf monoids.*

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "\mathcal{T}(\varsigma^*)"] \arrow[dd, "\pi"'] & \boldsymbol{\Sigma} \arrow[r, "\mathcal{T}(\pi_+^*)"] \arrow[dd, "\pi_{\mathbf{E}_+^*}"'] \arrow[dddrr, "\Upsilon^*"] & \overrightarrow{\boldsymbol{\Sigma}} \arrow[rr, "\vec{\beta}^*"] \arrow[dd, "\pi_{\mathbf{L}_+^*}"] & & \mathbb{L} \arrow[d, "s", "\cong"'] \\
& & & & \mathbb{L}^* \arrow[dd, "\vec{\beta}"] \\
\mathbf{E} \arrow[r, "\mathcal{S}(\varsigma^*)"] \arrow[dddrrr, "\cong"'] & \boldsymbol{\Pi} \arrow[r, "\mathcal{S}(\pi_+^*)"] \arrow[ddrr, "\psi"', "\cong"] & \overrightarrow{\boldsymbol{\Pi}} \arrow[ddrr, "\Upsilon"] & & \\
& & & \overrightarrow{\boldsymbol{\Pi}}^* \arrow[r, "\pi_{\mathbf{L}_+}^*"] \arrow[d, "\mathcal{S}^\vee(\pi_+)"] & \overrightarrow{\boldsymbol{\Sigma}}^* \arrow[d, "\mathcal{T}^\vee(\pi_+)"] \\
& & & \boldsymbol{\Pi}^* \arrow[r, "\pi_{\mathbf{E}_+}^*"'] \arrow[d, "\mathcal{S}^\vee(\varsigma)"] & \boldsymbol{\Sigma}^* \arrow[d, "\mathcal{T}^\vee(\varsigma)"] \\
& & & \mathbf{E}^* \arrow[r, "\pi^*"'] & \mathbf{L}^*
\end{tikzcd} \tag{12.14}$$

*Moreover, the diagram is self-dual, with duality acting by reflection across the diagonal.*

A small part of this diagram is given by diagram (8.34). We now make some comments on the proof. Let us first only worry about the underlying species. In this situation, the above result is a special case of [12, Theorem 5.6.1] which was stated for any finite Coxeter group. Letting the Coxeter group be the symmetric group recovers the above result. Motivation for (12.14) from purely geometric considerations (see Chapter 10) is given in [12, Chapter 5].

Let us now consider the Hopf monoid structure. We need to check that each object is a Hopf monoid and each map is a morphism of Hopf monoids. This can be verified by means of very similar arguments to those given in [12, Chapter 6] for proving [12, Theorem 6.1.3]. Alternatively, as shown in Table 12.1, each object is the value of some functor such as $\mathcal{T}$, $\mathcal{S}$, etc. These functors take values in the category of Hopf monoids, so it follows that each object is a Hopf monoid. Further, each map can be obtained through the freeness or cofreeness properties of these Hopf monoids (except for $s$, which has been considered in Proposition 12.12). It follows then that each map is a morphism of Hopf monoids. The maps are described in explicit terms in Section 12.8.2 below. One may also proceed directly and employ these descriptions to prove that they are morphisms of Hopf monoids (one may appeal to duality to reduce the number of verifications).

The maps in diagram (12.14) are described in Section 12.8.2 in terms of the linear bases introduced in Section 12.1. The way in which they arise as universal morphisms is briefly indicated in Section 12.8.3.

We touch upon a $q$-analogue of diagram (12.14) in Section 12.8.4; see (12.20).

**12.8.1. Universal objects.** In Chapter 11, we constructed the functors $\mathcal{T}$, $\mathcal{S}$, $\mathcal{T}^\vee$ and $\mathcal{S}^\vee$, which were the species analogues of the tensor and symmetric algebra functors. In particular, we showed that the Hopf monoids $\mathbf{L}$ and $\mathbf{E}$, and their duals, can be obtained by evaluating these functors on the positive species $\mathbf{X}$. As a result, these Hopf monoids satisfy certain universal properties. For example, $\mathbf{L}$ is the free monoid on one generator, and so on. We now show that most of the Hopf monoids in (12.14) can be obtained in the above manner.

**Proposition 12.58.** *There are isomorphisms of Hopf monoids*

$$\mathbf{\Sigma}^* \cong \mathcal{T}^\vee(\mathbf{E}_+), \quad \overrightarrow{\mathbf{\Sigma}}^* \cong \mathcal{T}^\vee(\mathbf{L}_+), \quad \mathbf{\Pi}^* \cong \mathcal{S}^\vee(\mathbf{E}_+) \quad \textit{and} \quad \overrightarrow{\mathbf{\Pi}}^* \cong \mathcal{S}^\vee(\mathbf{L}_+).$$

The first isomorphism identifies the basis element $M_G$ of $\mathbf{\Sigma}^*$ with the distinguished basis element of the 1-dimensional space $\mathbf{E}_+(G)$ (see Notation 11.1). The second isomorphism identifies

$$M_{(G,D)} \in \overrightarrow{\mathbf{\Sigma}}^* \longleftrightarrow D_1 \otimes \cdots \otimes D_k \in \mathbf{L}(G),$$

where $b_G(D) = (D_1, \dots, D_k)$ is as in (10.58). The other isomorphisms are defined similarly, in terms of the $m$ bases of $\mathbf{\Pi}^*$ and $\overrightarrow{\mathbf{\Pi}}^*$.

Proof. We recall that the product on $\mathbf{q}$ gives rise to the quasi-shuffle product on $\mathcal{T}^\vee(\mathbf{q})$ (Section 11.4.3). The coproduct on $\mathcal{T}^\vee(\mathbf{q})$ is given by deconcatenation and for that one only requires $\mathbf{q}$ to be a positive species (Definitions 11.17). It is straightforward to see that for $\mathbf{q} = \mathbf{E}_+$, this agrees with the combinatorial description of $\mathbf{\Sigma}^*$ given by Proposition 12.20. The remaining checks are similar. □

It is well known that the Hopf algebra of quasi-symmetric functions satisfies a certain universal property [10, Theorem 4.1]. The same is true of symmetric functions. Proposition 12.58 provides a species analogue of these facts. It also reveals a parallel between the Hopf monoids $\mathbf{\Pi}^*$ and $\mathbf{\Sigma}^*$. This was also clear from their combinatorial and geometric definitions. A similar remark applies to $\overrightarrow{\mathbf{\Pi}}^*$ and $\overrightarrow{\mathbf{\Sigma}}^*$. Dually:

**Proposition 12.59.** *There are isomorphisms of Hopf monoids*

$$\mathbf{\Sigma} \cong \mathcal{T}(\mathbf{E}_+^*), \quad \overrightarrow{\mathbf{\Sigma}} \cong \mathcal{T}(\mathbf{L}_+^*), \quad \mathbf{\Pi} \cong \mathcal{S}(\mathbf{E}_+^*) \quad \textit{and} \quad \overrightarrow{\mathbf{\Pi}} \cong \mathcal{S}(\mathbf{L}_+^*).$$

The isomorphisms are defined as in Proposition 12.58, using the $H$ and $h$ bases.

**12.8.2. The morphisms.** We now describe the various maps in (12.14).

The map $s\colon \mathbb{L} \to \mathbb{L}^*$ is the switch map $s_q$ defined in (12.9) with $q = 1$. It sends

$$K_{(D,C)} \mapsto F_{(C,D)}.$$

The morphism $\psi\colon \mathbf{\Pi} \to \mathbf{\Pi}^*$ sends

$$h_Y \mapsto \sum_X (X \vee Y)!\, m_X.$$

It is shown in Propositions 12.12 and 12.48 that $s$ and $\psi$ are self-dual morphisms of Hopf monoids. Duality is as indicated under Notation 12.1.

The vertical maps are instances of the abelianization (Section 11.6.2). The map $\pi\colon \mathbf{L} \to \mathbf{E}$ is the morphism of (8.31). The map

$$\pi_{\mathbf{E}_+^*}\colon \mathbf{\Sigma} \to \mathbf{\Pi} \qquad \text{sends} \qquad H_F \mapsto h_{\mathrm{supp}(F)}, \tag{12.15}$$

and the map

$$\pi_{\mathbf{L}_+^*}\colon \overrightarrow{\mathbf{\Sigma}} \to \overrightarrow{\mathbf{\Pi}} \qquad \text{sends} \qquad H_{(F,C)} \mapsto h_{\mathrm{supp}(F,C)}, \tag{12.16}$$

where supp denotes the support map (Section 10.1.5). In other words, $\pi_{\mathbf{E}_+}$ sends a set composition to the set partition obtained by forgetting the order among the blocks, and $\pi_{\mathbf{L}_+}$ sends a linear set composition to the linear set partition obtained by forgetting the order among the blocks (keeping the order within each block).

The duals of the preceding maps are as follows. The map $\pi^*\colon \mathbf{E}^* \to \mathbf{L}^*$ is the morphism of (8.33). The map

$$\pi_{\mathbf{E}_+}^*\colon \mathbf{\Pi}^* \to \mathbf{\Sigma}^* \qquad \text{sends} \qquad m_Y \mapsto \sum_{G\colon \mathrm{supp}(G)=Y} M_G,$$

and the map

$$\pi_{\mathbf{L}_+}^*\colon \overrightarrow{\mathbf{\Pi}}^* \to \overrightarrow{\mathbf{\Sigma}}^* \qquad \text{sends} \qquad m_M \mapsto \sum_{(G,D)\colon \mathrm{supp}(G,D)=M} M_{(G,D)}.$$

We turn to the morphisms obtained by evaluation of the functors $\mathcal{T}$ and $\mathcal{S}$ on the maps $\pi_+$ and $\varsigma$. The map $\pi_+\colon \mathbf{L}_+ \to \mathbf{E}_+$ is the positive part of (8.31), while $\varsigma\colon \mathbf{E}_+ \to \mathbf{X}$ is the canonical isomorphism on singletons and zero otherwise. It follows that

$$\mathcal{T}(\varsigma^*)\colon \mathbf{L} \to \mathbf{\Sigma} \qquad \text{sends} \qquad l \mapsto H_l$$

and

$$\mathcal{S}(\varsigma^*)\colon \mathbf{E} \to \mathbf{\Pi} \qquad \text{sends} \qquad *_I \mapsto h_I,$$

where we identify $I$ with the partition of $I$ into singletons. The next maps involve the base maps of Section 10.1.5. First,

$$\mathcal{T}(\pi_+^*)\colon \mathbf{\Sigma} \to \overrightarrow{\mathbf{\Sigma}} \qquad \text{sends} \qquad H_F \mapsto \sum_{C\colon F \leq C} H_{(F,C)}.$$

The sum is over all chambers $C$ containing $F$, or equivalently, over all linear orders that refine the set composition $F$. Since $\mathrm{base}(F,C) = F$, this may also be seen as the sum over all directed faces with base $F$. Similarly,

$$\mathcal{S}(\pi_+^*)\colon \mathbf{\Pi} \to \overrightarrow{\mathbf{\Pi}} \qquad \text{sends} \qquad h_X \mapsto \sum_{L\colon \mathrm{base}(L)=X} h_L.$$

The sum is over all directed flats with base $X$, or equivalently, over all linear partitions $L$ obtained by ordering elements within each block of $X$.

The dual maps are as follows. We have that

$$\mathcal{T}^\vee(\varsigma)\colon \mathbf{\Sigma}^* \to \mathbf{L}^* \quad \text{sends} \quad M_F \mapsto \begin{cases} F^* & \text{if } F \text{ is a linear order,} \\ 0 & \text{otherwise,} \end{cases}$$

while

$$\mathcal{S}^\vee(\varsigma)\colon \mathbf{\Pi}^* \to \mathbf{E}^* \quad \text{sends} \quad m_X \mapsto \begin{cases} *_I & \text{if } X \text{ is the partition of } I \text{ into singletons,} \\ 0 & \text{otherwise.} \end{cases}$$

The maps $\mathcal{T}^\vee(\pi_+)$ and $\mathcal{S}^\vee(\pi_+)$ are the linearizations of the base maps. Explicitly,

$$\mathcal{T}^\vee(\pi_+)\colon \overrightarrow{\mathbf{\Sigma}}^* \to \mathbf{\Sigma}^* \quad \text{sends} \quad M_{(F,C)} \mapsto M_F$$

and

$$\mathcal{S}^\vee(\pi_+)\colon \overrightarrow{\mathbf{\Pi}}^* \to \mathbf{\Pi}^* \qquad \text{sends} \qquad m_L \mapsto m_{\mathrm{base}(L)}.$$

We turn to the morphism $\vec{\beta}$. Recall the descent map from Section 10.7.2. It maps a pair of chambers $(C, D)$ to a face $\mathrm{Des}(C, D)$ of $D$. Thus, $(\mathrm{Des}(C, D), D)$ is a directed face. This allows us to define the map $\vec{\beta}$ as follows:

$$\vec{\beta}\colon \mathbb{L}^* \to \overrightarrow{\mathbf{\Sigma}}^* \qquad \text{sends} \qquad F_{(C,D)} \mapsto F_{(\mathrm{Des}(C,D),D)}.$$

It follows from (12.4) and (10.42) that

$$\vec{\beta}(F_{(C,D)}) = \sum_{G:\mathrm{Des}(C,D)\leq G\leq D} M_{(G,D)} = \sum_{G:GC=D} M_{(G,D)}.$$

It may also be shown that

$$\vec{\beta}(M_{(C,D)}) = \begin{cases} M_{(G,D)} & \text{if } C = \overline{G}D, \\ 0 & \text{otherwise,} \end{cases}$$

where $G = \mathrm{Des}(C, D)$. The proof is similar to that of [12, Lemma 5.6.2] .

Dually,

$$\vec{\beta}^*\colon \overrightarrow{\mathbf{\Sigma}} \to \mathbb{L} \quad \text{sends} \quad K_{(F,C)} \mapsto \sum_{D\colon \mathrm{Des}(D,C)=F} K_{(D,C)}$$

and we have

$$\vec{\beta}^*(H_{(F,C)}) = \sum_{D\colon \mathrm{Des}(D,C)\leq F} K_{(D,C)} = \sum_{D\colon FD=C} K_{(D,C)}$$

and also

$$\vec{\beta}^*(H_{(F,C)}) = H_{(\overline{F}C,C)}.$$

Finally, we come to the morphism $\Upsilon$. It is defined as follows:

$$\Upsilon\colon \overrightarrow{\mathbf{\Pi}} \to \mathbf{\Sigma}^* \qquad \text{sends} \qquad h_L \mapsto \sum_{F:F\subseteq\Psi(L)} M_F, \tag{12.17}$$

where the sum is over all faces $F$ contained in the cone $\Psi(L)$ associated to the directed flat $L$ as in Section 10.9.3. Explicitly, given a linear set partition $\{l^1, \dots, l^r\}$ of $I$ with $l^j$ a linear order on $S_j$,

$$\Upsilon(h_{\{l^1,\dots,l^r\}}) = \sum_F M_F, \tag{12.18}$$

where the sum is over those compositions $F = F^1|\cdots|F^k$ of $I$ such that, for each $i$ and $j$, $(F^1 \cup \cdots \cup F^i) \cap S_j$ is an initial segment of $l^j$. Note that the above map is induced from the pairing between directed flats and faces

$$\overrightarrow{\mathbf{\Pi}}[I] \times \mathbf{\Sigma}[I] \to \Bbbk$$

given by

$$\langle h_L, H_F \rangle = \begin{cases} 1 & \text{if } F \subseteq \Psi(L), \\ 0 & \text{otherwise.} \end{cases}$$

Dually,

$$\Upsilon^* \colon \mathbf{\Sigma} \to \overrightarrow{\mathbf{\Pi}}^* \qquad \text{sends} \qquad H_F \mapsto \sum_{L:\, F \subseteq \Psi(L)} m_L.$$

This completes the description of the morphisms in diagram (12.14). One other map is worth-mentioning. We let $\beta$ denote the composite

$$\mathbf{L}^* \xrightarrow{\vec{\beta}} \overrightarrow{\mathbf{\Sigma}}^* \xrightarrow{\mathcal{T}^\vee(\pi_+)} \mathbf{\Sigma}^*.$$

Explicitly, $\beta$ is given by

$$\beta(F_{(C,D)}) = \sum_{G\colon \mathrm{Des}(C,D) \le G \le D} M_G = \sum_{G\colon GC = D} M_G. \tag{12.19}$$

The dual map $\beta^* \colon \mathbf{\Sigma} \to \mathbf{L}$ is then given by

$$\beta^*(H_F) = \sum_{(C,D)\colon \mathrm{Des}(D,C) \le F \le C} K_{(D,C)} = \sum_{(C,D)\colon FD = C} K_{(D,C)}.$$

The map $\beta^*$ is the same as the map considered in (10.44).

**12.8.3. Universality of the morphisms.** We saw in Propositions 12.58 and 12.59 that most of the Hopf monoids in (12.14) arise via universal constructions. With this viewpoint, all the morphisms in (12.14) then arise from the universality of these Hopf monoids. We now explain this briefly.

**Convention 12.60.** Later in this section, as well as throughout Chapter 13, we encounter several morphisms of species of the form

$$\mathbf{p} \to \mathbf{E},$$

where $\mathbf{E}$ is the exponential species (Example 8.3) and $\mathbf{p}$ is one of several species. For ease of notation, we will describe these morphisms in terms of functionals

$$f_I \colon \mathbf{p}[I] \to \Bbbk,$$

one for each finite set $I$. The convention followed is that the $I$-component of the morphism is then

$$\mathbf{p}[I] \to \mathbf{E}[I], \qquad x \mapsto f_I(x) \cdot *_I,$$

where $*_I$ is the distinguished basis element of $\mathbf{E}[I]$.

TABLE 12.2. Universal maps.

| cofree Hopf monoid | morphism of monoids $\zeta$ | | morphism of Hopf monoids $\hat{\zeta}$ |
|---|---|---|---|
| $\overrightarrow{\mathbf{\Sigma}}^*$ | $\mathbf{L\!L}^*_+ \to \mathbf{L}_+$ | $F_{(C,D)} \mapsto \begin{cases} D & \text{if } C = D \\ 0 & \text{otherwise} \end{cases}$ | $\vec{\beta}$ |
| $\overrightarrow{\mathbf{\Sigma}}^*$ | $\overrightarrow{\mathbf{\Pi}}^*_+ \to \mathbf{L}_+$ | $m_{\{l^1,\dots,l^k\}} \mapsto \begin{cases} l^1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$ | $\pi^*_{\mathbf{L}_+}$ |
| $\mathbf{\Sigma}^*$ | $\mathbf{L\!L}^*_+ \to \mathbf{E}_+$ | $F_{(C,D)} \mapsto \begin{cases} 1 & \text{if } C = D \\ 0 & \text{otherwise} \end{cases}$ | $\beta$ |
| $\mathbf{\Sigma}^*$ | $\overrightarrow{\mathbf{\Pi}}_+ \to \mathbf{E}_+$ | $h_{\{l^1,\dots,l^k\}} \mapsto 1$ | $\Upsilon$ |
| $\mathbf{\Sigma}^*$ | $\overrightarrow{\mathbf{\Sigma}}^*_+ \to \mathbf{E}_+$ | $M_{(F,D)} \mapsto \begin{cases} 1 & \text{if } F \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$ | $\mathcal{T}^\vee(\pi_+)$ |
| $\mathbf{\Sigma}^*$ | $\mathbf{\Pi}^*_+ \to \mathbf{E}_+$ | $m_{\{X^1,\dots,X^k\}} \mapsto \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$ | $\pi^*_{\mathbf{E}_+}$ |

Consider $\mathbf{\Sigma}^*$, the cofree Hopf monoid on the nonunital monoid $\mathbf{E}_+$ (Proposition 12.58). The counit of the corresponding adjunction

$$\xi(\mathbf{E}_+) \colon \mathbf{\Sigma}^*_+ \to \mathbf{E}_+$$

is given by

$$M_{F^1|\cdots|F^l} \mapsto \begin{cases} 1 & \text{if } l = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In geometric language, $M_F$ maps to 1 if $F$ is the empty face of $\Sigma[I]$, and to 0 otherwise. It is easy to see directly that $\xi(\mathbf{E}_+)$ is a morphism of nonunital monoids but not of noncounital comonoids, as expected. The cofreeness of $\mathbf{\Sigma}^*$ explains the abundance of morphisms of Hopf monoids to $\mathbf{\Sigma}^*$. For any Hopf monoid $\mathbf{q}$, a morphism of nonunital monoids $\zeta \colon \mathbf{q}_+ \to \mathbf{E}_+$ determines a morphism of Hopf monoids $\hat{\zeta} \colon \mathbf{q} \to \mathbf{\Sigma}^*$ (Theorem 11.23).

Similar statements apply to $\overrightarrow{\mathbf{\Sigma}}^*$ with $\mathbf{E}_+$ replaced by $\mathbf{L}_+$. According to Proposition 12.58, it is the cofree Hopf monoid on the nonunital monoid $\mathbf{L}_+$. This again explains the abundance of morphisms of Hopf monoids to $\overrightarrow{\mathbf{\Sigma}}^*$. The counit of the corresponding adjunction

$$\xi(\mathbf{L}_+) \colon \overrightarrow{\mathbf{\Sigma}}^*_+ \to \mathbf{L}_+,$$

is given by

$$M_{(F,D)} \mapsto \begin{cases} D & \text{if } F \text{ is the empty face,} \\ 0 & \text{otherwise.} \end{cases}$$

Table 12.2 shows the morphisms of nonunital monoids either to $\mathbf{L}_+$ or to $\mathbf{E}_+$ which determine the various morphisms of Hopf monoids to $\overrightarrow{\mathbf{\Sigma}}^*$ and $\mathbf{\Sigma}^*$ which occur in diagram (12.14). We provide a couple of illustrations.

**Example 12.61.** The union of two linear set compositions is another linear set composition. Therefore, the map of species

$$\zeta\colon \overrightarrow{\mathbf{\Pi}} \to \mathbf{E}$$

given by

$$\zeta(h_L) := 1$$

is a morphism of monoids. Here we are following Convention 12.60. The universal property of Theorem 11.23 then yields a morphism of Hopf monoids

$$\hat{\zeta}\colon \overrightarrow{\mathbf{\Pi}} \to \mathcal{T}^{\vee}(\mathbf{E}_+) = \mathbf{\Sigma}^*.$$

We now use (11.18) to calculate this map. Let $\{l^1, \ldots, l^r\}$ be a linear set partition of $I$, with $l^j$ a linear order on $S_j$, and $F = F^1|\cdots|F^k$ a composition of $I$. It follows from Proposition 12.55 that the component of the iterated coproduct

$$\Delta_{F^1,\ldots,F^k}(h_{\{l^1,\ldots,l^r\}}) \in \overrightarrow{\mathbf{\Pi}}[F^1] \otimes \cdots \otimes \overrightarrow{\mathbf{\Pi}}[F^k]$$

is 0 unless $(F^1 \cup \cdots \cup F^i) \cap S_j$ is an initial segment of $l^j$ for each $i$ and $j$. In this case, $\Delta_{F^1,\ldots,F^k}$ simply restricts $h_{\{l^1,\ldots,l^r\}}$ to each block of $F$. It follows that $\hat{\zeta} = \Upsilon$, the map in (12.18).

**Example 12.62.** Consider the Hadamard product (8.7) of the species $\mathbf{L}$ with its dual and the canonical map

$$\mathbb{L}^* = \mathbf{L} \times \mathbf{L}^* \to \mathbf{E}$$

given by evaluation. It is easy to see that this is a morphism of monoids, but not of comonoids. The universal property of Theorem 11.23 then yields a morphism of Hopf monoids

$$\hat{\zeta}\colon \mathbb{L}^* \to \mathcal{T}^{\vee}(\mathbf{E}_+) = \mathbf{\Sigma}^*.$$

We now use (11.18) to calculate this map. Let $(C, D)$ be a pair of linear orders (chambers) on $I$ and $F = F^1|\cdots|F^k$ a composition of $I$. It follows from Definition 12.7 that the component of the iterated coproduct

$$\Delta_{F^1,\ldots,F^k}(F_{(C,D)}) \in \mathbb{L}^*[F^1] \otimes \cdots \otimes \mathbb{L}^*[F^k]$$

is 0 unless $F$ is a face of $D$ and $b_F(FC) = b_F(D)$, where $b_F$ refers to the break map (10.58). Since the break map is an isomorphism, the second condition can be rewritten simply as $FC = D$. It follows that $\hat{\zeta} = \beta$, the map in (12.19).

So far, we have discussed morphisms to $\mathbf{\Sigma}^*$ and $\overrightarrow{\mathbf{\Sigma}}^*$. Morphisms to $\mathbf{\Pi}^*$ and $\overrightarrow{\mathbf{\Pi}}^*$ can be understood similarly. More precisely, Proposition 12.58 says that $\mathbf{\Pi}^*$ and $\overrightarrow{\mathbf{\Pi}}^*$ are the cocommutative cofree Hopf monoids on $\mathbf{E}_+$ and $\mathbf{L}_+$ respectively. The counits of the corresponding adjunctions, namely

$$\xi(\mathbf{E}_+)\colon \mathbf{\Pi}^*_+ \to \mathbf{E}_+ \quad \text{and} \quad \xi(\mathbf{L}_+)\colon \overrightarrow{\mathbf{\Pi}}^*_+ \to \mathbf{L}_+,$$

coincide with the maps defined in Table 12.2. One then uses Theorem 11.27 to construct the morphisms to $\mathbf{\Pi}^*$ and $\overrightarrow{\mathbf{\Pi}}^*$.

The remaining morphisms in (12.14) can be obtained either by dualizing the morphisms that we have discussed so far, or by directly using Proposition 12.59 and proceeding as we did in the cofree case.

Several additional examples of a combinatorial nature which illustrate these universal properties are given in Chapter 13.

**12.8.4. A $q$-analogue.** Recall from the preceding sections that the Hopf monoids $\mathbf{L}$, $\mathbf{IL}$, $\overrightarrow{\mathbf{\Sigma}}$, and $\mathbf{\Sigma}$, and their duals can be deformed using a parameter $q$. So far in this section, we have only discussed the $q = 1$ (or undeformed) case. We now briefly touch upon the $q$-analogue of the preceding discussion.

To start with, we have the following generalization of (a part of) Propositions 12.58 and 12.59.

**Proposition 12.63.** *There are isomorphisms of $q$-Hopf monoids*

$$\mathbf{\Sigma}_q^* = \mathcal{T}_q^\vee(\mathbf{E}_+), \quad \overrightarrow{\mathbf{\Sigma}}_q^* = \mathcal{T}_q^\vee(\mathbf{L}_+), \quad \mathbf{\Sigma}_q = \mathcal{T}_q(\mathbf{E}_+^*), \quad \textit{and} \quad \overrightarrow{\mathbf{\Sigma}}_q = \mathcal{T}_q(\mathbf{L}_+^*).$$

The functors $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ are defined in Section 11.7. Roughly speaking, these are deformations of $\mathcal{T}$ and $\mathcal{T}^\vee$ via the Schubert cocycle on faces. This is precisely how the Hopf monoids $\mathbf{\Sigma}$, and so on were deformed. This explains why the above result holds.

We now turn to the relationships between these $q$-Hopf monoids.

**Theorem 12.64.** *The following is a self-dual diagram of $q$-Hopf monoids.*

$$\begin{tikzcd} \mathbf{L}_q \arrow[r] & \mathbf{\Sigma}_q \arrow[r] & \overrightarrow{\mathbf{\Sigma}}_q \arrow[r] & \mathbf{IL}_q \arrow[d, "s_q"] \\ \mathbf{L}_q^* & \mathbf{\Sigma}_q^* \arrow[l] & \overrightarrow{\mathbf{\Sigma}}_q^* \arrow[l] & \mathbf{IL}_q^* \arrow[l] \end{tikzcd} \tag{12.20}$$

This may be viewed as the $q$-analogue of Theorem 12.57. The duality functor acts by reflection in the horizontal line drawn in the center. The horizontal maps are the same as in diagram (12.14). The vertical map $s_q$ is the switch map (12.9). This map is an isomorphism (Proposition 12.12), so $\mathbf{IL}_q$ is self-dual. Further, in Propositions 12.6, 12.26 and 12.38, we showed that for generic values of $q$, $\mathbf{L}_q$, $\mathbf{\Sigma}_q$ and $\overrightarrow{\mathbf{\Sigma}}_q$ are self-dual. It is straightforward to check that the maps used to establish the self-duality of these objects are precisely the appropriate composites in (12.20).

These self-duality results explain why there are no $q$-analogues of $\mathbf{E}$, $\mathbf{\Pi}$ and $\overrightarrow{\mathbf{\Pi}}$. Thus Theorem 12.64 is the correct $q$-analogue of Theorem 12.57.

**Question 12.65.** Consider the map $\mathbf{\Sigma} \to \mathbf{\Sigma}^*$ obtained from diagram (12.14). Unlike the generic case, this map is far from being an isomorphism. In fact, we know that the image of this map yields a self-dual Hopf monoid which is isomorphic to $\mathbf{\Pi}$ or $\mathbf{\Pi}^*$.

Now consider the composite map $\overrightarrow{\mathbf{\Sigma}} \to \overrightarrow{\mathbf{\Sigma}}^*$. As the preceding map, it is a self-dual morphism of Hopf monoids. Hence the image of this map is a self-dual Hopf monoid. The question is to describe this Hopf monoid in explicit terms. Note that this Hopf monoid differs from $\overrightarrow{\mathbf{\Pi}}$ or $\overrightarrow{\mathbf{\Pi}}^*$, since these are not self-dual (Section 12.7).

**Question 12.66.** The previous question shows that the degeneracies in (12.20) are not fully understood even for $q = 1$. We now ask the same question but for $q = -1$.

The image of the map $\mathbf{L}_{-1} \to \mathbf{L}_{-1}^*$ is understood; it is the self-dual $(-1)$-Hopf monoid of the signed exponential species (Section 9.3). The corresponding commutative diagram is (9.21). We ask for similar results for $\mathbf{\Sigma}_{-1}$ and $\overrightarrow{\mathbf{\Sigma}}_{-1}$.

**12.8.5. Some concluding remarks.** In this chapter, we studied various Hopf monoids. This study was done separately for each Hopf monoid without making use of the inter-relationships between them, which are provided by diagrams (12.14) and (12.20). While this approach has its advantages, it is clearly inefficient. We provide an example to illustrate this point.

We know that antipodes commute with morphisms of Hopf monoids. Hence, the antipode formula for $\overrightarrow{\mathbf{\Sigma}}$, along with the morphisms relating it with $\mathbf{\Sigma}$, $\overrightarrow{\mathbf{\Pi}}$ and $\mathbf{\Pi}$, can be used to quickly derive antipode formulas for the latter. However, in the present exposition, we computed antipode formulas for each of them separately using Takeuchi's formula.

**Question 12.67.** The right-hand sides in the antipode formulas, for example Theorems 12.17 and 12.18, make sense for any central hyperplane arrangement, but the Hopf monoids exist only for the Coxeter complex of type $A$. What meaning do these formulas have in general?

CHAPTER 13

# Hopf Monoids from Combinatorics

Hopf monoids in species abound. In this chapter, we discuss a number of examples of Hopf monoids that arise naturally in combinatorics. In one form or another, many of these ideas go back to the paper by Joni and Rota [179]; we believe that they find a most natural and useful formulation in the context of species.

Each of these Hopf monoids is based on a particular combinatorial structure (specified by a species), such as posets, graphs, or matroids. For each of these there is a natural way to decompose a given combinatorial structure on a set $I = S \sqcup T$ into combinatorial structures on subsets $S$ and $T$. The coproducts of these Hopf monoids arise in this manner. A companion procedure for merging structures gives rise to the product.

Often, there are natural procedures for transforming a combinatorial structure on a set into a different combinatorial structure on the same set, which are compatible with the operations alluded to above. We discuss several examples of this kind; they give rise to morphisms of Hopf monoids. In particular, the universal properties of Chapter 11 allow us to construct from minimal principles a number of such morphisms. The target of these morphisms are usually the Hopf monoids $\mathbf{L}^*$ of linear orders and $\mathbf{\Sigma}^*$ of set compositions.

Applying the Fock functors to these Hopf monoids yields many well-known Hopf algebras, including the Hopf algebras of symmetric and quasi-symmetric functions. In the same vein we obtain morphisms of Hopf algebras that associate a (quasi) symmetric function to a given combinatorial structure on a set. Since they arise from morphisms of species, isomorphic structures have the same associated function. In other words, these functions are invariants of the combinatorial structures. These ideas are the subject of Chapter 17.

A Hopf monoid is a finer structure than that of the corresponding Hopf algebras: the latter arise from the former by means of the Fock functors, but in general one cannot recover the Hopf monoid from the Hopf algebra(s). On the other hand, the necessary ingredients for the construction of a Hopf monoid are often present, in some form, in the construction of the Hopf algebras. For this reason, many examples we present in this chapter owe a lot to earlier work on Hopf algebras by many authors. For specific references to the Hopf algebra literature, see Chapter 17.

Table 13.1 lists the main Hopf monoids discussed in this chapter. In addition, in Section 13.7.3 we discuss a monoid based on certain simplicial complexes.

We work over a field $\Bbbk$ of characteristic 0.

## 13.1. Posets

A partial order on a set $I$ is a reflexive, antisymmetric and transitive relation on $I$. It is also called a *poset*. The elements of $I$ are the *vertices* of the poset. Apart from the fact that the vertex set $I$ is always assumed to be finite, in this section

TABLE 13.1. Hopf monoids from combinatorics.

| Hopf monoid | Linear Basis | Section |
|---|---|---|
| $\mathbf{P}$ | posets | 13.1.1 |
| $\mathbf{O}$ | preposets | 13.1.6 |
| $\mathbf{G}$ | simple graphs | 13.2.1 |
| $\mathbf{F}$ | rooted forests | 13.3.1 |
| $\overrightarrow{\mathbf{F}}$ | planar rooted forests | 13.3.2 |
| $\mathbf{R}$ | relations | 13.4.1 |
| $\mathbf{Q}$ | equivalence relations | 13.4.3 |
| $\mathbf{sgP}$ | set-graded posets | 13.6.4 |
| $\mathbf{swP}$ | set-weighted posets | 13.6.8 |
| $\mathbf{C}$ | closure operators | 13.8.1 |
| $\mathbf{M}$ | matroids | 13.8.2 |
| $\mathbf{cG}$ | convex geometries | 13.8.3 |
| $\mathbf{T}$ | topologies | 13.8.6 |

we work with arbitrary posets. We also briefly consider preposets (reflexive and transitive relations).

**13.1.1. The Hopf monoid of posets.** Given a finite set $I$, let $\mathbf{P}[I]$ be the vector space with basis the set of all partial orders on $I$. For instance, $\mathbf{P}[\{a,b\}]$ is 3-dimensional, spanned by the posets $\{a < b\}$, $\{b < a\}$, and $\{a, b\}$ (no relations). It is customary to represent posets by their Hasse diagrams. As an example, the poset

$$\begin{tikzcd}[arrows=-] s \arrow[dr] & & h \arrow[dl] & u \arrow[d] \\ & i \arrow[d] & & n \\ & v & & \end{tikzcd} \qquad \in \mathbf{P}[\{v,i,s,h,n,u\}].$$

We agree that $\mathbf{P}[\emptyset] = \Bbbk$, spanned by the empty poset. This defines the species $\mathbf{P}$ of *posets*.

We proceed to turn $\mathbf{P}$ into a Hopf monoid, following Gessel [144] and Malvenuto [255]. In the formulas below, we regard a partial order $p$ on $I$ as a subset of $I \times I$, that is, as a relation on the set $I$.

Fix a decomposition $I = S \sqcup T$. The corresponding component $\mu_{S,T}$ of the product is

$$\begin{aligned} \mathbf{P}[S] \otimes \mathbf{P}[T] &\to \mathbf{P}[I] \\ p_1 \otimes p_2 &\mapsto p_1 \sqcup p_2. \end{aligned} \tag{13.1}$$

This is the (disjoint) union of the sets $p_1 \subseteq S \times S$ and $p_2 \subseteq T \times T$, so that in $p_1 \sqcup p_2$ there are no relations between the elements of $S$ and $T$.

The restriction of $p$ to $S$ is

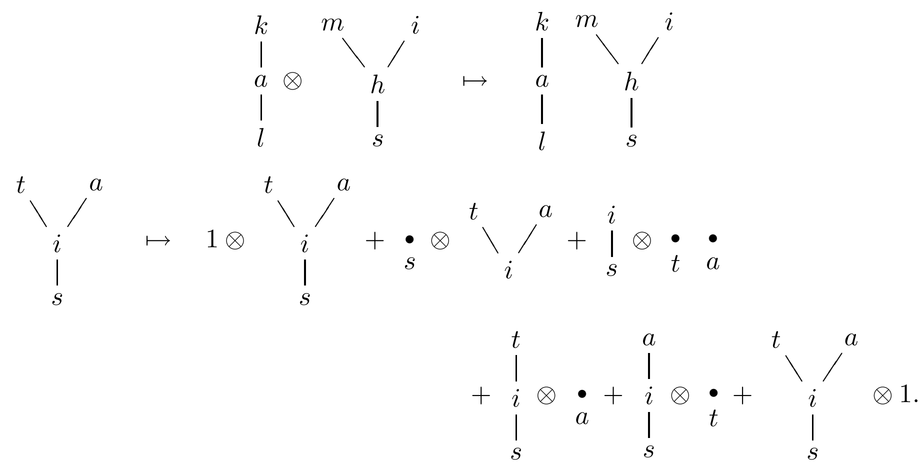
$$p|_S := p \cap (S \times S).$$

We say that $S$ is a *lower set* of $p$ if $p \cap (T \times S) = \emptyset$, that is, if no element of $T$ is less than an element of $S$. In this case, we also say that $T$ is an *upper set* of $p$.

For the coproduct, we set the component $\Delta_{S,T}$ to be

$$\begin{gathered} \mathbf{P}[I] \to \mathbf{P}[S] \otimes \mathbf{P}[T] \\ p \mapsto \begin{cases} p|_S \otimes p|_T & \text{if } S \text{ is a lower set of } p, \\ 0 & \text{otherwise.} \end{cases} \end{gathered} \tag{13.2}$$

For example,



The Hopf monoid axioms are straightforward, but we pause to verify the compatibility between the product and the coproduct in detail, since this well-illustrates how the essential combinatorial facts are faithfully and simply expressed in the language of species. According to (8.18), we have to verify the commutativity of the following diagram.

$$\begin{tikzcd} \mathbf{P}[A] \otimes \mathbf{P}[B] \otimes \mathbf{P}[C] \otimes \mathbf{P}[D] \arrow[rr, "\mathrm{id}_A \otimes \beta_{B,C} \otimes \mathrm{id}_D"] & & \mathbf{P}[A] \otimes \mathbf{P}[C] \otimes \mathbf{P}[B] \otimes \mathbf{P}[D] \arrow[d, "\mu_{A,C} \otimes \mu_{B,D}"] \\ \mathbf{P}[S] \otimes \mathbf{P}[T] \arrow[u, "\Delta_{A,B} \otimes \Delta_{C,D}"] \arrow[r, "\mu_{S,T}"'] & \mathbf{P}[I] \arrow[r, "\Delta_{S',T'}"'] & \mathbf{P}[S'] \otimes \mathbf{P}[T'], \end{tikzcd}$$

where $S \sqcup T = I = S' \sqcup T'$ are two decompositions of a finite set $I$ and $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, $D = T \cap T'$ as in Figure 8.1. Let $p_1$ and $p_2$ be partial orders on $S$ and $T$, respectively. One immediately sees that commutativity boils down to the following two facts:

$$A \text{ is a lower set of } p_1 \text{ and } C \text{ is a lower set of } p_2 \iff S' \text{ is a lower set of } p_1 \sqcup p_2;$$
$$(p_1 \sqcup p_2)|_{S'} = p_1|_A \sqcup p_2|_C \qquad \text{and} \qquad (p_1 \sqcup p_2)|_{T'} = p_1|_B \sqcup p_2|_D.$$

The Hopf monoid $\mathbf{P}$ is commutative but not cocommutative.

Let us briefly discuss the dual Hopf monoid $\mathbf{P}^*$. Let $\{p^*\}$ denote the basis of $\mathbf{P}^*[I]$ dual to the basis $\{p\}$ of $\mathbf{P}[I]$, with $p$ running over all partial orders on $I$.

The product is

$$\begin{aligned}\mathbf{P}^*[S] \otimes \mathbf{P}^*[T] &\to \mathbf{P}^*[I] \\ p_1^* \otimes p_2^* &\mapsto \sum_{x \subseteq S \times T} (p_1 \cup x \cup p_2)^*.\end{aligned}$$

Equivalently, the above sum is over those posets $p$ for which $p|_S = p_1$, $p|_T = p_2$, and no element of $T$ is less than an element of $S$. In other words, the Hasse diagram of $p$ is the union of the Hasse diagrams of $p_1$ and $p_2$ together with some edges going from $S$ to $T$.

The coproduct is

$$\begin{aligned}\mathbf{P}^*[I] &\to \mathbf{P}^*[S] \otimes \mathbf{P}^*[T] \\ p^* &\mapsto \begin{cases} (p|_S)^* \otimes (p|_T)^* & \text{if } p \cap \big((S \times T) \cup (T \times S)\big) = \emptyset, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus, the component of the coproduct is 0 precisely if there is a relation between elements of $S$ and $T$.

**13.1.2. Lower decompositions.** Let $\mathbf{\Sigma}^*$ be the Hopf monoid of set compositions discussed in Section 12.4. We construct a morphism of Hopf monoids $\mathbf{P} \to \mathbf{\Sigma}^*$ as an illustration of the universal property of $\mathbf{\Sigma}^*$. More elaborate examples of such universal morphisms are discussed throughout this chapter.

The product of two posets is another poset. Therefore, the map of species

$$\eta \colon \mathbf{P} \to \mathbf{E}$$

given by

$$\eta(p) := 1$$

is a morphism of monoids. Here we are following Convention 12.60. The universal property of Theorem 11.23 then yields a morphism of Hopf monoids

$$\hat{\eta} \colon \mathbf{P} \to \mathcal{T}^\vee(\mathbf{E}_+) = \mathbf{\Sigma}^*.$$

The identification of the cofree Hopf monoid on $\mathbf{E}_+$ with the Hopf monoid $\mathbf{\Sigma}^*$ comes from Proposition 12.58.

We now use (11.18) to calculate this map. Let $p$ be a poset on $I$ and $F = F^1|\cdots|F^k$ a composition of $I$. It follows from (13.2) that the iterated coproduct

$$\Delta_{F^1,\ldots,F^k}(p) \in \mathbf{P}[F^1] \otimes \cdots \otimes \mathbf{P}[F^k]$$

is 0 unless

$$\emptyset \subset F^1 \subset F^1 \cup F^2 \subset \cdots \subset F^1 \cup F^2 \cup \cdots \cup F^k$$

is a chain of lower sets in the poset $p$. In this case, $\Delta_{F^1,\ldots,F^k}$ simply restricts $p$ to each block of $F$. It follows that:

$$\hat{\eta}(p) = \sum_F M_F, \tag{13.3}$$

where the sum is over those compositions $F = F^1|\cdots|F^k$ of $I$ such that each $F^1 \cup \cdots \cup F^i$ is a lower set of $p$.

**13.1.3. Linear extensions of posets.** The theory of linear extensions of posets and poset partitions was developed by Stanley [336], [341, Section 4.5]. We discuss how these combinatorial notions emerge naturally from minimal principles.

Everything will follow from the consideration of a very simple map

$$\omega\colon \mathbf{P}_+ \to \mathbf{X}$$

from the (positive part of the) species of posets to the species $\mathbf{X}$, characteristic of singletons (8.3). We let $\omega$ be the map which identifies the basis elements of $\mathbf{P}[I]$ and $\mathbf{X}[I]$ when $I$ is a singleton (and which is zero otherwise). Up to a constant, this is the unique morphism of species from $\mathbf{P}_+$ to $\mathbf{X}$. Clearly, $\omega$ is a morphism of positive monoids (the product on $\mathbf{X}$ is zero).

Recall the Hopf monoid $\mathbf{L}^*$ of linear orders from Example 8.24. We know from Example 11.24 that $\mathbf{L}^* = \mathcal{T}^\vee(\mathbf{X})$ is the cofree Hopf monoid on $\mathbf{X}$. The universal property of Theorem 11.23 yields a morphism of Hopf monoids

$$\widehat{\omega}\colon \mathbf{P} \to \mathbf{L}^*.$$

We now use (11.18) to calculate this map. Let $p$ be a poset on $I$ and $F = F^1|\cdots|F^k$ a composition of $I$. We saw in the previous example that the iterated coproduct $\Delta_{F^1,\ldots,F^k}(p)$ is nonzero only if the blocks of $F$ give rise to a chain of lower sets in the poset $p$. In addition, the only compositions that contribute to (11.18) are those for which each block is a singleton, since otherwise $\omega$ is 0. Compositions satisfying both of these conditions are the same thing as linear orders on $I$ which are compatible with the partial order $p$; in other words, they are *linear extensions* of $p$. Thus

$$\widehat{\omega}(p) = \sum_{l \in \mathcal{L}(p)} l^* \tag{13.4}$$

where $\mathcal{L}(p)$ denotes the set of linear extensions of $p$.

**13.1.4. Poset partitions.** By further elementary manipulations, we will next arrive at the notion of poset partition.

Consider the evaluation map

$$\mathbb{L}^* = \mathbf{L} \times \mathbf{L}^* \to \mathbf{E}.$$

According to Example 12.62, applying the universal property of $\mathbf{\Sigma}^* = \mathcal{T}^\vee(\mathbf{E}_+)$ to this morphism of monoids yields the morphism of Hopf monoids $\beta : \mathbb{L}^* \to \mathbf{\Sigma}^*$ defined in (12.19). On the other hand, composing the evaluation map with $\mathrm{id} \times \widehat{\omega}$ we obtain another morphism of monoids

$$\begin{tikzcd} \mathbf{L} \times \mathbf{P} \arrow[rr, "\mathrm{id}\times\widehat{\omega}"] \arrow[dr, "\zeta"'] & & \mathbb{L}^* \arrow[dl] \\ & \mathbf{E}. & \end{tikzcd}$$

Explicitly, using Convention 12.60,

$$\zeta(l,p) = \begin{cases} 1 & \text{if } l \in \mathcal{L}(p), \\ 0 & \text{otherwise.} \end{cases}$$

Applying the universal property of Theorem 11.23 to the morphism of monoids $\zeta$ we obtain a commutative diagram of Hopf monoids

$$\begin{tikzcd} \mathbf{L} \times \mathbf{P} \arrow[rr, "\mathrm{id}\times\widehat{\omega}"] \arrow[dr, "\hat{\zeta}"'] & & \mathbb{L}^* \arrow[dl, "\beta"] \\ & \mathcal{T}^\vee(\mathbf{E}_+) = \mathbf{\Sigma}^*. & \end{tikzcd}$$

Let $p$ be a partial order on $I$ and $l$ a linear order on $I$. Using (11.18) one readily finds that

$$\hat{\zeta}(l,p) = \sum_F M_F \tag{13.5}$$

the sum being over those compositions $F = F^1|\cdots|F^k$ of $I$ satisfying two conditions:

- each $F^1 \cup \cdots \cup F^i$ is a lower set of $p$, $i = 1, \ldots, k$;
- the restriction of $l$ to each $F^i$ is a linear extension of the restriction of $p$.

These conditions may be equivalently formulated as follows:

- if $a \in F^i$, $b \in F^j$, and $a \le b$ in $p$, then $i \le j$;
- if in addition to the above we have $a > b$ in $l$, then $i < j$.

Such compositions $F$ correspond precisely to the notion of *poset partitions* (of $p$ with respect to $l$) in the literature. This notion can be captured nicely using Tits projection maps (10.13) as follows.

**Proposition 13.1.** *A set composition $F$ is a partition of the poset $p$ with respect to $l$ if and only if $F \cdot l$ is a linear extension of $p$.*

**13.1.5. Posets among other Hopf monoids.** The Hopf monoid $\overrightarrow{\mathbf{\Pi}} = \mathcal{S}(\mathbf{L}_+^*)$ of linear set partitions is studied in Section 12.7.2.

Consider the map of species

$$\mathbf{L}_+^* \xrightarrow{\nu} \mathbf{P}_+$$

which views a linear order as a partial order. A lower set of a linear order is an initial segment. Therefore, the above map is a morphism of positive comonoids (the coproducts are described in Example 8.24 and (13.2)). Since $\mathbf{P}$ is commutative, the map extends uniquely to a morphism of Hopf monoids

$$\overrightarrow{\mathbf{\Pi}} = \mathcal{S}(\mathbf{L}_+^*) \xrightarrow{\hat{\nu}} \mathbf{P}$$

(Theorem 11.14). It is explicitly given as in the example below.

$$\{v|i|s, 1|2, h|n|u\} \quad \mapsto \quad \begin{array}{ccc} s & & u \\ | & 2 & | \\ i & | & n \\ | & 1 & | \\ v & & h \end{array}$$

This map is injective. It identifies linear set partitions with those posets that are unions of linear orders.

There is a commutative diagram of monoids

$$\begin{tikzcd} \overrightarrow{\mathbf{\Pi}} \arrow[rr, "\hat{\nu}"] \arrow[dr] & & \mathbf{P} \arrow[dl, "\eta"] \\ & \mathbf{E} & \end{tikzcd}$$

with the morphism $\overrightarrow{\mathbf{\Pi}} \to \mathbf{E}$ as in Example 12.61 and $\eta$ as in Section 13.1.2. By universality, it gives rise to the following commutative diagram of Hopf monoids.

$$\begin{tikzcd} \overrightarrow{\mathbf{\Pi}} \arrow[rr, "\hat{\nu}"] \arrow[dr, "\Upsilon"'] & & \mathbf{P} \arrow[dl, "\hat{\eta}"] \\ & \mathbf{\Sigma}^* & \end{tikzcd} \tag{13.6}$$

The morphism $\Upsilon$ intervenes in diagram (12.14); explicitly, it is given by (12.18).

Recall the Hopf monoid $\overrightarrow{\mathbf{\Sigma}} = \mathcal{T}(\mathbf{L}_+^*)$ of linear set compositions (Section 12.5). Proceeding along the same lines as above, one constructs a morphism of Hopf monoids

$$\overrightarrow{\mathbf{\Sigma}} = \mathcal{T}(\mathbf{L}_+^*) \xrightarrow{\vec{\nu}} \mathbf{L} \times \mathbf{P}$$

given as in the example below.

$$H_{v|i|s|1|2|h|n|u} \quad \mapsto \quad v|i|s|1|2|h|n|u \otimes \begin{pmatrix} s & & u \\ | & 2 & | \\ i & | & n \\ | & 1 & | \\ v & & h \end{pmatrix}$$

The morphisms constructed in this section and the preceding ones fit in the commutative diagram below.

$$\begin{tikzcd} \overrightarrow{\mathbf{\Sigma}} = \mathcal{T}(\mathbf{L}_+^*) \arrow[r, "\vec{\nu}"] \arrow[d, "\pi_{\mathbf{L}_+^*}"'] & \mathbf{L} \times \mathbf{P} \arrow[r, "\mathrm{id} \times \widehat{\omega}"] \arrow[d, "\pi \times \mathrm{id}"] & \mathbf{L} \times \mathbf{L}^* \arrow[d, "\pi \times \mathrm{id}"] \\ \overrightarrow{\mathbf{\Pi}} = \mathcal{S}(\mathbf{L}_+^*) \arrow[r, "\hat{\nu}"'] & \mathbf{P} \arrow[r, "\widehat{\omega}"'] & \mathbf{L}^* \end{tikzcd} \tag{13.7}$$

The map $\pi_{\mathbf{L}_+^*}$ is as in (12.16). The top horizontal composite is also the composite of the map $\vec{\beta}^*$ and the switch map $s$ appearing in diagram (12.14). The map $\pi$ is defined in (8.31).

**13.1.6. Preposets.** A *preposet* is a relation which is transitive and reflexive. Preposets are also called *preorders*. The two main examples of preposets are equivalence relations and posets. The former are symmetric while the latter are antisymmetric. In fact, every preposet may be broken into its symmetric and antisymmetric parts as follows. Given a preposet $r$ on $I$, define an equivalence relation on $I$ by

$$i \sim j \iff (i,j) \in r \text{ and } (j,i) \in r. \tag{13.8}$$

The relation $r$ then induces a partial order on the set of equivalence classes $I/\sim$. In other words, a preposet is the same as an equivalence relation on the underlying set along with a partial order on the set of equivalence classes.

Let $r$ be a preposet on $I = S \sqcup T$. We say that $S$ is a *lower set* of $r$ if $r \cap (T \times S) = \emptyset$. This is the same definition as for posets (Section 13.1.1).

Let $\mathbf{O}$ be the species of preposets. The operations (13.1) and (13.2) can be defined for preposets. They turn $\mathbf{O}$ into a Hopf monoid and $\mathbf{P}$ is a Hopf submonoid.

The map

$$\mathbf{O} \to \mathbf{P}, \quad r \mapsto \begin{cases} r & \text{if } r \text{ is a poset,} \\ 0 & \text{otherwise,} \end{cases} \tag{13.9}$$

is a morphism of Hopf monoids, split by the inclusion:

$$\mathbf{P} \leftrightarrows \mathbf{O}.$$

Note that, as species,

$$\mathbf{O} = \mathbf{P} \circ \mathbf{E}_+ .$$

## 13.2. Simple graphs

All our graphs have finitely many vertices and edges and are simple, that is, contain no loops and no multiple edges. In particular, a graph with vertex set $I$ is precisely a symmetric irreflexive relation on $I$.

**13.2.1. The Hopf monoid of simple graphs.** Let $\mathbf{G}[I]$ be the vector space with basis the set of simple graphs with vertex set $I$. Restricting attention to simple graphs ensures that each space $\mathbf{G}[I]$ is finite-dimensional, but is not an essential requirement otherwise. We agree that $\mathbf{G}[\emptyset] = \Bbbk$, spanned by the empty graph. This defines the species $\mathbf{G}$ of simple graphs.

The restriction of a graph $g$ with vertex set $I$ to a subset $S \subseteq I$ has the same meaning as for posets (Section 13.1.1):

$$g|_S := g \cap (S \times S).$$

We turn $\mathbf{G}$ into a Hopf monoid, following Schmitt [322, Example 3.3.(3)]. Fix a decomposition $I = S \sqcup T$. The corresponding component $\mu_{S,T}$ of the product is

$$\begin{aligned} \mathbf{G}[S] \otimes \mathbf{G}[T] &\to \mathbf{G}[I] \\ g_1 \otimes g_2 &\mapsto g_1 \sqcup g_2, \end{aligned} \tag{13.10}$$

where $g_1 \sqcup g_2$ denotes disjoint union (no edges between $g_1$ and $g_2$). The coproduct is an instance of Schmitt's comonoid construction (Section 8.7.8). We set the component $\Delta_{S,T}$ to be

$$\begin{aligned} \mathbf{G}[I] &\to \mathbf{G}[S] \otimes \mathbf{G}[T] \\ g &\mapsto g|_S \otimes g|_T. \end{aligned} \tag{13.11}$$

The Hopf monoid $\mathbf{G}$ is both commutative and cocommutative.

For the dual Hopf monoid $\mathbf{G}^*$, the product is

$$\begin{aligned} \mathbf{G}^*[S] \otimes \mathbf{G}^*[T] &\to \mathbf{G}^*[I] \\ (g_1)^* \otimes (g_2)^* &\mapsto \sum_g g^*. \end{aligned}$$

The sum is over all those graphs $g$ with vertex set $I$ for which $g|_S = g_1$ and $g|_T = g_2$. In other words, an edge of $g$ is either an edge of $g_1$, or an edge of $g_2$, or it connects an element of $S$ to an element of $T$. The coproduct is

$$\begin{aligned} \mathbf{G}^*[I] &\to \mathbf{G}^*[S] \otimes \mathbf{G}^*[T] \\ g^* &\mapsto \begin{cases} (g|_S)^* \otimes (g|_T)^* & \text{if } g \cap \big((S \times T) \cup (T \times S)\big) = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, the component of the coproduct is 0 unless there are no edges between elements of $S$ and $T$.

**13.2.2. Graph coloring.** A graph is *discrete* if it has no edges. We start from the map

$$\zeta\colon \mathbf{G} \to \mathbf{E}.$$

defined by

$$\zeta(g) := \begin{cases} 1 & \text{if } g \text{ is discrete,} \\ 0 & \text{otherwise.} \end{cases} \tag{13.12}$$

As always, Convention 12.60 is enforced.

The universal property of Theorem 11.27 yields a morphism of Hopf monoids

$$\hat{\zeta}\colon \mathbf{G} \to \mathcal{S}^\vee(\mathbf{E}_+) = \mathbf{\Pi}^*.$$

The theorem applies since $\mathbf{G}$ is cocommutative. The identification of the cofree cocommutative Hopf monoid on $\mathbf{E}_+$ with the Hopf monoid $\mathbf{\Pi}^*$ comes from Proposition 12.58.

In order to describe this map in explicit terms, let us recall the notion of *stable partitions* from [339, p. 170]. A partition $X = \{X^1, \dots, X^k\}$ of the vertex set of $g$ is said to be stable if the restriction $g|_{X^i}$ is discrete for all $i$. Such partitions correspond to proper colorings of the graph in the evident manner.

An easy application of formula (11.18) gives

$$\hat{\zeta}(g) = \sum_X m_X, \tag{13.13}$$

where the sum is over all stable partitions of the vertex set.

More generally, for a parameter $q \in \Bbbk$, one may consider the morphism of monoids

$$\zeta_q\colon \mathbf{G} \to \mathbf{E}$$

given by

$$\zeta_q(g) := q^{e(g)} \tag{13.14}$$

where $e(g)$ denotes the number of edges in $g$. This reduces to the previous version when $q = 0$. Application of formula (11.18) gives a deformation of (13.13).

**13.2.3. From graphs to posets.** There is a canonical morphism from the Hopf monoid of graphs (Section 13.2.1) to that of posets (Section 13.1.1).

Recall that we can view both posets and graphs with vertex set $I$ as relations on $I$. Given such a poset $p$, let $H(p)$ denote its Hasse diagram and let $C(p)$ denote its *comparability graph*: there is an edge in $C(p)$ joining two elements of $I$ if one is less than the other according to $p$. We view both $H(p)$ and $C(p)$ as (undirected) graphs with vertex set $I$.

Following ideas of Stanley [338], we define

$$\rho\colon \mathbf{G} \to \mathbf{P}$$

by

$$\rho(g) := \sum_{H(p)\subseteq g\subseteq C(p)} p.$$

This can be viewed as a sum over the set of *acyclic orientations* of $g$. Each acyclic orientation of $g$ gives rise to a partial order $p$ on the set of vertices of $g$ by declaring that one vertex is less than another if there is a directed path from the former to the latter. Then $H(p) \subseteq g \subseteq C(p)$ and any such $p$ arises in this manner from a unique acyclic orientation of $g$.

It is not difficult to see that $\rho$ is a morphism of Hopf monoids. Moreover, it fits in the following commutative diagram of Hopf monoids.

$$\begin{tikzcd} \mathbf{G} \arrow[r, "\rho"] \arrow[d, "\zeta_1"'] & \mathbf{P} \arrow[d, "\widehat{\omega}"] \\ \mathbf{E}^* \arrow[r, "\pi^*"'] & \mathbf{L}^* \end{tikzcd}$$

The vertical maps are those in (13.4) and (13.14); the latter with $q = 1$. The map $\pi\colon \mathbf{L} \to \mathbf{E}$ is defined in (8.31). Its dual $\pi^*$ sends the basis element of $\mathbf{E}^*[I]$ to

$$\sum_{l\in\mathbf{L}[I]} l^*.$$

The commutativity of this diagram is best seen by making use of the universal property of $\mathbf{L}^*$ (cofreeness). Since all the maps are morphisms of comonoids (in fact, of Hopf monoids), it suffices to check that the diagram commutes after composing with

$$\mathbf{L}^* \to \mathbf{X}.$$

This means we are reduced to the case of a graph with a single vertex, in which case the statement is trivial.

The combinatorial fact encoded in the commutativity of this diagram is not entirely trivial. It states that given a simple graph $g$ with vertex set $I$ and a linear order $l$ on $I$, there exists a unique acyclic orientation of $g$ such that if there is an oriented edge $a \to b$, then $a < b$ in $l$. (To construct it, orient each edge from the smallest vertex to the biggest.)

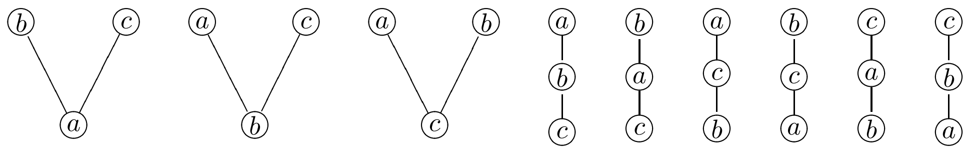
## 13.3. Rooted trees and forests

We briefly discuss two Hopf monoids based on rooted forests and on planar rooted forests which give rise to the Connes–Kreimer Hopf algebras. We also explain how they relate to other Hopf monoids discussed so far.

**13.3.1. The Hopf monoid of rooted forests.** A *rooted tree* is an acyclic, connected graph with a distinguished vertex called the *root*. A *rooted forest* is a graph whose connected components are rooted trees.

If the vertex set of a rooted tree (forest) is $I$, we say it is a rooted tree (forest) *on* $I$. We draw rooted trees and forests with the root at the bottom. The *ancestors* of a vertex $a$ in a rooted forest are the vertices along the path from $a$ down to the root of the tree to which $a$ belongs (including $a$ and the root). The *children* of a vertex $a$ are the vertices which have $a$ as its immediate ancestor.

If $|I| = n$, the number of rooted trees on $I$ is $n^{n-1}$ [343, Proposition 5.3.2]. Here are all 9 rooted trees on $\{a, b, c\}$.



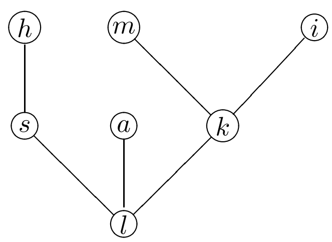
Let $\mathbf{a}[I]$ be the vector space with basis the set of rooted trees on $I$, and $\mathbf{F}[I]$ the vector space with basis the set of rooted forests on $I$. This defines species $\mathbf{a}$ and $\mathbf{F}$. The species $\mathbf{a}$ is positive, so we may consider $\mathcal{S}(\mathbf{a})$, the free commutative monoid on $\mathbf{a}$ (Section 11.3.1). The basis elements of $\mathcal{S}(\mathbf{a})[I]$ can be identified with rooted forests on $I$ and thus $\mathcal{S}(\mathbf{a})$ can be identified with $\mathbf{F}$. The product is disjoint union.

Given a rooted forest $f$, we say that a subset $S$ of its vertex set is *$f$-admissible* if whenever $a \in S$, every ancestor of $a$ is also in $S$.

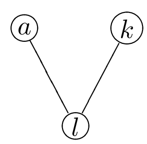
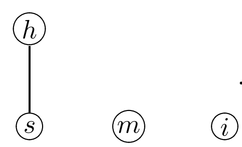
We define a comonoid structure on $\mathbf{F}$ as follows. Fix a decomposition $I = S \sqcup T$. The corresponding component of the coproduct is

$$\mathbf{F}[I] \to \mathbf{F}[S] \otimes \mathbf{F}[T], \quad f \mapsto \begin{cases} f|_S \otimes f|_T & \text{if } S \text{ is } f\text{-admissible,} \\ 0 & \text{otherwise.} \end{cases} \tag{13.15}$$

The restrictions are taken as for graphs (Section 13.2.1). For instance, if

(13.16) $t =$  and $S = \{l, a, k\}$,

then $S$ is $t$-admissible and

$\Delta_{S,T}(t) =$  $\otimes$  .

Together with the standard (free commutative) monoid structure on $\mathbf{F} = \mathcal{S}(\mathbf{a})$, this turns the species of rooted forests into a Hopf monoid.

Note that $\mathbf{a}$ is not a subcomonoid; the coproduct sends $\mathbf{a}$ to $\mathbf{a} \cdot \mathcal{S}(\mathbf{a})$. In particular, this Hopf monoid structure of $\mathcal{S}(\mathbf{a})$ does not arise from the construction of Section 11.3.2.

**13.3.2. The Hopf monoid of planar rooted forests.** A rooted tree is *planar* if for each vertex, the set of its children is given a linear order. If $|I| = n$, there are $n!\, C_{n-1}$ planar rooted trees on $I$, where

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

is the $n$-th Catalan number [343, Example 6.2.8, Exercise 6.19.e].

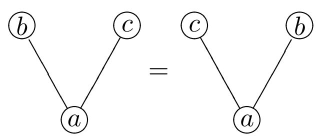
A planar rooted forest is an ordered sequence of planar rooted trees. In our diagrams, the children of each vertex in a planar rooted tree and the trees in a planar rooted forest are drawn from left to right in the given order.

Let $\vec{\mathbf{a}}[I]$ be the vector space with basis the set of planar rooted trees on $I$, and $\overrightarrow{\mathbf{F}}[I]$ the vector space with basis the set of plnar rooted forests on $I$. This defines species $\vec{\mathbf{a}}$ and $\overrightarrow{\mathbf{F}}$. The species $\vec{\mathbf{a}}$ is positive, so we may consider $\mathcal{T}(\vec{\mathbf{a}})$, the free monoid on $\vec{\mathbf{a}}$ (Section 11.2.1). The basis elements of $\mathcal{T}(\vec{\mathbf{a}})[I]$ can be identified with planar rooted forests on $I$ and thus $\mathcal{T}(\vec{\mathbf{a}})$ can be identified with $\overrightarrow{\mathbf{F}}$. The product is left-to-right concatenation of planar forests.

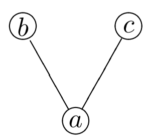
The restrictions of a rooted forest to an admissible subset of vertices and to its complement inherit a left-to-right order, and are planar rooted forests themselves. The coproduct (13.15) of $\mathbf{a}$ can therefore be lifted to $\vec{\mathbf{a}}$, by means of the same definition. This turns $\overrightarrow{\mathbf{F}} = \mathcal{T}(\vec{\mathbf{a}})$ into a Hopf monoid.

The species $\vec{\mathbf{a}}$ is not a subcomonoid of $\mathcal{T}(\vec{\mathbf{a}})$; the Hopf monoid structure of the latter does not arise from the construction of Section 11.2.5.

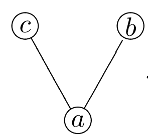
**13.3.3. From rooted forests to posets and linear orders.** By forgetting all linear orders involved in a planar rooted tree or forest (those on the sets of children of a vertex in a planar rooted tree, and that among the trees in a planar rooted forest) one obtains the *underlying* rooted tree or forest. For instance, the rooted tree



underlies both planar rooted trees



and



.

This defines maps of species

$$\upsilon \colon \vec{\mathbf{a}} \twoheadrightarrow \mathbf{a} \quad \text{and} \quad \upsilon \colon \overrightarrow{\mathbf{F}} = \mathcal{T}(\vec{\mathbf{a}}) \twoheadrightarrow \mathcal{S}(\mathbf{a}) = \mathbf{F},$$

the latter being a morphism of Hopf monoids.

A rooted forest on $I$ defines a partial order on $I$ in which $a \leq b$ if $a$ is an ancestor of $b$. This gives rise to a map of species

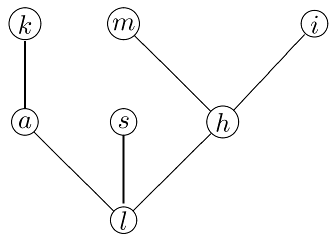
$$\phi \colon \mathbf{F} \hookrightarrow \mathbf{P}$$

from the species of rooted forests to that of posets. The forest can be recovered as the Hasse diagram of the poset, so $\phi$ is injective.

An admissible subset of vertices of a rooted forest $f$ is precisely a lower set of the poset $\phi(f)$. It follows that $\phi$ is a morphism of comonoids. It is also a morphism of monoids, the product being disjoint union on both sides. Therefore, $\phi$ is a morphism of Hopf monoids. In this manner, the Hopf monoid of rooted forests identifies with the submonoid of $\mathbf{P}$ generated by $\mathbf{a}$.

The set of vertices of a planar rooted forest carries a partial order and a linear order as follows. Recall that the trees in such a forest are linearly ordered, as is each set of siblings (children of a given vertex). We say that a vertex $a$ is *to the left* of a vertex $b$ if $a$ belongs to a tree that precedes the tree to which $b$ belongs, or if they belong to the same tree and there are siblings $a'$ and $b'$ such that $a'$ is an ancestor of $a$, $b'$ is an ancestor of $b$, and $a'$ precedes $b'$. The relation "to the left of" defines a partial order on the set of vertices. There is also a linear order for which $i \leq j$ if either $i$ is an ancestor of $j$ or $i$ is to the left of $j$. This is the familiar *depth-first order*.

Below we show a tree $t$, the left of partial order (drawn from left to right), and the depth-first order.



$$
\begin{tikzcd}
k \arrow[dr] & & m \arrow[r] & i \\
a \arrow[r] & s \arrow[ur] \arrow[r] & h & \\
& l & &
\end{tikzcd}
\qquad\qquad l|a|k|s|h|m|i.
$$

Consider the map of species

$$\delta\colon \overrightarrow{\mathbf{F}} \to \mathbf{L}$$

which sends a planar rooted forest $f$ to the depth-first order $\delta(f)$ on its vertex set. The map $\delta$ is a morphism of monoids but not of comonoids. However, combining it with the composite

$$\overrightarrow{\mathbf{F}} \xrightarrow{\upsilon} \mathbf{F} \xrightarrow{\phi} \mathbf{P}$$

we obtain a morphism of Hopf monoids

$$\vec{\phi}\colon \overrightarrow{\mathbf{F}} \to \mathbf{L} \times \mathbf{P}, \qquad f \mapsto \delta(f) \otimes \phi\big(\upsilon(f)\big). \tag{13.17}$$

There are other canonical linear orders associated to a planar rooted tree, but the depth-first order is the only one for which the above construction of $\vec{\phi}$ results in a morphism of Hopf monoids.

There is a commutative diagram

$$
\begin{tikzcd}
\overrightarrow{\mathbf{F}} \arrow[r, "\vec{\phi}"] \arrow[d, "\upsilon"'] & \mathbf{L} \times \mathbf{P} \arrow[d, "\pi \times \mathrm{id}"] \\
\mathbf{F} \arrow[r, "\phi"'] & \mathbf{P}
\end{tikzcd}
$$

where $\pi\colon \mathbf{L} \to \mathbf{E}$ is defined in (8.31).

The morphism $\vec{\phi}$ is injective. Indeed, the poset corresponding to a forest is the disjoint union of the posets corresponding to the trees in the forest, and the linear order is the concatenation of the depth-first orders of the trees, from left to right. We thus recover the underlying rooted forest from the poset and planarity from the linear order.

In this manner, the Hopf monoid of planar rooted forests identifies with the submonoid of $\mathbf{L} \times \mathbf{P}$ generated by $\vec{\mathbf{a}}$.

**13.3.4. Forests among other Hopf monoids.** The Hopf monoids of rooted forests are related to various other Hopf monoids discussed in this chapter and in Chapter 12. There is a commutative diagram of Hopf monoids as follows.

$$\begin{tikzcd}
\overrightarrow{\mathbf{\Sigma}} \arrow[r, "\vec{\lambda}"] \arrow[d, "\mathrm{supp}"'] & \overrightarrow{\mathbf{F}} \arrow[r, "\vec{\phi}"] \arrow[d, "\upsilon"'] & \mathbf{L}\times\mathbf{P} \arrow[r, "\mathrm{id}\times\widehat{\omega}"] \arrow[d, "\pi\times\mathrm{id}"] & \mathbf{L}\times\mathbf{L}^* \arrow[d, "\pi\times\mathrm{id}"] \\
\overrightarrow{\mathbf{\Pi}} \arrow[r, "\lambda"'] & \mathbf{F} \arrow[r, "\phi"'] & \mathbf{P} \arrow[r, "\widehat{\omega}"'] & \mathbf{L}^*
\end{tikzcd} \tag{13.18}$$

The square in the center is discussed in Section 13.3.3. The map $\widehat{\omega}\colon \mathbf{P}\to\mathbf{L}^*$ is as in (13.4); it sends a poset to the sum of its linear extensions.

We explain the left square in (13.18). Let $\vec{\lambda}\colon \mathbf{L}^*_+ \to \vec{\mathbf{a}}$ be the map with components

$$\mathbf{L}^*_+[I]\to\vec{\mathbf{a}}[I], \qquad (l^1|l^2|\cdots|l^n)^* \mapsto \begin{matrix} l^n \\ \vdots \\ l^2 \\ | \\ l^1 \end{matrix}$$

A subset $S$ of $I$ is initial for the linear order $l$ if and only if it is admissible for the tree $\vec{\lambda}(l^*)$. Therefore, $\vec{\lambda}\colon \mathbf{L}^*_+ \to \overrightarrow{\mathbf{F}}$ is a morphism of comonoids. We let $\lambda := \upsilon\vec{\lambda}\colon \mathbf{L}^*_+ \to \mathbf{a} \to \mathbf{F}$. We extend $\lambda$ to $\overrightarrow{\mathbf{\Pi}} = \mathcal{S}(\mathbf{L}^*_+)$ and $\vec{\lambda}$ to $\overrightarrow{\mathbf{\Sigma}} = \mathcal{T}(\mathbf{L}^*_+)$ as morphisms of monoids. By universality, these extensions exist and are unique. Since $\lambda$ and $\vec{\lambda}$ are morphisms of comonoids, their extensions are morphisms of Hopf monoids. We continue to denote them by $\lambda$ and $\vec{\lambda}$.

The composites

$$\overrightarrow{\mathbf{\Sigma}} \xrightarrow{\vec{\phi}\vec{\lambda}} \mathbf{L}\times\mathbf{P} \qquad \text{and} \qquad \overrightarrow{\mathbf{\Pi}} \xrightarrow{\phi\lambda} \mathbf{P}$$

are the maps $\hat{\nu}$ and $\vec{\nu}$ discussed in Section 13.1.5. Thus, diagram (13.18) refines (13.7).

**13.3.5. A $q$-Hopf monoid of planar rooted forests.** Let $f$ be a planar rooted forest on $I$ and $S \subseteq I$ an $f$-admissible subset of vertices. Write $I = S \sqcup T$. Define

$$\mathrm{sch}_{S,T}(f) := |\{(a,b)\in S\times T \mid b \text{ is to the left of } a \text{ in } f\}|.$$

For example, if $t$ and $S$ are as in (13.16), then

$$\mathrm{sch}_{S,T}(t) = |\{(a,s),(a,h),(k,s),(k,h)\}| = 4.$$

Fix a scalar $q\in\Bbbk$. The following defines a comonoid structure on $\overrightarrow{\mathbf{F}}$.

$$\overrightarrow{\mathbf{F}}[I]\to\overrightarrow{\mathbf{F}}[S]\otimes\overrightarrow{\mathbf{F}}[T], \quad f\mapsto \begin{cases} q^{\mathrm{sch}_{S,T}(f)}\, f|_S\otimes f|_T & \text{if } S \text{ is } f\text{-admissible,} \\ 0 & \text{otherwise.}\end{cases}$$

Together with the standard (free) monoid structure on $\overrightarrow{\mathbf{F}} = \mathcal{T}(\vec{\mathbf{a}})$ this results in a $q$-Hopf monoid that we denote by $\overrightarrow{\mathbf{F}}_q$.

All maps in the top row of diagram (13.18) are compatible with the $q$-deformations. Recall the $q$-Hopf monoid $\mathbf{L}_q$ from Definition 9.13 and the Schubert cocycle (9.12). The map $\delta\colon \overrightarrow{\mathbf{F}} \to \mathbf{L}$ of Section 13.3.3 satisfies

$$\mathrm{sch}_{S,T}(f) = \mathrm{sch}_{S,T}\big(\delta(f)\big) = \mathrm{dist}\big(\delta(f), (S|T)\delta(f)\big)$$

whenever $S$ is an $f$-admissible subset of vertices. It follows that

$$\vec{\phi}\colon \overrightarrow{\mathbf{F}}_q \to \mathbf{L}_q \times \mathbf{P}$$

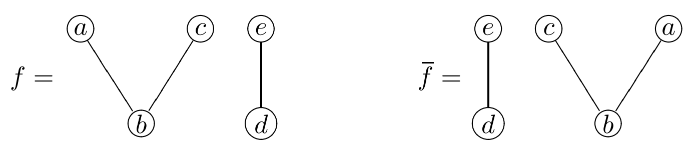
is a morphism of $q$-Hopf monoids, where $\vec{\phi}$ is as in (13.17). The $q$-Hopf monoid $\overrightarrow{\mathbf{\Sigma}}_q$ is discussed in Section 12.5.3. In order to prove that $\vec{\lambda}$ is a morphism of $q$-Hopf monoids, it suffices to check that $\vec{\lambda}\colon \mathbf{L}_+^* \to \overrightarrow{\mathbf{F}}_q$ is a morphism of comonoids. This follows from the argument given in Section 13.3.4, plus the fact that if $l$ is a linear order on $I$ and $S$ is an initial segment, then $\mathrm{sch}_{\vec{\lambda}(l^*)}(S) = 0$.

In summary, there are morphisms of $q$-Hopf monoids

$$\overrightarrow{\mathbf{\Sigma}}_q = \mathcal{T}_q(\mathbf{L}_+^*) \xrightarrow{\;\vec{\lambda}\;} \overrightarrow{\mathbf{F}}_q \xrightarrow{\;\vec{\phi}\;} \mathbf{L}_q \times \mathbf{P} \xrightarrow{\mathrm{id}\times\widehat{\omega}} \mathbf{L}_q \times \mathbf{L}^*$$

**13.3.6. Antipode formulas.** We provide antipode formulas for $\overrightarrow{\mathbf{F}}_q$ and $\mathbf{F}$. They extend known results for the corresponding Hopf algebras, details are given in Section 17.5.4. We begin with some preliminary definitions.

Let $f$ be a planar rooted forest on $I$. Its opposite $\overline{f}$ is the planar rooted forest obtained from $f$ by reversing all orders (the order on the planar rooted trees as well as the order on the children of a vertex). For example,




**Definition 13.2.** Let $f$ be a planar rooted forest on $I$, and let $F$ be a composition of $I$. We say that $F$ is a depth-first composition of $f$ if the following holds:

- The first part, say $S$, of $F$ is an admissible subset of the first planar rooted tree of $f$, and
- $F|_{I\setminus S}$ is a depth-first composition of $f|_{I\setminus S}$.

Note that the depth-first linear order $\delta(f)$ of $f$ is a depth-first composition of $f$ and it is the only linear order with this property. This is the motivation behind our terminology.

If $t_1$, $t_2$, ... are the planar rooted trees of $f$ in that order, then a depth-first composition of $f$ is a depth-first composition of $t_1$, followed by a depth-first composition of $t_2$, and so on.

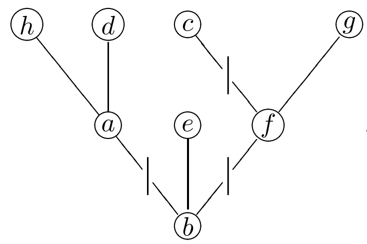
**Definition 13.3.** A subset of edges of a planar rooted forest is called a *cut*. A cut $c$ of a planar rooted forest $f$ determines a planar rooted forest $W(f,c)$ and a depth-first composition $\delta(f,c)$ of $f$ as follows.

Remove the edges of $f$ specified by $c$, and consider the resulting connected components (these are planar trees). Let $t$ be the connected component containing the root of the leftmost planar tree of $f$, and let $S$ be its vertex set. Define

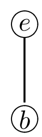
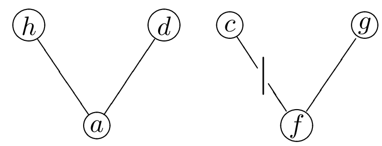
$$W(f,c) := t \cdot W(f|_T, c|_T) \qquad \text{and} \qquad \delta(f,c) := S \cdot \delta(f|_T, c|_T)$$

where $T$ is the complement of $S$ in the vertex set of $f$, and $c|_T$ is the restriction of $c$ to the edges of $f|_T$.

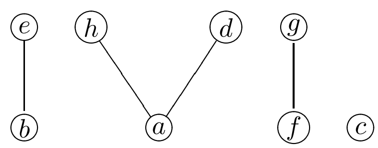
For example, if

$(f,c) =$  ,

then $S = \{b, e\}$,

$t =$  and $(f|_T, c|_T) =$  .

Proceeding with the recursion we find

$W(f,c) =$  and $\delta(f,c) = be|ahd|fg|c.$

It is straightforward to check that for fixed $f$, $c \mapsto \delta(f,c)$ is a bijective correspondence between cuts of $f$ and depth-first compositions of $f$.

Note also that if $c$ consists of all edges of $f$, then $\delta(f,c) = \delta(f)$.

**Theorem 13.4.** *The $I$-component of the antipode of $\overrightarrow{\mathbf{F}}_q$ is*

$$\mathrm{s}_I(f) = \sum_c (-1)^{\deg c} q^{\operatorname{dist}\left(\delta(f), \delta\left(W(\overline{f}, c)\right)\right)} \, W(\overline{f}, c),$$

*where the sum is over all cuts $c$ of $\overline{f}$ (and not of $f$), and $\deg c$ is the number of edges in $c$ plus one.*

Proof. Applying Takeuchi's formula (8.27), we obtain:

$$\mathrm{s}_I(f) = \sum_g \Bigg( \sum_{H \in D(f,g)} (-1)^{\deg(H)} \Bigg) q^{\operatorname{dist}(\delta(f),\delta(g))} \, g,$$

where

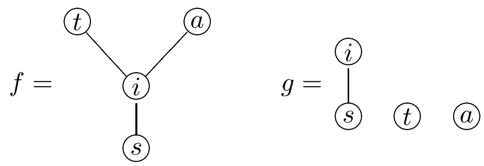
$$D(f,g) = \{H \mid \mu_H \Delta_H(f) = g\},$$

and $\mu_H$ and $\Delta_H$ are the iterated product and coproduct (11.33). For $H \in D(f,g)$, we have

$$\operatorname{dist}\big(\delta(f), H\delta(f)\big) = \operatorname{dist}\big(\delta(f), \delta(g)\big),$$

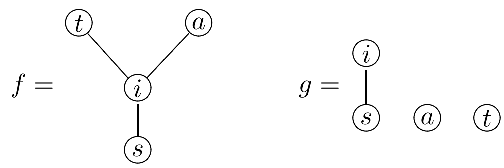
and this explains why the $q$-term can be pulled out of the parenthesis.

For example,



$$D(f, g) = \{si|ta, si|t|a\}.$$

One may check that, in general, the set $D(f, g)$ is a Boolean poset. It follows that the sum in parenthesis is zero unless $D(f, g)$ is a singleton, in which case the singleton element is a depth-first composition of $\overline{f}$ (and not of $f$). For example,



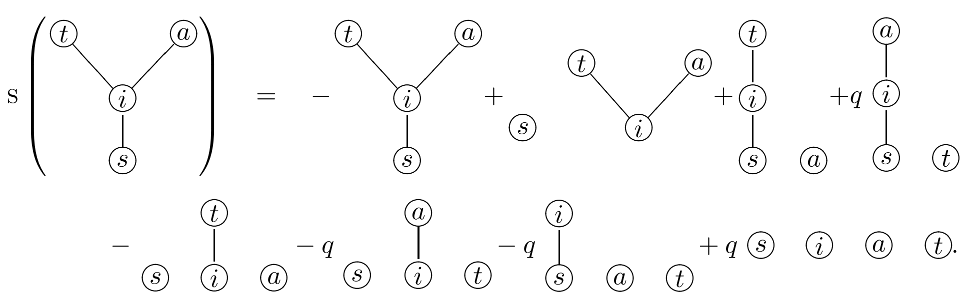
$$D(f, g) = \{si|a|t\}.$$

This yields

$$\mathrm{S}_I(f) = \sum_G (-1)^{\deg(G)} q^{\mathrm{dist}\left(\delta(f), G\delta(f)\right)} \mu_G \Delta_G(f),$$

where the sum is over all depth-first compositions $G$ of $\overline{f}$. The result follows by using the correspondence between depth-first compositions and cuts. $\square$

We illustrate the antipode formula on an example.



We now restrict some of the above terminology to rooted forests. A subset of edges of a rooted forest is called a *cut*. Let $f$ be a rooted forest. Removing from $f$ the edges specified by a cut $c$ yields a rooted forest which we denote by $W(f, c)$.

**Theorem 13.5.** *The $I$-component of the antipode of* $\mathbf{F}$ *is*

$$\mathrm{S}_I(f) = \sum_c (-1)^{\deg c}\, W(f, c),$$

*where the sum is over all cuts $c$ of $f$ and* $\deg c$ *is the number of edges in $c$ plus one.*

Proof. This follows from Theorem 13.4 by setting $q = 1$ and using the quotient map $\upsilon\colon \overrightarrow{\mathbf{F}} \twoheadrightarrow \mathbf{F}$. $\square$

Table 13.2. Hopf monoids from relations.

| Hopf monoid | Linear basis | Type of relation |
|---|---|---|
| **R** | relations | arbitrary |
| **O** | preposets | transitive and reflexive |
| **P** | posets | transitive, antisymmetric and reflexive |
| **Q** | equivalence relations | transitive, symmetric and reflexive |
| **G** | simple graphs | symmetric and irreflexive |

## 13.4. Relations

We introduce a Hopf monoid of relations whose coproduct depends on a parameter $q \in \Bbbk$. When $q = 0$, this Hopf monoid contains the Hopf monoid of (pre)posets of Section 13.1.1 and 13.1.6; when $q = 1$, it contains the Hopf monoid of graphs of Section 13.2.1. We also discuss a connection between the Hopf submonoid of equivalence relations and the Hopf monoids of set partitions of Section 12.6. These are summarized in Table 13.2.

**13.4.1. The Hopf monoid of relations.** Let $\mathbf{R}[I]$ denote the vector space with basis the set of all relations on the finite set $I$ (subsets $r$ of $I \times I$). This defines the species $\mathbf{R}$ of relations. We proceed to turn it into a Hopf monoid. The structure will depend on a fixed parameter $q \in \Bbbk$.

Given a decomposition $I = S \sqcup T$ and a relation $r$ on $I$, let

$$e_{S,T}(r) := |r \cap (T \times S)| = |\{(t,s) \in r \;\; s \in S, t \in T\}|. \tag{13.19}$$

As in Sections 13.1.1 and 13.8.1, the restriction of a relation $r$ on the set $I$ to a subset $S$ is

$$r|_S := r \cap (S \times S).$$

The coproduct is given by

$$\mathbf{R}[I] \to \mathbf{R}[S] \otimes \mathbf{R}[T], \qquad r \mapsto q^{e_{S,T}(r)}\, r|_S \otimes r|_T. \tag{13.20}$$

The product is given by

$$\mathbf{R}[S] \otimes \mathbf{R}[T] \to \mathbf{R}[I], \qquad r_1 \otimes r_2 \mapsto r_1 \sqcup r_2. \tag{13.21}$$

In $r_1 \sqcup r_2$ on $I$ there are no relations between the elements of $S$ and $T$. In other words, the set $r_1 \sqcup r_2 \subseteq I \times I$ is the (disjoint) union of the sets $r_1 \subseteq S \times S$ and $r_2 \subseteq T \times T$.

Let us discuss the Hopf monoid axioms.

In the case $q = 1$, the axioms boil down to simple properties of restriction and disjoint unions.

To obtain the general case, we note that for every decomposition $I = R \sqcup S \sqcup T$ we have

$$e_{R \sqcup S,T}(r) + e_{R,S}(r|_{R \sqcup S}) = e_{R,S \sqcup T}(r) + e_{S,T}(r|_{S \sqcup T}).$$

Indeed, both sides count the number of elements in the set

$$r \cap \big((T \times R) \sqcup (T \times S) \sqcup (S \times R)\big).$$

This says that the family of maps $e_{S,T}$ is a 2-cocycle (Section 9.6.1).

In addition, given two decompositions $S \sqcup T = I = S' \sqcup T'$, $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, $D = T \cap T'$ (as in Figure 8.1), and relations $r_1$ on $S$ and $r_2$ on $T$, we have

$$e_{S',T'}(r_1 \sqcup r_2) = e_{A,B}(r_1) + e_{C,D}(r_2).$$

This follows from the equality of sets

$$(r_1 \sqcup r_2) \cap (T' \times S') = \big(r_1 \cap (B \times A)\big) \sqcup \big(r_2 \cap (D \times C)\big),$$

which holds since in $r_1 \sqcup r_2$ there are no relations between elements of $S$ and $T$. This says that the 2-cocycle $e_{S,T}$ is multiplicative of twist 0 (Section 9.6.3). Therefore (Proposition 9.21), the above definitions turn $\mathbf{R}$ into a Hopf monoid.

We use $\mathbf{R}_q$ to denote the species of relations endowed with the above Hopf monoid structure. It is a deformation of the Hopf monoid $\mathbf{R}_1$, in the sense of Section 9.6.4.

We remark that the 2-cocycle $e_{S,T}$ is not a 2-coboundary (Section 9.6.1). Indeed, any 2-coboundary on $\mathbf{R}_1$ is symmetric on $S$ and $T$, but $e_{S,T} \neq e_{T,S}$. Thus, the Hopf monoid $\mathbf{R}_q$ is a nontrivial deformation of $\mathbf{R}_1$.

**13.4.2. Posets and graphs as relations.** Posets and simple graphs with vertex set $I$ (Sections 13.1 and 13.2) are special kinds of relations on $I$. Moreover, both posets and simple graphs are closed under disjoint unions and restrictions. Thus, the species of posets and of simple graphs are Hopf submonoids of $\mathbf{R}_q$. We denote them by $\mathbf{P}_q$ and $\mathbf{G}_q$, respectively.

Let $p$ be a poset and $g$ a simple graph, both with vertex set $I$. According to (13.19), for any decomposition $I = S \sqcup T$ we have

$$e_{S,T}(p) = |\{(t,s) \in T \times S : t \leq s \text{ in } p\}|$$

and

$$e_{S,T}(g) = |\{(t,s) \in T \times S : \text{there is an edge between } t \text{ and } s \text{ in } g\}|.$$

The Hopf monoid of simple graphs of Section 13.2.1 is $\mathbf{G}_1$. The product of the Hopf monoid $\mathbf{G}_0$ is still given by (13.10); its coproduct is given by

$$\mathbf{G}_0[I] \to \mathbf{G}_0[S] \otimes \mathbf{G}_0[T],$$
$$g \mapsto \begin{cases} \displaystyle\sum_{I=S\sqcup T} g|_S \otimes g|_T & \text{if there are no edges between } S \text{ and } T, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the Hopf monoid of posets of Section 13.1.1 is $\mathbf{P}_0$. This follows by comparing (13.2) and (13.20), and noting that

$$e_{S,T}(p) = 0 \iff S \text{ is a lower set of } p.$$

The deformation of the Hopf monoid of simple graphs is only interesting if $q = 0$. This is because

$$e(g) = e(g|_S) + e(g|_T) + e_{S,T}(g),$$

where $e(G)$ is the number of edges of $G$. This says that $e_{S,T}$ is a 2-coboundary, and then $\mathbf{G}_q \cong \mathbf{G}$ for $q \neq 0$ by Proposition 9.22.

Recall that a preposet is a reflexive and transitive relation. The discussion for posets above extends to preposets. The species of preposets is closed under disjoint unions and restrictions. This gives rise to a Hopf submonoid $\mathbf{O}_q$ of $\mathbf{R}_q$

which contains the Hopf monoid of posets $\mathbf{P}_q$. The Hopf monoid of preposets of Section 13.1.6 is $\mathbf{O}_0$.

**13.4.3. Equivalence relations.** Equivalence relations are closed under disjoint unions and restrictions. Thus, the species of equivalence relations is a Hopf submonoid of $\mathbf{R}_q$. We denote it by $\mathbf{Q}_q$. It is in fact a Hopf submonoid of $\mathbf{O}_q$, the Hopf monoid of preposets of Section 13.4.2.

Given a relation $r$ on $I$, let

$$e_I(r) := |\{(i,j) \in r \mid i \neq j\}|.$$

In other words, $e_I(r)$ is the cardinality of the set $r \subseteq I \times I$ minus the diagonal of $I \times I$. Note that for any symmetric relation $r$, $e_I(r)$ is an even integer. Moreover, in this case we have

$$e(r) = e(r|_S) + e(r|_T) + 2 \cdot e_{S,T}(r).$$

This says that on symmetric relations the 2-cocycle $e_{S,T}$ is the coboundary of the 1-cochain $\frac{1}{2}e_I$.

It follows from Proposition 9.22 that for any $q \neq 0$, the map

$$\mathbf{Q}_1 \to \mathbf{Q}_q, \qquad r \mapsto q^{-\frac{1}{2}e(r)}\, r$$

is an isomorphism of Hopf monoids.

**13.4.4. Set partitions as relations.** Partitions of a set $I$ and equivalence relations on $I$ are equivalent notions. Given a partition $X$ of $I$, let $r_X$ be the corresponding equivalence relation: $(a,b) \in r_X$ if and only if $a$ and $b$ belong to the same block of $X$.

The Hopf monoids of set partitions $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ were defined in Section 12.6. The Hopf monoid $\mathbf{Q}_q$ interpolates between the two in the following sense.

First of all, we have $\mathbf{\Pi} \cong \mathbf{Q}_1$ via $h_X \mapsto r_X$.

Consider the case $q = 0$. Let $I = S \sqcup T$ be a decomposition and $X$ a partition of $I$. Then

$$e_{S,T}(r_X) = 0 \iff S \text{ (and } T\text{) is a union of blocks of } X.$$

It follows from the formulas in Proposition 12.42 that the map

$$\mathbf{\Pi}^* \to \mathbf{Q}_0, \qquad p_X \mapsto r_X$$

is an isomorphism of Hopf monoids.

We know from Proposition 12.48 that $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ are isomorphic. Combining the above isomorphisms we see that $\mathbf{Q}_q \cong \mathbf{\Pi}$ for all $q$.

**13.4.5. Boolean algebras and set partitions.** A *Boolean algebra* on a finite set $I$ is a collection of subsets of $I$ which contains $\emptyset$ and is closed under unions, intersections and complements.

Let $I = S \sqcup T$ be a disjoint decomposition. Given Boolean algebras $b_1$ on $S$ and $b_2$ on $T$, define $b_1 \sqcup b_2$ to be the Boolean algebra on $I$ consisting of those subsets which can be written as a union of a subset in $b_1$ and a subset in $b_2$. Similarly, given a Boolean algebra $b$ on $I$ and a subset $S \in b$, define $b|_S$ to be the Boolean algebra on $S$ consisting of those subsets of $S$ which lie in $b$.

Let $\mathbf{B}[I]$ denote the vector space with basis the set of all Boolean algebras on $I$. This defines a species $\mathbf{B}$. It is a Hopf monoid with product

$$\mathbf{B}[S] \otimes \mathbf{B}[T] \to \mathbf{B}[I], \qquad b_1 \otimes b_2 \mapsto b_1 \sqcup b_2$$

and coproduct

$$\mathbf{B}[I] \to \mathbf{B}[S] \otimes \mathbf{B}[T], \qquad b \mapsto \begin{cases} b|_S \otimes b|_T & \text{if } S \in b, \\ 0 & \text{otherwise.} \end{cases}$$

The Hopf monoid $\mathbf{B}$ is commutative, cocommutative, and self-dual.

To any Boolean algebra $b$ on $I$, one can associate a partition $X(b)$ of $I$ consisting of the minimum nonempty subsets which belong to $b$. This sets up a bijection between Boolean algebras and set partitions. Using Proposition 12.42, one can see that the map $b \mapsto p_{X(b)}$ defines an isomorphism $\mathbf{B} \to \mathbf{\Pi}^*$ of Hopf monoids.

## 13.5. Combinatorics and geometry

The goal of this section is to formalize the correspondence between combinatorial and geometric objects. A summary of examples is given in Table 13.3. We have seen some of these correspondences in Chapter 10, particularly in Section 10.9.1. In this section, we note that preposets and posets also fit into the above formalism, with the corresponding geometric objects being cones and top-dimensional cones.

**13.5.1. Relations and the braid arrangement.** Let $I$ be a finite set. Recall that the braid arrangement in $\mathbb{R}^I$ is defined by the hyperplanes $\{x_i = x_j\}$ as $i \neq j$ vary over $I$ (Section 10.2.1).

We define correspondences between relations on the set $I$ and subsets of $\mathbb{R}^I$. Given a relation $r$ on $I$, let

$$\Psi(r) := \bigcap_{(i,j)\in r} \{x_i \le x_j\}. \tag{13.22}$$

Given a subset $R \subseteq \mathbb{R}^I$, let

$$\Phi(R) := \big\{(i,j) \in I \times I \mid R \subseteq \{x_i \le x_j\}\big\}. \tag{13.23}$$

**Proposition 13.6.** *Both maps $\Psi$ and $\Phi$ are inclusion-reversing. Moreover, for any relation $r$ on $I$ and subset $R$ of $\mathbb{R}^I$, we have*

$$r \subseteq \Phi(R) \iff \Psi(r) \supseteq R.$$

TABLE 13.3. Combinatorics and geometry.

| Combinatorics | Geometry |
|---|---|
| posets | top-dimensional cones |
| preposets | cones |
| linear orders | chambers |
| equivalence relations, or set partitions | flats |
| linear set partitions | directed flats |
| set compositions | faces |
| linear set compositions | directed faces |

The proof is straightforward. The result says that $\Psi$ and $\Phi$ define a *Galois connection* in the sense of [317]. The following result describes the associated closure operators $\Psi\Phi$ and $\Phi\Psi$. Closure operators are discussed in Section 13.8.

**Proposition 13.7.** *Let $r$ be a relation on $I$ and $R$ a subset of $\mathbb{R}^I$.*

(i) *The set $\Psi(r)$ is a cone of the braid arrangement.*
(ii) *$\Psi\Phi(R)$ is the smallest cone of the braid arrangement that contains $R$.*
(iii) *The relation $\Phi(R)$ is reflexive and transitive (a preposet).*
(iv) *$\Phi\Psi(r)$ is the reflexive and transitive closure of $r$.*

Proof. For any $i \in I$, the set $\{x_i \leq x_i\} = \mathbb{R}^I$ can be omitted from (13.22) without altering the resulting intersection. Thus, $\Psi(r)$ is an intersection of half-spaces, that is, a cone of the braid arrangement (Section 10.2.4). This proves (i).

Assertion (iv) follows from Farkas' lemma [382, Proposition 1.9]. The remaining assertions are straightforward. □

It follows from the proposition that the closed sets of the Galois connection identify with preposets on the one hand, and with cones of the braid arrangement on the other.

**Corollary 13.8.** *The maps $\Psi$ and $\Phi$ restrict to inverse correspondences between the following classes of relations on $I$ and of subsets of $\mathbb{R}^I$ associated to the braid arrangement.*

(i) *Preposets and cones.*
(ii) *Posets and top-dimensional cones.*
(iii) *Equivalence relations and flats.*
(iv) *Linear orders and chambers.*

Proof. A Galois connection always restricts to inverse correspondences on closed sets. The first assertion thus follows from Proposition 13.7.

The subset corresponding to a linear order $l$ is the chamber

$$\Psi(l) = \{x_{l^1} \leq \cdots \leq x_{l^n}\}.$$

As mentioned in Section 10.2.2, every chamber is of this form. This proves (iv).

To prove (ii), note that all chambers corresponding to linear extensions of a poset $p$ are contained in $\Psi(p)$. In particular, $\Psi(p)$ has nonempty interior and is thus a top-dimensional cone. Conversely, if both $(i,j)$ and $(j,i)$ belong to a preposet $p$, then $\Psi(p)$ is contained in the hyperplane

$$\{x_i \leq x_j\} \cap \{x_j \leq x_i\} = \{x_i = x_j\},$$

and therefore is not top-dimensional.

Similarly, if $r$ is a symmetric relation, then

$$\Psi(r) = \bigcap_{i \sim_r j} \{x_i = x_j\},$$

where $i \sim_r j$ is as in (13.8). Thus, $\Psi(r)$ is a flat, and as explained in Section 10.2.3, every flat is of this form. This proves (iii). □

**Remark 13.9.** In Corollary 13.8, assertions (ii) and (iii) together are equivalent to assertion (i): A preposet is equivalent to an equivalence relation together with a partial order on the equivalence classes. Similarly, a cone is equivalent to a flat together with a top-dimensional cone in that flat.

**Example 13.10.** Let $I = \{v, i, s, h, n, u\}$. We show a poset on $I$ and the corresponding top-dimensional cone.

$$\begin{tikzcd}[column sep=small, row sep=small] s & & h & \\ & i \arrow[ul, dash] \arrow[ur, dash] & & u \\ & v \arrow[u, dash] & & n \arrow[u, dash] \end{tikzcd} \qquad \longleftrightarrow \qquad \{(x_i)_{i\in I} \in \mathbb{R}^I \mid x_v \leq x_i,\ x_i \leq x_s,\ x_i \leq x_h,\ x_n \leq x_u\}.$$

We remark that the correspondences $\Psi$ and $\Phi$ (and hence the bijections of Corollary 13.8) are (iso)morphisms of species.

**Remark 13.11.** Corollary 13.8, along with additional related results, appear in a recent paper of Reiner, Postnikov and Williams [299, Section 3]; see in particular their Proposition 3.5. The correspondence between posets and top-dimensional cones in the braid arrangement goes back to work of Reiner [308, 309]. He also explores this correspondence for other Coxeter types.

**13.5.2. Geometric description of morphisms involving posets.** In Section 13.1.5 we discussed various morphisms of Hopf monoids involving the Hopf monoid of posets $\mathbf{P}$, such as

$$\overrightarrow{\mathbf{\Pi}} \xrightarrow{\hat{\nu}} \mathbf{P} \xrightarrow{\widehat{\omega}} \mathbf{L}^*$$

We now describe these maps in geometric terms.

The map $\widehat{\omega}$ sends a poset $p$ to the sum of all chambers which belong to $\Psi(p)$, the top-dimensional cone associated to $p$ as in (13.22).

Now consider the map $\hat{\nu}$ from the Hopf monoid of directed flats $\overrightarrow{\mathbf{\Pi}}$. In Section 10.9.3 we associated a top-dimensional cone $\Psi(L)$ to a given directed flat $L$. It is not hard to see that this is precisely the cone $\Psi(p)$, where $p := \hat{\nu}(L)$. In other words, $\Psi(L) = \Psi\big(\hat{\nu}(L)\big)$.

We turn to the morphisms $\Upsilon\colon \overrightarrow{\mathbf{\Pi}} \to \mathbf{\Sigma}^*$ and $\hat{\eta}\colon \mathbf{P} \to \mathbf{\Sigma}^*$ in diagram (13.6).

The map $\Upsilon$ is given by

$$h_L \mapsto \sum_{F:\, F \subseteq \Psi(L)} M_F.$$

as in (12.17). The letters $h$ and $M$ refer to the $h$ and the $M$ bases; they are written for book-keeping purposes. Thus, the map $\Upsilon$ sends a directed flat $L$ to the sum of all faces contained in the cone $\Psi(L)$. A combinatorial description is given in (12.18).

The map $\hat{\eta}\colon \mathbf{P} \to \mathbf{\Sigma}^*$ is given by

$$p \mapsto \sum_{F:\, F \subseteq \Psi(p)} F.$$

In other words, $\hat{\eta}$ sends a poset $p$ to the sum of all faces contained in the top-dimensional cone $\Psi(p)$. This corresponds to the combinatorial description of $\hat{\eta}$ given in (13.3).

## 13.6. Set-graded posets

All posets considered in this section are finite and have a bottom element $\hat{0}$ and a top element $\hat{1}$. We use capital letters such as $P$ to distinguish these posets from the posets $p$ of Section 13.1.

We use the following standard terminology. A *chain* in a poset $P$ is a totally ordered subset of $P$. The *length* of a chain $\{x_0 < x_1 < \cdots < x_n\}$ is $n$. A chain is *maximal* if it is not properly contained in any other chain. Such a chain must have $x_0 = \hat{0}$ and $x_n = \hat{1}$. A chain is *saturated* if it is maximal in a poset of the form

$$[x, y] := \{z \in P \mid x \leq z \leq y\}.$$

Such a chain must have $x_0 = x$ and $x_n = y$. We let $H(P)$ denote the set of edges in the Hasse diagram of a poset $P$, that is, the covering relation on the set of vertices defined by $P$.

**13.6.1. Graded posets.** Recall that a poset $P$ is *graded* if all maximal chains have the same length. The *rank* of $x \in P$ is the length of a saturated chain from $\hat{0}$ to $x$. The rank of $P$ is the rank of $\hat{1}$. A morphism of graded posets is an order-preserving map between posets of the same rank which preserves the grading.

Given a natural number $n$, we let $C_n$ denote the chain

$$C_n := \{0 < 1 < \cdots < n\}$$

consisting of the first $n+1$ natural numbers. This is a graded poset of rank $n$.

Let $P$ be a graded poset of rank $n$. We may define a map $\varphi \colon P \to C_n$ by

$$\varphi(x) := \operatorname{rank}(x).$$

The map $\varphi$ is a morphism of graded posets, that is, it is order and rank-preserving. In particular, it preserves covering relations. Conversely, given a poset $P$, if there is a map $\varphi \colon P \to C_n$ that preserves covering relations, then $P$ is graded. To summarize:

**Proposition 13.12.** *There is an equivalence between graded posets of rank $n$ and pairs $(P, \varphi)$ where $\varphi \colon P \to C_n$ is a map that preserves covering relations.*

**13.6.2. Set-graded posets.**

**Definition 13.13.** Let $I$ be a finite set. An *$I$-graded* poset is a pair $(P, \lambda)$ where $P$ is a poset (with $\hat{0}$ and $\hat{1}$) and $\lambda \colon H(P) \to I$ is a function such that for each maximal chain

$$C = \{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}\},$$

$\lambda$ restricts to a bijection

$$\{(x_{i-1}, x_i) \in H(P) \mid i = 1, \ldots, n\} \xrightarrow{\cong} I.$$

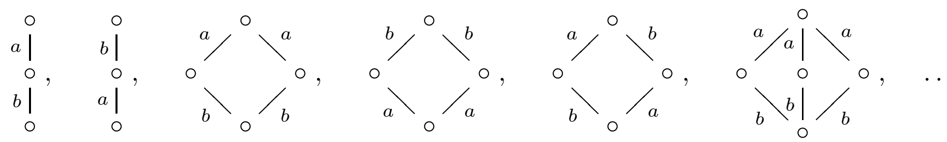
In other words, the edges of the Hasse diagram are labeled by elements of $I$ in such a way that along each maximal chain every element of $I$ appears exactly once.

We say that $\lambda$ is an *$I$-labeling* of the edges of $P$. When $I$ and $\lambda$ are understood or not specified, we may simply say that the poset $P$ is *set-graded*.

Two $I$-graded posets $(P, \lambda)$ and $(P', \lambda')$ are isomorphic if there is an isomorphism of posets $f \colon P \to P'$ such that

$$\begin{tikzcd} H(P) \arrow[rr, "H(f)"] \arrow[dr, "\lambda"'] & & H(P') \arrow[dl, "\lambda'"] \\ & I & \end{tikzcd}$$

commutes. For example, let $I = \{a, b\}$. Then the first few isomorphism classes of $I$-graded posets are



(there are infinitely many).

**Remark 13.14.** If $(P, \lambda)$ is $I$-graded and $|I| = n$, then all maximal chains in $P$ have length $n$, so $P$ is graded of rank $n$. Conversely, any graded poset of rank $n$ admits several $I$-labelings. For instance, we may choose a linear order $l^1|\cdots|l^n$ on $I$ and label all edges of $H(P)$ between ranks $i-1$ and $i$ with $l^i$.

We discuss some basic facts about the structure of set-graded posets.

Let $(P, \lambda)$ be an $I$-graded poset. Given any saturated chain $C$ in $P$, let $S(C) \subseteq I$ denote the set of labels along $C$.

Take $x \leq y \in P$ and let $C_1$ and $C_2$ be two saturated chains from $x$ to $y$. We claim that $S(C_1) = S(C_2)$. To see this, choose any saturated chains $B$ from $\hat{0}$ to $x$ and $D$ from $y$ to $\hat{1}$. Then $B \cup C_1 \cup D$ and $B \cup C_2 \cup D$ are maximal chains (from $\hat{0}$ to $\hat{1}$) in $P$. Then, we have disjoint decompositions

$$S(B) \sqcup S(C_1) \sqcup S(D) = I = S(B) \sqcup S(C_2) \sqcup S(D).$$

It follows that $S(C_1) = S(C_2)$.

Therefore, we may use $S(x, y)$ to denote the set of labels of any saturated chain from $x$ to $y$, and the poset $[x, y]$ is $S(x, y)$-labeled under the restriction of $\lambda$ to $H([x, y])$.

The preceding discussion also shows that for any $x \in P$ we have a disjoint decomposition

$$I = S(\hat{0}, x) \sqcup S(x, \hat{1}).$$

More generally, suppose we are given a chain

$$C := \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$$

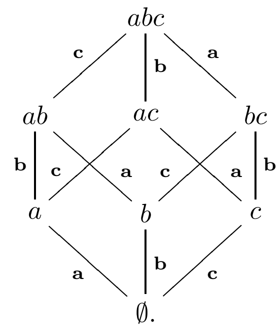
in $P$ (not necessarily maximal). Iterating the preceding observation we see that $I = S(x_0, x_1) \sqcup \cdots \sqcup S(x_{k-1}, x_k)$. Let

$$\lambda(C) := S(x_0, x_1)|\cdots|S(x_{k-1}, x_k) \tag{13.24}$$

be the resulting composition of $I$. Note that if the chain $C$ is maximal, then $\lambda(C)$ is a linear order on $I$.

### 13.6.3. Set-graded posets and Boolean posets.

Recall the Boolean poset $2^I$ consisting of subsets of $I$ ordered by inclusion. It is $I$-graded: the edge $(x, y)$ is labeled by the unique element in $y \setminus x$. For example, for $I = \{a, b, c\}$:




Let $(P, \lambda)$ be an $I$-graded poset. According to the discussion in Section 13.6.2, $P$ is graded of rank $n$, and the set $S(\hat{0}, x)$ of labels of any saturated chain from $\hat{0}$ to $x$ is well-defined. Therefore, we may define a map $\varphi\colon P \to 2^I$ by

$$\varphi(x) := S(\hat{0}, x). \tag{13.25}$$

As before, $\varphi$ is a morphism of graded posets. We refer to $\varphi$ as the *set-grading* of $P$.

Conversely, given a graded poset of rank $n$ and a morphism of graded posets $\varphi\colon P \to 2^I$, we can define an $I$-labeling $\lambda\colon H(P) \to I$ by

$$\lambda(x, y) := \text{the unique element in } \varphi(y) \setminus \varphi(x)$$

for any covering relation $(x, y)$. These constructions are inverse of each other. Thus:

**Proposition 13.15.** *There is an equivalence between $I$-graded posets and pairs $(P, \varphi)$ where $\varphi\colon P \to 2^I$ is a morphism of graded posets.*

A comparison with Proposition 13.12 clarifies the relation between graded and set-graded posets. Let $|I| = n$. The canonical morphism of graded posets $2^I \to C_n$ is responsible for the fact that any set-graded poset is graded. The construction of Remark 13.14 is explained as follows. The choice of a linear order on $I$ is equivalent to the choice of a section of graded posets $C_n \to 2^I$. This allows us to turn any graded poset into a set-graded poset.

The preceding remarks suggest that one may view Boolean posets as a species analogue of chains, and set-graded posets as a species analogue of graded posets.

**Example 13.16.** Let $\Bbbk I$ be the vector space with basis $I$ over a field $\Bbbk$ and $L(I)$ the poset of vector subspaces of $\Bbbk I$, ordered by inclusion. It is a graded poset in which the rank of a subspace is its dimension. Choose a linear order $l := l^1 | \cdots | l^n$ on $I$. The *Schubert symbol* is the map

$$\varphi_l\colon L(I) \to 2^I$$

defined as follows [275, §6]. Given $V \in L(I)$, let $s := \dim V$ and set

$$\varphi_l(V) := \{l^{i_1}, \ldots, l^{i_s}\} \text{ where } i_j := \min\{i \in [n] \mid \dim(V \cap \Bbbk\{l^1, \ldots, l^i\}) = j\}.$$

The Schubert symbol is a morphism of graded posets. In this manner, $(L(I), \varphi_l)$ is an $I$-graded poset.

We mention in passing that the *Grassmannian* of $s$-planes is the set

$$\mathrm{Gr}_s(I) := \{V \in L(I) \mid \dim V = s\}.$$

If $S \subseteq I$ has cardinality $s$, the fiber

$$\{V \in \mathrm{Gr}_s(I) \mid \varphi_l(V) = S\}$$

is called a *Schubert cell*.

**13.6.4. The Hopf monoid of set-graded posets.** Let $\mathbf{sgP}[I]$ be the (infinite-dimensional) vector space with basis the set of isomorphism classes of $I$-graded posets. We proceed to turn the species $\mathbf{sgP}$ into a Hopf monoid.

Before defining the product, note that the covering relations in the Cartesian product $P \times Q$ of two posets are of the form

$$(x, y) < (x', y) \text{ where } x < x' \text{ is a covering relation in } P,$$

or of the form

$$(x, y) < (x, y') \text{ where } y < y' \text{ is a covering relation in } Q.$$

Now, given $I = S \sqcup T$, the corresponding component of the product is

$$\begin{gathered}\mathbf{sgP}[S] \otimes \mathbf{sgP}[T] \to \mathbf{sgP}[I],\\ (P, \lambda) \otimes (Q, \mu) \mapsto (P \times Q, \rho),\end{gathered}$$

where $\rho\colon H(P \times Q) \to I$ is

$$\rho\big((x, y), (x', y')\big) = \begin{cases} \lambda(x, x') & \text{if } y = y',\\ \mu(y, y') & \text{if } x = x'.\end{cases}$$

For any maximal chain $C$ in $P \times Q$, there are maximal chains $X$ in $P$ and $Y$ in $Q$ such that $\rho(C)$ is a shuffle of $\lambda(X)$ and $\mu(Y)$. In particular, $S(C) = S(X) \cup S(Y) = S \cup T = I$. Thus, the product is well-defined.

To define the coproduct we make use of the observations in Section 13.6.2 and we set

$$\begin{aligned}&\mathbf{sgP}[I] \to \mathbf{sgP}[S] \otimes \mathbf{sgP}[T],\\ &(P, \lambda) \mapsto \sum_{x \in \varphi^{-1}(S)} \big([\hat{0}, x], \lambda|_{H([\hat{0},x])}\big) \otimes \big([x, \hat{1}], \lambda|_{H([x,\hat{1}])}\big),\end{aligned} \tag{13.26}$$

where $\varphi\colon P \to 2^I$ is the set-grading of $P$ (13.25). Note that if $\varphi(x) = S$, then $S(\hat{0}, x) = S$ and $S(x, \hat{1}) = T$, so the restrictions of $\lambda$ turn $[\hat{0}, x]$ into an $S$-graded poset and $[x, \hat{1}]$ into a $T$-graded poset. The fiber of $\varphi$ over $S$ may be empty, in which case the corresponding component of the coproduct is 0.

The species $\mathbf{sgP}$ is a Hopf monoid. It is commutative but not cocommutative.

**13.6.5. Flags.** We proceed as in Section 13.1.2. The morphism of monoids

$$\eta\colon \mathbf{sgP} \to \mathbf{E}$$

defined by

$$\eta(P, \lambda) := 1,$$

gives rise via the universal property of Theorem 11.23 to a morphism of Hopf monoids

$$\hat{\eta}\colon \mathbf{sgP} \to \mathcal{T}^\vee(\mathbf{E}_+) = \mathbf{\Sigma}^*.$$

Formula (11.18) yields the following explicit description for $\hat{\eta}$:

$$\hat{\eta}(P, \lambda) = \sum_F f_F(P, \lambda)\, M_F, \tag{13.27}$$

where the coefficient $f_F(P,\lambda)$ counts the number of chains $C$ in $P$ such that $\lambda(C) = F$, and $\lambda(C)$ is as in (13.24). Thus, $\hat{\eta}$ is simply an enumerator of chains.

**13.6.6. Chains and descents.** We start from the unique morphism of species

$$\omega\colon \mathbf{sgP}_+ \to \mathbf{X}$$

which sends the basis element $\begin{smallmatrix}\circ\\ a\,|\\ \circ\end{smallmatrix}$ of $\mathbf{sgP}[\{a\}]$ to the distinguished basis element of $\mathbf{X}[\{a\}]$. In the course of this and the following sections, we will see how elementary applications of universal properties take us from this simple starting point to several natural combinatorial notions pertaining to set-graded posets.

First, as in Section 13.1.3, we may use the universal property of $\mathbf{L}^*$ (cofreeness) to derive a canonical morphism of Hopf monoids

$$\widehat{\omega}\colon \mathbf{sgP} \to \mathbf{L}^*,$$

which has the following explicit form:

$$\widehat{\omega}(P,\lambda) = \sum_C \lambda(C)^*. \tag{13.28}$$

The sum is over all maximal chains $C$ of $P$, and $\lambda(C)$ stands for the linear order on $I$ corresponding to $C$ under $\lambda$, as in (13.24).

Continuing as in Section 13.1.4, combining $\widehat{\omega}$ with evaluation we obtain a commutative diagram of monoids

$$\begin{tikzcd}
\mathbf{L} \times \mathbf{sgP} \arrow[rr, "\mathrm{id}\times\widehat{\omega}"] \arrow[dr, "\zeta"'] & & \mathbf{I\!L}^* \arrow[dl] \\
& \mathbf{E}. &
\end{tikzcd}$$

Explicitly,

$$\zeta\big(l,(P,\lambda)\big) = \text{number of maximal chains } C \text{ in } P \text{ such that } \lambda(C) = l. \tag{13.29}$$

Here we are following Convention 12.60.

The universal property of Theorem 11.23 then yields a commutative diagram of Hopf monoids

$$\begin{tikzcd}
\mathbf{L} \times \mathbf{sgP} \arrow[rr, "\mathrm{id}\times\widehat{\omega}"] \arrow[dr, "\hat{\zeta}"'] & & \mathbf{I\!L}^* \arrow[dl, "\beta"] \\
& \mathcal{T}^\vee(\mathbf{E}_+) = \mathbf{\Sigma}^*. &
\end{tikzcd} \tag{13.30}$$

Formula (11.18) yields the following explicit description for $\hat{\zeta}$:

$$\hat{\zeta}\big(l,(P,\lambda)\big) = \sum_F f_F\big(l,(P,\lambda)\big)\, M_F \tag{13.31}$$

where the coefficient $f_F\big(l,(P,\lambda)\big)$ counts the number of maximal chains $C$ in $P$ satisfying two conditions:

$\lambda(C)$ refines $F$ and the restrictions of $l$ and of $\lambda(C)$ to each block of $F$ coincide.

These conditions can be succinctly expressed as:

$$F\cdot l = \lambda(C), \quad \text{or equivalently,} \quad \mathrm{Des}\big(l,\lambda(C)\big) \le F \le \lambda(C).$$

Here $F \cdot l$ is the projection of the chamber $l$ onto the face $F$ (10.13), and the equivalence follows from (10.42).

The morphism of Hopf monoids $\hat{\zeta}$ is thus an enumerator of descents.

**13.6.7. EL posets.** The comparison between two canonical maps $\mathbf{L} \times \mathbf{sgP} \to \mathbf{E}$ will lead us to the consideration of this interesting class of posets.

Recall the morphism $\eta \colon \mathbf{sgP} \to \mathbf{E}$ of Section 13.6.5. Let us use the same symbol to denote the map

$$\eta \colon \mathbf{L} \times \mathbf{sgP} \to \mathbf{E}, \qquad \big(l, (P, \lambda)\big) \mapsto 1. \tag{13.32}$$

Since the product of two basis elements of $\mathbf{L} \times \mathbf{sgP}$ is another basis element, $\eta$ is a morphism of monoids.

On the other hand, we also have the morphism of monoids

$$\zeta \colon \mathbf{L} \times \mathbf{sgP} \to \mathbf{E}$$

of Section 13.6.6.

Consider now the species $\ker(\zeta - \eta)$. Since both $\zeta$ and $\eta$ are morphisms of monoids, the species $\ker(\zeta - \eta)$ is a submonoid of $\mathbf{L} \times \mathbf{sgP}$.

We pause to discuss some general facts. If $\mathbf{k}$ is a subspecies of a comonoid $\mathbf{p}$, there exists a unique largest subcomonoid of $\mathbf{p}$ contained in $\mathbf{k}$. This is called the subcomonoid *cogenerated* by $\mathbf{k}$. In addition, if $\mathbf{p}$ is a bimonoid and $\mathbf{k}$ is a submonoid, then the subcomonoid cogenerated by $\mathbf{k}$ is a subbimonoid of $\mathbf{p}$. The analogous facts for vector spaces are familiar (see for instance [8, Section 6]) and the proofs for species are similar.

We proceed. Define $\mathbf{elP}$ as the subcomonoid of $\mathbf{L} \times \mathbf{sgP}$ cogenerated by the species $\ker(\zeta - \eta)$. According to the above, it is a subbimonoid of $\mathbf{L} \times \mathbf{sgP}$, and since the latter is connected, so is the former. Thus, $\mathbf{elP}$ is a Hopf submonoid of $\mathbf{L} \times \mathbf{sgP}$. This definition of $\mathbf{elP}$, natural from the algebraic point of view, does not immediately reveal its combinatorial significance. We will soon see that it contains an important class of posets.

**Definition 13.17.** Let $I$ be a finite set, $l$ a linear order on $I$, and $(P, \lambda)$ an $I$-graded poset. We say that $\big(l, (P, \lambda)\big)$ is EL if in every interval of $P$ there exists exactly one maximal chain $C$ such that $\lambda(C) = l|_C$.

Suppose $\big(l, (P, \lambda)\big)$ is EL, $I = [n]$ and $l$ is the canonical linear order on $[n]$. Then the labels along each maximal chain of $P$ must form a permutation of $[n]$, and the identity permutation must occur exactly once. In the literature, such posets are called $\mathrm{S}_n$ *EL-labeled*. They constitute a very interesting class of posets. Results of Stanley [337] and Liu [231] state that any finite *supersolvable* lattice admits an $\mathrm{S}_n$ EL-labeling. Conversely, McNamara has shown that if a finite lattice admits a symmetric EL-labeling, then it is supersolvable [266]. More recently, McNamara and Thomas have given a similar characterization for arbitrary $\mathrm{S}_n$ EL-labeled posets [267].

It follows immediately from (13.29) and (13.32) that if $\big(l, (P, \lambda)\big)$ is EL, then

$$\big(l, (P, \lambda)\big) \in \ker(\zeta - \eta).$$

On the other hand, since the defining condition of EL posets refers to *all* intervals of the underlying poset, it is clear that the subspecies of $\mathbf{sgP}$ spanned by the EL posets is a subcomonoid. Therefore, it must be contained in the subcomonoid cogenerated by $\ker(\zeta - \eta)$. Thus, every EL poset $\big(l, (P, \lambda)\big)$ belongs to $\mathbf{elP}$.

We close the section with a simple consequence of our considerations.

By construction, the maps $\zeta$ and $\eta$ agree on $\mathbf{elP}$. Therefore, the same holds for the morphisms

$$\mathbf{elP} \to \mathbf{\Sigma}^*$$

they give rise to via the universal property of $\mathbf{\Sigma}^*$. Comparing (13.27) and (13.31) we deduce that

$$f_F(P,\lambda) = f_F\big(l,(P,\lambda)\big)$$

for any EL poset $\big(l,(P,\lambda)\big)$ and every composition $F$. In other words, for such posets, the enumerator of flags and the enumerators of descents agree. A closely related result (which follows) appears in [341, Theorem 3.13.2] and [50, Theorem 2.7].

**13.6.8. Set-weighted posets.** There is a larger Hopf monoid than that of set-graded posets. Namely, we use posets $P$ as follows:

- $P$ has $\hat{0}$ and $\hat{1}$,
- each edge is labeled by a nonempty subset of $I$,
- the labels along any maximal chain form a composition of $I$.

We refer to these as *set-weighted* posets. Set-graded posets are the special case in which all labels are singletons. It follows that given $x$ and $y$ in $P$, the union of the subsets along a saturated chain from $x$ to $y$ is independent of the chosen chain. Call this set $S(x,y)$.

There is an equivalence between $I$-weighted posets and morphisms of posets $\varphi\colon P \to 2^I$ that preserve bottom and top elements and are such that if $x<y$, then $\varphi(x)$ is a proper subset of $\varphi(y)$. To pass from the former situation to the latter, take $\varphi(x) = S(\hat{0},x)$.

Note that set-weighted posets need not be graded in the usual sense. As an example, take any subposet of $2^I$ which contains $\emptyset$ and $I$.

Let $\mathbf{swP}[I]$ be the (infinite-dimensional) vector space with basis the set of isomorphism classes of $I$-weighted posets. The Hopf monoid structure on $\mathbf{swP}$ is defined in the same way as for $\mathbf{sgP}$, and $\mathbf{swP}$ contains the latter as a Hopf submonoid.

Given a maximal chain $C$ in a set-weighted poset $P$, we may consider $\lambda(C)$ as in (13.24), which is now a set composition. Recall that two compositions have the same support (Section 10.1.5) if they are reorderings of each other. Consider the subspecies $\widetilde{\mathbf{swP}}$ of $\mathbf{swP}$ linearly spanned by those set-weighted posets $(P,\lambda)$ such that for any maximal chains $C$ and $D$ of $P$,

$$\operatorname{supp}\lambda(C) = \operatorname{supp}\lambda(D).$$

It is a Hopf submonoid of $\mathbf{swP}$ and it contains $\mathbf{sgP}$. In addition, the map

$$\widetilde{\mathbf{swP}} \to \mathbf{sgP}, \quad P \mapsto \begin{cases} P & \text{if } P \text{ is set-graded,} \\ 0 & \text{otherwise,} \end{cases} \tag{13.33}$$

is a morphism of Hopf monoids, split by the inclusion:

$$\mathbf{sgP} \rightleftarrows \widetilde{\mathbf{swP}}.$$

Note that, as species,

$$\widetilde{\mathbf{swP}} = \mathbf{sgP} \circ \mathbf{E}_+.$$

TABLE 13.4. Analogies: from numbers to sets.

| natural number $n$ | finite set $I$ |
|---|---|
| chain $C_n$ | Boolean poset $2^I$ |
| graded poset | set-graded poset |
| simplex $\Delta_{[n-1]}$ | Coxeter complex $\Sigma[I]$ |
| balanced simplicial complex | set-balanced simplicial complex |

## 13.7. Set-balanced simplicial complexes

The notion of set-balanced simplicial complex is the species analogue of the more familiar notion of balanced simplicial complexes. The parallel is similar (and related) to that between graded and set-graded posets (Section 13.6). These and similar analogies are summarized in Table 13.4. After discussing these notions, we introduce an interesting monoid based on the species of set-balanced simplicial complexes. We close the section by relating the monoids of set-graded posets and of set-balanced simplicial complexes via the order complex construction.

**13.7.1. Balanced simplicial complexes.** We will use the terminology on simplicial complexes introduced in Section 10.3.1.

Recall that a balanced simplicial complex is a pair $(k, \varphi)$ where $k$ is a simplicial complex and $\varphi\colon V \to [n]$ is a function that restricts to a bijection

$$C \xrightarrow{\cong} [n]$$

for each maximal face $C$ of $k$. This implies that $k$ is pure of dimension $n-1$.

Given a balanced simplicial complex $(k, \varphi)$ of dimension $n-1$, the map

$$k \to \Delta_{[n]}, \qquad F \mapsto \varphi(F),$$

is a nondegenerate simplicial map. Conversely, given a pure simplicial complex $k$ of dimension $n-1$, any nondegenerate map $k \to \Delta_{[n]}$ turns $k$ into a balanced simplicial complex.

**Proposition 13.18.** *A balanced simplicial complex of dimension $n-1$ is equivalent to a pure simplicial complex $k$ of dimension $n-1$ equipped with a nondegenerate simplicial map $k \to \Delta_{[n]}$.*

**13.7.2. Set-balanced simplicial complexes.** This motivates the following definition, in which we replace the simplex $\Delta_{[n]}$ by the Coxeter complex $\Sigma[I]$ (Section 10.3).

**Definition 13.19.** Let $I$ be a finite set. An *$I$-balanced simplicial complex* is a pair $(k, \varphi)$ where $k$ is a pure simplicial complex of dimension $|I|-2$ and

$$\varphi\colon k \to \Sigma[I]$$

is a nondegenerate simplicial map. When $I$ and $\varphi$ are understood or not specified, we may simply say that the complex $k$ is *set-balanced*.

Two $I$-balanced simplicial complexes $(k, \varphi)$ and $(k', \varphi')$ are isomorphic if there is a simplicial isomorphism $f\colon k \to k'$ such that

$$\begin{tikzcd} k \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & k' \arrow[dl, "\varphi'"] \\ & \Sigma[I] & \end{tikzcd}$$

commutes.

**Remark 13.20.** Let $n := |I|$. Recall that the Coxeter complex $\Sigma[I]$ is pure and of dimension $n-2$. Further, it is balanced via the type map (10.12).

A fortiori, any $I$-balanced complex is also balanced when endowed with the nondegenerate simplicial map

$$k \xrightarrow{\varphi} \Sigma[I] \to \Delta_{[n-1]}.$$

Conversely, any balanced simplicial complex $(k, \varphi)$ of dimension $n-2$ can be turned into an $I$-balanced complex in many ways. For instance, choose a linear order $l^1|\cdots|l^n$ on $I$ and let $C_l$ be the corresponding chamber in $\Sigma[I]$. The simplices $\Delta_{[n-1]}$ and $\Delta_{C_l}$ are isomorphic by means of the map

$$\Delta_{[n-1]} \to \Delta_{C_l}, \qquad i \mapsto l^1 \cdots l^i \,|\, l^{i+1} \cdots l^n.$$

Therefore, $k$ is $I$-balanced by means of the nondegenerate simplicial map

$$k \xrightarrow{\varphi} \Delta_{[n-1]} \xrightarrow{\cong} \Delta_{C_l} \hookrightarrow \Sigma[I].$$

**Example 13.21.** Let $P$ be a poset with bottom $\hat{0}$ and top $\hat{1}$ such that $\hat{0} < \hat{1}$. The *reduced order complex* of $P$ is

$$\Delta(P) := \{C \subseteq P \setminus \{\hat{0}, \hat{1}\} \mid C \text{ is a chain in } P\}.$$

This is a simplicial complex with vertex set $P \setminus \{\hat{0}, \hat{1}\}$.

Suppose that $P$ is graded of rank $n$. Equivalently, by Proposition 13.12, there is a morphism of graded posets $P \to C_n$ where $C_n = \{0 < 1 < \cdots < n\}$. There is then a nondegenerate simplicial map

$$\Delta(P) \to \Delta(C_n) = \Delta_{[n-1]}$$

which turns $\Delta(P)$ into a balanced complex (of dimension $n-2$). On vertices, the function

$$P \setminus \{\hat{0}, \hat{1}\} \to [n-1]$$

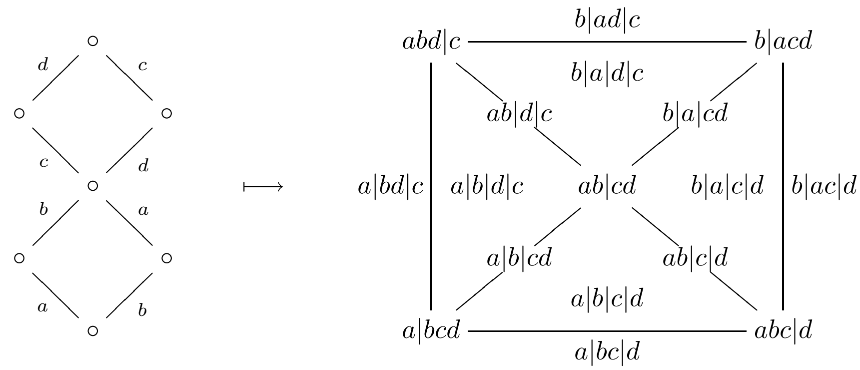
sends an element of $P$ to its rank.

Now suppose that $P$ is $I$-graded. Equivalently, by Proposition 13.15, there is a morphism of graded posets $P \to 2^I$. There is then a nondegenerate simplicial map

$$\Delta(P) \to \Delta(2^I) = \Sigma[I]$$

which turns $\Delta(P)$ into an $I$-balanced complex.

Consider the following example.




The order complex of the set-graded poset on the left is a set-balanced simplicial complex consisting of four triangles attached at a common vertex, with the labelings on the faces as shown. Note that it is an embedding into the Coxeter complex shown in Figure 10.4.

Now consider the poset $L(I)$ of subspaces in $\Bbbk I$ (Example 13.16). The simplicial complex $\Delta\big(L(I)\big)$ is a *building of type* $A$ [3, Sections 4.3 and 9.2]. Once we choose a linear order on $I$, the poset $L(I)$ is graded by means of the Schubert symbol. Therefore, $\Delta\big(L(I)\big)$ is set-balanced. The map

$$\Delta\big(L(I)\big) \to \Sigma[I]$$

is a *retraction* of the building onto an *apartment* [3, Section 4.4].

**Remark 13.22.** We may generalize the above setup by using simplicial complexes $k$ with a nondegenerate map $\varphi\colon k \to \Sigma[I]$, without conditions on the dimension of $k$. Such complexes need not be balanced or even pure. The order complex map takes set-weighted posets (Section 13.6.8) to this class of simplicial complexes.

**13.7.3. The monoid of set-balanced simplicial complexes.** Let $\mathbf{K}[I]$ be the (infinite-dimensional) vector space with basis the set of isomorphism classes of $I$-balanced simplicial complexes. This defines a species $\mathbf{K}$ which is connected, since there is only one $\emptyset$-balanced simplicial complex. We proceed to turn the species $\mathbf{K}$ into a monoid.

In order to define the product, fix a decomposition $I = S \sqcup T$ into nonempty subsets and two (isomorphism classes of) set-balanced simplicial complexes $(k_1, \varphi_1) \in \mathbf{K}[S]$ and $(k_2, \varphi_2) \in \mathbf{K}[T]$. Then $G := S|T$ is a vertex of the Coxeter complex $\Sigma[I]$ and we may consider the join map (10.57)

$$j_G\colon \Sigma[S] \times \Sigma[T] \xrightarrow{\cong} \mathrm{Star}_{\Sigma[I]}(G).$$

Together with the projection product (10.16), this allows us to define a new complex $k$ as follows.

$$k := \{(K_1, K_2, F) \mid K_1 \in k_1,\ K_2 \in k_2,\ F \in \Sigma[I],\ GF = j_G\big(\varphi_1(K_1), \varphi_2(K_2)\big)\}.$$

As in Proposition 12.20, the latter condition simply states that $F$ is a quasi-shuffle of the faces $\varphi_1(K_1)$ and $\varphi_2(K_2)$ (Section 10.1.6). Thus, the faces of $k$ are triples consisting of a face from each $k_i$ and a quasi-shuffle of their images under the maps $\varphi_i$.

We endow $k$ with the partial order of the Cartesian product $k_1 \times k_2 \times \Sigma[I]$. In other words, given $(J_1, J_2, E)$ and $(K_1, K_2, F) \in k$, we define

$$(J_1, J_2, E) \leq (K_1, K_2, F) \quad \text{if } J_1 \leq K_1,\ J_2 \leq K_2, \text{ and } E \leq F.$$

We show that $k$ is a simplicial complex by checking conditions (10.11a)–(10.11c). The triple consisting of the empty faces of $k_1$, $k_2$ and $\Sigma[I]$ is the minimum element of $k$. Next, suppose two faces $(J_1, J_2, E)$ and $(K_1, K_2, F)$ have an upper bound. Then, for each $i$, $J_i$ and $K_i$ are faces of a simplex of $k_i$, and since $\varphi_i$ is an isomorphism when restricted to a simplex, we have $\varphi_i(J_i \vee K_i) = \varphi_i(J_i) \vee \varphi_i(K_i)$. Therefore,

$$\begin{aligned} j_G\big(\varphi_1(J_1 \vee K_1), \varphi_2(J_2 \vee K_2)\big) &= j_G\big(\varphi_1(J_1) \vee \varphi_1(K_1), \varphi_2(J_2) \vee \varphi_2(K_2)\big) \\ &= j_G\big(\varphi_1(J_1), \varphi_2(J_2)\big) \vee j_G\big(\varphi_1(K_1), \varphi_2(K_2)\big) \\ &= GE \vee GF = G(E \vee F). \end{aligned}$$

(We used that $j_G$ is an isomorphism of complexes and property (vii) of projection maps in Proposition 10.1.) Thus,

$$(J_1 \vee K_1, J_2 \vee K_2, E \vee F)$$

is a face of $k$, and it is clearly the least upper bound of the given faces. Finally, we check that the poset of subfaces of a face $(K_1, K_2, F)$ is Boolean. We claim that a subface $(J_1, J_2, E)$ is uniquely determined by $E$: indeed, since the map $j_G$ is bijective, the faces $\varphi_i(J_i)$ are uniquely determined by $E$, and then since the maps $\varphi_i$ are bijective when restricted to $\Delta_{K_i}$, the faces $J_i$ are uniquely determined. Thus, the poset of subfaces of $(K_1, K_2, F)$ in $k$ is isomorphic to the poset of subfaces of $F$ in $\Sigma[I]$, so it is Boolean.

We turn $k$ into an $I$-balanced simplicial complex by defining $\varphi\colon k \to \Sigma[I]$ by

$$\varphi(K_1, K_2, F) := F.$$

The following pull-back diagram summarizes the definition of $(k, \varphi)$.

$$\begin{tikzcd} k \arrow[r, dashed] \arrow[dd, dashed, "\varphi"'] & k_1 \times k_2 \arrow[d, "(\varphi_1,\varphi_2)"] \\ & \Sigma[S] \times \Sigma[T] \arrow[d, "j_G", "\cong"'] \\ \Sigma[I] \arrow[r, "p_G"'] & \mathrm{Star}_{\Sigma[I]}(S|T) \end{tikzcd}$$

The projection map $p_G$ is as in (10.23).

We define the product of $\mathbf{K}$ by

$$\mathbf{K}[S] \otimes \mathbf{K}[T] \to \mathbf{K}[I], \qquad (k_1, \varphi_1) \otimes (k_2, \varphi_2) \mapsto (k, \varphi).$$

With this product, the species $\mathbf{K}$ is a commutative monoid.

As an example, take $k_1 := \Sigma[S]$ and $k_2 := \Sigma[T]$, viewed as set-balanced complexes by means of the identity maps. Since the map $j_G$ is bijective, a triple $(K_1, K_2, F) \in k$ is in this case uniquely determined by $F$. Therefore, $k \cong \Sigma[I]$ as set-balanced complexes, and the product of the monoid $\mathbf{K}$ is such that

$$(\Sigma[S], \mathrm{id}) \otimes (\Sigma[T], \mathrm{id}) \mapsto (\Sigma[I], \mathrm{id}).$$

As explained in Example 13.21, the (reduced) order complex of an $I$-graded poset is an $I$-balanced simplicial complex. This defines a morphism of species

$$\Delta\colon \mathbf{sgP} \to \mathbf{K}.$$

It is easy to see, comparing the definition here with the one in Section 13.6.4, that $\Delta$ is a morphism of monoids.

**Question 13.23.** Is it possible to define a coproduct on $\mathbf{K}$ that turns it into a Hopf monoid in such a manner that $\Delta$ is a morphism of Hopf monoids?

A positive answer may be obtained by restricting the class of set-balanced simplicial complexes. For instance, we may impose the following restriction: For each face $F$ of $k$, there is a commutative diagram of simplicial complexes

$$\begin{tikzcd}
\mathrm{Star}_k(F) \arrow[r, "b_F", "\cong"'] \arrow[d, "\varphi"'] & k_1 \times k_2 \times \cdots \times k_r \arrow[d, "(\varphi_1,\varphi_2,\dots,\varphi_r)"] \\
\mathrm{Star}_{\Sigma[I]}\big(\varphi(F)\big) \arrow[r, "\cong", "b_{\varphi(F)}"'] & \Sigma[S^1] \times \Sigma[S^2] \times \cdots \times \Sigma[S^r]
\end{tikzcd}$$

where $\varphi(F) = S^1|S^2|\cdots|S^r$. The existence of the $k_i$'s, $\varphi_i$'s and the map $b_F$ is a part of the requirement.

In this situation, a suitable coproduct can be defined. Note also that the order complex of a set-graded poset satisfies this requirement.

## 13.8. Closures, matroids, convex geometries, and topologies

Matroids were introduced by Whitney [374] as an abstraction of the notion of linear independence. Convex geometries were introduced by Edelman [110] and Jamison [177], independently.

Good references for matroids are the books by Crapo and Rota [87], Oxley [287] and Welsh [373], and for convex geometries and related notions, the surveys of Björner and Ziegler [53] and Edelman and Jamison [111].

Matroids, convex geometries, and topologies are special kinds of closure operators. Only a few basic concepts pertaining to these notions are needed for our purposes, which allow us to provide a self-contained discussion.

### 13.8.1. Closure operators.

Let $2^I$ denote the set of subsets of a finite set $I$.

**Definition 13.24.** A *closure operator* on $I$ is a map

$$c\colon 2^I \to 2^I$$

such that for every $A, B \in 2^I$, we have:

- $A \subseteq c(A)$;
- if $A \subseteq c(B)$, then $c(A) \subseteq c(B)$.

It follows from these axioms that

- $c\big(c(A)\big) = c(A)$;
- if $A \subseteq B$, then $c(A) \subseteq c(B)$.

The set $I$ is the *ground set* of $c$. An element $a \in I$ is a *loop* if

$$a \in c(\emptyset).$$

It is a *coloop* if

$$a \notin c(I \setminus \{a\}).$$

Loops and coloops play a role in subsequent sections.

Let $I = S \sqcup T$ be a disjoint decomposition. Given closure operators $c_1$ and $c_2$ with ground sets $S$ and $T$, their *direct sum*

$$c_1 \oplus c_2 \colon 2^I \to 2^I$$

is the operator defined by

$$(c_1 \oplus c_2)(C) := c_1(S \cap C) \cup c_2(T \cap C).$$

The direct sum $c_1 \oplus c_2$ is a closure operator with ground set $I$.

Given a closure operator $c$ with ground set $I$ and subsets $A \subseteq B \subseteq I$, the *minor*

$$c_{A:B} \colon 2^{B \setminus A} \to 2^{B \setminus A}$$

is the operator given by

$$c_{A:B}(C) := c(C \cup A) \cap B \setminus A.$$

The minor $c_{A:B}$ is a closure operator with ground set $B \setminus A$. The minors $c_{\emptyset:B}$ and $c_{A:I}$ are called the *restriction* of $c$ to $B$ and the *contraction* of $A$ from $c$, respectively.

Let $\mathbf{C}[I]$ denote the vector space with basis the set of all closure operators with ground set $I$. This defines a species $\mathbf{C}$. It is a Hopf monoid with product

$$\mathbf{C}[S] \otimes \mathbf{C}[T] \to \mathbf{C}[I], \qquad c_1 \otimes c_2 \mapsto c_1 \oplus c_2 \tag{13.34}$$

and coproduct

$$\mathbf{C}[I] \to \mathbf{C}[S] \otimes \mathbf{C}[T], \qquad c \mapsto c_{\emptyset:S} \otimes c_{S:I}. \tag{13.35}$$

The Hopf monoid $\mathbf{C}$ is commutative but not cocommutative.

The above construction originates in the work of Joni and Rota [179, Section XVII].

### 13.8.2. Matroids.

**Definition 13.25.** A *matroid* with ground set $I$ is a closure operator $m\colon 2^I \to 2^I$ that satisfies the *Mac Lane–Steinitz exchange axiom*:

- if $a \in m(A \cup \{b\})$ and $a \notin m(A)$, then $b \in m(A \cup \{a\})$,

for every $A \in 2^I$ and $a, b \in I$.

Let $\mathbf{M}[I]$ denote the vector subspace of $\mathbf{C}[I]$ spanned by all matroids with ground set $I$. The class of matroids is closed under direct sums and minors; therefore, the species $\mathbf{M}$ is a Hopf submonoid of $\mathbf{C}$.

Recall the notion of loops and coloops associated to a closure operator. In the context of matroids, coloops are sometimes called *isthmuses*.

A matroid $m$ with ground set $I$ is *split* if there exists a subset $L \subseteq I$ such that

$$m(A) = A \cup L$$

for every $A \subseteq I$. In this case $L$ is unique: $L = m(\emptyset)$; moreover, $L$ is the set of loops and $I \setminus L$ the set of isthmuses. Conversely, if every element of the ground set is either a loop or an isthmus, then $m$ is split.

Define $\zeta\colon \mathbf{M} \to \mathbf{E}$ by

$$\zeta(m) := \begin{cases} 1 & \text{if } m \text{ is split,} \\ 0 & \text{otherwise.} \end{cases} \tag{13.36}$$

Then $\zeta$ is a morphism of monoids. By universality (Theorem 11.23), it gives rise to a morphism of Hopf monoids

$$\hat{\zeta}\colon \mathbf{M} \to \mathbf{\Sigma}^*.$$

### 13.8.3. Convex geometries.

**Definition 13.26.** A *convex geometry* with ground set $I$ is a closure operator $g\colon 2^I \to 2^I$ that satisfies the *antiexchange axiom*:

- if $a \in g(A \cup \{b\})$, $a \neq b$ and $a, b \notin g(A)$, then $b \notin g(A \cup \{a\})$,

for every $A \in 2^I$ and $a, b \in I$.

Let $\mathbf{cG}[I]$ denote the vector subspace of $\mathbf{C}[I]$ spanned by all convex geometries with ground set $I$. The class of convex geometries is closed under direct sums and minors; therefore, the species $\mathbf{cG}$ is a Hopf submonoid of $\mathbf{C}$.

Convex geometries are discussed in [53, Section 8.7.A]. See [53, Figure 8.9] for an illustration of the antiexchange axiom. Convex geometries are in duality with *antimatroids*, a special class of *greedoids*. Both notions are carefully reviewed in [53]; we do not employ them here.

*Warning.* An antimatroid, and more generally a greedoid, has an associated *rank closure* [53, Section 8.4.B], but in general this is not a closure operator and should not be confused with the convex geometry dual to an antimatroid.

**13.8.4. A deformation.** We discuss a deformation of the Hopf monoid of closure operators that arose in conversation with Bill Schmitt. It gives rise to deformations of the Hopf monoids of matroids and of convex geometries. We are interested in the case when the deformation parameter is set to 0, for reasons that will be clear shortly.

We make use of the theory of cocycle deformations developed in Section 9.6.

Given a decomposition $I = S \sqcup T$, let

$$\ell_{S,T}(c) := \big|\big(c(S) \setminus c(\emptyset)\big) \cap T\big|.$$

The statistic $\ell_{S,T}(c)$ counts the number of loops created in contracting $S$ from $c$. More precisely, $\ell_{S,T}(c)$ is the number of loops of $c_{S:I}$ minus the number of loops of $c$ that belong to $T$.

Let $\ell_I(c)$ denote the number of loops of a closure operator $c$ with ground set $I$. We have

$$\ell(c) + \ell_{S,T}(c) = \ell(c_{\emptyset:S}) + \ell(c_{S:I}).$$

The family of maps $\ell_I$ is a 1-cochain on $\mathbf{C}$ with values on $\mathbb{N}$, in the sense of Section 9.6.1. The previous identity shows that the family $\ell_{S,T}$ is the corresponding 2-coboundary. Moreover, the 1-cochain $\ell_I$ is multiplicative, and hence the 2-cochain $\ell_{S,T}$ is multiplicative of twist 0 (Section 9.6.3). Therefore by Proposition 9.21, we may deform the coproduct of the Hopf monoid of closure operators as follows. We fix $q \in \Bbbk$ and define a coproduct by

$$\mathbf{C}[I] \to \mathbf{C}[S] \otimes \mathbf{C}[T], \qquad c \mapsto q^{\ell_{S,T}(c)}\, c_{\emptyset:S} \otimes c_{S:I}. \tag{13.37}$$

The coproduct (13.37) is compatible with the product (13.34), and with this structure the species of closure operators is a Hopf monoid.

We denote this Hopf monoid by $\mathbf{C}_q$. The species of matroids is a Hopf submonoid of $\mathbf{C}_q$ that we denote by $\mathbf{M}_q$. Setting $q = 1$ recovers the Hopf monoids of Sections 13.8.1 and 13.8.2.

Since the deformation is with respect to a 2-coboundary, the only other interesting case is $q = 0$. Indeed, by Proposition 9.22, the map

$$\mathbf{C} \to \mathbf{C}_q, \qquad c \mapsto q^{\ell(c)}\, c$$

is a morphism of Hopf monoids for any $q \in \Bbbk$. It follows that

$$\mathbf{C}_q \cong \mathbf{C}$$

for any $q \neq 0$. The case $q = 0$ is of interest and is discussed next.

**13.8.5. Loopless closure operators.** Let $c$ be a closure operator with ground set $I$. A subset $F \subseteq I$ is *closed* if $c(F) = F$, or equivalently if $F = c(A)$ for some $A \subseteq I$.

The closure operator $c$ is *loopless* if the empty set is closed: $c(\emptyset) = \emptyset$. Let $\overline{\mathbf{C}}$ denote the species spanned by loopless closure operators. The direct sum of two loopless closures is again loopless, so $\overline{\mathbf{C}}$ is a submonoid of $\mathbf{C}$.

Suppose $A \subseteq B \subseteq I$. The set

$$c_{A:B}(\emptyset) = c(A) \cap (B \setminus A)$$

may be nonempty even if $c$ is loopless. For this reason, the class of loopless closures is not closed under minors, and $\overline{\mathbf{C}}$ is not a subcomonoid of $\mathbf{C}$ under the coproduct (13.35). Note that

$$c_{A:B} \text{ is loopless} \iff A \text{ is closed.}$$

Suppose $c$ is loopless. Given a decomposition $I = S \sqcup T$, we have that

$$\ell_{S,T}(c) = 0 \iff S \text{ is closed.}$$

Hence, if we set $q = 0$ in the coproduct formula (13.37), we obtain

$$c \mapsto \begin{cases} c_{\emptyset:S} \otimes c_{S:I} & \text{if } S \text{ is closed,} \\ 0 & \text{otherwise.} \end{cases} \tag{13.38}$$

In this formula, both minors $c_{\emptyset:S}$ and $c_{S:I}$ are loopless closures. Therefore, the species $\overline{\mathbf{C}}$ is a Hopf submonoid of the Hopf monoid $\mathbf{C}_0$, with coproduct given by (13.38).

The species $\overline{\mathbf{M}}$ of loopless matroids and $\overline{\mathbf{cG}}$ of loopless convex geometries are Hopf submonoids of $\overline{\mathbf{C}}$. The closed sets of a matroid are called *flats*. The closed sets of a convex geometry are called *convex sets*; they are the complements of the *feasible sets* of the dual antimatroid [53, Proposition 8.7.3]. Flats and convex sets intervene in the coproducts of $\overline{\mathbf{M}}$ and $\overline{\mathbf{cG}}$, according to (13.38).

Björner and Ziegler define the contraction of a subset $T$ from an antimatroid only when $T$ is feasible [53, Section 8.4.D]. Restriction and contraction for convex geometries and antimatroids are related as follows. Let $c$ be a convex geometry and $c^\perp$ the dual antimatroid. Given a decomposition $I = S \sqcup T$,

$$(c_{\emptyset:S})^\perp = (c^\perp)_{T:I} \qquad \text{and} \qquad (c_{S:I})^\perp = (c^\perp)_{\emptyset:T},$$

the former when $T$ is feasible, or equivalently when $S$ is closed.

For a convex geometry $c$, the set of loops and the set of coloops are complementary. This follows from [53, Proposition 8.2.8.i]. Thus, $c$ is loopless if and only if every element is a coloop. In this case, the antimatroid $c^\perp$ is said to be *full* [53, p. 292].

It follows from this discussion that the coproduct (13.38) can be reformulated in terms of restriction and contraction for full antimatroids.

### 13.8.6. Topologies.

**Definition 13.27.** A closure operator $t\colon 2^I \to 2^I$ is *topological* if

- $t(\emptyset) = \emptyset$;
- $t(A \cup B) = t(A) \cup t(B)$.

These axioms, together with those in Definition 13.24 are known as *Kuratowski's axioms*. As is well-known [178, Section I.1] or [193, Theorem 1.8], topological closures with ground set $I$ and *topologies* on $I$ are equivalent notions: given a topological closure $t$, the subsets of the form $t(A)$ are the closed sets of a topology on the set $I$; conversely, any topology on $I$ arises in this manner from a unique topological closure operator.

Let $\mathbf{T}[I]$ denote the subspace of $\mathbf{C}[I]$ spanned by all topological closures with ground set $I$. This defines the species $\mathbf{T}$ of topologies. The direct sum of two topological closures is another topological closure (corresponding to the disjoint union of the topological spaces); hence, the species $\mathbf{T}$ is a submonoid of $\mathbf{C}$ under the product (13.34). For the same reason as for loopless closures $\mathbf{T}$ is not a subcomonoid of $\mathbf{C}$, but it is a Hopf submonoid of the Hopf monoid $\mathbf{C}_0$ (and in fact of $\overline{\mathbf{C}}$).

A topological closure $t$ is *Kolmogorov* if it satisfies the $T_0$-*separation axiom*:

- if $a \in t(\{b\})$ and $b \in t(\{a\})$, then $a = b$.

The species $\mathbf{kT}$ of Kolmogorov topologies is a Hopf submonoid of $\mathbf{T}$.

A Kolmogorov closure is always a loopless convex geometry. We thus obtain the commutative diagram of Hopf monoids below.

$$\begin{tikzcd}
\mathbf{T} \arrow[r, hook] & \overline{\mathbf{C}} \\
\mathbf{kT} \arrow[u, hook] \arrow[r, hook] & \overline{\mathbf{cG}} \arrow[u, hook]
\end{tikzcd}$$

## 13.9. The Birkhoff transform

In this section, we discuss the Birkhoff transform which relates the Hopf monoid of posets to the Hopf monoid of set-graded posets, and the Hopf monoid of preposets to that of set-weighted posets. We also explain how the Hopf monoids of loopless convex geometries, loopless closure operators and topological closures fit into this framework.

**13.9.1. From (pre)posets to (set weighted) set-graded posets.** Let $p$ be a poset with vertex set $I$, as in Section 13.1. The *Birkhoff transform* of $p$ is the poset $J(p)$ whose elements are the lower sets of $p$, ordered by inclusion. The poset has a bottom element $\emptyset$ and a top element $I$.

Note that if $S$ is a lower set of $p$ and $a$ is any minimal element of $I \setminus S$, then $S \cup \{a\}$ is another lower set in $p$. Letting

$$\lambda(S, S \cup \{a\}) := a$$

defines an $I$-labeling on $J(p)$, thus turning it into an $I$-graded poset. Thus, we may view the Birkhoff transform as a map

$$J \colon \mathbf{P} \to \mathbf{sgP}.$$

The labeling on $J(p)$ will be omitted from the notation. A classical result of Birkhoff states that a finite poset is of the form $J(p)$ for some poset $p$ if and only it is a *distributive lattice* [341, Theorem 3.4.1].

Let $S$ be a lower set of a poset $p$. The lower sets contained in $S$ are the lower sets of $p|_S$. A lower set in a disjoint union $p_1 \sqcup p_2$ is of the form $S_1 \sqcup S_2$ where each $S_i$ is a lower set in $p_i$. It follows that $J$ is a morphism of Hopf monoids.

The following diagram commutes trivially:

$$\begin{tikzcd}
\mathbf{P} \arrow[rr, "J"] \arrow[dr, "\omega"'] & & \mathbf{sgP} \arrow[dl, "\omega"] \\
& \mathbf{X}. &
\end{tikzcd}$$

Since the maps $\widehat{\omega}$ and $\hat{\zeta}$ were canonically constructed from the maps $\omega$ (in Sections 13.1.3 and 13.6.6), it follows that the these diagrams commute as well:

$$\begin{tikzcd}
\mathbf{P} \arrow[rr, "J"] \arrow[dr, "\widehat{\omega}"'] & & \mathbf{sgP} \arrow[dl, "\widehat{\omega}"] \\
& \mathbf{L}^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{L} \times \mathbf{P} \arrow[rr, "\mathrm{id} \times J"] \arrow[dr, "\hat{\zeta}"'] & & \mathbf{L} \times \mathbf{sgP} \arrow[dl, "\hat{\zeta}"] \\
& \mathbf{\Sigma}^*. &
\end{tikzcd}$$

The commutativity of the first diagram expresses the (obvious) fact that maximal chains in $J(p)$ are in one-to-one correspondence with linear extensions of $p$ via $C \mapsto \lambda(C)$. From the second diagram we recover the elementary fact that the enumerator of poset partitions and the enumerator of descents are related by the Birkhoff transform. More precisely, from (13.5) and (13.31) we deduce that

$$f_F\big(l, J(p)\big) = \begin{cases} 1 & \text{if } F \cdot l \text{ is a linear extension of } p, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the commutativity of

$$\begin{tikzcd}
\mathbf{P} \arrow[rr, "J"] \arrow[dr, "\eta"'] & & \mathbf{sgP} \arrow[dl, "\eta"] \\
& \mathbf{E} &
\end{tikzcd}
\quad \text{implies that of} \quad
\begin{tikzcd}
\mathbf{P} \arrow[rr, "J"] \arrow[dr, "\hat{\eta}"'] & & \mathbf{sgP} \arrow[dl, "\hat{\eta}"] \\
& \mathbf{\Sigma}^*. &
\end{tikzcd}$$

Using (13.27) and (13.3), we deduce that for any composition $F = F^1|\cdots|F^k$ of $I$,

$$f_F\big(J(p)\big) = \begin{cases} 1 & \text{if } F^1 \cup \cdots \cup F^i \text{ is a lower set of } p \text{ for every } i,\\ 0 & \text{otherwise.}\end{cases}$$

Recall the Hopf monoid $\mathbf{O}$ of preposets from Section 13.1.6. As above, let $J(r)$ denote the set of lower sets of a preposet $r$ on $I$, ordered by inclusion. We use the inclusion of $J(r)$ into $2^I$ to turn $J(r)$ into a set-weighted poset (Section 13.6.8).

Since the coproduct of $\mathbf{O}$ is given in terms of lower sets, the map

$$J\colon \mathbf{O} \to \mathbf{swP}$$

is a morphism of Hopf monoids. The commutative diagram

$$\begin{tikzcd} \mathbf{O} \arrow[r, "J"] & \mathbf{swP} \\ \mathbf{P} \arrow[u, hook] \arrow[r, "J"'] & \mathbf{sgP} \arrow[u, hook] \end{tikzcd}$$

expresses the fact that the Birkhoff transform of preposets extends that of posets.

Consider the projections $\mathbf{O} \twoheadrightarrow \mathbf{P}$ and $\widetilde{\mathbf{swP}} \twoheadrightarrow \mathbf{sgP}$ of (13.9) and (13.33). Given a preposet $r$, let $X$ be the set partition whose blocks are the classes for the equivalence relation (13.8). The labels along a maximal chain of $J(r)$ form a set composition with support $X$, independently of the chain. It follows that $J(r) \in \widetilde{\mathbf{swP}}$. In addition, since $J(r)$ is set-graded only if $r$ is a poset, the projections commute with the Birkhoff transform:

$$\begin{tikzcd} \mathbf{O} \arrow[r, "J"] \arrow[d, two heads] & \widetilde{\mathbf{swP}} \arrow[d, two heads] \\ \mathbf{P} \arrow[r, "J"'] & \mathbf{sgP}. \end{tikzcd}$$

**13.9.2. Preposets and topologies.** Given a preposet $r$ on $I$, define a map $t_r\colon 2^I \to 2^I$ by

$$t_r(A) := \{x \in I : \text{there is } a \in A \text{ such that } (x,a) \in r\}.$$

The map $t_r$ is a closure operator: the axioms in Definition 13.24 follow from reflexivity and transitivity of $r$. Moreover, $t_r$ is topological, since it clearly verifies Kuratowski's axioms (Definition 13.27). Conversely, given a topological closure $t$ on $I$, define a relation $r_t$ on $I$ by

$$(a,b) \in r \iff a \in t(\{b\}).$$

Then $r_t$ is a preposet on $I$.

These assignments define inverse correspondences between preposets on $I$ and topologies on $I$. (The fact that $t = t_{r_t}$ relies on the finiteness of $I$.) Moreover, they restrict to inverse correspondences between posets and Kolmogorov topologies.

Recall the Hopf monoids $\mathbf{O}$ and $\mathbf{T}$ of preposets and topologies. The closed sets of the topology $t_r$ are the lower sets of $r$. Therefore, the map

$$\mathbf{O} \to \mathbf{T}, \qquad r \mapsto t_r$$

is an isomorphism of Hopf monoids. It restricts to an isomorphism of Hopf monoids

$$\mathbf{P} \to \mathbf{kT}$$

and we obtain the commutative diagram below.

$$\begin{tikzcd} \mathbf{O} \arrow[r, "\cong"] & \mathbf{T} \\ \mathbf{P} \arrow[u, hook] \arrow[r, "\cong"'] & \mathbf{kT} \arrow[u, hook] \end{tikzcd}$$

**13.9.3. Convex geometries and set-graded posets.** Given a convex geometry $g$ with ground set $I$, let $P_g \subseteq 2^I$ denote the poset of convex sets of $g$ (Section 13.8.5), ordered by inclusion. The following result is a consequence of [53, Proposition 8.7.2]. We provide a proof for completeness.

**Lemma 13.28.** *Let $F$ and $G$ be convex sets of a convex geometry $g$ and suppose that $F$ covers $G$ in $P_g$. Then there exists an element $b \in I$ such that $F = G \cup \{b\}$.*

PROOF. Let $b$ be any element in $F \setminus G$. Since there is no closed set properly contained between $G$ and $F$, we must have $g\big(G \cup \{b\}\big) = F$. Suppose there is an element $a \in F \setminus \big(G \cup \{b\}\big)$. Then $a \in g\big(G \cup \{b\}\big)$, $a \neq b$, and $a, b \neq g(G) = G$. By the antiexchange axiom, $b \notin g\big(G \cup \{a\}\big) = F$, a contradiction. Thus, $F = G \cup \{b\}$. $\square$

**Remark 13.29.** The result of Lemma 13.28 holds for arbitrary greedoids. This follows from the discussion in [53, p. 289].

The ground set $I$ is the top element of $P_g$. If the convex geometry $g$ is loopless, then the empty set is the bottom element of $P_g$. It follows from Lemma 13.28 that in this case $P_g$ is $I$-graded by means of the inclusion $P_g \hookrightarrow 2^I$ (Proposition 13.15). For the rest of this section, we deal only with loopless convex geometries.

We compare the products of the Hopf monoids $\overline{\mathbf{cG}}$ of convex geometries and $\mathbf{sgP}$ of set-graded posets. A convex set of the direct sum of two convex geometries $g_1$ and $g_2$ consists of a pair of convex sets of each. Therefore,

$$P_{g_1 \oplus g_2} = P_{g_1} \times P_{g_2}$$

as set-graded posets.

We compare the coproducts (13.38) and (13.26). Given a decomposition $I = S \sqcup T$ with $S$ convex for $g$, the convex sets of $g_{\emptyset:S}$ are the subsets of $S$ which are convex for $g$. Hence, the interval $[\emptyset, S]$ in $P_g$ is the same $S$-graded poset as $P_{g_{\emptyset:S}}$. The convex sets $F$ of $g_{S:I}$ are in bijection with the supersets $G$ of $S$ which are convex for $g$ under the maps

$$F \mapsto S \cup F, \quad G \mapsto G \cap T.$$

Hence, the interval $[S, I]$ in $P_g$ is isomorphic to $P_{g_{S:I}}$ as $T$-graded posets.

It follows from this discussion that the map

$$\overline{\mathbf{cG}} \to \mathbf{sgP}, \qquad g \mapsto P_g$$

is a morphism of Hopf monoids. A result of Edelman identifies the the image of this map: a finite poset is of the form $P_g$ for some convex geometry $g$ if and only if it is a *meet-distributive lattice* [53, Theorem 8.7.6], [110, Theorem 3.3].

**13.9.4. Loopless closures and set-weighted posets.** We extend the preceding discussion to loopless closure operators and set-weighted posets (Section 13.6.8).

Let $I$ be a finite set. Given a closure operator $c$ on $I$, let $P_c \subseteq 2^I$ denote the poset of closed sets of $c$ (Section 13.8.5), ordered by inclusion. (This extends the definition of $P_g$ for a convex geometry $g$.) The ground set $I$ is always closed. If $c$ is loopless, then $\emptyset$ is closed as well. Hence, in this case $P_c$ is a subposet of $2^I$ containing $\emptyset$ and $I$; in particular, $P_c$ is an $I$-weighted poset. Moreover, the map

$$\overline{\mathbf{C}} \to \mathbf{swP}$$

is a morphism of Hopf monoids, by the same argument as for convex geometries. By construction, diagram

$$\begin{tikzcd} \overline{\mathbf{C}} \arrow[r] & \mathbf{swP} \\ \overline{\mathbf{cG}} \arrow[r] \arrow[u, hook] & \mathbf{sgP} \arrow[u, hook] \end{tikzcd}$$

commutes.

We turn to the image of the top map. For any closure operator $c$, the poset $P_c$ is a lattice (for which the meet is intersection of subsets) [49, Corollary in Section V.1]. Conversely, any finite lattice $L$ is of the form $P_c$ for some loopless closure $c$. To see this, given $L$ define $I := L \setminus \{\hat{0}\}$ and $c$ by

$$c\colon 2^I \to 2^I, \quad S \mapsto \{x \in L \setminus \{\hat{0}\} \mid x \leq \sup(S \cup \{\hat{0}\})\},$$

where sup denotes the join of all elements in the (nonempty) set $S \cup \{\hat{0}\}$. It is easy to see that $c$ is a loopless closure on $I$ and that

$$L \to P_c, \quad x \mapsto \{y \in L \setminus \{\hat{0}\} \mid y \leq x\}$$

is an isomorphism of lattices.

In summary, a finite poset is of the form $P_c$ for some loopless closure $c$ if and only if it is a lattice. This and more general results appear in [88, Remarks 7.4]. We thank Nathan Reading for help with these remarks.

Recall that the species $\overline{\mathbf{M}}$ of loopless matroids is a Hopf submonoid of $\overline{\mathbf{C}}$. The image of the map $\overline{\mathbf{M}} \to \mathbf{swP}$ consists of *geometric lattices*: a finite poset is of the form $P_m$ for some loopless matroid $m$ if and only if it is a geometric lattice [287, Theorem 1.7.5]. The poset $P_m$ is the *lattice of flats* of the matroid $m$.

**13.9.5. The Birkhoff transform factored through closure operators.** The Birkhoff transform on posets factors through the Hopf monoid of loopless convex geometries. Similarly, the Birkhoff transform on preposets factors through the Hopf monoid of loopless closure operators. Indeed, the maps constructed in the preceding sections fit in a commutative diagram as follows.

$$\begin{tikzcd} \mathbf{O} \arrow[r, "\cong"] & \mathbf{T} \arrow[r, hook] & \overline{\mathbf{C}} \arrow[r] & \mathbf{swP} \\ \mathbf{P} \arrow[r, "\cong"'] \arrow[u, hook] & \mathbf{kT} \arrow[r, hook] \arrow[u, hook] & \overline{\mathbf{cG}} \arrow[r] \arrow[u, hook] & \mathbf{sgP} \arrow[u, hook] \end{tikzcd}$$

The composites along the bottom and along the top are the Birkhoff transform (of posets and of preposets) of Section 13.9.1. As discussed in the preceding sections, the image of the map from $\mathbf{P}$ to $\mathbf{sgP}$ consists of finite distributive lattices, that of the map from $\overline{\mathbf{cG}}$ to $\mathbf{sgP}$ consists of finite meet-distributive lattices, and that of the map from $\overline{\mathbf{C}}$ to $\mathbf{swP}$ consists of arbitrary finite lattices. In addition, the image of the map from $\overline{\mathbf{M}} \subseteq \overline{\mathbf{C}}$ to $\mathbf{swP}$ consists of finite geometric lattices.

CHAPTER 14

# Hopf Monoids in Colored Species

Let $r$ denote a fixed positive integer. In this chapter, we consider $r$-colored species, which are a higher dimensional generalization of species. We recover species by letting $r = 1$. Roughly speaking, colored species correspond to multigraded vector spaces in the same way as species correspond to graded vector spaces.

Recall that for any fixed scalar $q$, there is a lax braiding $\beta_q$ on the category of species. Further, this lax braiding is a braiding if $q \neq 0$, and a symmetry if $q = \pm 1$. The corresponding situation in higher dimensions is quite interesting. For any fixed square matrix $Q$ of size $r$, there is a lax braiding $\beta_Q$ on the category of $r$-colored species. Further, this lax braiding is a braiding if all entries of $Q$ are nonzero, and a symmetry if $Q$ is log-antisymmetric (2.32). By letting $r = 1$ and $Q = [q]$, we recover the one-dimensional results.

We begin this chapter by defining the braided monoidal category of colored species in Section 14.1 as briefly noted above. In Section 14.2, we consider $Q$-Hopf monoids, which are Hopf monoids in this category. These are the basic objects of interest in this chapter. In Section 14.3, we introduce the colored exponential species. Apart from being a basic example of a $Q$-Hopf monoid, it also plays an important role in the general theory. Section 14.4 deals with colored versions of the Hadamard and signature functors. In Section 14.5, we introduce and study the colored linear order species. This is another important basic example. In Section 14.6, we discuss questions related to universality (freeness and cofreeness) of $Q$-Hopf monoids. In particular, we introduce colored versions of the free monoid functor and the related functors considered in Chapter 11, and show that the colored exponential and linear order species are universal objects. We conclude by looking at colored analogues of examples arising from the geometry of the Coxeter complex, which were discussed in Chapter 12.

## 14.1. Colored species

The basic theory of species of several variables is contained in Joyal's original work [181, Section 5]. These objects are called *multisort species* in [40, Section 2.4]. The equivalent notion of *colored species* was used by Méndez and Nava in [270] to provide a combinatorial framework for *plethysm*. We use the terminology of [270].

**14.1.1. Colored species.** An *$r$-colored set* is a pair $(I, f)$ where $I$ is a finite set and $f\colon I \to [r]$ is a function. A morphism of colored sets from $(I, f)$ to $(I', f')$ is a bijection $\sigma\colon I \to I'$ such that

$$\begin{tikzcd} I \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & I' \arrow[dl, "f'"] \\ & {[r]} & \end{tikzcd}$$

commutes.

One may view $(I,f)$ as a set $I$, each of whose elements is colored by one of $r$ colors: $i \in I$ is colored by $f(i) \in [r]$. Let $\mathsf{Set}^{(r)}$ be the category of $r$-colored sets. Note that $\mathsf{Set}^{(1)} = \mathsf{Set}^{\times}$, the category of finite sets and bijections.

**Definition 14.1.** An *$r$-colored species* is a functor $\mathsf{Set}^{(r)} \to \mathsf{Vec}$. A morphism of $r$-colored species is a natural transformation of functors.

Let $\mathsf{Sp}^{(r)}$ denote the category of $r$-colored species. The image of a colored set $(I,f)$ under a colored species $\mathbf{p}$ is denoted $\mathbf{p}[I,f]$. We say that $\mathbf{p}[I,f]$ is the space of **p**-*structures* on the colored set $(I,f)$, or the $(I,f)$-*component* of $\mathbf{p}$. If $I = [n]$, this is shortened to $\mathbf{p}[n,f]$.

Thus, a colored species consists of a family of vector spaces $\mathbf{p}[I,f]$, one for each colored set $(I,f)$, together with linear maps

$$\mathbf{p}[\sigma] \colon \mathbf{p}[I,f] \to \mathbf{p}[I',f'],$$

one for each morphism $\sigma \colon (I,f) \to (I',f')$, such that

$$\mathbf{p}[\mathrm{id}_{I,f}] = \mathrm{id}_{\mathbf{p}[I,f]} \quad \text{and} \quad \mathbf{p}[\tau\sigma] = \mathbf{p}[\tau]\mathbf{p}[\sigma]$$

whenever $(I,f) \xrightarrow{\sigma} (I',f') \xrightarrow{\tau} (I'',f'')$ are composable morphisms.

Similarly, a morphism of species $\alpha \colon \mathbf{p} \to \mathbf{q}$ consists of a family of linear maps

$$\alpha_{I,f} \colon \mathbf{p}[I,f] \to \mathbf{q}[I,f],$$

one for each colored set $(I,f)$, such that for each morphism $\sigma \colon (I,f) \to (I',f')$, the diagram

$$\begin{tikzcd} \mathbf{p}[I,f] \arrow[r, "\alpha_{I,f}"] \arrow[d, "\mathbf{p}[\sigma]"'] & \mathbf{q}[I,f] \arrow[d, "\mathbf{q}[\sigma]"] \\ \mathbf{p}[I',f'] \arrow[r, "\alpha_{I',f'}"'] & \mathbf{q}[I',f'] \end{tikzcd} \tag{14.1}$$

commutes.

**Remark 14.2.** Given $\mathsf{d} = (d^1, \ldots, d^r) \in \mathbb{N}^r$, let $n_{\mathsf{d}} = d^1 + \cdots + d^r$ and consider the function $f_{\mathsf{d}} \colon [n_{\mathsf{d}}] \to [r]$ that maps the first $d^1$ elements to 1, the next $d^2$ elements to 2, and so on. The automorphism group of the $r$-colored set $[n_{\mathsf{d}}, f_{\mathsf{d}}]$ is the standard parabolic subgroup

$$\mathrm{S}_{\mathsf{d}} := \mathrm{S}_{d^1} \times \cdots \times \mathrm{S}_{d^r}$$

of $\mathrm{S}_{n_{\mathsf{d}}}$. Since any $r$-colored set is isomorphic to a $[n_{\mathsf{d}}, f_{\mathsf{d}}]$ for exactly one $\mathsf{d} \in \mathbb{N}^r$, it follows that any $r$-colored species $\mathbf{q}$ is completely determined by the sequence

$$(\mathbf{q}[n_{\mathsf{d}}, f_{\mathsf{d}}])_{\mathsf{d} \in \mathbb{N}^r}$$

of $\mathrm{S}_{\mathsf{d}}$-modules.

**Notation 14.3.** It is convenient to employ the notation

$$(I,f) = (S,g) \sqcup (T,h).$$

This means that $I = S \sqcup T$ is a decomposition, and $f$ restricted to $S$ is $g$, and $f$ restricted to $T$ is $h$. In this situation, we write $f = g \sqcup h$. Similarly, we write

$$(I,f) = (R,k) \sqcup (S,g) \sqcup (T,h)$$

for a decomposition into three parts, and so on.

**14.1.2. The Cauchy product on colored species.** There is a monoidal structure on the category of colored species. Given $r$-colored species $\mathbf{p}$ and $\mathbf{q}$, define a new $r$-colored species $\mathbf{p}\cdot\mathbf{q}$ by

$$(\mathbf{p}\cdot\mathbf{q})[I,f] := \bigoplus_{(I,f)=(S,g)\sqcup(T,h)} \mathbf{p}[S,g]\otimes\mathbf{q}[T,h]. \tag{14.2}$$

A morphism $\sigma\colon (I,f)\to(I',f')$ of $r$-colored sets induces maps

$$\mathbf{p}[\sigma|_{S,g}]\otimes\mathbf{p}[\sigma|_{T,h}]\colon \mathbf{p}[S,g]\otimes\mathbf{q}[T,h]\to\mathbf{p}[S',g']\otimes\mathbf{q}[T',h']$$

where

$$(I',f')=(S',g')\sqcup(T',h')$$

is the corresponding decomposition: $S'=\sigma(S)$ and $T'=\sigma(T)$, and $g'=f'|_{S'}$ and $h'=f'|_{T'}$. This turns $\mathbf{p}\cdot\mathbf{q}$ into a functor $\mathsf{Set}^{(r)}\to\mathsf{Vec}$, that is, an $r$-colored species.

This operation generalizes (8.6). So following the terminology for $r=1$, we continue to refer to it as the Cauchy product. It turns $\mathsf{Sp}^{(r)}$ into a monoidal category. The unit object is the colored species $\mathbf{1}_{(r)}$ with

$$\mathbf{1}_{(r)}[I,f] := \begin{cases} \Bbbk & \text{if } I \text{ is empty,}\\ 0 & \text{otherwise.}\end{cases}$$

**14.1.3. Braidings for the Cauchy product.** In much the same way as for multigraded vector spaces (2.60), one can define a lax braiding on $r$-colored species for each matrix $Q$ of size $r$. We employ the braid coefficients defined in Section 10.13.3 for this purpose. We also use some terminology on matrices from Section 2.2.5.

The components $\beta_{S,g,T,h}$ of the braiding $\beta_Q\colon \mathbf{p}\cdot\mathbf{q}\to\mathbf{q}\cdot\mathbf{p}$ are given by

$$\begin{aligned}\mathbf{p}[S,g]\otimes\mathbf{q}[T,h] &\to \mathbf{q}[T,h]\otimes\mathbf{p}[S,g]\\ x\otimes y &\mapsto \mathrm{brd}^Q_{S,T,f}\; y\otimes x,\end{aligned} \tag{14.3}$$

where $\mathrm{brd}^Q_{S,T,f}$ is the braid coefficient (10.104). Note that for $r=1$ and $Q=[q]$, this specializes to (9.1).

The map $\beta_Q$ is a lax braiding. It is a braiding precisely if all entries of $Q$ are nonzero. In this case,

$$(\beta_Q)^{-1} = \beta_{Q^{-t}}. \tag{14.4}$$

It follows that $\beta_Q$ is a symmetry precisely if $Q$ is log-antisymmetric (2.32).

Now let $A$ be an integer square matrix of size $r$, and let $q$ be an invertible scalar. The components of the braiding $\beta_{A,q}\colon \mathbf{p}\cdot\mathbf{q}\to\mathbf{q}\cdot\mathbf{p}$ are given by

$$x\otimes y\mapsto q^{\mathrm{brd}^A_{S,T,f}}\; y\otimes x. \tag{14.5}$$

If $Q$ and $A$ are related by (2.33), then $\beta_Q=\beta_{A,q}$.

**14.1.4. Examples of colored species.** There are several ways of constructing colored species from ordinary species. We mention two. Given an ordinary species $\mathbf{p}$, define $r$-colored species $\mathbf{p}_{(r)}$ and $\mathbf{p}^{(r)}$ by

$$\mathbf{p}_{(r)}[I,f] := \mathbf{p}[I] \qquad\text{and}\qquad \mathbf{p}^{(r)}[I,f] := \bigotimes_{i=1}^{r}\mathbf{p}[f^{-1}(i)].$$

Let $1_{r,r}$ be the matrix all of whose entries are 1, and $I_{r,r}$ be the identity matrix.

**Proposition 14.4.** *The functors*

$$(-)_{(r)} \colon (\mathsf{Sp}, \cdot, \beta_q) \to (\mathsf{Sp}^{(r)}, \cdot, \beta_{q\,1_{r,r}})$$

*and*

$$(-)^{(r)} \colon (\mathsf{Sp}, \cdot, \beta_q) \to (\mathsf{Sp}^{(r)}, \cdot, \beta_{1_{r,r}+(q-1)I_{r,r}})$$

*are bistrong monoidal.*

The proof is straightforward. There are more general functors $\mathsf{Sp}^{(s)} \to \mathsf{Sp}^{(r)}$ whenever $s$ divides $r$ that we do not discuss.

Many useful examples of colored species arise from these constructions. For instance, the colored species $\mathbf{1}_{(r)}$ above is obtained by applying the first functor to the species $\mathbf{1}$. We discuss some similar examples next. The species $\mathbf{X}$ characteristic of singletons (8.3) yields $\mathbf{X}_{(r)}$ which takes value $\Bbbk$ on every colored set $(I, f)$ in which $I$ is a singleton, and is zero otherwise. Similarly, the exponential species yields the species $\mathbf{E}_{(r)}$ which takes value $\Bbbk$ on every colored set; the linear order species yields $\mathbf{L}_{(r)}$ which on the colored set $(I, f)$ has a basis consisting of linear orders on $I$.

**14.1.5. The duality functor.** Just as for multigraded vector spaces and species, one can define the dual of any colored species by taking the dual of each component. This gives rise to the colored version of the duality functor on species.

The duality functor on the category of finite-dimensional colored species

$$(-)^* \colon ((\mathsf{Sp}^{(r)})^{\mathrm{op}}, \cdot, \beta_Q^{\mathrm{op}}) \to (\mathsf{Sp}^{(r)}, \cdot, \beta_{Q^t}) \tag{14.6}$$

is bistrong.

The duality functor commutes with the functors $(-)_{(r)}$ and $(-)^{(r)}$ of Proposition 14.4.

## 14.2. $Q$-Hopf monoids

In Section 14.1, we defined the monoidal category of $r$-colored species. Further, for any square matrix $Q$ of size $r$, we defined a lax braiding $\beta_Q$ on this category. In this section, we begin by discussing monoids and comonoids in this category, and then go over to bimonoids and Hopf monoids. Since the latter depend on $Q$, we refer to them as $Q$-bimonoids and $Q$-Hopf monoids.

We employ the notations of Table 14.1 to denote the relevant categories. It is understood that the tensor product is the Cauchy product and the braiding is $\beta_Q$. Similar notations will be used to denote various related categories.

TABLE 14.1. Categories of colored (co, bi, Hopf) monoids in species.

| Category | Description |
|---|---|
| $\mathsf{Mon}(\mathsf{Sp}^{(r)})$ | Colored monoids |
| $\mathsf{Comon}(\mathsf{Sp}^{(r)})$ | Colored comonoids |
| $\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}^{(r)})$ | Colored commutative monoids |
| ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{Sp}^{(r)})$ | Colored cocommutative comonoids |
| $Q$-$\mathsf{Bimon}(\mathsf{Sp}^{(r)})$ | Colored bimonoids |
| $Q$-$\mathsf{Hopf}(\mathsf{Sp}^{(r)})$ | Colored Hopf monoids |

**14.2.1. Colored monoids and comonoids.** A *colored monoid* is a monoid in $(\mathsf{Sp}^{(r)}, \cdot)$. We make this notion explicit. A colored monoid consists of a linear map

$$\mu_{S,g,T,h} \colon \mathbf{p}[S,g] \otimes \mathbf{p}[T,h] \to \mathbf{p}[I,f]$$

for each decomposition $(I,f) = (S,g) \sqcup (T,h)$, and a linear map

$$\iota_{\emptyset,*} \colon \Bbbk \to \mathbf{p}[\emptyset,*],$$

where $*$ denotes the unique map $\emptyset \to [r]$. The following diagrams must commute.

- For each morphism $\sigma \colon (I,f) \to (I',f')$ and each decomposition $(I,f) = (S,g) \sqcup (T,h)$,

$$\begin{tikzcd}
\mathbf{p}[S,g] \otimes \mathbf{p}[T,h] \arrow[r, "\mu_{S,g,T,h}"] \arrow[d, "\mathbf{p}[\sigma|_{S,g}] \otimes \mathbf{p}[\sigma|_{T,h}]"'] & \mathbf{p}[I,f] \arrow[d, "\mathbf{p}[\sigma]"] \\
\mathbf{p}[S',g'] \otimes \mathbf{p}[T',h'] \arrow[r, "\mu_{S',g',T',h'}"'] & \mathbf{p}[I',f'],
\end{tikzcd} \tag{14.7}$$

  where $(I',f') = (S',g') \sqcup (T',h')$ is the corresponding decomposition.

- For each decomposition $(I,f) = (R,k) \sqcup (S,g) \sqcup (T,h)$,

$$\begin{tikzcd}
\mathbf{p}[R,k] \otimes \mathbf{p}[S,g] \otimes \mathbf{p}[T,h] \arrow[r, "\mathrm{id}_{R,k} \otimes \mu_{S,g,T,h}"] \arrow[d, "\mu_{R,k,S,g} \otimes \mathrm{id}_{T,h}"'] & \mathbf{p}[R,k] \otimes \mathbf{p}[(S,g) \sqcup (T,h)] \arrow[d, "\mu_{R,k,(S,g)\sqcup(T,h)}"] \\
\mathbf{p}[(R,k) \sqcup (S,g)] \otimes \mathbf{p}[T,h] \arrow[r, "\mu_{(R,k)\sqcup(S,g),T,h}"'] & \mathbf{p}[(R,k) \sqcup (S,g) \sqcup (T,h)].
\end{tikzcd} \tag{14.8}$$

- Finally, for each $(I,f)$,

$$\begin{tikzcd}
\mathbf{p}[I,f] & \mathbf{p}[\emptyset,*] \otimes \mathbf{p}[I,f] \arrow[l, "\mu_{\emptyset,*,I,f}"'] & \mathbf{p}[I,f] \otimes \mathbf{p}[\emptyset,*] \arrow[r, "\mu_{I,f,\emptyset,*}"] & \mathbf{p}[I,f] \\
& \Bbbk \otimes \mathbf{p}[I,f] \arrow[ul, equal] \arrow[u, "\iota_{\emptyset,*} \otimes \mathrm{id}_{I,f}"'] & \mathbf{p}[I,f] \otimes \Bbbk. \arrow[u, "\mathrm{id}_{I,f} \otimes \iota_{\emptyset.*}"] \arrow[ur, equal] &
\end{tikzcd} \tag{14.9}$$

A colored monoid is commutative if diagram

$$\begin{tikzcd}
\mathbf{p}[S,g] \otimes \mathbf{p}[T,h] \arrow[rr, "\beta_{S,g,T,h}"] \arrow[dr, "\mu_{S,g,T,h}"'] & & \mathbf{p}[T,h] \otimes \mathbf{p}[S,g] \arrow[dl, "\mu_{T,h,S,g}"] \\
& \mathbf{p}[I,f] &
\end{tikzcd} \tag{14.10}$$

commutes, for all decompositions $(I,f) = (S,g) \sqcup (T,h)$. Note that this notion depends on the braiding.

Let $\mathbf{p}$ and $\mathbf{p}'$ be colored monoids. A morphism $\alpha$ between them consists of a linear map

$$\alpha_{I,f} \colon \mathbf{p}[I,f] \to \mathbf{p}'[I,f]$$

for each colored set $(I, f)$, satisfying (14.1) and such that the following diagrams commute.

$$\begin{tikzcd} \mathbf{p}[S,g] \otimes \mathbf{p}[T,h] \arrow[r, "\alpha_{S,g} \otimes \alpha_{T,h}"] \arrow[d, "\mu_{S,g,T,h}"'] & \mathbf{p}'[S,g] \otimes \mathbf{p}'[T,h] \arrow[d, "\mu'_{S,g,T,h}"] \\ \mathbf{p}[I,f] \arrow[r, "\alpha_{I,f}"'] & \mathbf{p}'[I,f] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{p}[\emptyset,*] \arrow[rr, "\alpha_{\emptyset,*}"] & & \mathbf{p}'[\emptyset,*] \\ & \Bbbk \arrow[ul, "\iota_{\emptyset,*}"] \arrow[ur, "\iota'_{\emptyset,*}"'] & \end{tikzcd} \tag{14.11}$$

Colored (cocommutative) comonoids and morphisms between them are defined dually. We use $\Delta_{S,g,T,h}$ for the components of the coproduct and $\epsilon_{\emptyset,*}$ for the component of the counit.

**14.2.2. *Q*-Hopf monoids.** Consider the lax braided monoidal category of colored species $(\mathsf{Sp}^{(r)}, \cdot, \beta_Q)$. (We write lax braided instead of braided to include the case when one or more entries of the matrix $Q$ are zero.) Bimonoids and Hopf monoids in this category are known as *Q-bimonoids* and *Q-Hopf monoids* respectively.

A $Q$-bimonoid can be explicitly described as follows. First, for any pair of decompositions

$$(S, g) \sqcup (T, h) = (I, f) = (S', g') \sqcup (T', h')$$

of a colored set $(I, f)$, diagram

$$\begin{tikzcd} \mathbf{p}[A,a] \otimes \mathbf{p}[B,b] \otimes \mathbf{p}[C,c] \otimes \mathbf{p}[D,d] \arrow[rr, "\mathrm{id}_{A,a} \otimes \beta_{B,b,C,c} \otimes \mathrm{id}_{D,d}"] & & \mathbf{p}[A,a] \otimes \mathbf{p}[C,c] \otimes \mathbf{p}[B,b] \otimes \mathbf{p}[D,d] \arrow[d, "\mu_{A,a,C,c} \otimes \mu_{B,b,D,d}"] \\ \mathbf{p}[S,g] \otimes \mathbf{p}[T,h] \arrow[u, "\Delta_{A,a,B,b} \otimes \Delta_{C,c,D,d}"] \arrow[r, "\mu_{S,g,T,h}"'] & \mathbf{p}[I,f] \arrow[r, "\Delta_{S',g',T',h'}"'] & \mathbf{p}[S',g'] \otimes \mathbf{p}[T',h'] \end{tikzcd} \tag{14.12}$$

must commute, where $A$, $B$, $C$, and $D$ are as in Lemma 8.7, and $a$, $b$, $c$ and $d$ are the restrictions of $f$ to these sets. In addition, diagrams

$$\begin{tikzcd} \mathbf{p}[\emptyset,*] \otimes \mathbf{p}[\emptyset,*] \arrow[r, "\epsilon_{\emptyset,*} \otimes \epsilon_{\emptyset,*}"] \arrow[d, "\mu_{\emptyset,*,\emptyset,*}"'] & \Bbbk \otimes \Bbbk \arrow[d, equal] \\ \mathbf{p}[\emptyset,*] \arrow[r, "\epsilon_{\emptyset,*}"'] & \Bbbk \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "\iota_{\emptyset,*}"] \arrow[d, equal] & \mathbf{p}[\emptyset,*] \arrow[d, "\Delta_{\emptyset,*,\emptyset,*}"] \\ \Bbbk \otimes \Bbbk \arrow[r, "\iota_{\emptyset,*} \otimes \iota_{\emptyset,*}"'] & \mathbf{p}[\emptyset,*] \otimes \mathbf{p}[\emptyset,*] \end{tikzcd} \tag{14.13}$$

$$\begin{tikzcd} & \mathbf{p}[\emptyset,*] \arrow[dr, "\epsilon_{\emptyset,*}"] & \\ \Bbbk \arrow[ur, "\iota_{\emptyset,*}"] \arrow[rr, equal] & & \Bbbk \end{tikzcd} \tag{14.14}$$

must commute as well.

A $Q$-Hopf monoid is a $Q$-bimonoid equipped with an antipode. Explicitly, the antipode $\mathrm{s}$ consists of a linear map

$$\mathrm{s}_{I,f} \colon \mathbf{h}[I, f] \to \mathbf{h}[I, f]$$

for each colored set $(I, f)$, commuting with bijections, and such that for each nonempty set $I$ the composites

$$\begin{tikzcd}
\mathbf{h}[I,f] \arrow[r, "\oplus\Delta_{S,g,T,h}"] & \displaystyle\bigoplus_{(S,g)\sqcup(T,h)=(I,f)} \mathbf{h}[S,g]\otimes\mathbf{h}[T,h] \arrow[d, "\oplus \mathrm{id}_{S,g}\otimes \mathrm{s}_{T,h}"] \\
\mathbf{h}[I,f] & \displaystyle\bigoplus_{(S,g)\sqcup(T,h)=(I,f)} \mathbf{h}[S,g]\otimes\mathbf{h}[T,h] \arrow[l, "\oplus\mu_{S,g,T,h}"]
\end{tikzcd} \tag{14.15}$$

and

$$\begin{tikzcd}
\mathbf{h}[I,f] \arrow[r, "\oplus\Delta_{S,g,T,h}"] & \displaystyle\bigoplus_{(S,g)\sqcup(T,h)=(I,f)} \mathbf{h}[S,g]\otimes\mathbf{h}[T,h] \arrow[d, "\oplus\, \mathrm{s}_{S,g}\otimes \mathrm{id}_{T,h}"] \\
\mathbf{h}[I,f] & \displaystyle\bigoplus_{(S,g)\sqcup(T,h)=(I,f)} \mathbf{h}[S,g]\otimes\mathbf{h}[T,h] \arrow[l, "\oplus\mu_{S,g,T,h}"]
\end{tikzcd} \tag{14.16}$$

are zero, and for which the following diagrams commute

$$\begin{tikzcd}
\mathbf{h}[\emptyset,*]\otimes\mathbf{h}[\emptyset,*] \arrow[rr, "\mathrm{id}_{\emptyset,*}\otimes \mathrm{s}_{\emptyset,*}"] & & \mathbf{h}[\emptyset,*]\otimes\mathbf{h}[\emptyset,*] \arrow[d, "\mu_{\emptyset,*,\emptyset,*}"] \\
\mathbf{h}[\emptyset,*] \arrow[u, "\Delta_{\emptyset,*,\emptyset,*}"] \arrow[r, "\epsilon_{\emptyset,*}"'] & \Bbbk \arrow[r, "\iota_{\emptyset,*}"'] & \mathbf{h}[\emptyset,*] \\
\mathbf{h}[\emptyset,*]\otimes\mathbf{h}[\emptyset,*] \arrow[rr, "\mathrm{s}_{\emptyset,*}\otimes \mathrm{id}_{\emptyset,*}"] & & \mathbf{h}[\emptyset,*]\otimes\mathbf{h}[\emptyset,*] \arrow[d, "\mu_{\emptyset,*,\emptyset,*}"] \\
\mathbf{h}[\emptyset,*] \arrow[u, "\Delta_{\emptyset,*,\emptyset,*}"] \arrow[r, "\epsilon_{\emptyset,*}"'] & \Bbbk \arrow[r, "\iota_{\emptyset,*}"'] & \mathbf{h}[\emptyset,*].
\end{tikzcd} \tag{14.17}$$

Note that the $(\emptyset,*)$-component of a $Q$-Hopf monoid is a Hopf algebra. Conversely, a $Q$-bimonoid for which the $(\emptyset,*)$-component is a Hopf algebra is automatically a $Q$-Hopf monoid. This generalizes Proposition 8.10.

A *connected $Q$-bimonoid* is a $Q$-bimonoid whose $(\emptyset,*)$-component is $\Bbbk$. It follows that a connected $Q$-bimonoid is always a $Q$-Hopf monoid. In this case, the antipode is given similar to (8.27) by using nonempty decompositions of colored sets.

**14.2.3. Connected and positive colored species.** Connected and positive species were discussed in Sections 8.9 and 9.1.3. We now briefly discuss their colored versions.

A colored species $\mathbf{q}$ is *connected* if there is a specified isomorphism

$$\mathbf{q}[\emptyset, *] \xrightarrow{\cong} \Bbbk.$$

A morphism of connected colored species is a morphism of colored species whose $(\emptyset, *)$-component commutes with these specified isomorphisms. The Cauchy product of connected colored species is again connected; this yields the monoidal category of connected colored species. A *connected colored monoid (comonoid, bimonoid)* is a monoid (comonoid, bimonoid) in this category. A connected colored bimonoid is the same as a connected $Q$-bimonoid as defined above.

A colored species $\mathbf{q}$ is *positive* if $\mathbf{q}[\emptyset, *] = 0$. Let $\mathsf{Sp}_+^{(r)}$ be the category of positive colored species. The Cauchy product turns it into a nonunital monoidal category. We denote the category of nonunital monoids and noncounital comonoids by

$$\mathsf{Mon}(\mathsf{Sp}_+^{(r)}) \qquad \text{and} \qquad \mathsf{Comon}(\mathsf{Sp}_+^{(r)})$$

respectively. The modified Cauchy product (8.55) turns $\mathsf{Sp}_+^{(r)}$ into a braided monoidal category, which we denote by $(\mathsf{Sp}_+, \odot, \beta_Q)$. (Co)monoids in this category are equivalent to non(co)unital (co)monoids considered above. We also refer to them as positive colored (co)monoids. Further, we define a *positive colored bimonoid (Hopf monoid)* to be a bimonoid (Hopf monoid) in $(\mathsf{Sp}_+, \odot, \beta_Q)$. We also refer to these as positive $Q$-bimonoids ($Q$-Hopf monoids).

One can show as in Proposition 8.44 that a connected colored (co, bi, Hopf) monoid is equivalent to a positive colored (co, bi, Hopf) monoid.

**14.2.4. Duality for $Q$-Hopf monoids.** The dual of a finite-dimensional $Q$-Hopf monoid is a $Q^t$-Hopf monoid. This is a consequence of the fact (14.6) that the duality functor interchanges the braiding $\beta_Q$ with the braiding $\beta_{Q^t}$.

**Convention 14.5.** In this and subsequent sections, we encounter colored versions of the geometric $q$-Hopf monoids of Chapter 12. If $\mathbf{H}$ denotes any one of the Hopf monoids in Table 12.1, then we use $\mathbf{H}_Q$ to denote a corresponding $Q$-Hopf monoid defined in this chapter. We convene that $\mathbf{H}_Q^*$ is the $Q$-Hopf monoid that corresponds to $\mathbf{H}^*$. In general, this is *not* the dual of the $Q$-Hopf monoid $\mathbf{H}_Q$. In fact, the dual of $\mathbf{H}_Q$ is a $Q^t$-Hopf monoid. For the Hopf monoids considered in this chapter, it turns out that

$$(\mathbf{H}_Q)^* \cong \mathbf{H}_{Q^t}^* \tag{14.18}$$

as $Q^t$-Hopf monoids. In particular, if $Q$ is symmetric, then $\mathbf{H}_Q^*$ is the dual of $\mathbf{H}_Q$.

## 14.3. The colored exponential species

Let $Q$ be a log-antisymmetric matrix (2.32); in particular, all entries of $Q$ are nonzero. In this section, we define the colored exponential species $\mathbf{E}_Q$ and equip it with a $Q$-Hopf monoid structure. This example simultaneously generalizes the exponential species $\mathbf{E}$ and the signed exponential species $\mathbf{E}^-$ (Example 8.15 and Section 9.3).

**14.3.1. A symmetric-exterior algebra.** Let $(I, f)$ be a colored set. Let $n := |I|$. Take the quotient of the tensor algebra on $\Bbbk I$ by the ideal generated by the relations: For $i, j \in I$,

$$i \otimes j = q_{f(j)f(i)}\, j \otimes i.$$

Since $Q$ is log-antisymmetric, this relation is equivalent to

$$j \otimes i = q_{f(i)f(j)}\, i \otimes j.$$

We use $\wedge$ to denote the product in the quotient. Define $\mathrm{Det}^Q_f(\Bbbk I)$ to be the one-dimensional subspace of this quotient which is spanned by the element

$$i_1 \wedge \cdots \wedge i_n,$$

where $I = \{i_1, \ldots, i_n\}$, with its elements written in some order. Changing the order of the elements in the wedge product incurs a scalar.

Let us see how this works for $r = 1$. If $Q = [1]$, then $\mathrm{Det}^Q_f(\Bbbk I)$ is the one-dimensional subspace of the symmetric algebra on $\Bbbk I$ spanned by the monomial of degree $n$ in which each element of $I$ appears exactly once. If $Q = [-1]$, then $\mathrm{Det}^Q_f(\Bbbk I)$ is the same as $\mathrm{Det}(\Bbbk I)$ (as defined in Section 9.3). It is the highest nonzero degree component of the exterior algebra on $\Bbbk I$, and is necessarily one-dimensional.

We now illustrate the $r = 2$ case. Let $I = \{a, m, p, e, r\}$ and let $f \colon I \to [2]$ be given by $f(a) = f(e) = 1$ and $f(m) = f(p) = f(r) = 2$. We represent the 2-colored set $(I, f)$ by

$$(I, f) = \{a, m, p, e, r\}, \tag{14.19}$$

with blue denoting color 1 and red denoting color 2. Then

$$p \wedge r \wedge e \wedge m \wedge a = q_{12}\, p \wedge e \wedge r \wedge m \wedge a \qquad \in \mathrm{Det}^Q_f(\Bbbk I).$$

The interchange of $r$ and $e$ gave the scalar $q_{12}$.

**14.3.2. The colored exponential species as a *Q*-Hopf monoid.** The colored exponential species is defined as follows. Let

$$\mathbf{E}_Q[I, f] := \mathrm{Det}^Q_f(\Bbbk I).$$

We now turn $\mathbf{E}_Q$ into a $Q$-Hopf monoid. The product is concatenation and the coproduct is deshuffling. Details follow.

The product is given by

$$\begin{gathered}\mathbf{E}_Q[S, g] \otimes \mathbf{E}_Q[T, h] \to \mathbf{E}_Q[I, f]\\ (l^1 \wedge \cdots \wedge l^s) \otimes (m^1 \wedge \cdots \wedge m^t) \mapsto l^1 \wedge \cdots \wedge l^s \wedge m^1 \wedge \cdots \wedge m^t,\end{gathered}$$

where $S = \{l^1, \ldots, l^s\}$ and $T = \{m^1, \ldots, m^t\}$.

The coproduct is given by

$$\begin{gathered}\mathbf{E}_Q[I, f] \to \mathbf{E}_Q[S, g] \otimes \mathbf{E}_Q[T, f]\\ l^1 \wedge \cdots \wedge l^i \mapsto (-1)^{\mathrm{sch}^Q_{S,T,f}(l)}\ (l^{i_1} \wedge \cdots \wedge l^{i_s}) \otimes (l^{j_1} \wedge \cdots \wedge l^{j_t})\end{gathered}$$

where $\{i_1 < \cdots < i_s\} = S$, and $\{j_1 < \cdots < j_t\} = T$, and $l = l^1|\cdots|l^i$, and $\mathrm{sch}^Q_{S,T,f}(l)$ is the (multiplicative) weighted Schubert cocycle (10.102). We note that the coproduct is well-defined (independent of which linear order on $I$ is used).

The following examples (with notation as set above) should clarify these definitions.

$$(p \wedge r \wedge e) \otimes (m \wedge a) \mapsto p \wedge r \wedge e \wedge m \wedge a.$$

$$\begin{aligned} r \wedge a \wedge m \mapsto 1 \otimes (r \wedge a \wedge m) + r \otimes (a \wedge m) + q_{12}\, a \otimes (r \wedge m) + q_{21}q_{22}\, m \otimes (r \wedge a) \\ + (r \wedge a) \otimes m + q_{21}\,(r \wedge m) \otimes a + q_{12}q_{22}\,(a \wedge m) \otimes r + (r \wedge a \wedge m) \otimes 1. \end{aligned}$$

This turns $\mathbf{E}_Q$ into a $Q$-Hopf monoid. The antipode is given by

$$\mathrm{s}_{I,f} \colon \mathbf{E}_Q[I,f] \to \mathbf{E}_Q[I,f] \qquad x \mapsto (-1)^{|I|}x.$$

Observe that $\mathbf{E}_Q$ is commutative and cocommutative (in the colored sense). Further,

$$(\mathbf{E}_Q)^* = \mathbf{E}_{Q^t}$$

as $Q^t$-Hopf monoids. Note that this is in agreement with (14.18). Thus, if $Q$ is symmetric, then $\mathbf{E}_Q$ is self-dual (over any field). In this case, since $\mathbf{E}_Q$ is only defined when $Q$ is log-antisymmetric, the entries of $Q$ are either 1 or $-1$.

**14.3.3. Specializations.** We now discuss some special cases. Let $1_{r,r}$ be the matrix all of whose entries are 1, and $I_{r,r}$ be the identity matrix. Observe that

$$\mathbf{E}_{[1]} = \mathbf{E} \qquad \text{and} \qquad \mathbf{E}_{[-1]} = \mathbf{E}^-. \tag{14.20}$$

As a generalization of the first identity in (14.20), note that

$$\mathbf{E}_{1_{r,r}} \cong \mathbf{E}_{(r)} \cong \mathbf{E}^{(r)},$$

where the $Q$-Hopf monoids on the right are obtained by applying the bistrong functors of Proposition 14.4 to $\mathbf{E}$. In this case, all entries of the matrix $Q$ are 1; so the braiding simply interchanges the factors. The explicit description of $\mathbf{E}_{1_{r,r}}$ is simple and as follows:

$$\mathbf{E}_{1_{r,r}}[I,f] = \Bbbk$$

on every colored set $(I,f)$, and let $*_{(I,f)}$ denote the canonical basis element $1 \in \Bbbk$. Given a decomposition $(I,f) = (S,g) \sqcup (T,h)$, the corresponding components of the product and coproduct of $\mathbf{E}_{1_{r,r}}$ are

$$*_{(S,g)} \otimes *_{(T,h)} \mapsto *_{(I,f)} \qquad \text{and} \qquad *_{(I,f)} \mapsto *_{(S,g)} \otimes *_{(T,h)}.$$

As a generalization of the second identity in (14.20), note that

$$\mathbf{E}_{-1_{r,r}} \cong (\mathbf{E}^-)_{(r)} \qquad \text{and} \qquad \mathbf{E}_{1_{r,r}-2I_{r,r}} \cong (\mathbf{E}^-)^{(r)}.$$

The first matrix has all entries $-1$, while the second matrix has diagonal entries $-1$ and off-diagonal entries 1. For example, let $I = \{a,h,i,k,l,m,s\}$ and let $f \colon I \to [3]$ be given by $f(l) = f(a) = f(k) = 1$, $f(s) = f(h) = 2$ and $f(m) = f(i) = 3$. We represent the 3-colored set $(I,f)$ by

$$(I,f) = \{a,h,i,k,l,m,s\},$$

with blue denoting color 1, red denoting color 2, and green denoting color 3. Then the second isomorphism on this colored set sends

$$l \wedge a \wedge k \wedge s \wedge h \wedge m \wedge i \mapsto (l \wedge a \wedge k) \otimes (s \wedge h) \otimes (m \wedge i).$$

## 14.4. The colored Hadamard and signature functors

The Hadamard and signature functors are studied in Section 9.4. In this section, we introduce colored versions and discuss their monoidal properties.

**14.4.1. The Hadamard functor.** Recall that for ordinary species, the Hadamard functor is bilax with respect to the Cauchy product. More precisely, there exist structure transformations $\varphi$ and $\psi$ such that $(\times, \varphi, \psi)$ is bilax (Propositions 8.58 and 9.5). This result can be extended to colored species.

Given $r$-colored species $\mathbf{p}$ and $\mathbf{q}$, define a new $r$-colored species $\mathbf{p} \times \mathbf{q}$ by

$$(\mathbf{p} \times \mathbf{q})[I, f] := \mathbf{p}[I, f] \otimes \mathbf{q}[I, f]. \tag{14.21}$$

This operation generalizes (8.7). So following the terminology for $r = 1$, we continue to refer to it as the Hadamard product. It turns $\mathsf{Sp}^{(r)}$ into a monoidal category. The unit object is the colored species $\mathbf{E}_{(r)}$ which is $\Bbbk$ on every colored set.

The structure transformations $\varphi$ and $\psi$ can be defined for colored species in a similar way. This leads to the following straightforward generalization.

**Proposition 14.6.** *The functor*

$$(\times, \varphi, \psi)\colon (\mathsf{Sp}^{(r)} \times \mathsf{Sp}^{(r)}, \cdot, \beta_P \times \beta_Q) \to (\mathsf{Sp}^{(r)}, \cdot, \beta_{P \times Q})$$

*is a normal braided bilax monoidal functor.*

We remark that the analogous statement for multigraded vector spaces does not hold just as in the $r = 1$ case (Remark 8.65).

**Corollary 14.7.** *If $\mathbf{h}_1$ is a $P$-bimonoid and $\mathbf{h}_2$ is a $Q$-bimonoid, then $\mathbf{h}_1 \times \mathbf{h}_2$ is a $(P \times Q)$-bimonoid. Further, if $\mathbf{h}_1$ and $\mathbf{h}_2$ are (co)commutative, then so is $\mathbf{h}_1 \times \mathbf{h}_2$.*

The same statement holds for Hopf monoids.

As an example, if $P$ and $Q$ are log-antisymmetric matrices, then

$$\mathbf{E}_P \times \mathbf{E}_Q \xrightarrow{\cong} \mathbf{E}_{P \times Q} \tag{14.22}$$

as $(P \times Q)$-bimonoids. In particular,

$$\mathbf{E}_Q \times \mathbf{E}_{Q^t} \cong \mathbf{E}_{1_{r,r}}$$

as $1_{r,r}$-bimonoids. We illustrate the isomorphism (14.22) with an example. On the component of the 2-colored set $(I, f)$ of (14.19), the map is given by

$$(p \wedge r \wedge e \wedge m \wedge a) \otimes (p \wedge r \wedge e \wedge m \wedge a) \mapsto p \wedge r \wedge e \wedge m \wedge a.$$

Note that the choice of the matrices makes this map well-defined.

**14.4.2. Self-duality of the Hadamard functor.** We now restrict attention to finite-dimensional species. Note that we have not claimed that the Hadamard functor is self-dual. This point requires more care.

Consider the following situation

$$\begin{tikzcd} \mathsf{Sp}^{(r)} \times \mathsf{Sp}^{(r)} \arrow[r, bend left, "*"] & \mathsf{Sp}^{(r)} \times \mathsf{Sp}^{(r)} \arrow[l, bend left, "*"] \end{tikzcd} \quad \text{where } (\mathbf{p}, \mathbf{q})^* := (\mathbf{q}^*, \mathbf{p}^*).$$

This is an instance of (3.44). With this setup, the contragredient of the Cauchy product is itself, and the contragredient of the braiding $\beta_P \times \beta_Q$ is $\beta_{Q^t} \times \beta_{P^t}$. It follows that

$$(\mathsf{Sp}^{(r)} \times \mathsf{Sp}^{(r)}, \cdot, \beta_{Q^t} \times \beta_Q)$$

is a self-dual braided monoidal category, and further the Hadamard functor

$$(\times, \varphi, \psi)\colon (\mathsf{Sp}^{(r)} \times \mathsf{Sp}^{(r)}, \cdot, \beta_{Q^t} \times \beta_Q) \to (\mathsf{Sp}^{(r)}, \cdot, \beta_{Q^t \times Q})$$

is self-dual.

This leads to the following consequences.

**Corollary 14.8.** *If* $\mathbf{h}$ *is a finite-dimensional* $Q$*-Hopf monoid, then* $\mathbf{h}^* \times \mathbf{h}$ *is a self-dual* $(Q^t \times Q)$*-Hopf monoid.*

**Corollary 14.9.** *If* $\mathbf{h}_1$ *and* $\mathbf{h}_2$ *are finite-dimensional* $P$*- and* $Q$*-Hopf monoids respectively, then*

$$(\mathbf{h}_1 \times \mathbf{h}_2)^* \cong \mathbf{h}_1^* \times \mathbf{h}_2^*$$

*as* $(P^t \times Q^t)$*-Hopf monoids.*

**14.4.3. The colored signature functor.** Using the colored exponential species, we now define a colored analogue of the signature functor (9.10).

Let $Q$ be a log-antisymmetric matrix. Define the *colored signature functor* to be the functor

$$(-)_Q \colon \mathsf{Sp}^{(r)} \to \mathsf{Sp}^{(r)}$$

that sends a $r$-colored species $\mathbf{p}$ to the $r$-colored species

$$\mathbf{p}_Q := \mathbf{p} \times \mathbf{E}_Q. \tag{14.23}$$

**Proposition 14.10.** *The colored signature functor gives rise to a bistrong monoidal functor*

$$(\mathsf{Sp}, \cdot, \beta_P) \to (\mathsf{Sp}, \cdot, \beta_{P\times Q}).$$

Proof. The functor is bilax since $\mathbf{E}_Q$ is a $Q$-bimonoid and the Hadamard functor is bilax (Proposition 14.6). Further, each component of the product and coproduct of $\mathbf{E}_Q$ is bijective, so the structure morphisms of the colored signature functor are invertible, and hence it is bistrong. □

Since $\mathbf{E}_{1_{r,r}}$ is the unit for the Hadamard product, it follows that the functor $(-)_{1_{r,r}}$ is the identity. Similarly, $(-)_{[-1]}$ is the signature functor. Further, it follows from (14.22) that if $P$ and $Q$ are log-antisymmetric, then

$$\big((-)_P\big)_Q = (-)_{P\times Q}.$$

In particular, if $Q$ is log-antisymmetric and symmetric (that is, entries of $Q$ are 1 or $-1$), then $(-)_Q$ is an involution.

On finite-dimensional colored species, the contragredient of the signature functor $(-)_Q$ is $(-)_{Q^t}$.

**Proposition 14.11.** *The following diagram commutes (up to isomorphism) as bilax monoidal functors.*

$$\begin{tikzcd}
(\mathsf{Sp}^{(r)} \times \mathsf{Sp}^{(r)}, \cdot, \beta_P \times \beta_R) \arrow[r, "\times"] \arrow[d, "{(\mathrm{id},(-)_Q)}"'] & (\mathsf{Sp}^{(r)}, \cdot, \beta_{P\times R}) \arrow[d, "(-)_Q"] \\
(\mathsf{Sp}^{(r)} \times \mathsf{Sp}^{(r)}, \cdot, \beta_P \times \beta_{R\times Q}) \arrow[r, "\times"'] & (\mathsf{Sp}^{(r)}, \cdot, \beta_{P\times R\times Q})
\end{tikzcd}$$

This yields a colored generalization of Proposition 9.12. It can be proved in the same way.

## 14.5. The colored linear order species

One of the first examples of Hopf monoids discussed in this monograph is the Hopf monoid $\mathbf{L}$ of linear orders (Example 8.16). Later in Section 9.5, we constructed a one-parameter deformation $\mathbf{L}_q$ using the Schubert cocycle. In Section 12.2, we explained the same construction from a geometric viewpoint using the break and join maps $b_K$ and $j_K$ (10.57), and the gallery metric on chambers.

Now let $Q$ be any matrix of size $r$. In this section we show that one can construct a $Q$-deformation, with $Q = [q]$ recovering $\mathbf{L}_q$. We provide both a combinatorial and geometric description for $\mathbf{L}_Q$, an explicit formula for its antipode, a sufficient condition on $Q$ for its self-duality, and the value taken on it by the signature functor.

**14.5.1. The $Q$-Hopf monoids on linear orders.** The colored linear order species is defined as follows. Given a colored set $(I, f)$, we set

$$\mathbf{L}_Q[I, f] := \mathbf{L}[I].$$

In other words, $\mathbf{L}_Q[I, f]$ is the vector space with basis the set of linear orders on $I$. This makes no use of the colors. We proceed to turn $\mathbf{L}_Q$ into a $Q$-Hopf monoid. This structure will make use of the colors.

Fix a decomposition $(I, f) = (S, g) \sqcup (T, h)$. The coproduct is given by

$$\begin{aligned} (\mathbf{L}_Q)[I, f] &\to (\mathbf{L}_Q)[S, g] \otimes (\mathbf{L}_Q)[T, h] \\ C &\mapsto \operatorname{dist}^Q_f(C, KC)\, C_1 \otimes C_2, \end{aligned}$$

where $K$ is the vertex $S|T$, $\operatorname{dist}^Q_f$ is the weighted distance (10.75), and $C_1$ and $C_2$ are defined by $b_K(KC) = (C_1, C_2)$.

The product is given by

$$\begin{aligned} (\mathbf{L}_Q)[S, g] \otimes (\mathbf{L}_Q)[T, h] &\to (\mathbf{L}_Q)[I, f] \\ C_1 \otimes C_2 &\mapsto j_K(C_1, C_2), \end{aligned}$$

where $K$ is the vertex $S|T$ of $\Sigma[I]$.

In combinatorial terms, the product is concatenation and the coproduct is deshuffling:

$$\begin{aligned} \mathbf{L}_Q[S, g] \otimes \mathbf{L}_Q[T, h] &\to \mathbf{L}_Q[I, f] & \qquad \mathbf{L}_Q[I, f] &\to \mathbf{L}_Q[S, g] \otimes \mathbf{L}_Q[T, h] \\ l_1 \otimes l_2 &\mapsto l_1 \cdot l_2 & l &\mapsto \operatorname{sch}^Q_{S,T,f}(l)\, l|_S \otimes l|_T, \end{aligned}$$

where $\operatorname{sch}^Q_{S,T,f}(l)$ is the (multiplicative) weighted Schubert cocycle (10.102). The connection between the two coproducts can be made using (10.103).

We illustrate the $r = 2$ case. Let $(I, f)$ be the 2-colored set given in (14.19). An element of $\mathbf{L}_Q[I, f]$ is a linear order on $I$, so as an example, we have

$$p|r|e|m|a \in \mathbf{L}_Q[I, f].$$

We illustrate the product and coproduct using this notation.

$$p|r|e \otimes m|a \mapsto p|r|e|m|a.$$

$$\begin{aligned} r|a|m \mapsto 1 \otimes r|a|m + r \otimes a|m + q_{12}\, a \otimes r|m + q_{21}q_{22}\, m \otimes r|a \\ + r|a \otimes m + q_{21}\, r|m \otimes a + q_{12}q_{22}\, a|m \otimes r + r|a|m \otimes 1. \end{aligned}$$

If all entries of $Q$ are equal to $q$, that is, $Q = q1_{r,r}$, then the $Q$-Hopf monoid $\mathbf{L}_Q$ can be viewed as the image of the $q$-Hopf monoid $\mathbf{L}_q$ under one of the bistrong functors of Proposition 14.4, namely:

$$\mathbf{L}_{q1_{r,r}} = (\mathbf{L}_q)_{(r)}.$$

We make a couple of remarks as to why $\mathbf{L}_Q$ is a $Q$-Hopf monoid. The coassociativity of the coproduct follows from the cocycle condition (10.112), or equivalently, (10.113). The product-coproduct compatibility follows from the multiplicative property of the cocycle (10.114), or equivalently, (10.115).

**Proposition 14.12.** *The antipode of* $\mathrm{S}\colon \mathbf{L}_Q \to \mathbf{L}_Q$ *is given by*

$$\begin{aligned} &\mathbf{L}_Q[I,f] \to \mathbf{L}_Q[I,f] \\ &\mathrm{S}_{I,f}(C) = (-1)^{\deg(C)} \operatorname{dist}^Q_f(C,\overline{C})\,\overline{C}. \end{aligned}$$

For example,

$$\mathrm{S}(r|a|m) = -q_{12}q_{21}q_{22}\, m|a|r.$$

The proof of the above result is similar to that of Proposition 12.3.

One may similarly define a colored version of the Hopf monoid $\mathbf{L}^*_q$. We denote it by $\mathbf{L}^*_Q$, following Convention 14.5. It is a $Q$-Hopf monoid. Its structure maps are as follows.

Fix a decomposition $[I,f] = [S,g] \sqcup [T,h]$. The coproduct is given by

$$\begin{aligned} \mathbf{L}^*_Q[I,f] &\to \mathbf{L}^*_Q[S,g] \otimes \mathbf{L}^*_Q[T,h] \\ D^* &\mapsto \begin{cases} D^*_1 \otimes D^*_2 & \text{if } K = S|T \text{ is a vertex of } D, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $D_1$ and $D_2$ are defined by $b_K(D) = (D_1, D_2)$.

The product is given by

$$\begin{aligned} \mathbf{L}^*_Q[S,g] \otimes \mathbf{L}^*_Q[T,h] &\to \mathbf{L}^*_Q[I,f] \\ D^*_1 \otimes D^*_2 &\mapsto \sum_{D:KD=j_K(D_1,D_2)} \operatorname{dist}^Q_f(KD,D)\, D^*. \end{aligned}$$

The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum.

**Proposition 14.13.** *The antipode of* $\mathrm{S}\colon \mathbf{L}^*_Q \to \mathbf{L}^*_Q$ *is given by*

$$\begin{aligned} &\mathbf{L}^*_Q[I,f] \to \mathbf{L}^*_Q[I,f] \\ &\mathrm{S}_{I,f}(D^*) = (-1)^{\deg(D)} \operatorname{dist}^Q_f(D,\overline{D})\,\overline{D}^*. \end{aligned}$$

A comparison with Propositions 12.2–12.5 shows that the descriptions of $\mathbf{L}_Q$ and $\mathbf{L}^*_Q$ are obtained from those of $\mathbf{L}_q$ and $\mathbf{L}^*_q$ by replacing

$$q^{\operatorname{dist}(C,D)} \quad \text{with} \quad \operatorname{dist}^Q_f(C,D).$$

We point out that the order in which the chambers are written is crucial in the multi-dimensional case, since the weighted distance function is not symmetric in general. It is symmetric precisely when the matrix $Q$ is symmetric (10.77).

**14.5.2. Relating $\mathbf{L}_Q$ and $\mathbf{L}^*_Q$.** Recall from Proposition 12.6 that the Hopf monoid $\mathbf{L}_q$ is self-dual if $q$ is not a root of unity. We now discuss to what extent this holds for the $Q$-Hopf monoid $\mathbf{L}_Q$.

First of all, we remark that $\mathbf{L}^*_Q$ is not the dual of $\mathbf{L}_Q$ in general. In fact, it follows from the above definitions that

$$\mathbf{L}^*_Q = (\mathbf{L}_{Q^t})^*$$

as $Q$-Hopf monoids. In particular, if $Q$ is symmetric, then $\mathbf{L}_Q$ and $\mathbf{L}^*_Q$ are duals of each other. These statements are in agreement with (14.18).

Self-duality of $\mathbf{L}_q$ is replaced by the following statement.

**Proposition 14.14.** *The map $\mathbf{L}_Q \to \mathbf{L}^*_Q$ given by*

$$\mathbf{L}_Q[I,f] \to \mathbf{L}^*_Q[I,f] \qquad\qquad C \mapsto \sum_D \mathrm{dist}^Q_f(C,D)\, D^*$$

*is a morphism of $Q$-Hopf monoids. Further, if no monomial in the $q_{ij}$'s equals* 1*, then it is an isomorphism.*

In particular, if $Q$ is symmetric and no monomial in the $q_{ij}$'s equals 1, then the Hopf monoid $\mathbf{L}_Q$ is self-dual.

PROOF. It can be directly checked that the above map is a morphism of colored monoids and comonoids. Both checks make use of (10.84). It then follows that the above map is a morphism of $Q$-Hopf monoids. The next observation is that the above map is induced from the bilinear form on chambers given in (10.134). It is shown in Lemma 10.33 that if no monomial in the $q_{ij}$'s equals 1, then this bilinear form is nondegenerate. The result follows. $\square$

If $Q$ is log-antisymmetric, then the image of the above map is one-dimensional. This follows from Proposition 10.22. More details regarding this situation are given below.

**14.5.3. The value of the signature.** Let $Q$ be a log-antisymmetric matrix. Then there is an isomorphism of $(P \times Q)$-Hopf monoids

$$\mathbf{L}_P \times \mathbf{E}_Q \to \mathbf{L}_{P\times Q}.$$

We illustrate this map by an example. On the component of the 2-colored set $(I,f)$ of (14.19), the map is given by

$$(p|r|e|m|a) \otimes (p \wedge r \wedge e \wedge m \wedge a) \mapsto p|r|e|m|a.$$

The above result also holds with $\mathbf{L}^*_Q$ instead of $\mathbf{L}_Q$. To summarize, the values of $\mathbf{L}_P$ and $\mathbf{L}^*_P$ under the signature functor are:

$$(\mathbf{L}_P)_Q \cong \mathbf{L}_{P\times Q} \qquad \text{and} \qquad (\mathbf{L}^*_P)_Q \cong \mathbf{L}^*_{P\times Q} \tag{14.24}$$

respectively.

Applying the first bistrong functor of Proposition 14.4 to the morphism $\mathbf{L} \to \mathbf{E}$ of (8.31) yields a morphism

$$\mathbf{L}_{1_{r,r}} = \mathbf{L}_{(r)} \to \mathbf{E}_{(r)} = \mathbf{E}_{1_{r,r}}$$

of $1_{r,r}$-Hopf monoids. Now applying the signature functor to this morphism yields a morphism

$$\mathbf{L}_Q \to \mathbf{E}_Q$$

of $Q$-Hopf monoids, which sends a linear order on $I$ to the wedge of the elements of $I$ written using that linear order. For example, on the component of the 2-colored set $(I, f)$ of (14.19),

$$p|r|e|m|a \mapsto p \wedge r \wedge e \wedge m \wedge a.$$

More generally, applying the first bistrong functor of Proposition 14.4 followed by the signature functor to diagram (8.34) yields the following commutative diagram of $Q$-Hopf monoids.

$$\begin{tikzcd} \mathbf{L}_Q \arrow[r] \arrow[d] & \mathbf{L}^*_Q \\ \mathbf{E}_Q \arrow[r, "\mathrm{id}"'] & \mathbf{E}^*_Q \arrow[u] \end{tikzcd} \tag{14.25}$$

The top horizontal map is the same as in Proposition 14.14. It is worth pointing out that the context in that proposition was completely general whereas here we are only dealing with the log-antisymmetric case.

## 14.6. The colored free and cofree Hopf monoids

In this section, we discuss the colored generalizations of the functors $\mathcal{T}_q$, $\mathcal{S}$ and $\Lambda$ and their contragredients which were discussed in Chapter 11. These, along with their universal properties, are summarized in Table 14.2.

We first define these functors and state (without proof) their universal properties. We then discuss the colored norm transformation between $\mathcal{T}_Q$ and $\mathcal{T}^\vee_Q$, give sufficient conditions on $Q$ under which it is an isomorphism, and show that if $Q$ is log-antisymmetric, then its image is precisely $\mathcal{S}_Q$, or equivalently, $\mathcal{S}^\vee_Q$. We also provide antipode formulas for $Q$-Hopf monoids which arise as values of these functors. Examples include the colored exponential species (Section 14.3) and the colored linear order species (Section 14.5). More examples are given in Section 14.7.

**Notation 14.15.** Let $F = F^1|\cdots|F^k \vDash I$ be a composition and $f\colon I \to [r]$ a function. We write

$$\mathbf{q}(F, f) := \mathbf{q}[F^1, f|_{F^1}] \otimes \cdots \otimes \mathbf{q}[F^k, f|_{F^k}].$$

Similarly, given a partition $X \vdash I$ and a function $f\colon I \to [r]$, we write

$$\mathbf{q}(X, f) := \bigotimes_{S \in X} \mathbf{q}[S, f|_S].$$

TABLE 14.2. Universal $Q$-objects.

| $Q$-Hopf monoid | Universal property | Specializations |
|---|---|---|
| $\mathcal{T}_Q(\mathbf{q})$ | free | $\mathcal{T}_q(\mathbf{q})$ for $Q = [q]$ |
| $\mathcal{T}^\vee_Q(\mathbf{q})$ | cofree | $\mathcal{T}^\vee_q(\mathbf{q})$ for $Q = [q]$ |
| $\mathcal{S}_Q(\mathbf{q})$ | free commutative | $\mathcal{S}(\mathbf{q})$ for $Q = [1]$, $\Lambda(\mathbf{q})$ for $Q = [-1]$ |
| $\mathcal{S}^\vee_Q(\mathbf{q})$ | cofree cocommutative | $\mathcal{S}^\vee(\mathbf{q})$ for $Q = [1]$, $\Lambda^\vee(\mathbf{q})$ for $Q = [-1]$ |

There are canonical identifications

$$\mathbf{q}(F, f) \otimes \mathbf{q}(G, g) \cong \mathbf{q}(F \cdot G, f \sqcup g), \tag{14.26}$$

$$\mathbf{q}(X, f) \otimes \mathbf{q}(Y, g) \cong \mathbf{q}(X \sqcup Y, f \sqcup g). \tag{14.27}$$

**14.6.1. The functor $\mathcal{T}_Q$.** We now proceed to define the functor

$$\mathcal{T}_Q \colon \mathsf{Comon}(\mathsf{Sp}_+^{(r)}) \to Q\text{-}\mathsf{Hopf}(\mathsf{Sp}^{(r)})$$

from the category of positive colored comonoids to the category of colored $Q$-Hopf monoids. Let $\mathbf{q}$ be a positive colored comonoid. Define

$$\mathcal{T}_Q(\mathbf{q})[I, f] := \bigoplus_{F \vDash I} \mathbf{q}(F, f).$$

The component $\mu_{S,g,T,h}$ of the product is the direct sum of the maps of the form (14.26) over all $F \vDash S$ and $G \vDash T$. The component $\Delta_{S,g,T,h}$ of the coproduct is the direct sum over all compositions $F \vDash I$ of the map

$$\mathbf{q}(F, f) \to \mathbf{q}(F|_S, f|_S) \otimes \mathbf{q}(F|_T, f|_T)$$

which is defined like the map in (11.12) using the components of the coproduct of $\mathbf{q}$ multiplied by the coefficient

$$\mathrm{sch}^Q_{S,T,f}(F),$$

the weighted multiplicative Schubert cocycle on faces (10.126).

The functor $\mathcal{T}_Q$ satisfies a universal property: $\mathcal{T}_Q(\mathbf{q})$ is the free $Q$-Hopf monoid on the positive colored comonoid $\mathbf{q}$. This provides a $Q$-analogue to Theorems 11.9 and 11.10.

**14.6.2. The functor $\mathcal{S}_Q$.** Let $Q$ be a log-antisymmetric matrix (2.32). Recall that this is precisely the condition under which the braiding $\beta_Q$ is a symmetry.

Let $\mathbf{q}$ be a positive colored species. Define

$$\mathcal{S}_Q(\mathbf{q})[I, f] := \bigoplus_{k \geq 0} (\mathbf{q}^{\cdot k}[I])_{\mathrm{S}_k},$$

where the action of the symmetric group $\mathrm{S}_k$ on $\mathbf{q}^{\cdot k}$ is induced by the symmetry $\beta_Q$. Equivalently,

$$\mathcal{S}_Q(\mathbf{q})[I, f] = \bigoplus_{X \vdash I} \Bigg( \bigoplus_{F : \mathrm{supp}(F) = X} \mathbf{q}[F, f] \Bigg)_{\mathrm{S}_k},$$

where $k$ is the number of blocks in $X$, and the action of $\mathrm{S}_k$ permutes the $k$ ordered tensor factors in $\mathbf{q}[F, f]$ using the symmetry $\beta_Q$. We now make this more explicit.

Let $X$ be a partition of $I$ with $k$ blocks, and let $f \colon I \to [r]$. Take the quotient of the tensor algebra on $\Bbbk X$ by the ideal generated by the relations: for $S, T \in X$,

$$S \otimes T = \mathrm{brd}^Q_{S,T,f}\, T \otimes S,$$

where $\mathrm{brd}^Q_{S,T,f}$ is the multiplicative braid coefficient (10.104). We use $\wedge$ to denote the product in the quotient. Define $\mathrm{Det}^Q_f(\Bbbk X)$ to be the one-dimensional subspace of this quotient which is spanned by the element

$$X^1 \wedge \cdots \wedge X^k,$$

where $X = \{X^1, \dots, X^k\}$, with its elements written in some order. Changing the order of the elements in the wedge product incurs a scalar. For example, for $X = \{S, T, U\}$,

$$S \wedge T \wedge U = \mathrm{brd}^{Q}_{S,T,f|_{S \sqcup T}} \; T \wedge S \wedge U$$

spans $\mathrm{Det}^{Q}_{f}(\Bbbk X)$. If $X$ is the partition of $I$ into singletons, then this space agrees with $\mathrm{Det}^{Q}_{f}(\Bbbk I)$, as defined in Section 14.3.

The main observation is that

$$\mathcal{S}_Q(\mathbf{q})[I, f] := \bigoplus_{X \vdash I} \mathbf{q}(X, f) \otimes \mathrm{Det}^{Q}_{f}(\Bbbk X).$$

So far, $\mathcal{S}_Q(\mathbf{q})$ is only a colored species. We now show that if $\mathbf{q}$ is a positive colored comonoid, then one can turn it into a commutative $Q$-Hopf monoid. In other words, there is a functor

$$\mathcal{S}_Q \colon \mathsf{Comon}(\mathsf{Sp}^{(r)}_{+}) \to Q\text{-}\mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp}^{(r)}).$$

Accordingly, let $\mathbf{q}$ be a positive colored comonoid. The component $\mu_{S,g,T,h}$ of the product in $\mathcal{S}_Q(\mathbf{q})$ is given by tensoring the map (14.27) with the map

$$\begin{gathered}
\mathrm{Det}^{Q}_{g}(\Bbbk X) \otimes \mathrm{Det}^{Q}_{h}(\Bbbk Y) \to \mathrm{Det}^{Q}_{g \sqcup h}\big(\Bbbk(X \sqcup Y)\big) \\
(X^1 \wedge \cdots \wedge X^k) \otimes (Y^1 \wedge \cdots \wedge Y^l) \mapsto X^1 \wedge \cdots \wedge X^k \wedge Y^1 \wedge \cdots \wedge Y^l,
\end{gathered}$$

and then summing over all $X \vdash S$ and $Y \vdash T$.

The component $\Delta_{S,g,T,h}$ of the coproduct is given by tensoring the map

$$\mathbf{q}(X, f) \xrightarrow{\cong} \mathbf{q}(X|_S, f|_S) \otimes \mathbf{q}(X|_T, f|_T)$$

(defined as in (11.15) using the coproduct of $\mathbf{q}$), with the map

$$\mathrm{Det}^{Q}_{f}(\Bbbk X) \to \mathrm{Det}^{Q}_{f|_S}(\Bbbk X|_S) \otimes \mathrm{Det}^{Q}_{f|_T}(\Bbbk X|_T)$$

defined by

$$\begin{aligned}
X^1 \wedge \cdots \wedge X^k \mapsto (-1)^{\mathrm{sch}^{Q}_{S,T,f}(X^1|\cdots|X^k)}\, &(X^1 \cap S) \wedge \cdots \wedge (X^k \cap S) \\
&\otimes (X^1 \cap T) \wedge \cdots \wedge (X^k \cap T),
\end{aligned}$$

and then summing over all $X \vdash I$. Here $\mathrm{sch}^{Q}_{S,T,f}(X^1|\cdots|X^k)$ is the weighted multiplicative Schubert cocycle on faces (10.126).

The functor $\mathcal{S}_Q$ satisfies a universal property: $\mathcal{S}_Q(\mathbf{q})$ is the free commutative $Q$-Hopf monoid on the positive colored comonoid $\mathbf{q}$. This provides a $Q$-analogue to Theorem 11.14.

In dimension one, there are only two log-antisymmetric matrices, namely $[1]$ and $[-1]$. In this situation, $\mathcal{S}_Q$ specializes to $\mathcal{S}$ and $\Lambda$ respectively.

**14.6.3. The colored abelianization.** There is a colored analogue of the abelianization, namely, for any log-antisymmetric matrix $Q$, there is a transformation

$$\pi_Q \colon \mathcal{T}_Q \Rightarrow \mathcal{S}_Q.$$

We refer to it as the *colored abelianization*. It is defined by summing the maps

$$\begin{aligned}
\mathbf{q}(F, f) &\xrightarrow{\cong} \mathbf{q}(X, f) \otimes \mathrm{Det}^{Q}_{f}(\Bbbk X) \\
x &\longmapsto \pi(x) \otimes (F^1 \wedge \cdots \wedge F^k),
\end{aligned} \tag{14.28}$$

where $F = F^1|\cdots|F^k$, and $X = \mathrm{supp}(F)$, and $\pi$ identifies the unbracketed tensor product with the unordered tensor product as in (11.23).

**14.6.4. The functors $\mathcal{T}_Q^\vee$ and $\mathcal{S}_Q^\vee$.** The functor

$$\mathcal{T}_Q^\vee\colon \mathsf{Mon}(\mathsf{Sp}_+^{(r)}) \to Q\text{-}\mathsf{Hopf}(\mathsf{Sp}^{(r)})$$

is defined by

$$\mathcal{T}_Q^\vee(\mathbf{q})[I,f] := \bigoplus_{F \vDash I} \mathbf{q}(F,f),$$

where $\mathbf{q}$ is a positive colored monoid. The coproduct of $\mathcal{T}_Q^\vee(\mathbf{q})$ is deconcatenation. The product is given in terms of quasi-shuffles, with a coefficient involving the Schubert cocycle on faces (as employed for the coproduct of the functor $\mathcal{T}_Q$). With this structure, $\mathcal{T}_Q^\vee(\mathbf{q})$ is a $Q$-Hopf monoid. We omit the details.

Note that $\mathcal{T}_Q(\mathbf{q})$ and $\mathcal{T}_Q^\vee(\mathbf{q})$ are identical as colored species but different as $Q$-Hopf monoids.

The functor $\mathcal{T}_Q^\vee$ satisfies a universal property: $\mathcal{T}_Q^\vee(\mathbf{q})$ is the cofree $Q$-Hopf monoid on the positive colored monoid $\mathbf{q}$. This provides a $Q$-analogue to Theorems 11.22 and 11.23.

When $Q$ is a log-antisymmetric matrix, the functor

$$\mathcal{S}_Q^\vee\colon \mathsf{Mon}(\mathsf{Sp}_+^{(r)}) \to Q\text{-}{}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{Sp}^{(r)})$$

and the natural transformation

$$\pi_Q^\vee\colon \mathcal{S}_Q^\vee \Rightarrow \mathcal{T}_Q^\vee.$$

can be defined similarly.

In dimension one, there are only two log-antisymmetric matrices, namely $[1]$ and $[-1]$. In this situation, $\mathcal{S}_Q^\vee$ specializes to $\mathcal{S}^\vee$ and $\Lambda^\vee$ respectively.

**14.6.5. Contragredients up to transpose.** Let us restrict the functors of the preceding sections to finite-dimensional colored species and consider their contragredients (Section 3.10). The functor $\mathcal{T}_Q^\vee$ is *not* the contragredient of $\mathcal{T}_Q$. In fact, $\mathcal{T}_Q^\vee(\mathbf{q})$ is a $Q$-Hopf monoid, while the contragredient $(\mathcal{T}_Q)^\vee(\mathbf{q})$ is a $Q^t$-Hopf monoid. In more detail, according to (3.45) and (14.6), the contragredient $(\mathcal{T}_Q)^\vee$ of $\mathcal{T}_Q$ is the following composite.

$$\mathsf{Mon}(\mathsf{Sp}_+^{(r)}) \xrightarrow{\ *\ } \mathsf{Comon}(\mathsf{Sp}_+^{(r)}) \xrightarrow{\mathcal{T}_Q} Q\text{-}\mathsf{Hopf}(\mathsf{Sp}^{(r)}) \xrightarrow{\ *\ } Q^t\text{-}\mathsf{Hopf}(\mathsf{Sp}^{(r)})$$

On the other hand, it follows from the definitions of $\mathcal{T}_Q$ and $\mathcal{T}_Q^\vee$ that

$$(\mathcal{T}_Q)^\vee = \mathcal{T}_{Q^t}^\vee.$$

Thus, $\mathcal{T}_Q^\vee$ is the contragredient of $\mathcal{T}_Q$ up to transposing the matrix $Q$. In particular, if $\mathbf{q}$ is a positive comonoid, then

$$\mathcal{T}_Q(\mathbf{q})^* = \mathcal{T}_{Q^t}(\mathbf{q}^*) \tag{14.29}$$

as $Q^t$-Hopf monoids. This is in agreement with (14.18).

Similarly, when $Q$ is log-antisymmetric, $\mathcal{S}_{Q^t}^\vee$ is the contragredient of $\mathcal{S}_Q$, and the transformation $\pi_{Q^t}^\vee\colon \mathcal{S}_{Q^t}^\vee \Rightarrow \mathcal{T}_{Q^t}^\vee$ is the contragredient of the colored abelianization $\pi_Q\colon \mathcal{T}_Q \Rightarrow \mathcal{S}_Q$.

**14.6.6. The norm transformation.** We now define a colored version of the norm transformation. For that purpose, we view

$$\mathcal{T}_Q, \mathcal{T}_Q^\vee \colon \mathsf{Sp}_+^{(r)} \to Q\text{-}\mathsf{Hopf}(\mathsf{Sp}^{(r)})$$

by viewing a positive colored species as a positive colored (co)monoid in the trivial way. The $Q$-norm transformation

$$\kappa_Q \colon \mathcal{T}_Q \Rightarrow \mathcal{T}_Q^\vee$$

is defined as follows. Fix a composition $F$ of $I$ and a function $f \colon I \to [r]$. For each set composition $G$ with the same support as $F$, consider the map

$$\mathbf{q}(F, f) \to \mathbf{q}(G, f)$$

which reorders the tensor factors and multiplies by the coefficient

$$\operatorname{dist}_f^Q(F, G), \quad \text{or equivalently,} \quad \prod_{(i,j)\in \operatorname{Inv}(F,G)} \ \prod_{s\in F^i,\, t\in F^j} q_{f(t)f(s)},$$

with notation as in (10.92). By summing over all such $G$, we obtain

$$\mathbf{q}(F, f) \to \bigoplus_G \mathbf{q}(G, f).$$

This is the $Q$-norm transformation. One can check that

$$\kappa_Q(\mathbf{q}) \colon \mathcal{T}_Q(\mathbf{q}) \to \mathcal{T}_Q^\vee(\mathbf{q})$$

is a natural morphism of $Q$-Hopf monoids. In addition:

**Proposition 14.16.** *When the norm is restricted to finite-dimensional species, we have*

$$(\kappa_Q)^\vee = \kappa_{Q^t}.$$

*In particular, if the matrix $Q$ is symmetric, then the $Q$-norm is self-dual.*

**Theorem 14.17.** *For the matrix $Q$, if no monomial in the $q_{ij}$'s equals one, then $\kappa_Q$ is an isomorphism.*

Proof. We generalize the proof of Theorem 11.35 as follows. Fix a partition $X$ of $I$, and a function $f \colon I \to [r]$. It is enough to show that each of the restrictions

$$\bigoplus_{F:\operatorname{supp}(F)=X} \mathbf{q}(F, f) \to \bigoplus_{F:\operatorname{supp}(F)=X} \mathbf{q}(F, f) \tag{14.30}$$

of $\kappa_Q$ is an isomorphism.

For this, let us first do the case when $\mathbf{q}$ is the positive colored exponential species. Then $\mathbf{q}(F, f) = \Bbbk$ and hence (14.30) can be viewed as a map

$$\mathbf{L}[X] \to \mathbf{L}[X]$$

where we recall that $\mathbf{L}[X]$ is the span of linear orders on $X$. Further, this map arises from the bilinear form discussed in Example 10.34, where we showed that if no monomial in the $q_{ij}$'s equals one, then $\kappa_Q$ is an isomorphism.

The general case follows as in the proof of Theorem 11.35 by fixing a basis and reducing to the above case. ∎

For $Q = 0_{r,r}$, the hypothesis of Theorem 14.17 is satisfied. Hence $\kappa_{0_{r,r}}$ is an isomorphism. In fact, in this case, the functors $\mathcal{T}_{0_{r,r}}$ and $\mathcal{T}^{\vee}_{0_{r,r}}$ are identical, and $\kappa_{0_{r,r}}$ is the identity transformation. In particular, $\mathcal{T}_{0_{r,r}}$ is a self-dual functor.

Now let $Q$ be a log-antisymmetric matrix. Such a matrix clearly fails the hypothesis of Theorem 14.17. The norm $\kappa_Q$, in this case, is far from being an isomorphism. The situation can be summarized as follows.

(14.31)
$$\begin{tikzcd} \mathcal{T}_Q \arrow[r, Rightarrow, "\kappa_Q"] \arrow[d, Rightarrow, "\pi_Q"'] & \mathcal{T}^{\vee}_Q \\ \mathcal{S}_Q \arrow[r, Rightarrow, "\mathrm{id}"'] & \mathcal{S}^{\vee}_Q \arrow[u, Rightarrow, "\pi^{\vee}_Q"'] \end{tikzcd}$$

The vertical maps are the colored abelianization and its contragredient up to transpose. In other words, $\mathcal{S}_Q$ is the image of the $Q$-norm transformation. This diagram simultaneously generalizes (11.26) and (11.31). Further, its contragredient yields the same diagram with $Q$ replaced by $Q^t$. In particular, $\mathcal{S}_Q$ is self-dual if the entries of $Q$ are either 1 or $-1$. This can also be seen as a consequence of the discussion in Section 3.11.6.

**14.6.7. Example.** The simplest instance of the preceding theory occurs when $\mathbf{q} = \mathbf{X}_{(r)}$ (Section 14.1.4). Now view it as a positive colored (co)monoid with the zero (co)product. Then

$$\mathcal{T}_Q(\mathbf{X}_{(r)}) = \mathbf{L}_Q \quad \text{and} \quad \mathcal{T}^{\vee}_Q(\mathbf{X}_{(r)}) = \mathbf{L}^*_Q.$$

This recovers the $Q$-Hopf monoids on colored linear order species (Section 14.5). They are the free and cofree $Q$-Hopf monoids on $\mathbf{X}_{(r)}$ respectively. Further, the morphism $\mathbf{L}_Q \to \mathbf{L}^*_Q$ induced by the colored norm coincides with the one defined in Proposition 14.14. This result can thus be seen as a consequence of Theorem 14.17.

Now let $Q$ be log-antisymmetric. Then

$$\mathcal{S}_Q(\mathbf{X}_{(r)}) = \mathcal{S}^{\vee}_Q(\mathbf{X}_{(r)}) = \mathbf{E}_Q.$$

This recovers the colored exponential species of Section 14.3. It is the free commutative as well as the cofree cocommutative $Q$-Hopf monoid on $\mathbf{X}_{(r)}$. More generally, applying the functors in diagram (14.31) to $\mathbf{X}_{(r)}$ yields the diagram (14.25) of $Q$-Hopf monoids.

More examples are given in Proposition 14.32.

**14.6.8. Antipode formulas.** We now give antipode formulas for $\mathcal{T}_Q(\mathbf{q})$ and $\mathcal{T}^{\vee}_Q(\mathbf{q})$. For this purpose, we first generalize the notation introduced in Section 11.8.1.

For any face $H = H^1|\cdots|H^k$, let

$$\mu_{H,f} := \mu_{H^1,f|_{H^1},\dots,H^k,f|_{H^k}} \qquad \text{and} \qquad \Delta_{H,f} := \Delta_{H^1,f|_{H^1},\dots,H^k,f|_{H^k}}$$

denote the components of the iterated product and coproduct. If $H$ has only one part, then $\mu_{H,f}$ and $\Delta_{H,f}$ are the identity. We proceed.

Let $F \leq G$ and let the compositions $G^i$'s be as in (11.32). Let $\mathbf{q}$ be a positive colored comonoid. Define

$$\Delta_{G/F,f} \colon \mathbf{q}(F,f) \to \mathbf{q}(G,f) \qquad \text{by} \qquad \Delta_{G/F,f} := \Delta_{G^1,f|_{G^1}} \otimes \cdots \otimes \Delta_{G^k,f|_{G^k}}.$$

Similarly, for a positive colored monoid $\mathbf{q}$, let

$$\mu_{F\backslash G,f}\colon \mathbf{q}(G,f) \to \mathbf{q}(F,f) \qquad \text{by} \qquad \mu_{F\backslash G,f} := \mu_{G^1,f|_{G^1}} \otimes \cdots \otimes \mu_{G^k,f|_{G^k}}.$$

For faces $F$ and $G$ having the same support, let

$$\beta_{G,F,f}\colon \mathbf{q}(F,f) \to \mathbf{q}(G,f)$$

be the map which reorders the tensor factors.

**Theorem 14.18.** *Let $F$ be a composition of $I$, and $f\colon I \to [r]$. Let $\mathbf{q}$ be a positive colored comonoid. The antipode $\mathrm{S}\colon \mathcal{T}_Q(\mathbf{q}) \to \mathcal{T}_Q(\mathbf{q})$ on the $(F,f)$-component is given by*

$$\mathbf{q}(F,f) \to \bigoplus_{G:G\vDash I} \mathbf{q}(G,f)$$
$$\mathrm{S}_{I,f}(x) = \mathrm{dist}^Q_f(F,\overline{F}) \sum_{G:\overline{F}\leq G} (-1)^{\deg(G)}\, \Delta_{G/\overline{F},f}\beta_{\overline{F},F,f}(x).$$

*Here $\deg(G)$ is the number of blocks in $G$, $\overline{F}$ is the opposite of $F$, and $\mathrm{dist}^Q_f(F,G)$ is as in (10.91).*

The proof proceeds along the lines of the proof of Theorem 11.38, the main step being the application of Lemma 11.37. Special cases of the above result include Proposition 14.12, and Theorems 14.29 and 14.31 (which are discussed later).

By dualizing Theorem 14.18 or by proceeding directly, one obtains:

**Theorem 14.19.** *Let $G$ be a composition of $I$, and $f\colon I \to [r]$. Let $\mathbf{q}$ be a positive colored monoid. The antipode $\mathrm{S}\colon \mathcal{T}^\vee_Q(\mathbf{q}) \to \mathcal{T}^\vee_Q(\mathbf{q})$ on the $(G,f)$-component is given by*

$$\mathbf{q}(G,f) \to \bigoplus_{F:F\vDash I} \mathbf{q}(F,f)$$
$$\mathrm{S}_{I,f}(x) = (-1)^{\deg(G)} \sum_{F:F\leq\overline{G}} \mathrm{dist}^Q_f(\overline{F},F)\, \beta_{F,\overline{F},f}\mu_{\overline{F}\backslash G,f}(x),$$

*where the notation is as in Theorem 14.18.*

Special cases of the above result include Proposition 14.13, and Theorems 14.28 and 14.30 (which are discussed later).

The colored abelianization and its contragredient can be used to derive antipode formulas for $\mathcal{S}_Q(\mathbf{q})$ and $\mathcal{S}^\vee_Q(\mathbf{q})$. Let $X \leq Y$ be partitions of $I$, and let $f\colon I \to [r]$. Fix $F$ to be any set composition with support $X$. Let $\mathbf{q}$ be a positive colored comonoid. Define $\Delta^Q_{Y/X,f}$ by the commutativity of the following diagram.

$$\begin{tikzcd}
\mathbf{q}(F,f) \arrow[rr, "\bigoplus_{G:F\leq G,\mathrm{supp}(G)=Y} \Delta_{G/F,f}"] \arrow[d, "(\pi_Q)_{\mathbf{q}}"'] & & \displaystyle\bigoplus_{G:\,F\leq G,\mathrm{supp}(G)=Y} \mathbf{q}(G,f) \arrow[d, "(\pi_Q)_{\mathbf{q}}"] \\
\mathbf{q}(X,f)\otimes \mathrm{Det}^Q_f(\Bbbk X) \arrow[rr, dashed, "\Delta^Q_{Y/X,f}"'] & & \mathbf{q}(Y,f)\otimes \mathrm{Det}^Q_f(\Bbbk Y)
\end{tikzcd}$$

The vertical maps are the colored abelianization.

Similarly, for a positive colored monoid $\mathbf{q}$, define $\mu^Q_{X\setminus Y,f}$ by the commutativity of the following diagram.

$$\begin{tikzcd}
\displaystyle\bigoplus_{G:\mathrm{supp}(G)=Y} \mathbf{q}(G,f) \arrow[rr, "\bigoplus \mu_{F\setminus G,f}"] & & \displaystyle\bigoplus_{F:\mathrm{supp}(F)=X} \mathbf{q}(F,f) \\
\mathbf{q}(Y,f)\otimes \mathrm{Det}^Q_f(\Bbbk Y) \arrow[u, "(\pi^\vee_Q)_{\mathbf{q}}"] \arrow[rr, dashed, "\mu^Q_{X\setminus Y,f}"'] & & \mathbf{q}(X,f)\otimes \mathrm{Det}^Q_f(\Bbbk X) \arrow[u, "(\pi^\vee_Q)_{\mathbf{q}}"']
\end{tikzcd}$$

The vertical maps are the contragredient up to transpose of the colored abelianization. The top horizontal map above is obtained by summing $\mu_{F\setminus G,f}$ over all $F$ with support $X$, and $G$ with support $Y$ such that $F \le G$. Note that there exist $G$ for which there is no corresponding $F$; these components map to zero.

**Theorem 14.20.** *Let $X$ be a partition of $I$, and $f\colon I \to [r]$. Let $\mathbf{q}$ be a positive colored comonoid. The antipode $\mathrm{s}\colon \mathcal{S}_Q(\mathbf{q}) \to \mathcal{S}_Q(\mathbf{q})$ on the $(X,f)$-component is given by*

$$\mathbf{q}(X,f) \to \bigoplus_{Y:Y\vdash I} \mathbf{q}(Y,f)$$
$$\mathrm{s}_{I,f}(x) = \sum_{Y:X\le Y} (-1)^{\deg(Y)}\, \Delta^Q_{Y/X,f}(x),$$

*where $\deg(Y)$ is the number of blocks in $Y$.*

The result follows by applying the colored abelianization to the antipode formula of Theorem 14.18. The antipode of the colored exponential species (Section 14.3) illustrates this theorem. Dually,

**Theorem 14.21.** *Let $Y$ be a partition of $I$, and $f\colon I \to [r]$. Let $\mathbf{q}$ be a positive colored monoid. The antipode $\mathrm{s}\colon \mathcal{S}^\vee_Q(\mathbf{q}) \to \mathcal{S}^\vee_Q(\mathbf{q})$ on the $(Y,f)$-component is given by*

$$\mathbf{q}(Y,f) \to \bigoplus_{X:X\vdash I} \mathbf{q}(X,f)$$
$$\mathrm{s}_{I,f}(x) = (-1)^{\deg(Y)} \sum_{X:X\le Y} \mu^Q_{X\setminus Y,f}(x).$$

## 14.7. Colored Hopf monoids from geometry

We now briefly revisit the $q$-Hopf monoids in Chapter 12, and show how they can be generalized to provide examples of $Q$-Hopf monoids. These are summarized in Table 14.3. The value of each of these $Q$-Hopf monoids on a colored set $(I,f)$ depends only on $I$ (and not on $f$) and is the same as in the one-dimensional case. For example, $\mathbf{\Sigma}_Q[I,f]$ is spanned by compositions of $I$, and so forth. The $M$, $F$, $H$ and $K$ bases are then defined on each colored component exactly as in Section 12.1.

The $Q$-Hopf monoid $\mathbf{L}_Q$ of linear orders was treated in Section 14.5. The goal of this section is to explain the rest of them. Those based on linear orders, set compositions, and linear set compositions can be obtained as values of the functors $\mathcal{T}_Q$ and $\mathcal{T}^\vee_Q$ considered in Section 14.6, while those based on pairs of linear orders

TABLE 14.3. $Q$-Hopf monoids.

| $Q$-Hopf monoid | | | $Q$-Hopf monoid | | |
|---|---|---|---|---|---|
| $\mathbf{L}_Q$ | $\mathcal{T}_Q(\mathbf{X}^*_{(r)})$ | | $\mathbf{L}^*_Q$ | $\mathcal{T}^\vee_Q(\mathbf{X}_{(r)})$ | |
| $\mathbf{\Sigma}_Q$ | $\mathcal{T}_Q\big((\mathbf{E}^*_{(r)})_+\big)$ | $H$ | $\mathbf{\Sigma}^*_Q$ | $\mathcal{T}^\vee_Q\big((\mathbf{E}_{(r)})_+\big)$ | $M$ |
| $\overrightarrow{\mathbf{\Sigma}}_Q$ | $\mathcal{T}_Q\big((\mathbf{L}^*_{(r)})_+\big)$ | $H, K$ | $\overrightarrow{\mathbf{\Sigma}}^*_Q$ | $\mathcal{T}^\vee_Q\big((\mathbf{L}_{(r)})_+\big)$ | $M, F$ |
| $\mathbf{IL}_Q$ | $\mathbf{L}^*_{1_{r,r}} \times \mathbf{L}_Q$ | $H, K$ | $\mathbf{IL}^*_Q$ | $\mathbf{L}_{1_{r,r}} \times \mathbf{L}^*_Q$ | $M, F$ |

can be obtained as values of the Hadamard functor of Section 14.4. The interrelationships between these $Q$-Hopf monoids is given in diagram (14.34).

The notation employed in Table 14.3 follows Convention 14.5. Recall that the dual of a $Q$-Hopf monoid is a $Q^t$-Hopf monoid. The objects shown are related as follows.

$$\mathbf{L}^*_Q = (\mathbf{L}_{Q^t})^*, \quad \mathbf{\Sigma}^*_Q = (\mathbf{\Sigma}_{Q^t})^*, \quad \overrightarrow{\mathbf{\Sigma}}^*_Q = (\overrightarrow{\mathbf{\Sigma}}_{Q^t})^*, \quad \mathbf{IL}_Q = (\mathbf{IL}^*_{Q^t})^*. \tag{14.32}$$

The first three of these isomorphisms of $Q$-Hopf monoids are special instances of (14.29).

**Remark 14.22.** Recall that the functors $\mathcal{S}_Q$ and $\mathcal{S}^\vee_Q$ evaluated on $\mathbf{X}_{(r)}$ yield the colored exponential species $\mathbf{E}_Q$ discussed in Section 14.3. One may also consider the values of these functors on $\mathbf{E}_{(r)}$ and $\mathbf{L}_{(r)}$. These would lead to colored analogues of the Hopf monoids $\mathbf{\Pi}$ of set partitions and $\overrightarrow{\mathbf{\Pi}}$ of linear set partitions. We do not discuss them here.

### 14.7.1. $Q$-Hopf monoids of pairs of chambers.

Recall the species $\mathbf{IL}$ of pairs of chambers (linear orders). Consider the colored species $\mathbf{IL}_Q$ defined by

$$\mathbf{IL}_Q[I, f] := \mathbf{IL}[I],$$

and let $K$ be its canonical basis. Let $\mathbf{IL}^*_Q$ be the dual colored species, and let $F$ be its canonical basis (dual to $K$). The $H$ and $M$ bases are then defined as in Section 12.1.

Since the species are linearized, $\mathbf{IL}_Q$ and $\mathbf{IL}^*_Q$ are isomorphic as colored species. We now proceed to turn these into $Q$-Hopf monoids. The structure maps, as expected, depend on $Q$.

**Definition 14.23.** Fix a decomposition $(I, f) = (S, g) \sqcup (T, h)$ into nonempty subsets. The coproduct of $\mathbf{IL}^*_Q$ is given by

$$\mathbf{IL}^*_Q[I, f] \to \mathbf{IL}^*_Q[S, g] \otimes \mathbf{IL}^*_Q[T, h]$$
$$F_{(C,D)} \mapsto \begin{cases} F_{(C_1,D_1)} \otimes F_{(C_2,D_2)} & \text{if } K = S|T \text{ is a vertex of } D, \\ 0 & \text{otherwise,} \end{cases}$$

where $C_1$, $C_2$, $D_1$ and $D_2$ are defined by $b_K(D) = (D_1, D_2)$ and $b_K(KC) = (C_1, C_2)$.

The product is given by

$$\mathbf{IL}^*_Q[S, g] \otimes \mathbf{IL}^*_Q[T, h] \to \mathbf{IL}^*_Q[I, f]$$
$$F_{(C_1,D_1)} \otimes F_{(C_2,D_2)} \mapsto \sum_{D:KD = j_K(D_1,D_2)} \mathrm{dist}^Q_f(KD, D)\, F_{(j_K(C_1,C_2),D)}.$$

The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum.

A comparison with Definition 12.7 shows that the only change is in the coproduct formula where $q^{\mathrm{dist}(KD,D)}$ has been replaced by $\mathrm{dist}_f^Q(KD, D)$.

We illustrate the definition with an example for $r = 2$. Following the notations of (14.19), consider the colored set $(I, f) = \{s, i, t, a\}$, with blue denoting color 1 and red denoting color 2. As an example,

$$(t|a|s|i, s|i|t|a) \in \mathbb{L}_Q^*[I, f].$$

We now illustrate the product and coproduct.

$$\begin{aligned} F_{(t|a|s|i,s|i|t|a)} \mapsto 1 \otimes F_{(t|a|s|i,s|i|t|a)} + F_{(s,s)} \otimes F_{(t|a|i,i|t|a)} \\ + F_{(s|i,s|i)} \otimes F_{(t|a,t|a)} + F_{(t|s|i,s|i|t)} \otimes F_{(a,a)} + F_{(t|a|s|i,s|i|t|a)} \otimes 1. \end{aligned}$$

$$\begin{aligned} F_{(s|i,s|i)} \otimes F_{(t|a,a|t)} \mapsto F_{(s|i|t|a,s|i|a|t)} + q_{21} F_{(s|i|t|a,s|a|i|t)} + q_{21}q_{22} F_{(s|i|t|a,a|s|i|t)} \\ + q_{21}^2 F_{(s|i|t|a,s|a|t|i)} + q_{21}^2 q_{22} F_{(s|i|t|a,a|s|t|i)} + q_{21}^2 q_{22}^2 F_{(s|i|t|a,a|t|s|i)}. \end{aligned}$$

The antipode formula for $\mathbb{L}_Q^*$ on the $F$ basis is as in Theorem 12.17, with

$$q^{\mathrm{dist}(C,D)} \quad \text{replaced by} \quad \mathrm{dist}_f^Q(C, D).$$

The formulas for the product, coproduct, and antipode of $\mathbb{L}_Q^*$ on the $M$ basis can be obtained through the same replacement from Theorems 12.13 and 12.18.

Dualizing the structure maps in Definition 14.23 and replacing $Q$ by $Q^t$ (14.32) one obtains the following.

**Proposition 14.24.** *Fix a decomposition $(I, f) = (S, g) \sqcup (T, h)$ into nonempty subsets. The product of $\mathbb{L}_Q$ is given by*

$$\begin{aligned} \mathbb{L}_Q[S, g] \otimes \mathbb{L}_Q[T, h] &\to \mathbb{L}_Q[I, f] \\ K_{(D_1,C_1)} \otimes K_{(D_2,C_2)} &\mapsto \sum_{D: KD = j_K(D_1,D_2)} K_{(D, j_K(C_1,C_2))}. \end{aligned}$$

*The vertex $K = S|T \in \Sigma[I]$ is fixed in the above sum.*

*The coproduct is given by*

$$\begin{aligned} &\mathbb{L}_Q[I, f] \to \mathbb{L}_Q[S, g] \otimes \mathbb{L}_Q[T, h] \\ &K_{(D,C)} \mapsto \begin{cases} \mathrm{dist}_f^Q(C, KC)\, K_{(D_1,C_1)} \otimes K_{(D_2,C_2)} & \text{if } K = S|T \text{ is a vertex of } D, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

*where the chambers $C_1$, $C_2$, $D_1$ and $D_2$ are defined by $b_K(D) = (D_1, D_2)$ and $b_K(KC) = (C_1, C_2)$.*

The formulas for the structure maps of $\mathbb{L}_Q$ on the $H$ basis are as in Theorem 12.15 with

$$q^{\mathrm{dist}(C,D)} \quad \text{replaced by} \quad \mathrm{dist}_f^Q(C, D).$$

We have the following $Q$-analogue of (12.8):

$$\mathbb{L}_Q := \mathbf{L}_{1_{r,r}}^* \times \mathbf{L}_Q \qquad \text{and} \qquad \mathbb{L}_Q^* := \mathbf{L}_{1_{r,r}} \times \mathbf{L}_Q^*.$$

**Proposition 14.25.** *Let $P$ be a square matrix none of whose entries are zero, and $Q$ be any other matrix of the same size. There is an isomorphism*

$$\mathbf{L}_P \times \mathbf{L}_Q^* \to \mathbf{L} \times \mathbf{L}_{P \times Q}^*$$

*of $(P \times Q)$-Hopf monoids whose $(I, f)$-component is given by*

$$(C, D^*) \mapsto \operatorname{dist}_f^P(C, D)\ (C, D^*).$$

PROOF. The product and coproduct are preserved because of (10.84). $\square$

**Corollary 14.26.** *Let $P, Q, P'$ and $Q'$ be square matrices of the same size none of whose entries are zero, and such that $P \times Q = P' \times Q'$. Then*

$$\mathbf{L}_P \times \mathbf{L}_Q^* \cong \mathbf{L}_{P'} \times \mathbf{L}_{Q'}^*$$

*as $(P \times Q)$-Hopf monoids.*

Let $s_Q \colon \mathbb{L}_Q \to \mathbb{L}_Q^*$ be the map defined by

$$K_{(D,C)} \mapsto \operatorname{dist}_f^Q(C, D)\ F_{(C,D)}. \tag{14.33}$$

We refer to $s_Q$ as the *switch map*. Proposition 14.25 and Corollary 14.26 imply:

**Proposition 14.27.** *Let $Q$ be a matrix none of whose entries are zero. Then the switch map $s_Q$ is an isomorphism of $Q$-Hopf monoids. In particular, if $Q$ is symmetric with nonzero entries, then $\mathbb{L}_Q$ is self-dual.*

Applying the signature functor to $\mathbb{L}_P$ yields $\mathbb{L}_{P\times Q}$, and to $\mathbb{L}_P^*$ yields $\mathbb{L}_{P\times Q}^*$. This follows from the corresponding results for $\mathbf{L}_P$ and $\mathbf{L}_P^*$.

**14.7.2. $Q$-Hopf monoids of faces.** Recall the species $\boldsymbol{\Sigma}$ of faces (set compositions). Consider the colored species $\boldsymbol{\Sigma}_Q$ defined by

$$\boldsymbol{\Sigma}_Q[I, f] := \boldsymbol{\Sigma}[I],$$

and let $H$ be its canonical basis. Let $\boldsymbol{\Sigma}_Q^*$ be the dual colored species, and let $M$ be its canonical basis (dual to $H$).

We turn $\boldsymbol{\Sigma}_Q^*$ into a $Q$-Hopf monoid as follows. Let $(I, f) = (S, g) \sqcup (T, h)$ be a decomposition. The coproduct is given by

$$\begin{gathered}
\boldsymbol{\Sigma}_Q^*[I, f] \to \boldsymbol{\Sigma}_Q^*[S, g] \otimes \boldsymbol{\Sigma}_Q^*[T, h] \\
M_G \mapsto \begin{cases} M_{G_1} \otimes M_{G_2} & \text{if } K = S|T \text{ is a vertex of } G, \\ 0 & \text{otherwise,} \end{cases}
\end{gathered}$$

where $b_K(G) = (G_1, G_2)$.

The product is given by

$$\begin{gathered}
\boldsymbol{\Sigma}_Q^*[S, g] \otimes \boldsymbol{\Sigma}_Q^*[T, h] \to \boldsymbol{\Sigma}_Q^*[I, f] \\
M_{G_1} \otimes M_{G_2} \mapsto \sum_{G : KG = j_K(G_1, G_2)} \operatorname{dist}_f^Q(K, G)\, M_G,
\end{gathered}$$

where the vertex $K = S|T$ is fixed, and $\operatorname{dist}_f^Q(K, G)$ is as in (10.91).

We illustrate the product and coproduct for $r = 2$ with blue denoting color 1 and red denoting color 2.

$$M_{vi|sh|nu} \mapsto 1 \otimes M_{vi|sh|nu} + M_{vi} \otimes M_{sh|nu} + M_{vi|sh} \otimes M_{nu} + M_{vi|sh|nu} \otimes 1.$$

$$\begin{aligned}
M_{la|ksh} \otimes M_{mi} \mapsto M_{la|ksh|mi} &+ q_{11} q_{12}^2 q_{21} q_{22}^2\, M_{la|mi|ksh} + q_{11}^3 q_{12}^2 q_{21}^3 q_{22}^2\, M_{mi|la|ksh} \\
&+ M_{la|kshmi} + q_{11} q_{12}^2 q_{21} q_{22}^2\, M_{lami|ksh}.
\end{aligned}$$

**Theorem 14.28.** *The antipode* $\mathrm{s}\colon \mathbf{\Sigma}^*_Q \to \mathbf{\Sigma}^*_Q$ *is given by*

$$\begin{gathered}\mathbf{\Sigma}^*_Q[I,f] \to \mathbf{\Sigma}^*_Q[I,f]\\ \mathrm{s}_{I,f}(M_G) = (-1)^{\deg(G)} \sum_{F:\,F\leq \overline{G}} \mathrm{dist}^Q_f(\overline{F},F)\, M_F,\end{gathered}$$

*where* $\deg(G)$ *is the number of blocks in* $G$, *and* $\overline{G}$ *denotes the opposite of* $G$.

For example,

$$\begin{aligned}\mathrm{s}\big(M_{mi|ksh|la}\big) = -M_{lakshmi} - q_{11}^3q_{12}^3q_{21}^2q_{22}^2\, M_{laksh|mi}\\ - q_{11}^4q_{12}^6\, M_{la|kshmi} - q_{11}^5q_{12}^7q_{21}^2q_{22}^2\, M_{la|ksh|mi}.\end{aligned}$$

The above result generalizes Theorem 12.21; the proof is essentially the same. It can also be seen as a special case of Theorem 14.19.

In view of (14.32), dualizing the above formulas and replacing $Q$ by $Q^t$ one obtains descriptions for the structure of the $Q$-Hopf monoid $\mathbf{\Sigma}_Q$. They are as follows.

Fix a decomposition $(I,f) = (S,g) \sqcup (T,h)$. The coproduct of $\mathbf{\Sigma}_Q$ is given by

$$\begin{gathered}\mathbf{\Sigma}_Q[I,f] \to \mathbf{\Sigma}_Q[S,g] \otimes \mathbf{\Sigma}_Q[T,h]\\ H_F \mapsto \mathrm{dist}^Q_f(F,K)\, H_{F_1} \otimes H_{F_2},\end{gathered}$$

where $K$ is the vertex $S|T$, and $F_1$ and $F_2$ are defined by $b_K(KF) = (F_1,F_2)$.

The product is given by

$$\begin{gathered}\mathbf{\Sigma}_Q[S,g] \otimes \mathbf{\Sigma}_Q[T,h] \to \mathbf{\Sigma}_Q[I,f]\\ H_{F_1} \otimes H_{F_2} \mapsto H_{j_K(F_1,F_2)}\end{gathered}$$

where the vertex $K$ is defined to be $S|T$.

**Theorem 14.29.** *The antipode* $\mathrm{s}\colon \mathbf{\Sigma}_Q \to \mathbf{\Sigma}_Q$ *is given by*

$$\begin{gathered}\mathbf{\Sigma}_Q[I,f] \to \mathbf{\Sigma}_Q[I,f]\\ \mathrm{s}_{I,f}(H_F) = \mathrm{dist}^Q_f(F,\overline{F}) \sum_{G:\,\overline{F}\leq G} (-1)^{\deg(G)}\, H_G.\end{gathered}$$

**14.7.3. *Q*-Hopf monoids of directed faces.** Recall the species $\overrightarrow{\mathbf{\Sigma}}$ of directed faces (linear set compositions). Consider the colored species $\overrightarrow{\mathbf{\Sigma}}_Q$ defined by

$$\overrightarrow{\mathbf{\Sigma}}_Q[I,f] := \overrightarrow{\mathbf{\Sigma}}[I],$$

and let $K$ be its canonical basis. Let $\overrightarrow{\mathbf{\Sigma}}^*_Q$ be the dual colored species, and let $F$ be its canonical basis (dual to $K$). The $H$ and $M$ bases are then defined as in Section 12.1.

We now describe the structure maps of $\overrightarrow{\mathbf{\Sigma}}^*_Q$ on the $M$ basis. Fix a decomposition $(I,f) = (S,g) \sqcup (T,h)$. The coproduct is given by

$$\begin{gathered}\overrightarrow{\mathbf{\Sigma}}^*_Q[I,f] \to \overrightarrow{\mathbf{\Sigma}}^*_Q[S,g] \otimes \overrightarrow{\mathbf{\Sigma}}^*_Q[T,h]\\ M_{(G,D)} \mapsto \begin{cases} M_{(G_1,D_1)} \otimes M_{(G_2,D_2)} & \text{if } K = S|T \text{ is a vertex of } G,\\ 0 & \text{otherwise,}\end{cases}\end{gathered}$$

where $G_1$, $G_2$, $D_1$ and $D_2$ are defined by $b_K(G) = (G_1,G_2)$ and $b_K(D) = (D_1,D_2)$.

The product is given by

$$\begin{aligned} &\overrightarrow{\mathbf{\Sigma}}^*_Q[S,g] \otimes \overrightarrow{\mathbf{\Sigma}}^*_Q[T,h] \to \overrightarrow{\mathbf{\Sigma}}^*_Q[I,f] \\ &M_{(G_1,D_1)} \otimes M_{(G_2,D_2)} \mapsto \sum_{G:\, KG = j_K(G_1,G_2)} \operatorname{dist}^Q_f(K,G)\, M_{(G, Gj_K(D_1,D_2))}, \end{aligned}$$

where the vertex $K = S|T$ is fixed, and $\operatorname{dist}^Q_f(K,G)$ is as in (10.91).

We illustrate the product and coproduct for $r = 2$ with blue denoting color 1 and red denoting color 2.

$$\begin{aligned} M_{(sh|iv|a,s|h|i|v|a)} \mapsto 1 \otimes M_{(sh|iv|a,s|h|i|v|a)} + M_{(sh,s|h)} \otimes M_{(iv|a,i|v|a)} \\ + M_{(sh|iv,s|h|i|v)} \otimes M_{(a,a)} + M_{(sh|iv|a,s|h|i|v|a)} \otimes 1. \\ M_{(ha,h|a)} \otimes M_{(r|i,r|i)} \mapsto M_{(ha|r|i,h|a|r|i)} + M_{(har|i,h|a|r|i)} + q_{21}^2\, M_{(r|ha|i,r|h|a|i)} \\ + q_{21}^2\, M_{(r|hai,r|h|a|i)} + q_{21}^2 q_{11}^2\, M_{(r|i|ha,r|i|h|a)}. \end{aligned}$$

**Theorem 14.30.** *The antipode* $\mathrm{s}\colon \overrightarrow{\mathbf{\Sigma}}^*_Q \to \overrightarrow{\mathbf{\Sigma}}^*_Q$ *is given by*

$$\begin{aligned} &\overrightarrow{\mathbf{\Sigma}}^*_Q[I,f] \to \overrightarrow{\mathbf{\Sigma}}^*_Q[I,f] \\ &\mathrm{s}_{I,f}(M_{(G,D)}) = (-1)^{\deg(G)} \sum_{F:\, F \leq \overline{G}} \operatorname{dist}^Q_f(\overline{F},F)\, M_{(F,FD)}, \end{aligned}$$

*where the notation is as in Theorem 14.28.*

For example,

$$\begin{aligned} \mathrm{s}\big(M_{m|i\big|k|s|h\big|l|a}\big) = -M_{m|i|k|s|h|l|a} - q_{11}^3 q_{12}^3 q_{21}^2 q_{22}^2\, M_{k|s|h|l|a\big|m|i} \\ - q_{11}^4 q_{12}^6\, M_{l|a\big|m|i|k|s|h} - q_{11}^5 q_{12}^7 q_{21}^2 q_{22}^2\, M_{l|a\big|k|s|h\big|m|i}. \end{aligned}$$

The above result generalizes Theorem 12.34. It can also be seen as a special case of Theorem 14.19.

In view of (14.32), dualizing the above formulas and replacing $Q$ by $Q^t$ one obtains descriptions for the structure of the $Q$-Hopf monoid $\overrightarrow{\mathbf{\Sigma}}_Q$. They are as follows.

Fix a decomposition $(I,f) = (S,g) \sqcup (T,h)$. The coproduct is given by

$$\begin{aligned} &\overrightarrow{\mathbf{\Sigma}}_Q[I,f] \to \overrightarrow{\mathbf{\Sigma}}_Q[S,g] \otimes \overrightarrow{\mathbf{\Sigma}}_Q[T,h] \\ &H_{(F,C)} \mapsto \begin{cases} \operatorname{dist}^Q_f(F,K)\, H_{(F_1,C_1)} \otimes H_{(F_2,C_2)} & \text{if } K = S|T \text{ satisfies } FK \leq C, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $b_K(KF) = (F_1,F_2)$, $b_K(KC) = (C_1,C_2)$.

The product is given by

$$\begin{aligned} &\overrightarrow{\mathbf{\Sigma}}_Q[S,g] \otimes \overrightarrow{\mathbf{\Sigma}}_Q[T,h] \to \overrightarrow{\mathbf{\Sigma}}_Q[I,f] \\ &H_{(F_1,C_1)} \otimes H_{(F_2,C_2)} \mapsto H_{(j_K(F_1,F_2), j_K(C_1,C_2))}, \end{aligned}$$

where the vertex $K$ is defined to be $S|T$.

**Theorem 14.31.** *The antipode* $\mathrm{S}\colon \overrightarrow{\mathbf{\Sigma}}_Q \to \overrightarrow{\mathbf{\Sigma}}_Q$ *is given by*

$$\overrightarrow{\mathbf{\Sigma}}_Q[I,f] \to \overrightarrow{\mathbf{\Sigma}}_Q[I,f]$$
$$\mathrm{S}_{I,f}(H_{(F,C)}) = \mathrm{dist}^Q_f(F,\overline{F}) \sum_{G\colon F\leq \overline{G},\, FG\leq C} (-1)^{\deg(G)}\, H_{(G,GC)}.$$

**14.7.4. Relating the *Q*-Hopf monoids.** We mentioned earlier that many of the $Q$-Hopf monoids under consideration are values of the functors $\mathcal{T}_Q$ and $\mathcal{T}_Q^\vee$, and hence can be viewed as universal objects. We now state this formally.

Let $\mathbf{X}_{(r)}$, $\mathbf{E}_{(r)}$ and $\mathbf{L}_{(r)}$ be the images of $\mathbf{X}$, $\mathbf{E}$ and $\mathbf{L}$ under one of the bistrong functors of Proposition 14.4. Their duals, which we denote $\mathbf{X}^*_{(r)}$, $\mathbf{E}^*_{(r)}$ and $\mathbf{L}^*_{(r)}$, are the images of $\mathbf{X}^*$, $\mathbf{E}^*$ and $\mathbf{L}^*$.

**Proposition 14.32.** *There are isomorphisms of $Q$-Hopf monoids*

$$\begin{aligned}
&\mathbf{L}_Q = \mathcal{T}_Q(\mathbf{X}^*_{(r)}), && \mathbf{\Sigma}_Q = \mathcal{T}_Q\big((\mathbf{E}^*_{(r)})_+\big), && \overrightarrow{\mathbf{\Sigma}}_Q = \mathcal{T}_Q\big((\mathbf{L}^*_{(r)})_+\big)\\
&\mathbf{L}^*_Q = \mathcal{T}_Q^\vee\big(\mathbf{X}_{(r)}\big), && \mathbf{\Sigma}^*_Q = \mathcal{T}_Q^\vee\big((\mathbf{E}_{(r)})_+)\big), && \overrightarrow{\mathbf{\Sigma}}^*_Q = \mathcal{T}_Q^\vee\big((\mathbf{L}_{(r)})_+\big).
\end{aligned}$$

We now explain the inter-relationships between the $Q$-Hopf monoids of Table 14.3. The following result generalizes Theorem 12.64.

**Theorem 14.33.** *The following is a diagram of $Q$-Hopf monoids.*

$$\begin{tikzcd}
\mathbf{L}_Q \arrow[r] & \mathbf{\Sigma}_Q \arrow[r] & \overrightarrow{\mathbf{\Sigma}}_Q \arrow[r] & \mathbb{L}_Q \arrow[d, "s_Q"] \\
\mathbf{L}^*_Q & \mathbf{\Sigma}^*_Q \arrow[l] & \overrightarrow{\mathbf{\Sigma}}^*_Q \arrow[l] & \mathbb{L}^*_Q \arrow[l]
\end{tikzcd} \tag{14.34}$$

The map $s_Q$ is defined in (14.33). The remaining morphisms are essentially the same as in (12.20). These can also be seen as instances of the universal property of $\mathcal{T}_Q$ and $\mathcal{T}_Q^\vee$. Applying the duality functor to (14.34) yields the same diagram but with $Q$ replaced by $Q^t$. If $Q$ is symmetric, then diagram (14.34) is self-dual.

**Question 14.34.** If $Q$ is symmetric and no monomial on the entries of $Q$ is 1, then $\mathbf{L}_Q$ is self-dual (Proposition 14.14). A similar result for $\mathbb{L}_Q$ is given in Proposition 14.27. Are there similar results for $\mathbf{\Sigma}_Q$ and $\overrightarrow{\mathbf{\Sigma}}_Q$? A positive answer to this question in the one-dimensional case is given in Propositions 12.26 and 12.38.

# Part III

# Fock Functors

CHAPTER 15

# From Species to Graded Vector Spaces

Stover described how to construct graded Hopf algebras from Hopf monoids in species [346, Section 14]. These constructions were then discussed in more detail by Patras, Reutenauer, and Schocker [291, 292, 293]. In this chapter we formulate these constructions and study their properties in categorical terms.

In Section 15.1, we define four monoidal functors from species to graded vector spaces, namely, $\mathcal{K}$, $\mathcal{K}^\vee$, $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$. We prove that the first two are bilax and the remaining two are bistrong. We refer to all of them collectively by the term *Fock functors*. For further distinction, we refer to $\mathcal{K}$ and $\mathcal{K}^\vee$ as *full Fock functors*, and $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ as *bosonic Fock functors*. As suggested by the terminology, there are also bistrong functors $\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^\vee_{-1}$ called *fermionic Fock functors*. These will be introduced and studied in Chapter 16. The Fock functors with their notations are summarized in Table 15.1. The motivation for our terminology comes from classical Fock spaces. These are treated in Chapter 19; see in particular Tables 19.1 and 19.2.

The monoidal properties of the Fock functors imply that the image of a Hopf monoid in species under any Fock functor is a graded Hopf algebra. This is how the categorical framework relates to Stover's constructions. This is explained in Section 15.2.

The categorical approach allows us to reduce the study of the relation between various properties of Hopf monoids and properties of the corresponding Hopf algebras, such as commutativity, duality, antipode, and primitive elements, to the study of general properties of the Fock functors. This is pursued in later sections of this chapter.

## 15.1. The Fock bilax monoidal functors

Recall that $(\mathsf{Sp}, \cdot)$ and $(\mathsf{gVec}, \cdot)$ denote respectively the categories of species and graded vector spaces under the Cauchy product. In this section, we first construct two bilax monoidal functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ from $(\mathsf{Sp}, \cdot)$ to $(\mathsf{gVec}, \cdot)$ and a natural transformation between them which is compatible with the bilax monoidal structure. We also relate them in a different manner via the linear order species.

TABLE 15.1. The Fock functors.

| Fock functor | Name |
|---|---|
| $\mathcal{K}$, $\mathcal{K}^\vee$ | Full Fock functor |
| $\overline{\mathcal{K}}$, $\overline{\mathcal{K}}^\vee$ | Bosonic Fock functor |
| $\overline{\mathcal{K}}_{-1}$, $\overline{\mathcal{K}}^\vee_{-1}$ | Fermionic Fock functor |

We also discuss two other bilax monoidal functors, namely $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$, which, as suggested by the notation, are related to $\mathcal{K}$ and $\overline{\mathcal{K}}$ through duality.

### 15.1.1. The bilax monoidal functors $\mathcal{K}$ and $\overline{\mathcal{K}}$.

**Definition 15.1.** Let

$$\mathcal{K}, \overline{\mathcal{K}} \colon \mathsf{Sp} \to \mathsf{gVec}$$

be the functors defined by

$$\mathcal{K}(\mathbf{q}) := \bigoplus_{n \geq 0} \mathbf{q}[n] \qquad \text{and} \qquad \overline{\mathcal{K}}(\mathbf{q}) := \bigoplus_{n \geq 0} \mathbf{q}[n]_{\mathrm{S}_n},$$

where $\mathbf{q}[n]_{\mathrm{S}_n}$ is the vector space of $\mathrm{S}_n$-coinvariants of $\mathbf{q}[n]$.

The quotient maps $\mathcal{K}(\mathbf{q}) \twoheadrightarrow \overline{\mathcal{K}}(\mathbf{q})$ define a natural transformation $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$, because a morphism of species $\mathbf{p} \to \mathbf{q}$ yields maps of $\mathrm{S}_n$-modules $\mathbf{p}[n] \to \mathbf{q}[n]$, which therefore factor through coinvariants.

We proceed to turn $\mathcal{K}$ into a bilax monoidal functor with respect to the Cauchy product on graded vector spaces (2.2) and species (8.6), that is,

$$\mathcal{K} \colon (\mathsf{Sp}, \cdot, \beta) \to (\mathsf{gVec}, \cdot, \beta).$$

Define maps

$$\begin{tikzcd} \mathcal{K}(\mathbf{p}) \cdot \mathcal{K}(\mathbf{q}) \arrow[r, shift left, "\varphi_{\mathbf{p},\mathbf{q}}"] & \mathcal{K}(\mathbf{p} \cdot \mathbf{q}) \arrow[l, shift left, "\psi_{\mathbf{p},\mathbf{q}}"] \end{tikzcd}$$

as follows. On the degree $n$ components of these graded vector spaces, we define maps

$$\begin{tikzcd} \displaystyle\bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[t] \arrow[r, shift left, "\varphi_{\mathbf{p},\mathbf{q}}"] & \displaystyle\bigoplus_{S \sqcup T = [n]} \mathbf{p}[S] \otimes \mathbf{q}[T] \arrow[l, shift left, "\psi_{\mathbf{p},\mathbf{q}}"] \end{tikzcd}$$

as the direct sum of the following maps:

$$\varphi_{\mathbf{p},\mathbf{q}} \colon \mathbf{p}[s] \otimes \mathbf{q}[t] \xrightarrow{\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}]} \mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t]$$
$$\psi_{\mathbf{p},\mathbf{q}} \colon \mathbf{p}[S] \otimes \mathbf{q}[T] \xrightarrow{\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}]} \mathbf{p}[|S|] \otimes \mathbf{q}[|T|],$$

with notations as defined in Notation 2.5. Note that the composite $\psi_{\mathbf{p},\mathbf{q}} \varphi_{\mathbf{p},\mathbf{q}}$ is the identity, but in general these maps are not invertible on the degree $n$ component.

It is also clear that $\mathcal{K}(\mathbf{1}) = \Bbbk$; hence $\mathcal{K}$ takes the unit object in $(\mathsf{Sp}, \cdot)$ to the unit object in $(\mathsf{gVec}, \cdot)$. We define $\varphi_0$ and $\psi_0$ to be the identity maps

$$\begin{tikzcd} \Bbbk \arrow[r, shift left, "\varphi_0"] & \mathcal{K}(\mathbf{1}). \arrow[l, shift left, "\psi_0"] \end{tikzcd}$$

We show in Theorem 15.3 below that $(\mathcal{K}, \varphi, \psi)$ is a bilax monoidal functor.

For the bilax structure of $\overline{\mathcal{K}}$, we define the maps $\overline{\varphi}$ and $\overline{\psi}$ by the commutativity of the diagram below.

$$\begin{tikzcd}
\mathcal{K}(\mathbf{p}) \cdot \mathcal{K}(\mathbf{q}) \arrow[r, shift left, "\varphi_{\mathbf{p},\mathbf{q}}"] \arrow[d, two heads] & \mathcal{K}(\mathbf{p} \cdot \mathbf{q}) \arrow[l, shift left, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, two heads] \\
\overline{\mathcal{K}}(\mathbf{p}) \cdot \overline{\mathcal{K}}(\mathbf{q}) \arrow[r, shift left, dashed, "\overline{\varphi}_{\mathbf{p},\mathbf{q}}"] & \overline{\mathcal{K}}(\mathbf{p} \cdot \mathbf{q}) \arrow[l, shift left, dashed, "\overline{\psi}_{\mathbf{p},\mathbf{q}}"]
\end{tikzcd} \tag{15.1}$$

**Proposition 15.2.** *The maps $\overline{\varphi}$ and $\overline{\psi}$ are well-defined and inverses of each other.*

PROOF. For the map $\overline{\psi}$, we need to consider the diagram

$$\begin{tikzcd}
\displaystyle\bigoplus_{S \sqcup T=[n]} \mathbf{p}[S] \otimes \mathbf{q}[T] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"] \arrow[d, two heads] & \displaystyle\bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[t] \arrow[d, two heads] \\
\Big(\displaystyle\bigoplus_{S \sqcup T=[n]} \mathbf{p}[S] \otimes \mathbf{q}[T]\Big)_{\mathrm{S}_n} \arrow[r, dashed] & \displaystyle\bigoplus_{s+t=n} \mathbf{p}[s]_{\mathrm{S}_s} \otimes \mathbf{q}[t]_{\mathrm{S}_t}
\end{tikzcd}$$

and show that the bottom horizontal map is well defined.

Let $\sigma \in \mathrm{S}_n$ be any permutation. For $S \sqcup T = [n]$, say $\sigma$ sends $S$ to $U$ and $T$ to $V$. This defines bijections $\sigma_1' \colon S \to U$ and $\sigma_2' \colon T \to V$. By standardizing the sets $S$, $T$, $U$ and $V$, we obtain two permutations $\sigma_1 \in \mathrm{S}_s$ and $\sigma_2 \in \mathrm{S}_t$, defined by the commutative diagrams

$$\begin{tikzcd}
S \arrow[r, "\mathrm{cano}"] \arrow[d, "\sigma_1'"'] & {[s]} \arrow[d, "\sigma_1"] \\
U \arrow[r, "\mathrm{cano}"'] & {[s]}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T \arrow[r, "\mathrm{cano}"] \arrow[d, "\sigma_1'"'] & {[t]} \arrow[d, "\sigma_2"] \\
V \arrow[r, "\mathrm{cano}"'] & {[t]}
\end{tikzcd}$$

where $s = |S| = |U|$ and $t = |T| = |V|$:

By functoriality, we then obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{p}[S] \otimes \mathbf{q}[T] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"] \arrow[d, "{\mathbf{p}[\sigma_1']\otimes\mathbf{q}[\sigma_2']}"'] & \mathbf{p}[s] \otimes \mathbf{q}[t] \arrow[d, "{\mathbf{p}[\sigma_1]\otimes\mathbf{q}[\sigma_2]}"] \\
\mathbf{p}[U] \otimes \mathbf{q}[V] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"'] & \mathbf{p}[s] \otimes \mathbf{q}[t].
\end{tikzcd}$$

This guarantees that $\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}]$ factors through coinvariants.

The argument for the map $\overline{\varphi}$ is similar. From $\psi_{\mathbf{p},\mathbf{q}}\varphi_{\mathbf{p},\mathbf{q}} = \mathrm{id}$ we deduce $\overline{\psi}_{\mathbf{p},\mathbf{q}}\overline{\varphi}_{\mathbf{p},\mathbf{q}} = \mathrm{id}$. The map $\varphi_{\mathbf{p},\mathbf{q}}\psi_{\mathbf{p},\mathbf{q}}$ is given by permutation actions, so it induces the identity map on coinvariants. Thus, $\overline{\varphi}$ and $\overline{\psi}$ are inverses. $\square$

**Theorem 15.3.** *The full Fock functor $(\mathcal{K}, \psi, \varphi)$ is bilax monoidal, the bosonic Fock functor $(\overline{\mathcal{K}}, \overline{\psi}, \overline{\varphi})$ is bistrong monoidal, and the transformation $\overline{\mathcal{K}} \Rightarrow \mathcal{K}$ is a morphism of bilax monoidal functors.*

Proof. We start by showing that $(\mathcal{K}, \varphi)$ is lax and $(\mathcal{K}, \psi)$ is colax by checking that $\varphi$ and $\psi$ satisfy the conditions described in Definitions 3.1 and 3.2.

*Naturality.* Let $f \colon \mathbf{p} \to \mathbf{p}'$ and $g \colon \mathbf{q} \to \mathbf{q}'$ be morphisms of species. For the naturality of $\psi$, the diagram

$$\begin{tikzcd}
\mathbf{p}[S] \otimes \mathbf{q}[T] \arrow[r, "{\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}]}"] \arrow[d, "f_S \otimes g_T"'] & \mathbf{p}[|S|] \otimes \mathbf{q}[|T|] \arrow[d, "f_{[|S|]} \otimes g_{[|T|]}"] \\
\mathbf{p}'[S] \otimes \mathbf{q}'[T] \arrow[r, "{\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}]}"'] & \mathbf{p}'[|S|] \otimes \mathbf{q}'[|T|]
\end{tikzcd}$$

must commute. This follows from the naturality of $f$ and $g$. The argument for the naturality of $\varphi$ is similar.

*Associativity.* If we follow the two directions in diagram (3.5) and its dual, the maps $\varphi$ and $\psi$ yield the following two unambiguous maps respectively

$$\mathbf{p}[s] \otimes \mathbf{q}[t] \otimes \mathbf{r}[u] \xrightarrow{\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}] \otimes \mathbf{r}[\mathrm{cano}]} \mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t] \otimes \mathbf{r}[s+t+1, s+t+u]$$
$$\mathbf{p}[S] \otimes \mathbf{q}[T] \otimes \mathbf{r}[U] \xrightarrow{\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}] \otimes \mathbf{r}[\mathrm{cano}]} \mathbf{p}[|S|] \otimes \mathbf{q}[|T|] \otimes \mathbf{r}[|U|];$$

hence they are associative.

*Unitality.* It is trivial to check that the unitality diagrams in (3.6) and their duals commute. All the maps in these diagrams are isomorphisms.

This shows that $(\mathcal{K}, \varphi)$ is lax and $(\mathcal{K}, \psi)$ is colax. We now check the braiding and unitality axioms in Definition 3.3.

*Braiding.* If we follow the two directions in diagram (3.11), the maps $\varphi$ and $\psi$ yield an unambiguous map

$$\mathcal{K}(\mathbf{p} \cdot \mathbf{q}) \cdot \mathcal{K}(\mathbf{r} \cdot \mathbf{s}) \to \mathcal{K}(\mathbf{p} \cdot \mathbf{r}) \cdot \mathcal{K}(\mathbf{q} \cdot \mathbf{s})$$

defined as follows.

Let $A$, $B$, $C$ and $D$ be sets such that $A \sqcup B = [m]$ and $C \sqcup D = [n]$, and $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$. The cano maps induce an isomorphism

$$(\mathbf{p}[A] \otimes \mathbf{q}[B]) \otimes (\mathbf{r}[C] \otimes \mathbf{s}[D]) \to (\mathbf{p}[a] \otimes \mathbf{r}[a+1, a+c]) \otimes (\mathbf{q}[b] \otimes \mathbf{s}[b+1, b+d]).$$

Varying $A$, $B$, $C$ and $D$, and then $m$ and $n$, and then taking direct sum gives the above map.

*Unitality.* It is trivial to check that the unitality diagrams in (3.12) and (3.13) commute. All the maps in these diagrams are isomorphisms.

This completes the proof that $(\mathcal{K}, \varphi, \psi)$ is bilax. The remaining claims follow immediately. The surjectivity of the vertical maps in diagram (15.1) and $(\mathcal{K}, \varphi, \psi)$ being bilax implies that $(\overline{\mathcal{K}}, \overline{\varphi}, \overline{\psi})$ is bilax. Since $\overline{\varphi}$ and $\overline{\psi}$ are inverse isomorphisms, we moreover have that $(\overline{\mathcal{K}}, \overline{\varphi}, \overline{\psi})$ is bistrong. The fact that $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ is a morphism of bilax functors follows by construction. $\square$

**Remark 15.4.** In Section 15.2 we employ the above result to construct graded Hopf algebras from Hopf monoids in $(\mathsf{Sp}, \cdot)$. The functor $\mathcal{K}$ is not bistrong. Since it is normal (it satisfies $\varphi_0\psi_0 = \mathrm{id}$), it cannot be Hopf lax either, by Proposition 3.60. Nevertheless, the image of a Hopf monoid under $\mathcal{K}$ is always a graded Hopf algebra (Theorem 15.12).

**15.1.2. The bilax monoidal functors $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$.** There are two other functors from species to graded vector spaces, namely the contragredients of $\mathcal{K}$ and $\overline{\mathcal{K}}$ (Section 3.10.4). We begin by describing them explicitly.

**Definition 15.5.** Let

$$\mathcal{K}^\vee, \overline{\mathcal{K}}^\vee : \mathsf{Sp} \to \mathsf{gVec}$$

be the functors defined by

$$\mathcal{K}^\vee(\mathbf{q}) := \bigoplus_{n\geq 0} \mathbf{q}[n] \qquad \text{and} \qquad \overline{\mathcal{K}}^\vee(\mathbf{q}) := \bigoplus_{n\geq 0} \mathbf{q}[n]^{\mathrm{S}_n},$$

where $\mathbf{q}[n]^{\mathrm{S}_n}$ is the vector space of $\mathrm{S}_n$-invariants of $\mathbf{q}[n]$.

The inclusion maps $\overline{\mathcal{K}}^\vee(\mathbf{q}) \hookrightarrow \mathcal{K}^\vee(\mathbf{q})$ define a natural transformation of functors $\overline{\mathcal{K}}^\vee \Rightarrow \mathcal{K}^\vee$.

We proceed to turn $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$ into bilax monoidal functors $(\mathsf{Sp}, \cdot, \beta) \to (\mathsf{gVec}, \cdot, \beta)$. For the bilax structure of $\mathcal{K}^\vee$, we define maps

$$\bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[t] \underset{\varphi^\vee_{\mathbf{p},\mathbf{q}}}{\overset{\psi^\vee_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} \bigoplus_{S\sqcup T=[n]} \mathbf{p}[S] \otimes \mathbf{q}[T] \tag{15.2}$$

as follows. The lax structure map $\psi^\vee_{\mathbf{p},\mathbf{q}}$ is the direct sum of the following maps, one for each $s, t$ and each summand in the target with $|S| = s$ and $|T| = t$:

$$\mathbf{p}[s] \otimes \mathbf{q}[t] \xrightarrow{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]} \mathbf{p}[S] \otimes \mathbf{q}[T]$$

The colax structure map $\varphi^\vee_{\mathbf{p},\mathbf{q}}$ is the direct sum of the following maps:

$$\mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t] \xrightarrow{\mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}]} \mathbf{p}[s] \otimes \mathbf{q}[t]$$

On the components for which $S \neq [s]$ (and $T \neq [s+1, s+t]$), the map $\varphi^\vee_{\mathbf{p},\mathbf{q}}$ is zero. Note that the composite $\varphi^\vee_{\mathbf{p},\mathbf{q}}\psi^\vee_{\mathbf{p},\mathbf{q}}$ is the identity but in general these maps are not invertible.

The structure maps of $\mathcal{K}^\vee$ restrict to invariants, as indicated below.

$$\begin{tikzcd}
\mathcal{K}^\vee(\mathbf{p}) \cdot \mathcal{K}^\vee(\mathbf{q}) \arrow[rr, shift left, "\psi^\vee_{\mathbf{p},\mathbf{q}}"] & & \mathcal{K}^\vee(\mathbf{p} \cdot \mathbf{q}) \arrow[ll, shift left, "\varphi^\vee_{\mathbf{p},\mathbf{q}}"] \\
\overline{\mathcal{K}}^\vee(\mathbf{p}) \cdot \overline{\mathcal{K}}^\vee(\mathbf{q}) \arrow[u, hook] \arrow[rr, shift left, dashed, "\overline{\psi}^\vee_{\mathbf{p},\mathbf{q}}"] & & \overline{\mathcal{K}}^\vee(\mathbf{p} \cdot \mathbf{q}) \arrow[u, hook] \arrow[ll, shift left, dashed, "\overline{\varphi}^\vee_{\mathbf{p},\mathbf{q}}"]
\end{tikzcd} \tag{15.3}$$

**Theorem 15.6.** *The full Fock functor $(\mathcal{K}^\vee, \psi^\vee, \varphi^\vee)$ is bilax monoidal, the bosonic Fock functor $(\overline{\mathcal{K}}^\vee, \overline{\psi}^\vee, \overline{\varphi}^\vee)$ is bistrong monoidal, and the transformation $\overline{\mathcal{K}}^\vee \Rightarrow \mathcal{K}^\vee$ is a morphism of bilax monoidal functors.*

The proof is similar to that of Theorem 15.3. As for $\mathcal{K}$, the functor $\mathcal{K}^\vee$ is not Hopf lax.

**Remark 15.7.** A species $\mathbf{q}$ being a functor $\mathsf{Set}^\times \to \mathsf{Vec}$ and the category $\mathsf{Vec}$ being complete and cocomplete, one may consider the limit and colimit of the functor $\mathbf{q}$ (Section A.3.5). We have

$$\operatorname{colim}\mathbf{q} = \bigoplus_{n\geq 0}\mathbf{q}[n]_{\mathrm{S}_n} \qquad \text{and} \qquad \lim\mathbf{q} = \prod_{n\geq 0}\mathbf{q}[n]^{\mathrm{S}_n}.$$

The former is the graded vector space $\overline{\mathcal{K}}(\mathbf{q})$ while the latter is a completion of the graded vector space $\overline{\mathcal{K}}^\vee(\mathbf{q})$. This "explains" why the functors $\overline{\mathcal{K}}(\mathbf{q})$ and $\overline{\mathcal{K}}^\vee(\mathbf{q})$ are better behaved than their counterparts $\mathcal{K}$ and $\mathcal{K}^\vee$ (as we will see).

**15.1.3. Relating the Fock functors.** We now show that in the finite-dimensional case, $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$ are indeed the (bilax) contragredients of $\mathcal{K}$ and $\overline{\mathcal{K}}$. This construction is discussed in Proposition 3.102. Note that $\psi^\vee$ stands for the lax structure and $\varphi^\vee$ stands for the colax structure of $\mathcal{K}^\vee$, as per the general notation in the contragredient construction.

**Proposition 15.8.** *On finite-dimensional species, the bilax functors* $(\mathcal{K}^\vee, \psi^\vee, \varphi^\vee)$ *and* $(\overline{\mathcal{K}}^\vee, \overline{\psi}^\vee, \overline{\varphi}^\vee)$ *are respectively isomorphic to the contragredients of* $(\mathcal{K}, \varphi, \psi)$ *and* $(\overline{\mathcal{K}}, \overline{\varphi}, \overline{\psi})$.

Proof. The contragredient of the functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ are the composites

$$\mathsf{Sp} \xrightarrow{(-)^*} \mathsf{Sp} \xrightarrow{\mathcal{K}} \mathsf{gVec} \xrightarrow{(-)^*} \mathsf{gVec}$$
$$\mathsf{Sp} \xrightarrow{(-)^*} \mathsf{Sp} \xrightarrow{\overline{\mathcal{K}}} \mathsf{gVec} \xrightarrow{(-)^*} \mathsf{gVec}$$

where the arrows labeled $(-)^*$ denote the duality functors on species and graded vector spaces. First note that there are canonical isomorphisms

$$\mathcal{K}^\vee(\mathbf{q}) \cong \mathcal{K}(\mathbf{q}^*)^* \qquad \text{and} \qquad \overline{\mathcal{K}}^\vee(\mathbf{q}) \cong \overline{\mathcal{K}}(\mathbf{q}^*)^* \tag{15.4}$$

given by the canonical identification $\mathbf{q}[n] \cong (\mathbf{q}[n]^*)^*$. Under this identification, one easily checks that

$$\psi^\vee_{\mathbf{p},\mathbf{q}} = (\psi_{\mathbf{p}^*,\mathbf{q}^*})^*, \quad \varphi^\vee_{\mathbf{p},\mathbf{q}} = (\varphi_{\mathbf{p}^*,\mathbf{q}^*})^*, \quad \overline{\psi}^\vee_{\mathbf{p},\mathbf{q}} = (\overline{\psi}_{\mathbf{p}^*,\mathbf{q}^*})^*, \quad \text{and} \quad \overline{\varphi}^\vee_{\mathbf{p},\mathbf{q}} = (\overline{\varphi}_{\mathbf{p}^*,\mathbf{q}^*})^*.$$

The right-hand sides are precisely the bilax structures of the contragredients (Proposition 3.102). The result follows. □

Note that the functors $\mathcal{K}^\vee$ and $\mathcal{K}$ coincide; however, their bilax structures are defined differently. We will see later that, in fact, they cannot be isomorphic as bilax functors (Example 15.17). In general, in the finite-dimensional case, properties of the functors $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$ can be derived from the corresponding properties of $\mathcal{K}$ and $\overline{\mathcal{K}}$ by means of Proposition 15.8 (and viceversa). For example, Theorems 15.3 and 15.6 imply each other.

Now let $\mathbf{L}$ be the Hopf monoid of linear orders (Example 8.16). The functor $\mathcal{K}$ can be expressed in terms of the functor $\overline{\mathcal{K}}$ and the Hopf monoid $\mathbf{L}$. To see this, consider the composite of functors

$$\mathsf{Sp} \xrightarrow{\mathbf{L}\times(-)} \mathsf{Sp} \xrightarrow{\overline{\mathcal{K}}} \mathsf{gVec}. \tag{15.5}$$

Since $\mathbf{L}$ is a bimonoid, the functor $\mathbf{L} \times (-)$ can be viewed as a bilax monoidal functor, as in Proposition 8.66. Now, since both the above functors are bilax, so is the composite, by Theorem 3.22.

**Proposition 15.9.** *There is an isomorphism of bilax monoidal functors*

$$\mathcal{K} \cong \overline{\mathcal{K}}\big(\mathbf{L} \times (-)\big). \tag{15.6}$$

Proof. Given a species $\mathbf{p}$, define a map of graded vector spaces

$$\mathcal{K}(\mathbf{p}) \to \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{p})$$

with components

$$\mathbf{p}[n] \to (\mathbf{L}[n] \otimes \mathbf{p}[n])_{\mathrm{S}_n}, \qquad\qquad x \mapsto \overline{C_{(n)} \otimes x},$$

where $C_{(n)} = 1|\cdots|n$ is the canonical linear order on $[n]$ and the overline denotes the projection to coinvariants.

Since $\mathbf{L}[n]$ is the regular representation of $\mathrm{S}_n$, by Lemma 2.18, this defines a natural isomorphism of functors. We need to show that it is a morphism of bilax monoidal functors. We check below that the colax structures are preserved; the verification for the lax structures is similar.

According to Theorem 3.22, the colax structure of $\overline{\mathcal{K}}\big(\mathbf{L} \times (-)\big)$ is given by the composite

$$\begin{array}{ccc}
\overline{\mathcal{K}}\big(\mathbf{L} \times (\mathbf{p} \cdot \mathbf{q})\big) \xrightarrow{\overline{\mathcal{K}}(\Delta \times \mathrm{id})} \overline{\mathcal{K}}\big((\mathbf{L} \cdot \mathbf{L}) \times (\mathbf{p} \cdot \mathbf{q})\big) & & \\
\Big\downarrow & & \\
\overline{\mathcal{K}}\big((\mathbf{L} \times \mathbf{p}) \cdot (\mathbf{L} \times \mathbf{q})\big) & \xrightarrow{\overline{\psi}} & \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{p}) \cdot \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{q}).
\end{array}$$

The map $\Delta$ is the coproduct of $\mathbf{L}$ as in Example 8.16 and the vertical map is $\overline{\mathcal{K}}$ applied to the colax structure of the Hadamard functor as in (8.73).

Take $x \in \mathbf{p}[S]$ and $y \in \mathbf{q}[T]$ with $S \sqcup T = [n]$. Applying the above sequence of maps to the element $\overline{C_{(n)} \otimes x \otimes y}$, we obtain

$$\begin{aligned}
\overline{C_{(n)} \otimes x \otimes y} &\mapsto \sum_{S_1 \sqcup T_1 = [n]} \overline{C_{(n)}|_{S_1} \otimes C_{(n)}|_{T_1} \otimes x \otimes y} \\
&\mapsto \overline{C_{(n)}|_S \otimes x \otimes C_{(n)}|_T \otimes y} \\
&\mapsto \overline{C_{(s)} \otimes \mathrm{cano}(x)} \otimes \overline{C_{(t)} \otimes \mathrm{cano}(y)}.
\end{aligned}$$

This matches the colax structure $\psi$ of $\mathcal{K}$. ☐

**Proposition 15.10.** *There is an isomorphism of bilax monoidal functors*

$$\mathcal{K}^{\vee} \cong \overline{\mathcal{K}}^{\vee}\big(\mathbf{L}^* \times (-)\big). \tag{15.7}$$

This may be proved directly, as in Proposition 15.9. In the finite-dimensional case, it also follows by applying the contragredient construction to (15.6):

$$\mathcal{K}^{\vee} \cong \overline{\mathcal{K}}^{\vee}\big(\big(\mathbf{L} \times (-)\big)^{\vee}\big) \cong \overline{\mathcal{K}}^{\vee}\big(\mathbf{L}^* \times (-)\big).$$

The last isomorphism follows as in (8.82).

**Remark 15.11.** There is an additional relation between the Fock functors: over a field of characteristic 0, the bilax monoidal functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee}$ are isomorphic. We show this in Proposition 15.21. However, over a field of positive characteristic they differ. The bilax monoidal functors $\mathcal{K}$ and $\mathcal{K}^{\vee}$ differ regardless of the field characteristic. These claims are justified in Section 15.3. There is nevertheless a natural transformation $\kappa\colon \mathcal{K} \Rightarrow \mathcal{K}^{\vee}$. This is studied in Section 15.4, where we explain how one may view the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee}$ as the coimage and the image of this transformation (when the field characteristic is 0); see Section 15.4.3. From this point of view, the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee}$ are determined by $\mathcal{K}$ and $\mathcal{K}^{\vee}$. On the other hand, (15.6) and (15.7) say that the latter are determined by the former also.

## 15.2. From Hopf monoids to Hopf algebras: Stover's constructions

In this section, we provide a categorical framework for Stover's constructions. We then explain how the different constructions relate to one another. Some elementary but illustrative examples will be given in the next section.

**15.2.1. Evaluating the Fock functors on Hopf monoids.** We apply general results on bilax functors to the bilax functors in Section 15.1 to construct graded Hopf algebras starting with Hopf monoids in species.

**Theorem 15.12.** *If* $\mathbf{h}$ *is a Hopf monoid in species, then* $\mathcal{K}(\mathbf{h}), \overline{\mathcal{K}}(\mathbf{h}), \mathcal{K}^{\vee}(\mathbf{h})$, *and* $\overline{\mathcal{K}}^{\vee}(\mathbf{h})$ *are graded Hopf algebras. This defines four functors from the category of Hopf monoids in species to the category of graded Hopf algebras.*

Proof. According to Theorems 15.3 and 15.6, the Fock functors are bilax monoidal. Hence, Proposition 3.31 implies that their values on $\mathbf{h}$ are graded bialgebras, and that the assignments are functorial. We need to check that they are Hopf algebras. The statement for $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee}$ follows from Proposition 3.50, since they are bistrong monoidal functors. The functors $\mathcal{K}$ and $\mathcal{K}^{\vee}$ are not bistrong, not even Hopf lax; so a separate argument is needed. We give it below for $\mathcal{K}$; the same argument applies to $\mathcal{K}^{\vee}$.

Since $\mathcal{K}(\mathbf{h})$ is a graded bialgebra, it is enough to show that $\mathcal{K}(\mathbf{h})_0$ is a Hopf algebra (Section 2.3.2). Note that $\mathcal{K}(\mathbf{h})_0$ is a subbialgebra of $\mathcal{K}(\mathbf{h})$. Further, by definition, the product and coproduct on $\mathcal{K}(\mathbf{h})_0$ are the $\emptyset$-component of the product and coproduct of $\mathbf{h}$. In other words, $\mathcal{K}(\mathbf{h})_0 = \mathbf{h}[\emptyset]$ as bialgebras. But recall that for any Hopf monoid $\mathbf{h}$, $\mathbf{h}[\emptyset]$ is a Hopf algebra (Proposition 8.10), so we are done. $\square$

Let $\mu$, $\iota$, $\Delta$, and $\epsilon$ be the structure maps of a Hopf monoid $\mathbf{h}$ in species. According to Proposition 3.31, the graded Hopf algebra $\mathcal{K}(\mathbf{h})$ of Theorem 15.12 has structure maps

$$\mathcal{K}(\mathbf{h}) \cdot \mathcal{K}(\mathbf{h}) \xrightarrow{\varphi_{\mathbf{h},\mathbf{h}}} \mathcal{K}(\mathbf{h} \cdot \mathbf{h}) \xrightarrow{\mathcal{K}(\mu)} \mathcal{K}(\mathbf{h})$$
$$\Bbbk \xrightarrow{\varphi_0} \mathcal{K}(\mathbf{1}) \xrightarrow{\mathcal{K}(\iota)} \mathcal{K}(\mathbf{h})$$
$$\mathcal{K}(\mathbf{h}) \xrightarrow{\mathcal{K}(\Delta)} \mathcal{K}(\mathbf{h} \cdot \mathbf{h}) \xrightarrow{\psi_{\mathbf{h},\mathbf{h}}} \mathcal{K}(\mathbf{h}) \cdot \mathcal{K}(\mathbf{h})$$
$$\mathcal{K}(\mathbf{h}) \xrightarrow{\mathcal{K}(\epsilon)} \mathcal{K}(\mathbf{1}) \xrightarrow{\psi_0} \Bbbk.$$

More explicitly, the component of degree $n$ of the coproduct of $\mathcal{K}(\mathbf{h})$ is the composite

$$\mathbf{h}[n] \xrightarrow{\quad \sum \Delta_{S,T} \quad} \bigoplus_{S \sqcup T=[n]} \mathbf{h}[S] \otimes \mathbf{h}[T] \xrightarrow{\sum \mathbf{h}[\mathrm{cano}] \otimes \mathbf{h}[\mathrm{cano}]} \bigoplus_{s+t=n} \mathbf{h}[s] \otimes \mathbf{h}[t]$$

and similarly for the other structure maps. Recall the notions of shifting and standardization from Notation 2.5. The above expression shows that the coproduct of $\mathcal{K}(\mathbf{h})$ always involves the notion of standardization. Similarly, the product of $\mathcal{K}(\mathbf{h})$ always involve the notions of shifting. Thus we obtain a conceptual explanation for the occurrence of these combinatorial procedures in the definition of several Hopf algebras of prominence in combinatorics, including all the Hopf algebras in [12, Theorem 6.1.3].

This construction of the graded bialgebra $\mathcal{K}(\mathbf{h})$ appears for the first time in work of Stover [346, Proposition 14.6.i], without reference to monoidal functors. Similarly, the other bialgebras in [346, Proposition 14.6.ii-iv] are the images of $\mathbf{h}$ under the bilax monoidal functors $\mathcal{K}^\vee$, $\overline{\mathcal{K}}$, and $\overline{\mathcal{K}}^\vee$, with the structure afforded by Proposition 3.31.

Stover's constructions have been taken up by Patras and Reutenauer [291] and Patras and Schocker [292, 293]. In these references, $\mathcal{K}(\mathbf{h})$ and $\mathcal{K}^\vee(\mathbf{h})$ are called the *cosymmetrized* and *symmetrized bialgebras* associated to $\mathbf{h}$, respectively [292, Definition 8 and Proposition 15]. Stover mentions that these constructions admit two versions, one with signs and the other without [346, Section 14.7]. Patras et al consider the unsigned version (as we do presently). The signed case will be dealt in Chapter 16.

If $\mathrm{s}$ is the antipode of $\mathbf{h}$, then $\overline{\mathcal{K}}(\mathrm{s})$ and $\overline{\mathcal{K}}^\vee(\mathrm{s})$ are the antipodes of $\overline{\mathcal{K}}(\mathbf{h})$ and $\overline{\mathcal{K}}^\vee(\mathbf{h})$, according to Proposition 3.50. On the other hand, since the functors $\mathcal{K}$ and $\mathcal{K}^\vee$ are not bistrong (not even Hopf lax; see Remark 15.4), the antipodes of $\mathcal{K}(\mathbf{h})$ and $\mathcal{K}^\vee(\mathbf{h})$ are not directly related to that of $\mathbf{h}$. Determining the antipodes of these Hopf algebras in explicit terms is often a challenging problem. We do not address this problem in this monograph.

**15.2.2. Relating the values of the Fock functors.** We now discuss various relations between the Hopf algebras constructed in Theorem 15.12.

Recall that $\mathbf{L}$ denotes the Hopf monoid of linear orders. For any species $\mathbf{h}$, there are canonical identifications

$$\mathbf{h}[n] \cong (\mathbf{L}[n] \otimes \mathbf{h}[n])_{\mathrm{S}_n} \cong (\mathbf{L}^*[n] \otimes \mathbf{h}[n])^{\mathrm{S}_n}. \tag{15.8}$$

Recall that the Hadamard product of Hopf monoids is another Hopf monoid (Corollary 8.59).

**Theorem 15.13.** *Let $\mathbf{h}$ be a Hopf monoid in species. There are natural isomorphisms of graded Hopf algebras*

$$\mathcal{K}(\mathbf{h}) \cong \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{h}) \qquad \text{and} \qquad \mathcal{K}^\vee(\mathbf{h}) \cong \overline{\mathcal{K}}^\vee(\mathbf{L}^* \times \mathbf{h}) \tag{15.9}$$

*given by the identifications (15.8). The maps*

$$\mathcal{K}(\mathbf{h}) \twoheadrightarrow \overline{\mathcal{K}}(\mathbf{h}) \qquad \text{and} \qquad \overline{\mathcal{K}}^\vee(\mathbf{h}) \hookrightarrow \mathcal{K}^\vee(\mathbf{h}) \tag{15.10}$$

*are natural morphisms of graded Hopf algebras. Further, if $\mathbf{h}$ is finite-dimensional, there are natural isomorphisms of graded Hopf algebras*

$$\mathcal{K}^\vee(\mathbf{h}) \cong \mathcal{K}(\mathbf{h}^*)^* \qquad \text{and} \qquad \overline{\mathcal{K}}^\vee(\mathbf{h}) \cong \overline{\mathcal{K}}(\mathbf{h}^*)^* \tag{15.11}$$

*given by the canonical identification* $\mathbf{h}[n] \cong (\mathbf{h}[n]^*)^*$.

PROOF. According to Theorems 15.3 and 15.6 and Proposition 15.8, the maps

$$\mathcal{K} \Rightarrow \overline{\mathcal{K}}, \quad \overline{\mathcal{K}}^\vee \Rightarrow \mathcal{K}^\vee, \quad \mathcal{K}^\vee \Rightarrow \mathcal{K}\big((-)^*\big)^* \quad \text{and} \quad \overline{\mathcal{K}}^\vee \Rightarrow \overline{\mathcal{K}}\big((-)^*\big)^*$$

are morphisms of bilax monoidal functors. Therefore, Proposition 3.32 implies that (15.11) and (15.10) are isomorphisms of graded bialgebras (hence also of Hopf algebras). Similarly, applying Proposition 3.32 to the isomorphisms (15.6) and (15.7) yields that (15.9) are isomorphisms of graded Hopf algebras. □

Recall from Remark 15.11 that over a field of characteristic 0, the bilax monoidal functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ are isomorphic. It follows from (15.11) that in this case

$$\overline{\mathcal{K}}(\mathbf{h}^*) \cong \overline{\mathcal{K}}(\mathbf{h})^*$$

as graded Hopf algebras. We come back to this point in more detail in Section 15.4.4.

In view of (15.11), the study of the general properties of the Hopf algebras $\mathcal{K}^\vee(\mathbf{h})$ and $\overline{\mathcal{K}}^\vee(\mathbf{h})$ can be reduced to that of the Hopf algebras $\mathcal{K}(\mathbf{h})$ and $\overline{\mathcal{K}}(\mathbf{h})$ (and viceversa).

## 15.3. Values of Fock functors on particular Hopf monoids

We illustrate Theorems 15.12 and 15.13 on the examples of Section 8.5. The findings in these examples will prove claims made in Remark 15.11. More elaborate examples are given in Chapter 17.

**Example 15.14.** Let $\Bbbk[x]$ and $\Bbbk\{x\}$ be the polynomial and divided power Hopf algebras in the variable $x$ (Example 2.3). Let $\mathbf{E}$ be the Hopf monoid of Example 8.15. We claim that

$$\mathcal{K}(\mathbf{E}) \cong \overline{\mathcal{K}}(\mathbf{E}) \cong \Bbbk[x] \qquad \text{and} \qquad \mathcal{K}^\vee(\mathbf{E}) \cong \overline{\mathcal{K}}^\vee(\mathbf{E}) \cong \Bbbk\{x\}.$$

Since $\mathbf{E}$ is one-dimensional in each degree, it is clear that $\mathcal{K}(\mathbf{E}) \cong \overline{\mathcal{K}}(\mathbf{E})$ and $\mathcal{K}^\vee(\mathbf{E}) \cong \overline{\mathcal{K}}^\vee(\mathbf{E})$ as Hopf algebras. We alter notation for purposes of this example and denote the element $*_{[n]} \in \mathbf{E}[n]$ by $x^n$. The claim involving $\Bbbk[x]$ follows from the computation below.

$$\begin{array}{ccccc}
\mathcal{K}(\mathbf{E}) \cdot \mathcal{K}(\mathbf{E}) & \xrightarrow{\varphi} & \mathcal{K}(\mathbf{E} \cdot \mathbf{E}) & \xrightarrow{\mathcal{K}(\mu)} & \mathcal{K}(\mathbf{E}) \\
x^s \otimes x^t & \longmapsto & x^{|[s]|} \otimes x^{|[s+1,s+t]|} & \longmapsto & x^{s+t}
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{K}(\mathbf{E}) & \xrightarrow{\mathcal{K}(\Delta)} & \mathcal{K}(\mathbf{E} \cdot \mathbf{E}) & \xrightarrow{\psi} & \mathcal{K}(\mathbf{E}) \cdot \mathcal{K}(\mathbf{E}) \\
x^n & \longmapsto & \displaystyle\sum_{S \sqcup T = [n]} x^{|S|} \otimes x^{|T|} & \longmapsto & \displaystyle\sum_{s+t=n} \binom{n}{s} x^s \otimes x^t
\end{array}$$

The claim involving $\Bbbk\{x\}$ follows from a similar computation to the above using the structure maps $\psi^\vee$ and $\varphi^\vee$. Alternatively, one can deduce it from the first claim by applying duality as below.

$$\mathcal{K}^\vee(\mathbf{E}) \cong \mathcal{K}(\mathbf{E}^*)^* \cong \mathcal{K}(\mathbf{E})^* \cong \Bbbk[x]^* \cong \Bbbk\{x\}$$

For the first equality, we used (15.11) and for the second equality, we used the self-duality of $\mathbf{E}$ (Example 8.22).

Over a field of positive characteristic, the graded Hopf algebras

$$\overline{\mathcal{K}}(\mathbf{E}) \cong \Bbbk[x] \qquad \text{and} \qquad \overline{\mathcal{K}}^{\vee}(\mathbf{E}) \cong \Bbbk\{x\}$$

are not isomorphic; therefore, the bilax monoidal functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee}$ are not isomorphic. For more in this direction, see Section 15.4.3.

**Example 15.15.** Let $\mathbf{E}^{\cdot 2}$ be the Hopf monoid of subsets of Example 8.17. Calculations similar to those in Example 15.14 show that

$$\mathcal{K}(\mathbf{E}^{\cdot 2}) \cong \Bbbk\langle x, y\rangle \qquad \text{and} \qquad \overline{\mathcal{K}}(\mathbf{E}^{\cdot 2}) \cong \Bbbk[x, y].$$

These are polynomial algebras in two variables. The square brackets indicate that the variables commute and the angle brackets that they do not. The coproduct is determined by declaring that $x$ and $y$ are primitive. The first isomorphism is defined by using the following fact. A monomial of degree $n$ in the noncommutative variables $x$ and $y$ corresponds to a decomposition $(S, T)$ of $[n]$: the positions of $x$ and $y$ define the subsets $S$ and $T$ respectively.

The canonical map $\mathcal{K}(\mathbf{E}^{\cdot 2}) \to \overline{\mathcal{K}}(\mathbf{E}^{\cdot 2})$ sends a polynomial in two noncommuting variables to the same polynomial (with the understanding that the variables now commute). It is a morphism of Hopf algebras.

**Example 15.16.** We now generalize the previous examples. Let $\mathbf{E}_V$ be the Hopf monoid of Example 8.18. Then

$$\mathcal{K}(\mathbf{E}_V) = \mathcal{T}(V), \quad \overline{\mathcal{K}}(\mathbf{E}_V) = \mathcal{S}(V), \quad \mathcal{K}^{\vee}(\mathbf{E}_V) = \mathcal{T}^{\vee}(V), \quad \text{and} \quad \overline{\mathcal{K}}^{\vee}(\mathbf{E}_V) = \mathcal{S}^{\vee}(V),$$

where the right-hand sides include the tensor algebra, the shuffle algebra and the symmetric algebra of $V$ (Section 2.6.1). These are to be viewed as graded Hopf algebras with $V$ belonging to the degree 1 component.

**Example 15.17.** Let $\mathbf{L}$ and $\mathbf{L}^*$ be the Hopf monoids of linear orders of Examples 8.16 and 8.24. Calculations similar to those in Example 15.14 show that

$$\overline{\mathcal{K}}(\mathbf{L}) \cong \overline{\mathcal{K}}^{\vee}(\mathbf{L}) \cong \Bbbk[x] \qquad \text{and} \qquad \overline{\mathcal{K}}(\mathbf{L}^*) \cong \overline{\mathcal{K}}^{\vee}(\mathbf{L}^*) \cong \Bbbk\{x\}$$

as graded Hopf algebras.

We now turn our attention to the functors $\mathcal{K}$ and $\mathcal{K}^{\vee}$. The Hopf monoid $\mathbf{L}$ and the associated Hopf algebra $\mathcal{K}(\mathbf{L})$ are studied by Patras and Reutenauer [291, Section 6]. The Hopf algebra $\mathcal{K}(\mathbf{L})$ is cocommutative but not commutative and is as follows. The degree $n$ component of $\mathcal{K}(\mathbf{L})$ has the set of linear orders on $[n]$ for a linear basis. To describe the product and coproduct explicitly, we setup a notation.

Let $I \to J$ be a bijection between finite sets, and let $l$ be a linear order on $I$. Then $\mathbf{L}[I \to J](l)$ denotes the linear order on $J$ obtained by transporting $l$ from $I$ to $J$ by means of the given bijection. In the situations we consider, $I$ and $J$ will be subsets of the integers and $I \to J$ will be the unique order-preserving bijection between them. Further, if $J = [n]$, then it is convenient to write

$$\operatorname{std}(l) := \mathbf{L}[I \to [n]](l) \in \mathbf{L}[n].$$

We refer to it as the *standardization* of $l$. Similarly if $I = [n]$, then it is convenient to write

$$\operatorname{sft}_J(l) := \mathbf{L}[[n] \to J](l) \in \mathbf{L}[J].$$

We refer to it as the *shifting* of $l$ to $J$.

We can now describe the product and coproduct of $\mathcal{K}(\mathbf{L})$. The coproduct is, for $l \in \mathbf{L}[n]$,

$$\Delta(l) = \sum_{[n]=S\sqcup T} \mathrm{std}(l|_S) \otimes \mathrm{std}(l|_T),$$

where, given $S \subseteq I$, $l|_S$ denotes the restriction of $l$ to $S$. For example,

$$\Delta(1|3|2) = (\,) \otimes 1|3|2 + 2(1 \otimes 1|2) + 1 \otimes 2|1 + 2(1|2 \otimes 1) + 2|1 \otimes 1 + 1|3|2 \otimes (\,),$$

where $(\,)$ stands for the empty list. It is the unique linear order on the empty set. The product of $l_1 \in \mathbf{L}[s]$ and $l_2 \in \mathbf{L}[t]$ is the linear order

$$l_1 * l_2 = l_1 \cdot \bigl(\mathrm{sft}_{[s+1,s+t]}(l_2)\bigr) \in \mathbf{L}[s+t]$$

where $\cdot$ denotes concatenation of linear orders. In other words, $l_1 * l_2$ is obtained by adding $s$ to each entry of $l_2$ and then placing it to the right of $l_1$. For example,

$$1|3|2 * 2|1 = 1|3|2|5|4.$$

Using (15.11), we obtain

$$\mathcal{K}^\vee(\mathbf{L}^*) \cong \mathcal{K}(\mathbf{L})^*.$$

We now describe the product and coproduct of the dual explicitly. For a linear order $l$ on $[n]$, let $l^* \in \mathbf{L}^*[n]$ denote the dual basis element. Let $l_1$ and $l_2$ be linear orders on $[s]$ and $[t]$ respectively. Then the product is given by

$$l_1^* * l_2^* = \sum_{[n]=S\sqcup T} \sum_{l} l^*,$$

where the first sum is over all decompositions with $|S| = s$ and $|T| = t$, and the second sum is over all shuffles $l$ of the linear orders $\mathrm{sft}_S(l_1)$ and $\mathrm{sft}_T(l_2)$ on $S$ and $T$ respectively. The coproduct is given by

$$\Delta(l^*) = \sum \mathrm{std}(l^1|\cdots|l^s)^* \otimes \mathrm{std}(l^{s+1}|\cdots|l^n)^*,$$

where $l = l^1|\cdots|l^n \in \mathbf{L}[n]$ and the sum is over those $s$ for which $l^1, \ldots, l^s$ are all less than $l^{s+1}, \ldots, l^n$. In other words, the sum is over all positions $s$ at which $l$ has a *global ascent*. Global ascents of $l$ correspond to global descents of the reverse linear order $\bar{l}$. The latter are defined in Section 10.7.1. It follows directly that the space of primitive elements is spanned by those linear orders which have no global ascents. For example,

$$\begin{aligned}1^* * 2|1^* &= (1|3|2^* + 3|1|2^* + 3|2|1^*) + (2|3|1^* + 3|2|1^* + 3|1|2^*)\\ &\qquad\qquad\qquad\qquad + (3|2|1^* + 2|3|1^* + 2|1|3^*)\\ &= 1|3|2^* + 2\,(2|3|1^*) + 2|1|3^* + 2\,(3|1|2^*) + 3\,(3|2|1^*).\end{aligned}$$

$$\Delta(1|3|2|5|4^*) = (\,)^* \otimes 1|3|2|5|4^* + 1|3|2^* \otimes 2|1^* + 1|3|2|5|4^* \otimes (\,)^*.$$

On the other hand, we have

$$\mathcal{K}(\mathbf{L}^*) \cong \mathrm{S}\Lambda \qquad \text{and} \qquad \mathcal{K}^\vee(\mathbf{L}) \cong \mathrm{S}\Lambda^* \cong \mathrm{S}\Lambda,$$

where $\mathrm{S}\Lambda$ is the graded Hopf algebra of permutations of Malvenuto and Reutenauer [255, 256]. This Hopf algebra is self-dual, free and cofree, and neither commutative nor cocommutative. The fact that $\mathcal{K}(\mathbf{L}^*) \cong \mathrm{S}\Lambda$ was first pointed out by Patras and Reutenauer [291, Proposition 16]; the second fact then follows by (15.11) and self-duality. The self-duality appears in [255, Section 5.2] and [256, Theorem 3.3]. The freeness was established by Poirier and Reutenauer [297]. For related ideas, see the works of Reutenauer [311], Patras and Reutenauer [290], Loday and

Ronco [242, 243], Duchamp, Hivert and Thibon [106, 107], Foissy [131] and Aguiar and Sottile [13, 14, 15].

We now proceed to describe $\mathrm{S}\Lambda$. Write $F_l$ for the basis element of $\mathcal{K}(\mathbf{L}^*)$ corresponding to $l^* \in \mathbf{L}^*[n]$. The coproduct is

$$\Delta(F_l) = \sum_{s=0}^{n} F_{\mathrm{std}(l^1|\cdots|l^s)} \otimes F_{\mathrm{std}(l^{s+1}|\cdots|l^n)},$$

where $l = l^1|\cdots|l^n \in \mathbf{L}[n]$, and the product is

$$F_{l_1} * F_{l_2} = \sum_{l} F_l,$$

where $l_1 \in \mathbf{L}[s]$, $l_2 \in \mathbf{L}[t]$, and the sum is over all linear orders $l \in \mathbf{L}[s+t]$ obtained by adding $s$ to the entries of $l_2$ and then shuffling them with the entries of $l_1$. For example,

$$\Delta(F_{1|3|4|2}) = F_{(\,)} \otimes F_{1|3|4|2} + F_1 \otimes F_{2|3|1} + F_{1|2} \otimes F_{2|1} + F_{1|2|3} \otimes F_1 + F_{1|3|4|2} \otimes F_{(\,)}$$

and

$$F_{2|1} * F_{1|2} = F_{2|1|3|4} + F_{2|3|1|4} + F_{2|3|4|1} + F_{3|2|1|4} + F_{3|2|4|1} + F_{3|4|2|1}.$$

The primitive elements of $\mathrm{S}\Lambda$ are harder to compute. The dimension of this space in degree $n$ is given by the number of permutations on $n$ letters with no global descents [14, 107, 297]. Thus, the dimension of the space of primitive elements of $\mathcal{K}(\mathbf{L}^*)$ and $\mathcal{K}^\vee(\mathbf{L}^*)$ is the same.

In view of (15.9), we have

$$\overline{\mathcal{K}}(\mathbf{L} \times \mathbf{L}^*) \cong \mathcal{K}(\mathbf{L}^*) \cong \mathrm{S}\Lambda.$$

Thus, $\mathrm{S}\Lambda$ can be viewed as the image of the Hopf monoid $\mathbf{L} \times \mathbf{L}^*$ under the functor $\overline{\mathcal{K}}$. This point of view is useful in light of the nice properties of $\overline{\mathcal{K}}$. For instance, the self-duality of $\mathbf{L} \times \mathbf{L}^*$ implies that of $\mathrm{S}\Lambda$. These ideas are explained in more detail in Section 17.2.

The Hopf monoid $\mathbf{L}$, as well as the associated Hopf algebra $\mathcal{K}(\mathbf{L})$, are cocommutative. We show in Section 15.5 that, in general, the functor $\mathcal{K}$ preserves cocommutativity. On the other hand, the Hopf monoid $\mathbf{L}^*$ is commutative but the Hopf algebra $\mathcal{K}(\mathbf{L}^*)$ is not. Hence $\mathcal{K}$ does not preserve commutativity. Exactly the reverse is true of $\mathcal{K}^\vee$.

In addition, note that $\mathcal{K}(\mathbf{L})$ and $\mathcal{K}(\mathbf{L}^*)$ cannot be dual Hopf algebras, since the first is cocommutative whereas the second in not commutative. A similar statement applies to $\mathcal{K}^\vee$. Hence $\mathcal{K}$ and $\mathcal{K}^\vee$ do not preserve duality. This is further studied in Section 15.4.

The Hopf algebras $\mathcal{K}(\mathbf{L})$ and $\mathcal{K}^\vee(\mathbf{L})$ are not isomorphic, since the former is cocommutative, while the latter is not. Therefore, the bilax monoidal functors $\mathcal{K}$ and $\mathcal{K}^\vee$ are not isomorphic.

Recall the morphism of Hopf monoids $\pi^* \colon \mathbf{E}^* \to \mathbf{L}^*$ of (8.33). Its image under $\mathcal{K}$ is the morphism of graded Hopf algebras

$$\Bbbk[x] \to \mathrm{S}\Lambda \qquad \text{given by} \qquad x^n \mapsto \sum_{l \in \mathbf{L}[n]} F_l.$$

The canonical map $\mathcal{K}(\mathbf{L}^*) \to \overline{\mathcal{K}}(\mathbf{L}^*)$ turns out to be the dual morphism of graded Hopf algebras

$$\mathrm{S}\Lambda \to \Bbbk\{x\}.$$

## 15.4. The norm transformation between full Fock functors

In this section, we study a morphism between the full Fock functors $\mathcal{K}$ and $\mathcal{K}^\vee$. It is called the norm transformation and denoted $\kappa$. We explain how the bosonic Fock functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ can be interpreted as the coimage and image of this morphism. Finally, we apply this circle of ideas to answer the questions on whether the Fock functors preserve duality.

**15.4.1. Relating the structure transformations of the full Fock functors.** Let $\mathrm{Sh}(s,t)$ denote the set of $(s,t)$-shuffle permutations (2.21).

**Lemma 15.18.** *The structure maps $\varphi$ and $\psi^\vee$, and $\psi$ and $\varphi^\vee$, are related by the formulas*

$$\psi^\vee(x \otimes y) = \sum_{\zeta \in \mathrm{Sh}\,(s,t)} \zeta\big(\varphi(x \otimes y)\big) \qquad \textit{for } x \in \mathbf{p}[s],\ y \in \mathbf{q}[t], \tag{15.12}$$

$$\psi(a \otimes b) = \sum_{\zeta \in \mathrm{Sh}\,(|S|,|T|)} \varphi^\vee\big(\zeta^{-1}(a \otimes b)\big) \qquad \textit{for } a \in \mathbf{p}[S],\ b \in \mathbf{q}[T]. \tag{15.13}$$

*In particular,*

$$\psi^\vee(x \otimes y) = \varphi(x \otimes y) \qquad \textit{if } s = 0 \textit{ or } t = 0,$$
$$\psi(a \otimes b) = \varphi^\vee(a \otimes b) \qquad \textit{if } S = \emptyset \textit{ or } T = \emptyset.$$

Proof. We explain (15.13). Let $s = |S|$ and $t = |T|$. Among all $(s,t)$-shuffle permutations there is one such that $\zeta([s]) = S$ and $\zeta([s+1, s+t]) = T$. For this shuffle $\zeta$ we have

$$\varphi^\vee\big(\zeta^{-1}(a \otimes b)\big) = \psi(a \otimes b);$$

for all other shuffles $\zeta$ we have $\varphi^\vee\big(\zeta^{-1}(a \otimes b)\big) = 0$. $\square$

**15.4.2. The norm transformation between full Fock functors.** We now define the norm transformation from $\mathcal{K}$ to $\mathcal{K}^\vee$ and show that it is a morphism of bilax monoidal functors. We then discuss the induced transformation between the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$.

**Definition 15.19.** For any species $\mathbf{p}$, let $\kappa_{\mathbf{p}} \colon \mathcal{K}(\mathbf{p}) \to \mathcal{K}^\vee(\mathbf{p})$ be the map of graded vector spaces given by

$$\mathbf{p}[n] \to \mathbf{p}[n] \qquad \kappa_{\mathbf{p}}(z) := \sum_{\sigma \in \mathrm{S}_n} \sigma \cdot z, \tag{15.14}$$

for any $z \in \mathbf{p}[n]$. This defines a natural transformation $\kappa \colon \mathcal{K} \Rightarrow \mathcal{K}^\vee$ which we call the *norm*.

Thus, the degree $n$ component of $\kappa_{\mathbf{p}}$ is the norm map $N_{\mathbf{p}[n]}$ of Section 2.5. The naturality of $\kappa$ follows from that of $N$.

**Proposition 15.20.** *The norm is a morphism of bilax monoidal functors*

$$\kappa \colon \mathcal{K} \Rightarrow \mathcal{K}^\vee.$$

PROOF. We first verify that norm is a morphism of colax functors. The diagrams in (3.15) are in this case

$$\begin{tikzcd} \mathcal{K}(\mathbf{p}\cdot\mathbf{q}) \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, "\kappa_{\mathbf{p}\cdot\mathbf{q}}"'] & \mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}) \arrow[d, "\kappa_{\mathbf{p}}\cdot\kappa_{\mathbf{q}}"] \\ \mathcal{K}^\vee(\mathbf{p}\cdot\mathbf{q}) \arrow[r, "\varphi^\vee_{\mathbf{p},\mathbf{q}}"'] & \mathcal{K}^\vee(\mathbf{p})\cdot\mathcal{K}^\vee(\mathbf{q}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{K}(\mathbf{1}) \arrow[dr, "\psi_0"] \arrow[dd, "\kappa_{\mathbf{1}}"'] & \\ & \Bbbk. \\ \mathcal{K}^\vee(\mathbf{1}) \arrow[ur, "\varphi_0^\vee"'] & \end{tikzcd}$$

The diagram on the right commutes trivially. For the diagram on the left, fix a decomposition $[n] = S \sqcup T$ and take $a \otimes b \in \mathbf{p}[S] \otimes \mathbf{q}[T]$. Let $s = |S|$ and $t = |T|$. Using (15.13) and the naturality of $\varphi^\vee$ we find

$$(\kappa_{\mathbf{p}}\cdot\kappa_{\mathbf{q}})\psi_{\mathbf{p},\mathbf{q}}(a\otimes b) = \sum_{\substack{\sigma\in \mathrm{S}_s\\ \tau\in\mathrm{S}_t}} \sum_{\zeta\in \mathrm{Sh}(s,t)} \varphi^\vee_{\mathbf{p},\mathbf{q}}\big((\sigma\times\tau)\cdot\zeta^{-1}\cdot(a\otimes b)\big).$$

Taking inverses in (2.22) and replacing $\sigma$, $\tau$ and $\rho$ by their inverses, we deduce

$$\begin{aligned}(\kappa_{\mathbf{p}}\cdot\kappa_{\mathbf{q}})\psi_{\mathbf{p},\mathbf{q}}(a\otimes b) &= \sum_{\rho\in\mathrm{S}_n} \varphi^\vee_{\mathbf{p},\mathbf{q}}\big(\rho\cdot(a\otimes b)\big)\\ &= \varphi^\vee_{\mathbf{p},\mathbf{q}}\kappa_{\mathbf{p}\cdot\mathbf{q}}(a\otimes b).\end{aligned}$$

Thus, the diagram on the left commutes and $\kappa$ is a morphism of colax functors. The proof can be summarized in the following commutative diagram

$$\begin{tikzcd} \mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"] \arrow[d, "{\mathbf{p}[\rho|_S]\otimes\mathbf{q}[\rho|_T]}"'] & \mathbf{p}[s]\otimes\mathbf{q}[t] \arrow[d, "{\mathbf{p}[\sigma]\otimes\mathbf{q}[\tau]}"] \\ \mathbf{p}[\rho(S)]\otimes\mathbf{q}[\rho(T)] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"'] & \mathbf{p}[s]\otimes\mathbf{q}[t] \end{tikzcd}$$

where $\rho = (\sigma\times\tau)\cdot\zeta^{-1}$ and $\zeta$ is the unique $(s,t)$-shuffle permutation which sends $[s]$ to $S$ and $[s+1, s+t]$ to $T$.

Similarly, using (15.12), one can show that $\kappa$ is a morphism of lax functors. On finite-dimensional species, this can be deduced from the above result plus self-duality of $\kappa$ (15.17), using Proposition 3.102. □

The image of the norm map $\kappa_{\mathbf{p}}\colon \mathcal{K}(\mathbf{p}) \to \mathcal{K}^\vee(\mathbf{p})$ consists of invariant elements, so it is contained in $\overline{\mathcal{K}}^\vee(\mathbf{p})$. Since the inclusion $\overline{\mathcal{K}}^\vee \hookrightarrow \mathcal{K}^\vee$ is a morphism of bilax monoidal functors, so is the resulting transformation

$$\mathcal{K} \Rightarrow \overline{\mathcal{K}}^\vee.$$

This transformation factors through coinvariants, giving rise to another morphism of bilax monoidal functors

$$\overline{\kappa}\colon \overline{\mathcal{K}} \Rightarrow \overline{\mathcal{K}}^\vee.$$

These fit in a commutative diagram as follows.

$$
\begin{tikzcd}
\mathcal{K} \arrow[r, Rightarrow, "\kappa"] \arrow[d, Rightarrow] & \mathcal{K}^{\vee} \\
\overline{\mathcal{K}} \arrow[r, Rightarrow, "\overline{\kappa}"'] & \overline{\mathcal{K}}^{\vee} \arrow[u, Rightarrow]
\end{tikzcd}
\tag{15.15}
$$

If the field characteristic is 0, then one can say more. Applying Lemma 2.20 we obtain:

**Proposition 15.21.** *The morphism of bistrong monoidal functors*

$$\overline{\kappa} \colon \overline{\mathcal{K}} \Rightarrow \overline{\mathcal{K}}^{\vee}$$

*is an isomorphism if the field characteristic is* 0*. In addition, regardless of the field characteristic, if the species* $\mathbf{p}$ *consists of flat* $\Bbbk \mathrm{S}_n$*-modules* $\mathbf{p}[n]$*, then*

$$\overline{\kappa}_{\mathbf{p}} \colon \overline{\mathcal{K}}(\mathbf{p}) \to \overline{\mathcal{K}}^{\vee}(\mathbf{p})$$

*is bijective.*

Applying Proposition 3.32 we obtain that for any Hopf monoid $\mathbf{h}$, the diagram of graded Hopf algebras

$$
\begin{tikzcd}
\mathcal{K}(\mathbf{h}) \arrow[r, "\kappa_{\mathbf{h}}"] \arrow[d, two heads] & \mathcal{K}^{\vee}(\mathbf{h}) \\
\overline{\mathcal{K}}(\mathbf{h}) \arrow[r, "\overline{\kappa}_{\mathbf{h}}"'] & \overline{\mathcal{K}}^{\vee}(\mathbf{h}) \arrow[u, hook]
\end{tikzcd}
\tag{15.16}
$$

commutes. In addition:

**Corollary 15.22.** *If the species* $\mathbf{h}$ *consists of flat* $\Bbbk \mathrm{S}_n$*-modules* $\mathbf{h}[n]$*, then*

$$\overline{\kappa}_{\mathbf{h}} \colon \overline{\mathcal{K}}(\mathbf{h}) \to \overline{\mathcal{K}}^{\vee}(\mathbf{h})$$

*is an isomorphism of graded Hopf algebras. This holds if the field characteristic is* 0*, for any* $\mathbf{h}$*.*

**Example 15.23.** For the Hopf monoid $\mathbf{E}$ of Examples 8.15 and 15.14, we have that $\kappa_{\mathbf{E}} \colon \mathcal{K}(\mathbf{E}) \to \mathcal{K}^{\vee}(\mathbf{E})$ is the map (2.11). It is an isomorphism of Hopf algebras in characteristic 0. More generally, for the Hopf monoid $\mathbf{E}_V$ of Examples 8.18 and 15.16, diagram (15.16) specializes to (2.66).

Suppose now that the species $\mathbf{p}$ is finite-dimensional. It follows from Proposition 15.8 that $\kappa$ is related to its contragredient (3.47) as follows.

$$
\begin{tikzcd}
\mathcal{K}^{\vee}(\mathbf{p})^* \arrow[r, "(\kappa_{\mathbf{p}})^*"] \arrow[d, equal] & \mathcal{K}(\mathbf{p})^* \arrow[d, equal] \\
\mathcal{K}(\mathbf{p}^*) \arrow[r, "\kappa_{\mathbf{p}^*}"'] & \mathcal{K}^{\vee}(\mathbf{p}^*)
\end{tikzcd}
\tag{15.17}
$$

This means that the norm transformation is self-dual (Definition 3.108). The same property holds for $\overline{\kappa}$. More generally, Lemma 2.22 yields:

**Proposition 15.24.** *On finite-dimensional species, diagram* (15.15) *is self-dual.*

**15.4.3. The image of the norm.** Let $\Im$ denote the (co)image of the norm transformation $\kappa\colon \mathcal{K} \Rightarrow \mathcal{K}^\vee$, in the sense of Section 3.11. It is a bilax monoidal functor

$$\Im\colon (\mathsf{Sp}, \cdot, \beta) \to (\mathsf{gVec}, \cdot, \beta).$$

Proposition 15.21 implies that, in characteristic 0, $\overline{\mathcal{K}}$, $\overline{\mathcal{K}}^\vee$, and $\Im$ are isomorphic bistrong monoidal functors. Thus, in this situation the bilax monoidal functors $\overline{\mathcal{K}}^\vee$ and $\overline{\mathcal{K}}$ are naturally associated to the morphism $\kappa$ (they are the image and coimage of $\kappa$, see Remark 3.117).

In general, $\overline{\mathcal{K}}$, $\overline{\mathcal{K}}^\vee$, and $\Im$ are three distinct bistrong monoidal functors related by morphisms of bistrong functors

$$\overline{\mathcal{K}} \Rightarrow \Im \Rightarrow \overline{\mathcal{K}}^\vee.$$

The fact that $\Im$ is bistrong follows from the fact that the first natural transformation is onto, or from the fact that the second one is into. The connection between all five functors is as in the following diagram.

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, Rightarrow, "\kappa"] \arrow[d, Rightarrow] & & \mathcal{K}^\vee \\
\overline{\mathcal{K}} \arrow[r, Rightarrow] & \Im \arrow[r, Rightarrow] & \overline{\mathcal{K}}^\vee \arrow[u, Rightarrow]
\end{tikzcd}$$

On finite-dimensional species, this diagram is self-dual. In particular, $\Im$ is a self-dual functor (regardless of the characteristic). This can be seen as a consequence of Proposition 3.119.

The distinction between $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ is illustrated in Example 15.23. It follows that, if the characteristic of $\Bbbk$ is $p$, then

$$\Im(\mathbf{E}) = \Bbbk[x]/(x^p).$$

Since a self-dual functor preserves self-dual objects (Proposition 3.107), it follows that $\Im(\mathbf{E})$ is a self-dual Hopf algebra. This was noted in Example 2.3.

**15.4.4. The Fock functors and duality.** Consider the question of whether the functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ preserve duality of Hopf monoids. We know from Example 15.17 that for $\mathcal{K}$ the answer is negative. On the other hand, the functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ are related through duality to the functors $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$, as given in Proposition 15.8. This may be rewritten as follows: if $\mathbf{p}$ is a finite-dimensional species, then

$$\mathcal{K}(\mathbf{p}^*) \cong \mathcal{K}^\vee(\mathbf{p})^* \qquad \text{and} \qquad \overline{\mathcal{K}}(\mathbf{p}^*) \cong \overline{\mathcal{K}}^\vee(\mathbf{p})^*.$$

Therefore, the above question is closely related to whether $\mathcal{K}$ and $\mathcal{K}^\vee$, and $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$, are isomorphic as bilax monoidal functors. This is a point we addressed in Section 15.4.2. When expressed in terms of duality, the answers take the following form.

Suppose $\mathbf{h}$ is a finite-dimensional Hopf monoid. By applying (15.16) to $\mathbf{h}^*$ together with the isomorphisms in (15.11), we obtain the commutative diagrams of graded Hopf algebras below.

$$\begin{tikzcd}
\mathcal{K}(\mathbf{h}^*) \arrow[r, "\kappa_{\mathbf{h}^*}"] \arrow[d, two heads] & \mathcal{K}(\mathbf{h})^* \\
\overline{\mathcal{K}}(\mathbf{h}^*) \arrow[r, "\overline{\kappa}_{\mathbf{h}^*}"'] & \overline{\mathcal{K}}(\mathbf{h})^* \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}^\vee(\mathbf{h})^* \arrow[r, "\kappa_{\mathbf{h}^*}"] \arrow[d, two heads] & \mathcal{K}^\vee(\mathbf{h}^*) \\
\overline{\mathcal{K}}^\vee(\mathbf{h})^* \arrow[r, "\overline{\kappa}_{\mathbf{h}^*}"'] & \overline{\mathcal{K}}^\vee(\mathbf{h}^*) \arrow[u, hook]
\end{tikzcd}$$

(The two diagrams are the same.) By applying Corollary 15.22 to $\mathbf{h}^*$ we obtain:

**Corollary 15.25.** *Let* $\mathbf{h}$ *be a finite-dimensional Hopf monoid. If* $\mathbf{h}^*$ *consists of flat* $\Bbbk\mathrm{S}_n$*-modules* $\mathbf{h}[n]^*$, *then* $\overline{\kappa}_{\mathbf{h}^*}$ *is an isomorphism of graded Hopf algebras. If in addition* $\mathbf{h}$ *is a self-dual Hopf monoid, then* $\overline{\mathcal{K}}(\mathbf{h})$ *is a self-dual graded Hopf algebra.*

Recall that over a field characteristic is 0 any $\mathrm{S}_n$-module is flat.

## 15.5. The Fock functors and commutativity

In this section, we discuss whether the Fock functors preserve commutative monoids or cocommutative comonoids.

**15.5.1. Are the Fock functors braided?** In light of the discussion in Section 3.4.4, one essentially has to study whether $\mathcal{K}$ and $\overline{\mathcal{K}}$ are braided viewed both as lax and colax functors.

**Proposition 15.26.** *The functor* $(\mathcal{K}, \psi)$ *is braided colax, but the functor* $(\mathcal{K}, \varphi)$ *is not braided lax. On the other hand, the functor* $(\overline{\mathcal{K}}, \overline{\varphi}, \overline{\psi})$ *is braided bilax.*

Proof. For the assertions about $\mathcal{K}$, we have to show that the left-hand diagram below commutes while the right-hand diagram does not.

$$\begin{tikzcd}
\mathcal{K}(\mathbf{p}\cdot\mathbf{q}) \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, "\mathcal{K}(\beta)"'] \arrow[dr, phantom, "\text{Yes}"] & \mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}) \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}"] \arrow[d, "\beta"] \arrow[dr, phantom, "\text{No}"] & \mathcal{K}(\mathbf{p}\cdot\mathbf{q}) \arrow[d, "\mathcal{K}(\beta)"] \\
\mathcal{K}(\mathbf{q}\cdot\mathbf{p}) \arrow[r, "\psi_{\mathbf{q},\mathbf{p}}"'] & \mathcal{K}(\mathbf{q})\cdot\mathcal{K}(\mathbf{p}) \arrow[r, "\varphi_{\mathbf{q},\mathbf{p}}"'] & \mathcal{K}(\mathbf{q}\cdot\mathbf{p})
\end{tikzcd}$$

We look at the degree $n$ part of the above diagram. The relevant portion is shown below.

$$\begin{tikzcd}
\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\text{Yes}"] & \mathbf{p}[s]\otimes\mathbf{q}[t] \arrow[r, "{\mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}]}"] \arrow[d, "\beta"] \arrow[dr, phantom, "\text{No}"] & \mathbf{p}[s]\otimes\mathbf{q}[s+1,s+t] \arrow[d, "\beta"] \\
\mathbf{q}[T]\otimes\mathbf{p}[S] \arrow[r, "{\mathbf{q}[\mathrm{cano}]\otimes\mathbf{p}[\mathrm{cano}]}"'] & \mathbf{q}[t]\otimes\mathbf{p}[s] \arrow[r, "{\mathbf{q}[\mathrm{id}]\otimes\mathbf{p}[\mathrm{cano}]}"'] & \displaystyle\bigoplus_{S\sqcup T=[n]} \mathbf{q}[T]\otimes\mathbf{p}[S]
\end{tikzcd}$$

The first diagram clearly commutes. The second diagram does not commute because following the two directions land us in

$$\mathbf{q}[t]\otimes\mathbf{p}[t+1,t+s] \qquad \text{and} \qquad \mathbf{q}[s+1,s+t]\otimes\mathbf{p}[s],$$

which are distinct components (unless $s$ or $t$ is zero).

This problem disappears for $\overline{\varphi}$ because there is an element of $\mathrm{S}_{s+t}$ which induces an isomorphism between the two components above, so commutativity is attained at the level of coinvariants. In other words, $\overline{\mathcal{K}}$ is braided bilax. Alternatively, this can be deduced by noting that $\overline{\mathcal{K}}$ is bistrong and applying Proposition 3.46. A third proof is given in Proposition 15.31. □

Propositions 3.35, 3.36 and 3.37 yield:

**Corollary 15.27.** *For any comonoid* (*Hopf monoid*) $\mathbf{h}$,

$$\mathcal{K}(\mathbf{h}^{\mathrm{cop}}) = \mathcal{K}(\mathbf{h})^{\mathrm{cop}}$$

*as comonoids* (*Hopf monoids*). *In particular,* $\mathcal{K}$ *takes cocommutative comonoids to cocommutative coalgebras. The functor* $\overline{\mathcal{K}}$ *preserves both commutativity and cocommutativity.*

This has immediate implications for the contragredients $\mathcal{K}^{\vee}$ and $\overline{\mathcal{K}}^{\vee}$. By Propositions 3.102 and 15.8:

**Proposition 15.28.** *The functor* $(\mathcal{K}^{\vee}, \psi^{\vee})$ *is braided lax, but the functor* $(\mathcal{K}^{\vee}, \varphi^{\vee})$ *is not braided colax. On the other hand, the functor* $(\overline{\mathcal{K}}^{\vee}, \overline{\psi}^{\vee}, \overline{\varphi}^{\vee})$ *is braided bilax.*

Hence $\mathcal{K}^{\vee}$ takes commutative monoids to commutative algebras and $\overline{\mathcal{K}}^{\vee}$ preserves both commutativity and cocommutativity. Proposition 4.13 gives that $\mathcal{K}^{\vee}$ takes Lie monoids to graded Lie algebras. Similarly, $\mathcal{K}$ takes Lie comonoids to graded Lie coalgebras, and $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee}$ preserve both Lie monoids and Lie comonoids.

### 15.5.2. The half-twist transformation from $\mathcal{K}$ to itself.

Let ${}^{b}\varphi$ and ${}^{b}\psi$ denote the conjugates of $\varphi$ and $\psi$ as in Definition 3.14. Since the braidings are symmetries in the present case, the side on which the exponent $b$ is written does not matter.

We saw that the colax monoidal functor $(\mathcal{K}, \psi)$ is braided, so ${}^{b}\psi = \psi$. On the other hand, the functor $\mathcal{K}$ is not braided lax. So it may not take commutative monoids to commutative algebras. An example of this kind was given in Example 15.17. The fact that $\mathcal{K}$ does not preserve commutativity can be stated formally by saying that the identity natural transformation

$$(\mathcal{K}, {}^{b}\varphi) \Rightarrow (\mathcal{K}, \varphi)$$

is *not* a morphism of lax monoidal functors (3.17). For the same reason, given a monoid $\mathbf{p}$, the identity map

$$\mathcal{K}(\mathbf{p}^{\mathrm{op}}) \to \mathcal{K}(\mathbf{p})^{\mathrm{op}}$$

need not be a morphism of graded algebras. This opens the possibility of there being two distinct algebras $\mathcal{K}(\mathbf{p}^{\mathrm{op}})$ and $\mathcal{K}(\mathbf{p})^{\mathrm{op}}$ associated to $\mathbf{p}$. However, this is not the case: it turns out that there is a nontrivial isomorphism of algebras

$$\mathcal{K}(\mathbf{p}^{\mathrm{op}}) \cong \mathcal{K}(\mathbf{p})^{\mathrm{op}}.$$

We explain this remarkable fact next.

**Definition 15.29.** For each $n$, let $\omega_n$ be the longest permutation in $\mathrm{S}_n$. It sends $i$ to $n+1-i$ for each $i$. Let $\theta\colon \mathcal{K} \Rightarrow \mathcal{K}$ be the natural transformation defined by the maps

$$\mathbf{p}[\omega_n]\colon \mathbf{p}[n] \to \mathbf{p}[n]$$

for each species $\mathbf{p}$ and each nonnegative integer $n$. For $n = 0, 1$, this map is the identity.

The definition of morphism of species (8.1) guarantees that $\theta$ is indeed a natural transformation. We call it the *half-twist transformation.*

**Proposition 15.30.** *The half-twist transformation is an isomorphism of bilax monoidal functors*

$$\theta\colon (\mathcal{K}, {}^b\varphi, {}^b\psi) \Rightarrow (\mathcal{K}, \varphi, \psi).$$

Proof. We first check that $\theta$ is a morphism of colax monoidal functors. The second diagram in (3.15) commutes trivially while the first diagram takes the following form.

$$\begin{tikzcd}
\mathcal{K}(\mathbf{p}\cdot\mathbf{q}) \arrow[r, "\mathcal{K}(\beta^{-1})"] \arrow[d, "\theta_{\mathbf{p}\cdot\mathbf{q}}"'] & \mathcal{K}(\mathbf{q}\cdot\mathbf{p}) \arrow[r, "\psi_{\mathbf{q},\mathbf{p}}"] & \mathcal{K}(\mathbf{q})\cdot\mathcal{K}(\mathbf{p}) \arrow[r, "\beta"] & \mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}) \arrow[d, "\theta_{\mathbf{p}}\cdot\theta_{\mathbf{q}}"] \\
\mathcal{K}(\mathbf{p}\cdot\mathbf{q}) \arrow[rrr, "\psi_{\mathbf{p},\mathbf{q}}"'] & & & \mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}).
\end{tikzcd}$$

The commutativity of this diagram boils down to the following diagram, where $S \sqcup T = [n]$ is a decomposition and $S' = \omega_n(S)$, $T' = \omega_n(T)$.

$$\begin{tikzcd}
\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[r, "\beta^{-1}"] \arrow[d, "(\mathbf{p}\cdot\mathbf{q})[\omega_n]"] & \mathbf{q}[T]\otimes\mathbf{p}[S] \arrow[r, "{\mathbf{q}[\mathrm{cano}]\otimes\mathbf{p}[\mathrm{cano}]}"] & \mathbf{q}[t]\otimes\mathbf{p}[s] \arrow[r, "\beta"] & \mathbf{p}[s]\otimes\mathbf{q}[t] \arrow[d, "{\mathbf{p}[\omega_s]\otimes\mathbf{q}[\omega_t]}"'] \\
\mathbf{p}[S']\otimes\mathbf{q}[T'] \arrow[rrr, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"'] & & & \mathbf{p}[s]\otimes\mathbf{q}[t]
\end{tikzcd}$$

The composite along the top is $\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}]$ (as encountered in the proof of Proposition 15.26). The commutativity of this diagram follows by functoriality from that of

$$\begin{tikzcd}
S\times T \arrow[r, "\mathrm{cano}\times\mathrm{cano}"] \arrow[d, "\omega_n|_S\times\omega_n|_T"'] & {[s]\times[t]} \arrow[d, "\omega_s\times\omega_t"] \\
S'\times T' \arrow[r, "\mathrm{cano}\times\mathrm{cano}"'] & {[s]\times[t].}
\end{tikzcd}$$

Similarly, to check that $\theta$ is a morphism of lax monoidal functors, one needs to check the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}) \arrow[r, "\beta^{-1}"] \arrow[d, "\theta_{\mathbf{p}}\cdot\theta_{\mathbf{q}}"'] & \mathcal{K}(\mathbf{q})\cdot\mathcal{K}(\mathbf{p}) \arrow[r, "\varphi_{\mathbf{q},\mathbf{p}}"] & \mathcal{K}(\mathbf{q}\cdot\mathbf{p}) \arrow[r, "\mathcal{K}(\beta)"] & \mathcal{K}(\mathbf{p}\cdot\mathbf{q}) \arrow[d, "\theta_{\mathbf{p}\cdot\mathbf{q}}"] \\
\mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}) \arrow[rrr, "\varphi_{\mathbf{p},\mathbf{q}}"'] & & & \mathcal{K}(\mathbf{p}\cdot\mathbf{q}).
\end{tikzcd}$$

This follows from that of

$$\begin{tikzcd}
{[s] \times [t]} \arrow[r, "\mathrm{cano} \times \mathrm{cano}"] \arrow[d, "\omega_s \times \omega_t"'] & {[t+1, t+s] \times [t]} \arrow[d, "\omega_n|_{[t+1,t+s]} \times \omega_n|_{[t]}"] \\
{[s] \times [t]} \arrow[r, "\mathrm{cano} \times \mathrm{cano}"'] & {[s] \times [s+1, s+t].}
\end{tikzcd}$$ □

**Proposition 15.31.** *The following is a commutative diagram of morphisms of bilax functors.*

$$\begin{tikzcd}
(\mathcal{K}, {}^b\varphi, {}^b\psi) \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow] & (\mathcal{K}, \varphi, \psi) \arrow[d, Rightarrow] \\
(\overline{\mathcal{K}}, {}^b\overline{\varphi}, {}^b\overline{\psi}) \arrow[r, Rightarrow, "\mathrm{id}"'] & (\overline{\mathcal{K}}, \overline{\varphi}, \overline{\psi})
\end{tikzcd}$$

*In particular,* $(\overline{\mathcal{K}}, \overline{\varphi}, \overline{\psi})$ *is braided bilax.*

Proof. We first note that the above is a commutative diagram of natural transformations. In other words, $\theta$ factors through the projection $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ and gives rise to the identity natural transformation on $\overline{\mathcal{K}}$. It then follows from Proposition 15.30 that the identity is a morphism of bilax functors and further that the diagram commutes as morphisms of bilax functors. □

**Corollary 15.32.** *For any Hopf monoid* $\mathbf{h}$, *the map*

$$\mathcal{K}(\mathbf{h}^{\mathrm{op}}) \to \mathcal{K}(\mathbf{h})^{\mathrm{op}}$$

*whose degree* $n$ *component is* $\mathbf{h}[\omega_n]$ *(Definition 15.29) is a natural isomorphism of Hopf algebras.*

Proof. Propositions 15.30, 3.32, and 3.34 imply that

$$\theta_{\mathbf{h}} \colon {}^{\mathrm{op,cop}}\mathcal{K}(\mathbf{h}^{\mathrm{op,cop}}) \to \mathcal{K}(\mathbf{h})$$

is an isomorphism of Hopf algebras. By applying $(-)^{\mathrm{op,cop}}$ (which is the inverse to ${}^{\mathrm{op,cop}}(-)$), we obtain that

$$\mathcal{K}(\mathbf{h}^{\mathrm{op,cop}}) \to \mathcal{K}(\mathbf{h})^{\mathrm{op,cop}}$$

is an isomorphism of Hopf algebras. The result now follows by replacing $\mathbf{h}$ by $\mathbf{h}^{\mathrm{cop}}$, and using Corollary 15.27 and the fact that both braidings are symmetries. □

**Example 15.33.** Consider the Hopf monoid $\mathbf{L}^*$ of Example 8.24. As explained in Example 15.17, we have $\mathcal{K}(\mathbf{L}^*) = \mathrm{S}\Lambda$. Since $\mathbf{L}^*$ is commutative, we obtain an isomorphism of Hopf algebras

$$\mathrm{S}\Lambda \to \mathrm{S}\Lambda^{\mathrm{op}} \qquad \text{given by} \qquad F_{l^1|\cdots|l^n} \mapsto F_{n+1-l^1|\cdots|n+1-l^n}.$$

For example, $F_{2|1|4|3} \mapsto F_{3|4|1|2}$.

**Example 15.34.** Consider the Hopf monoid $\mathbf{\Sigma}^*$ discussed in Section 12.4. Applying the functor $\mathcal{K}$ yields a Hopf algebra indexed by set compositions. This is the Hopf algebra $\mathrm{P}\Pi$ considered in [12, Section 6.2.4]. More information regarding this is given in Section 17.3.

The Hopf monoid $\mathbf{\Sigma}^*$ is commutative while the Hopf algebra $\mathcal{K}(\mathbf{\Sigma}^*)$ is not. Hence, we obtain an isomorphism of Hopf algebras

$$\mathcal{K}(\mathbf{\Sigma}^*) \to \mathcal{K}(\mathbf{\Sigma}^*)^{\mathrm{op}} \qquad \text{given by} \qquad M_F \mapsto M_{\omega_n(F)},$$

where $F$ and $\omega_n(F)$ are both compositions of $[n]$, the latter obtained from the former by replacing $i$ by $n+1-i$. For example, $M_{13|5|24|6} \mapsto M_{46|2|35|1}$.

## 15.6. The Fock functors and primitive elements

Let $\mathsf{gHopf}$ and $\mathsf{gLie}$ be the categories of graded Hopf algebras and graded Lie algebras respectively. Recall the classical functor

$$\mathcal{P}\colon \mathsf{gHopf} \to \mathsf{gLie},$$

which sends a Hopf algebra to its Lie algebra of primitive elements. The analogue of this functor for species, namely

$$\mathcal{P}\colon \mathsf{Hopf}(\mathsf{Sp}) \to \mathsf{Lie}(\mathsf{Sp}),$$

was defined in (11.39). The functors $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$ are better behaved with respect to $\mathcal{P}$ than the functors $\mathcal{K}$ and $\overline{\mathcal{K}}$. One reason is that the functor $\mathcal{K}^\vee$ being braided lax preserves Lie monoids while the functor $\mathcal{K}$ does not. We consider the diagram

$$\begin{tikzcd}
\mathsf{Hopf}(\mathsf{Sp}) \arrow[r, "\mathcal{P}"] \arrow[d] & \mathsf{Lie}(\mathsf{Sp}) \arrow[d] \\
\mathsf{gHopf} \arrow[r, "\mathcal{P}"'] & \mathsf{gLie}
\end{tikzcd}$$

with the vertical functors being either $\mathcal{K}^\vee$ or $\overline{\mathcal{K}}^\vee$.

**15.6.1. The main result.** For any Hopf monoid $\mathbf{h}$, $\mathcal{K}^\vee(\mathbf{h})$ can be viewed as a graded Lie algebra in two different ways. The first way is to view it as the image under $\mathcal{K}^\vee$ of the Lie monoid $\mathbf{h}$. The second way is to view the Hopf algebra $\mathcal{K}^\vee(\mathbf{h})$ as a graded Lie algebra. One checks that the two Lie structures coincide, the key being that $\mathcal{K}^\vee$ is braided lax. Since $\overline{\mathcal{K}}^\vee$ and $\overline{\mathcal{K}}$ are also braided lax, the same statement can be made for $\overline{\mathcal{K}}^\vee(\mathbf{h})$ and $\overline{\mathcal{K}}(\mathbf{h})$.

**Proposition 15.35.** *For any connected Hopf monoid* $\mathbf{h}$, *we have the following diagram of graded Lie algebras.*

$$\begin{tikzcd}
\mathcal{K}^\vee\big(\mathcal{P}(\mathbf{h})\big) \arrow[r, phantom, "\subseteq"] & \mathcal{P}\big(\mathcal{K}^\vee(\mathbf{h})\big) \arrow[r, phantom, "\subseteq"] & \mathcal{K}^\vee(\mathbf{h}) \\
\overline{\mathcal{K}}^\vee\big(\mathcal{P}(\mathbf{h})\big) \arrow[u, hook] \arrow[r, phantom, "="] & \mathcal{P}\big(\overline{\mathcal{K}}^\vee(\mathbf{h})\big) \arrow[u, hook] \arrow[r, phantom, "\subseteq"] & \overline{\mathcal{K}}^\vee(\mathbf{h}) \arrow[u, hook]
\end{tikzcd}$$

Proof. The inclusions in the second square are obvious. We check below the inclusion and equality in the first square (as graded vector spaces). Since all spaces involved are Lie subalgebras of $\mathcal{K}^\vee(\mathbf{h})$, the claim regarding the "Lie" part is automatic.

There are three coproducts one needs to keep track of; these are shown in the commutative diagram below.

$$\begin{tikzcd}
\mathbf{h}[n] \arrow[r, "\Delta"] & \displaystyle\bigoplus_{S\sqcup T=[n]} \mathbf{h}[S]\otimes\mathbf{h}[T] \arrow[r, "\varphi^\vee"] & \displaystyle\bigoplus_{s+t=n} \mathbf{h}[s]\otimes\mathbf{h}[t] \\
\mathbf{h}[n]^{\mathrm{S}_n} \arrow[u, hook] \arrow[r] & \Big(\displaystyle\bigoplus_{S\sqcup T=[n]} \mathbf{h}[S]\otimes\mathbf{h}[T]\Big)^{\mathrm{S}_n} \arrow[u, hook] \arrow[r, "\cong", "\overline{\varphi}^\vee"'] & \displaystyle\bigoplus_{s+t=n} \mathbf{h}[s]^{\mathrm{S}_s}\otimes\mathbf{h}[t]^{\mathrm{S}_t} \arrow[u, hook]
\end{tikzcd}$$

The coproduct on $\mathbf{h}$ is the map $\Delta$ above, the coproduct on $\mathcal{K}^\vee(\mathbf{h})$ is the composite $\varphi^\vee \circ \Delta$ of the top horizontal arrows, and the coproduct on $\overline{\mathcal{K}}^\vee(\mathbf{h})$ is the composite of the bottom horizontal arrows. It follows that

$$\Delta_+ = \varphi^\vee \circ \mathcal{K}^\vee(\Delta_+) \qquad \text{and} \qquad \Delta_+ = \overline{\varphi}^\vee \circ \overline{\mathcal{K}}^\vee(\Delta_+), \tag{15.18}$$

where the $\Delta_+$ in the right-hand sides refers to the positive part of the coproduct of $\mathbf{h}$ while the $\Delta_+$ in the left-hand sides refers to the positive part of the coproducts of $\mathcal{K}^\vee(\mathbf{h})$ and $\overline{\mathcal{K}}^\vee(\mathbf{h})$ respectively.

Recall that for a connected Hopf monoid $\mathbf{h}$ and $\Delta_+ \colon \mathbf{h}_+ \to \mathbf{h}_+ \cdot \mathbf{h}_+$, we have $\mathcal{P}(\mathbf{h}) = \ker \Delta_+$. The same result also holds for a connected graded Hopf algebra. The functoriality of $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$ and (15.18) now implies that

$$\mathcal{K}^\vee\big(\mathcal{P}(\mathbf{h})\big) \subseteq \mathcal{P}\big(\mathcal{K}^\vee(\mathbf{h})\big) \qquad \text{and} \qquad \overline{\mathcal{K}}^\vee\big(\mathcal{P}(\mathbf{h})\big) \subseteq \mathcal{P}\big(\overline{\mathcal{K}}^\vee(\mathbf{h})\big).$$

To complete the proof, we have to show that the second inclusion is an equality. Since the definition of the functor $\overline{\mathcal{K}}^\vee$ is in terms of invariants, we know that it is left exact. We now claim the following chain of equalities from which the result follows.

$$\overline{\mathcal{K}}^\vee\big(\mathcal{P}(\mathbf{h})\big) = \overline{\mathcal{K}}^\vee\big(\ker(\Delta_+)\big) = \ker\big(\overline{\mathcal{K}}^\vee(\Delta_+)\big) = \ker\big(\overline{\varphi}^\vee \circ \overline{\mathcal{K}}^\vee(\Delta_+)\big) = \mathcal{P}\big(\overline{\mathcal{K}}^\vee(\mathbf{h})\big)$$

The second equality holds because $\overline{\mathcal{K}}^\vee$ is left exact, and the third equality because $\overline{\varphi}^\vee$ is an isomorphism. □

Note that the functor $\mathcal{K}^\vee$ is also left exact. However, the argument given for $\overline{\mathcal{K}}^\vee$ fails for $\mathcal{K}^\vee$ because the map $\varphi^\vee$ is not an isomorphism.

**Proposition 15.36.** *For any connected Hopf monoid* $\mathbf{h}$, *we have the following diagram of graded vector spaces.*

$$\begin{tikzcd}
\mathcal{K}\big(\mathcal{P}(\mathbf{h})\big) \arrow[r, phantom, "\subseteq"] \arrow[d, two heads] & \mathcal{P}\big(\mathcal{K}(\mathbf{h})\big) \arrow[r, phantom, "\subseteq"] \arrow[d, two heads] & \mathcal{K}(\mathbf{h}) \arrow[d, two heads] \\
\overline{\mathcal{K}}\big(\mathcal{P}(\mathbf{h})\big) \arrow[r, phantom, "\subseteq"] & \mathcal{P}\big(\overline{\mathcal{K}}(\mathbf{h})\big) \arrow[r, phantom, "\subseteq"] & \overline{\mathcal{K}}(\mathbf{h})
\end{tikzcd}$$

*The bottom horizontal row is an inclusion of graded Lie algebras. Moreover, if* $\Bbbk$ *is a field of characteristic zero, then*

$$\overline{\mathcal{K}}\big(\mathcal{P}(\mathbf{h})\big) = \mathcal{P}\big(\overline{\mathcal{K}}(\mathbf{h})\big). \tag{15.19}$$

This result is similar to Proposition 15.35 and can be proved along the same lines. We note some differences. Since the functor $\mathcal{K}$ is not braided lax, the top horizontal row is only an inclusion of graded vector spaces. Since the definition of $\overline{\mathcal{K}}$ is in terms of coinvariants, this functor is right exact. Over a field of characteristic 0, it is also left exact, and (15.19) follows as in the proof of Proposition 15.35.

**15.6.2. Examples.** We illustrate the above results on some of our familiar examples. In particular, we will see that the results are optimal in the sense that all inclusions are strict in general. We write $\Bbbk x$ for the one-dimensional subspace spanned by the variable $x$ inside the space of polynomials in $x$. For convenience, we also use it to denote a graded vector space which is $\Bbbk$ in degree one and zero in all other components.

**Example 15.37.** Consider the Hopf monoid $\mathbf{E}$. According to Examples 11.44 and 15.14, the functor $\overline{\mathcal{K}}^\vee$ yields:

$$\overline{\mathcal{K}}^\vee\big(\mathcal{P}(\mathbf{E})\big) = \Bbbk x = \mathcal{P}(\Bbbk\{x\}) = \mathcal{P}\big(\overline{\mathcal{K}}^\vee(\mathbf{E})\big).$$

Note that $\mathcal{P}(\Bbbk\{x\})$ is always one-dimensional irrespective of the field characteristic. On the other hand, for the functor $\overline{\mathcal{K}}$, we get an inclusion

$$\overline{\mathcal{K}}\big(\mathcal{P}(\mathbf{E})\big) = \Bbbk x \subseteq \mathcal{P}(\Bbbk[x]) = \mathcal{P}\big(\overline{\mathcal{K}}(\mathbf{E})\big).$$

In characteristic 0, we have $\Bbbk x = \mathcal{P}(\Bbbk[x])$, but in characteristic $p$ the inclusion is strict: the primitive elements of $\Bbbk[x]$ are spanned by the monomials $x^{p^e}$ where $e \geq 0$.

More generally, for any Lie algebra $g$, the space of primitive elements of the universal enveloping algebra $\mathcal{U}(g)$ is the *restricted Lie subalgebra* of $\mathcal{U}(g)$ generated by $g$. (To get the previous result, let $g := \Bbbk x$.) This result is stated in the paper by Kharchenko [200, p. 69]. It also follows from [63, Exercises II.1.12 and II.3.4]. For the definition of restricted Lie algebras, see [175, Section V.7].

**Example 15.38.** Consider the Hopf monoid $\mathbf{L}$. According to Examples 11.44 and 15.17,

$$\mathcal{K}^\vee\big(\mathcal{P}(\mathbf{L})\big) = \mathcal{K}^\vee(\mathbf{Lie}) \subseteq \mathcal{P}(\mathrm{S}\Lambda) = \mathcal{P}\big(\mathcal{K}^\vee(\mathbf{L})\big),$$

and the dimension of the degree $n$ component of $\mathcal{P}(\mathrm{S}\Lambda)$ is the number of permutations in $\mathrm{S}_n$ with no global descents. Now one can conclude that the left-hand side is a proper Lie subalgebra of the right-hand side. This can be seen from a dimension count: the number of permutations on $n$ letters with no global descents is in general greater than $(n-1)!$, which is the dimension of $\mathbf{Lie}[n]$.

On the other hand, the functor $\overline{\mathcal{K}}^\vee$ yields an equality of Lie algebras:

$$\overline{\mathcal{K}}^\vee\big(\mathcal{P}(\mathbf{L})\big) = \bigoplus_n (\mathbf{Lie}[n])^{\mathrm{S}_n} = \mathcal{P}(\Bbbk[x]) = \mathcal{P}\big(\overline{\mathcal{K}}^\vee(\mathbf{L})\big).$$

The first equality says that the space

$$\bigoplus_n (\mathbf{Lie}[n])^{\mathrm{S}_n}$$

carries the structure of a Lie algebra. We denote its bracket by $*$. It can be made explicit using the lax structure of $\overline{\mathcal{K}}^\vee$ and the bracket of the Lie monoid $\mathbf{Lie}$. The second equality says that this Lie algebra is abelian and further its component in degree $p^e$ for $e \geq 0$ is $\Bbbk$, while all other components are zero (here $p$ is the field characteristic). This is a special case of a result of Fresse [136, Theorem 1.2.5] and [137, Proposition 1.2.16] which implies that $\overline{\mathcal{K}}^{\vee}(\mathbf{Lie})$ is the free restricted Lie algebra on one generator. Fresse's result implies more generally that $\overline{\mathcal{K}}^{\vee}_V(\mathbf{Lie})$ is the free restricted Lie algebra on $V$ (the functor $\overline{\mathcal{K}}^{\vee}_V$ is defined in Chapter 19).

For example, in characteristic 2, the invariants in degrees 1, 2 and 4 are spanned by

$$[1], \qquad [1\ 2] \qquad \text{and} \qquad \big[[1\ 2][3\ 4]\big] + \big[[1\ 3][2\ 4]\big] + \big[[1\ 4][2\ 3]\big].$$

We explicitly compute the $*$ product in two cases and check that it is zero.

$$[1] * [1\ 2] = \big[1[2\ 3]\big] + \big[2[1\ 3]\big] + \big[3[1\ 2]\big] = 0.$$

Note that for the product we shift up the indices of the second term and then sum over all shuffles. The middle term is the Jacobi identity and hence zero (this is why there are no degree 3 invariants).

$$[1\ 2]*[1\ 2] = \big[[1\ 2][3\ 4]\big]+\big[[1\ 3][2\ 4]\big]+\big[[1\ 4][2\ 3]\big]+\big[[2\ 3][1\ 4]\big]+\big[[2\ 4][1\ 3]\big]+\big[[3\ 4][1\ 2]\big].$$

The right-hand side is twice the degree 4 invariant and hence zero in characteristic 2.

For the functor $\overline{\mathcal{K}}$, we get an inclusion (strict in general):

$$\overline{\mathcal{K}}\big(\mathcal{P}(\mathbf{L})\big) = \bigoplus_n (\mathbf{Lie}[n])_{\mathrm{S}_n} \subseteq \mathcal{P}(\Bbbk[x]) = \mathcal{P}\big(\overline{\mathcal{K}}(\mathbf{L})\big).$$

The left-hand side is the free Lie algebra on one generator and is always one-dimensional, except in characteristic 2.

Now consider the Hopf monoid $\mathbf{L}^*$. We have

$$\mathcal{K}\big(\mathcal{P}(\mathbf{L}^*)\big) = \Bbbk x \subseteq \mathcal{P}(\mathrm{S}\Lambda) = \mathcal{P}\big(\mathcal{K}(\mathbf{L}^*)\big),$$
$$\mathcal{K}^{\vee}\big(\mathcal{P}(\mathbf{L}^*)\big) = \Bbbk x \subseteq \mathcal{P}\big(\mathcal{K}^{\vee}(\mathbf{L}^*)\big).$$

Both inclusions are strict. The graded dimension of the spaces on the right in both statements is the same. The dimension of the component of degree $n$ is the number of permutations in $\mathrm{S}_n$ with no global descents. For completeness, we also record the following.

$$\overline{\mathcal{K}}\big(\mathcal{P}(\mathbf{L}^*)\big) = \Bbbk x \subseteq \mathcal{P}(\Bbbk\{x\}) = \mathcal{P}\big(\overline{\mathcal{K}}(\mathbf{L}^*)\big),$$
$$\overline{\mathcal{K}}^{\vee}\big(\mathcal{P}(\mathbf{L}^*)\big) = \Bbbk x = \mathcal{P}(\Bbbk\{x\}) = \mathcal{P}\big(\overline{\mathcal{K}}^{\vee}(\mathbf{L}^*)\big).$$

More examples are given in Sections 17.2.5 and 17.3.3.

**Remark 15.39.** The vector space $\bigoplus_{n\geq 1}\mathbf{Lie}[n]$ carries a structure of Lie algebra (Lie subalgebra of $\mathcal{P}(\mathrm{S}\Lambda)$) and also (a different structure) of twisted Lie algebra, as mentioned in Section 11.9.1. Both structures are discussed in [11, Section 5.3]. The connection between the two becomes now clear: the twisted Lie algebra structure is an equivalent formulation of the Lie monoid structure of the species **Lie**, while the Lie algebra structure is the result of applying the functor $\mathcal{K}^{\vee}$ to this Lie monoid.

## 15.7. The full Fock functors and dendriform algebras

In this section we look at further monoidal properties of the Fock functors. The main result is that $\mathcal{K}^{\vee}$ is a Zinbiel-lax monoidal functor. We discuss some consequences involving dendriform algebra structures.

**15.7.1. The operadic monoidal properties of $\mathcal{K}^\vee$.** Let us restrict the full Fock functor $\mathcal{K}^\vee$ to the category of positive species (Section 8.9.2). Its image then lies in the category of positively graded vector spaces (Section 2.3.4):

$$\mathcal{K}^\vee \colon (\mathsf{Sp}_+, \cdot) \to (\mathsf{gVec}_+, \cdot).$$

We proceed to turn this functor into a Zinbiel lax monoidal functor, as in Definition 4.9. Note that the monoidal categories are nonunital, as in the situation of Notation 4.6. We first need to define a natural transformation

$$\gamma \colon \mathcal{K}^\vee(\mathbf{p}) \cdot \mathcal{K}^\vee(\mathbf{q}) \to \mathcal{K}^\vee(\mathbf{p} \cdot \mathbf{q}).$$

For this, we need maps

$$\mathbf{p}[s] \otimes \mathbf{q}[t] \to \bigoplus_{S \sqcup T = [n]} \mathbf{p}[S] \otimes \mathbf{q}[T] \tag{15.20}$$

where $s$ and $t$ are nonzero and $S$ and $T$ are nonempty. We define this to be the direct sum of the following maps, one for each summand in the target with $|S| = s$ and $|T| = t$ and $1 \in S$:

$$\mathbf{p}[s] \otimes \mathbf{q}[t] \xrightarrow{\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}]} \mathbf{p}[S] \otimes \mathbf{q}[T].$$

**Proposition 15.40.** *The functor $(\mathcal{K}^\vee, \gamma)$ is Zinbiel-lax monoidal.*

Proof. Both sides of (4.5) yield a map of the form

$$\mathbf{p}[s] \otimes \mathbf{q}[t] \otimes \mathbf{r}[u] \to \bigoplus_{S \sqcup T \sqcup U = [n]} \mathbf{p}[S] \otimes \mathbf{q}[T] \otimes \mathbf{r}[U].$$

It follows from (15.20) that the left-hand side of (4.5) is the sum of the maps

$$\mathbf{p}[s] \otimes \mathbf{q}[t] \otimes \mathbf{r}[u] \xrightarrow{\mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}] \otimes \mathbf{r}[\mathrm{cano}]} \mathbf{p}[S] \otimes \mathbf{q}[T] \otimes \mathbf{r}[U],$$

one for each summand in the target with $|S| = s$, $|T| = t$, $|U| = u$ and $1 \in S$. The right-hand side of (4.5) consists of the sum of the same maps, split according to whether $s + 1 \in T$ or $s + 1 \in U$. $\square$

Recall from (15.2) that the lax structure $\psi^\vee$ of $\mathcal{K}^\vee$ is given by a formula similar to that of $\gamma$, in which the sum is over *all* summands in the target with $|S| = s$ and $|T| = t$. Observe that $\gamma^b$, the conjugate of $\gamma$ by the braiding as given in Definition 3.14, has the same description as $\gamma$ except that the condition $1 \in S$ is replaced by $1 \notin S$. Therefore,

$$\psi^\vee = \gamma + \gamma^b. \tag{15.21}$$

Recall from Proposition 4.12 that associated to a Zinbiel-lax monoidal structure on a functor there is a braided lax monoidal structure on the same functor. It follows from the preceding observation that the braided lax monoidal structure on $\mathcal{K}^\vee$ associated to $\gamma$ is $\psi^\vee$. We thus recover (the nonunital version of) the result that $(\mathcal{K}^\vee, \psi^\vee)$ is braided lax monoidal (Proposition 15.28).

**15.7.2. The functor $\mathcal{K}^\vee$ and dendriform algebras.** We combine the general results on transformation of monoids under monoidal functors (Sections 4.1.3 and 4.4.4) and the fact that $\mathcal{K}^\vee$ is Zinbiel-lax in order to derive two constructions of graded dendriform and graded Zinbiel algebras.

**Proposition 15.41.** *Let* $\mathbf{p}$ *be a nonunital associative monoid in* $(\mathsf{Sp}_+, \cdot)$*. Then* $\mathcal{K}^\vee(\mathbf{p})$ *is a graded dendriform algebra. If* $\mathbf{p}$ *is commutative, then* $\mathcal{K}^\vee(\mathbf{p})$ *is in fact a graded Zinbiel algebra.*

Proof. Both statements follow from Proposition 4.15. □

We mention that dual results hold for the full Fock functor $\mathcal{K}$. If $\mathbf{p}$ is a noncounital comonoid in $(\mathsf{Sp}_+, \cdot)$, then $\mathcal{K}(\mathbf{p})$ is a graded dendriform coalgebra, and if $\mathbf{p}$ is cocommutative, then $\mathcal{K}(\mathbf{p})$ is in fact a graded Zinbiel coalgebra.

We illustrate Proposition 15.41 with two well-known examples.

**Example 15.42.** The positive decorated exponential species $(\mathbf{E}_V)_+$ is a nonunital commutative monoid in $\mathsf{Sp}_+$ (Example 8.18). It follows from Proposition 15.41 that

$$\mathcal{T}^\vee(V)_+ = \mathcal{K}^\vee\big((\mathbf{E}_V)_+\big)$$

is a (graded) Zinbiel algebra. As a nonunital associative algebra, it is the positive degree part of the shuffle algebra on $V$ (Section 2.6.1). The fact that $\mathcal{T}^\vee(V)_+$ is a Zinbiel algebra is well-known; in fact, it is the free Zinbiel algebra on $V$, see [238, Section 7.1] and [325, p. 19].

**Example 15.43.** The positive linear order species $\mathbf{L}_+$ (Example 8.16) is a nonunital monoid in $\mathsf{Sp}_+$. It follows from Proposition 15.41 that

$$\mathrm{S}\Lambda^*_+ = \mathcal{K}^\vee(\mathbf{L}_+)$$

is a graded dendriform algebra. Here $\mathrm{S}\Lambda^*$ is the dual of the Malvenuto–Reutenauer Hopf algebra, as explained in Example 15.17. The dendriform structure was introduced by Loday and Ronco in [243, Definition 4.4].

CHAPTER 16

# Deformations of Fock Functors

Hopf monoids in the category of species are richer than the graded Hopf algebras that correspond to them under the functors $\mathcal{K}$ and $\mathcal{K}^\vee$. In this chapter we show that one can construct $q$-deformations of those Hopf algebras starting from the *same* Hopf monoids. In fact, it is the bilax monoidal functors themselves that can be deformed. We call the deformed functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$ and refer to them as the $q$-Fock functors. Further, we show that these functors can also be applied to $p$-deformations of Hopf monoids in which case one obtains $pq$-deformations of the corresponding Hopf algebras.

We begin by explaining these ideas in Section 16.1. In Section 16.2, we study the norm transformation between $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$. This is a deformation of the norm transformation between $\mathcal{K}$ and $\mathcal{K}^\vee$. The $q$-norm, for generic values of $q$, behaves quite differently from the norm for $q = 1$. We show that if $q$ is not a root of unity and the field characteristic is zero, then the $q$-norm is an isomorphism.

Recall that for $q = 1$, in addition to the full Fock functors, we had also considered bosonic Fock functors. We had constructed them by taking invariants and coinvariants. Something similar can be done for $q = -1$. This leads to the fermionic Fock functors, which are studied in Section 16.3.

The theory for parameter values $\pm 1$ is in many ways special. In representation theory this corresponds to the fact that the symmetric group has two one-dimensional representations, namely the trivial and sign representations. In category theory this corresponds to the fact that among a family of braidings $\beta_q$ on graded vector spaces and species, only $q = \pm 1$ are symmetries.

In general, the image of the norm transformation yields a functor which we denote by $\Im_q$. We refer to it as the anyonic Fock functor. In characteristic 0, for $q = 1$, it recovers the bosonic Fock functors and for $q = -1$, it recovers the fermionic Fock functors. For $q = 0$, the norm transformation is the identity, and so $\mathcal{K}_0 = \mathcal{K}_0^\vee = \Im_0$. We refer to this as the free Fock functor. The different functors are summarized in Table 16.1. The entries complement those in Table 15.1.

The behavior of the deformed full Fock functor with respect to commutativity is studied in Section 16.4. The functor $\mathcal{K}_q$ is not braided colax in general. We show

TABLE 16.1. The deformed Fock functors.

| Fock functor | Name |
|---|---|
| $\mathcal{K}_q$, $\mathcal{K}_q^\vee$ | Deformed full Fock functor |
| $\Im_q$ | Anyonic Fock functor |
| $\mathcal{K}_0 = \mathcal{K}_0^\vee = \Im_0$ | Free Fock functor |

that conjugating the colax structure with the braidings yields the functor $\mathcal{K}_{q^{-1}}$. We also construct a $q$-analogue of the half-twist transformation. We conclude this chapter with Section 16.5 which contains some illustrative examples.

The constructions in this chapter will make use of the Schubert statistic and related notions from Section 2.2.

## 16.1. Deformations of the full Fock functors

Throughout this section, $p$ and $q$ are fixed scalars, possibly zero. Our goal is to construct bilax deformations $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$ of the full Fock functors of Chapter 15.

**16.1.1. The functor $\mathcal{K}_q$.** Recall the bilax monoidal functor $(\mathcal{K}, \varphi, \psi)$ of Section 15.1.1. We now proceed to construct a bilax monoidal functor

$$(\mathcal{K}, \varphi, \psi_q)\colon (\mathsf{Sp}, \cdot, \beta_p) \to (\mathsf{gVec}, \cdot, \beta_{pq}),$$

which is a deformation of the previous construction. The functor $\mathcal{K}$ and the natural transformation $\varphi$ are the same; namely

$$\mathcal{K}(\mathbf{q}) := \bigoplus_{n\geq 0} \mathbf{q}[n]$$

and

$$\varphi_{\mathbf{p},\mathbf{q}}\colon \mathcal{K}(\mathbf{p}) \cdot \mathcal{K}(\mathbf{q}) \to \mathcal{K}(\mathbf{p}\cdot\mathbf{q})$$

has components

$$\bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[t] \xrightarrow{\bigoplus \mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}]} \bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t] \subseteq \bigoplus_{S\sqcup T=[n]} \mathbf{p}[S] \otimes \mathbf{q}[T].$$

On the other hand, the natural transformation

$$(\psi_q)_{\mathbf{p},\mathbf{q}}\colon \mathcal{K}(\mathbf{p}\cdot\mathbf{q}) \to \mathcal{K}(\mathbf{p}) \cdot \mathcal{K}(\mathbf{q})$$

has components

$$\bigoplus_{S\sqcup T=[n]} \mathbf{p}[S] \otimes \mathbf{q}[T] \to \bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[t]$$

which map

$$x \otimes y \mapsto q^{\mathrm{sch}_n(S)}\, \mathbf{p}[\mathrm{cano}](x) \otimes \mathbf{q}[\mathrm{cano}](y),$$

where $\mathrm{sch}_n(S)$ is the Schubert statistic (2.13) and the canonical maps in question are

$$\mathrm{cano}\colon S \to [|S|] \quad \text{and} \quad \mathrm{cano}\colon T \to [|T|],$$

as in Notation 2.5. Thus, up to a power of $q$, the map $\psi_q$ is the same as the map $\psi$.

As for $\mathcal{K}$, we let $\varphi_0$ and $(\psi_q)_0$ be the identity maps

$$\Bbbk \xrightarrow{\varphi_0} \mathcal{K}(\mathbf{1}) \xrightarrow{(\psi_q)_0} \Bbbk.$$

**Theorem 16.1.** *The functor*

$$(\mathcal{K}, \varphi, \psi_q)\colon (\mathsf{Sp}, \cdot, \beta_p) \to (\mathsf{gVec}, \cdot, \beta_{pq})$$

*is bilax monoidal.*

Proof. In view of Theorem 15.3, the only diagrams in Definitions 3.1, 3.2 and 3.3 that require verification are those involving $\psi_q$. Moreover, Theorem 15.3 guarantees that these diagrams commute up to a power of $p$ and $q$; we only need to check that the two powers resulting from each diagram agree for both $p$ and $q$.

Note that the braiding and hence the parameter $p$ is relevant only to diagram (3.11) and we observe directly that the powers of $p$ agree. We now deal with powers of $q$. Diagram (3.13) involves no powers of $q$. For the coassociativity and counitality diagrams required for $(\mathcal{K}, \psi_q)$ to be colax (Definition 3.2) this is the case by the properties of the Schubert statistic given in (2.17) and (2.14), respectively. Similarly, properties (2.18) and (2.14) respectively guarantee that the powers of $q$ agree for diagrams (3.11) and (3.12) relating $\varphi$ and $\psi_q$. $\square$

We use $\mathcal{K}_q$ as an abbreviation for $(\mathcal{K}, \varphi, \psi_q)$. Since $\psi_1 = \psi$, we have $\mathcal{K}_1 = \mathcal{K}$. In this sense, $\mathcal{K}_q$ is a deformation of the bilax monoidal functor $\mathcal{K}$.

**16.1.2. The functor $\mathcal{K}_q^\vee$.** A deformation of the bilax monoidal functor $\mathcal{K}^\vee$ of Section 15.1.2 can be constructed too. Define a natural transformation

$$(\psi_q^\vee)_{\mathbf{p},\mathbf{q}} \colon \mathcal{K}^\vee(\mathbf{p}) \cdot \mathcal{K}^\vee(\mathbf{q}) \to \mathcal{K}^\vee(\mathbf{p} \cdot \mathbf{q})$$

with components

$$\mathbf{p}[s] \otimes \mathbf{q}[t] \to \bigoplus_{S \sqcup T = [n]} \mathbf{p}[S] \otimes \mathbf{q}[T]$$

which map

$$x \otimes y \mapsto \sum_{\substack{S \sqcup T = [n] \\ |S|=s, |T|=t}} q^{\mathrm{sch}_n(S)}\, \mathbf{p}[\mathrm{cano}](x) \otimes \mathbf{q}[\mathrm{cano}](y),$$

where the canonical maps in question are, for each term in the sum,

$$\mathrm{cano} \colon [s] \to S \qquad \text{and} \qquad \mathrm{cano} \colon [t] \to T.$$

Thus, up to a power of $q$, the map $\psi_q^\vee$ is the same as the map $\psi^\vee$ of (15.2). The natural transformation

$$\varphi^\vee_{\mathbf{p},\mathbf{q}} \colon \mathcal{K}^\vee(\mathbf{p} \cdot \mathbf{q}) \to \mathcal{K}^\vee(\mathbf{p}) \cdot \mathcal{K}^\vee(\mathbf{q})$$

is the same as that in (15.2), and we let $(\psi_q^\vee)_0$ and $\varphi_0^\vee$ be the identity maps

$$\Bbbk \xrightarrow{(\psi_q^\vee)_0} \mathcal{K}^\vee(\mathbf{1}) \xrightarrow{\varphi_0^\vee} \Bbbk.$$

It is straightforward to show that:

**Theorem 16.2.** *The functor*

$$(\mathcal{K}^\vee, \psi_q^\vee, \varphi^\vee) \colon (\mathsf{Sp}, \cdot, \beta_p) \to (\mathsf{gVec}, \cdot, \beta_{pq})$$

*is bilax monoidal.*

We abbreviate $(\mathcal{K}^\vee, \psi_q^\vee, \varphi^\vee)$ to $\mathcal{K}_q^\vee$. As suggested by the notation, in the finite-dimensional setting, the functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$ are related by the contragredient construction of Section 3.10: either one is obtained from the other by conjugating with the duality functors on species and on graded vector spaces.

**Proposition 16.3.** *On finite-dimensional species, the bilax functor $(\mathcal{K}^\vee, \psi_q^\vee, \varphi^\vee)$ is isomorphic to the contragredient of $(\mathcal{K}, \varphi, \psi_q)$.*

PROOF. This follows from Proposition 15.8 by noting in addition that the colax structure of $\mathcal{K}$ and the lax structure of $\mathcal{K}^\vee$ are deformed in exactly the same way by using the Schubert statistic. $\square$

In view of the above result, Theorems 16.1 and 16.2 are equivalent to each other in the finite-dimensional setting. We will see many statements of this type.

**16.1.3. The free Fock functor $\mathcal{K}_0$.** Consider the functor $(\mathcal{K}, \varphi, \varphi^\vee)$; that is, we mix the lax structure of $\mathcal{K}$ with the colax structure of $\mathcal{K}^\vee$. At first glance, this may seem a little strange; however we show below that this is very natural.

**Proposition 16.4.** *We have*

$$(\mathcal{K}, \varphi, \psi_0) = (\mathcal{K}^\vee, \psi_0^\vee, \varphi^\vee).$$

*In other words, $\varphi = \psi_0^\vee$ and $\psi_0 = \varphi^\vee$.*

This is a straightforward check. This says that

$$\mathcal{K}_0 = \mathcal{K}_0^\vee = (\mathcal{K}, \varphi, \varphi^\vee). \tag{16.1}$$

It follows from either of Theorems 16.1 or 16.2 that the above is a bilax monoidal functor from $(\mathsf{Sp}, \cdot, \beta_p)$ to $(\mathsf{gVec}, \cdot, \beta_0)$. Further, it is self-dual (Definition 3.105). We refer to it as the *free Fock functor*.

It is also possible to view this functor as a deformation of the bosonic and fermionic Fock functors. This point of view, which motivates our terminology, is explained in Section 16.3.5.

**16.1.4. Relating the deformed full Fock functors by the signature functor.** Recall the signature functor $(-)^-$ on species from Section 9.4.2. It is natural to ask what happens if one precomposes the deformed full Fock functor by the signature functor. The answer is given by the following result.

**Proposition 16.5.** *The following diagram commutes* (*up to isomorphism*) *as bilax monoidal functors.*

$$\begin{tikzcd} (\mathsf{Sp}, \cdot, \beta_p) \arrow[rr, "(-)^-"] \arrow[dr, "\mathcal{K}_{-q}"'] & & (\mathsf{Sp}, \cdot, \beta_{-p}) \arrow[dl, "\mathcal{K}_q"] \\ & (\mathsf{gVec}, \cdot, \beta_{-pq}) & \end{tikzcd} \tag{16.2}$$

*The same result holds with $\mathcal{K}$ replaced by $\mathcal{K}^\vee$.*

PROOF. We note that using the canonical linear order on $[n]$, there is an isomorphism of vector spaces

$$\mathbf{p}[n] \otimes \mathbf{E}^-[n] \xrightarrow{\cong} \mathbf{p}[n] \qquad x \otimes (1 \wedge \cdots \wedge n) \mapsto x \tag{16.3}$$

and hence

$$\mathcal{K}_q(\mathbf{p}^-) \cong \mathcal{K}_{-q}(\mathbf{p}).$$

This says that there is a natural isomorphism of functors

$$\mathcal{K}_q\big((-)^-\big) \cong \mathcal{K}_{-q}(-).$$

The parameter $q$ has played no part so far. We claim next that the above is in fact an isomorphism of bilax monoidal functors. The lax part is straightforward (the parameter $q$ still plays no role). The check for the colax part boils down to the identity

$$q^{\mathrm{sch}_n(S)}(-1)^{\mathrm{sch}_n(S)} = (-q)^{\mathrm{sch}_n(S)}.$$

This proves the first assertion.

Applying the contragredient construction to (16.2) and using Proposition 9.11 yields the same diagram, but with $\mathcal{K}^\vee$ instead of $\mathcal{K}$. □

**16.1.5. Constructing the deformed full Fock functors from the bosonic Fock functors.** It is interesting to note that $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ can, in fact, be used to construct the deformed functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$. The $q$-Hopf monoid $\mathbf{L}_q$ and its dual $\mathbf{L}_q^*$ defined in Section 9.5 play a crucial role in this construction as explained below. This generalizes Propositions 15.9 and 15.10 which give the undeformed case.

**Proposition 16.6.** *There are isomorphisms of bilax functors*

$$\mathcal{K}_q(-) \cong \overline{\mathcal{K}}(\mathbf{L}_q \times (-)) \qquad \textit{and} \qquad \mathcal{K}_q^\vee(-) \cong \overline{\mathcal{K}}^\vee(\mathbf{L}_q^* \times (-))$$

*from* $(\mathsf{Sp}, \cdot, \beta_p)$ *to* $(\mathsf{gVec}, \cdot, \beta_{pq})$.

Proof. If we set $p = 1$ and $q = 1$, then the first claim specializes to Proposition 15.9. For the general case, we define the above isomorphism exactly as in the proof of this proposition; hence it is independent of both $p$ and $q$. Note also that for both functors in question, the lax structure is independent of both $p$ and $q$ while the colax structure depends on $q$. In the latter case, the power of $q$ arising from the first functor is $\mathrm{sch}_n(S)$ and from the second functor is $\mathrm{sch}_{S,T}(C_{(n)})$, where $C_{(n)}$ is the canonical linear order on $[n]$ and $\mathrm{sch}_{S,T}$ is the Schubert cocycle which appears in Definition 9.13. The two powers are equal by (9.13).

The second claim can be proved in a similar manner. In the finite-dimensional setting, it follows from the first via the contragredient construction. □

Proposition 16.5 can be derived from Proposition 16.6 as follows.

$$\mathcal{K}_q\big((-)^-\big) \cong \overline{\mathcal{K}}(\mathbf{L}_q \times (-)^-) \cong \overline{\mathcal{K}}\big((\mathbf{L}_q)^- \times (-)\big) \cong \overline{\mathcal{K}}(\mathbf{L}_{-q} \times (-)) \cong \mathcal{K}_{-q}(-)$$

The first and last isomorphisms use the above result, the second isomorphism uses Proposition 9.12, and the third isomorphism uses (9.20).

**16.1.6. Evaluating the deformed full Fock functors.** We know that bilax functors preserve bimonoids. We record the implications of this statement along with examples for the deformed bilax functors considered above.

**Theorem 16.7.** *If* $\mathbf{h}$ *is a* $p$*-Hopf monoid, then* $\mathcal{K}_q(\mathbf{h})$ *and* $\mathcal{K}_q^\vee(\mathbf{h})$ *are* $pq$*-Hopf algebras.*

Proof. Proposition 3.31 and Theorems 16.1 and 16.2 say that $\mathcal{K}_q(\mathbf{h})$ and $\mathcal{K}_q^\vee(\mathbf{h})$ will take $p$-bimonoids to $pq$-bialgebras. For the Hopf part, the same argument as in the proof of Theorem 15.12 can be used. □

Letting both $p$ and $q$ to be 1 recovers Theorem 15.12. As explained in Section 15.2, Stover was the first to point out that signed or unsigned Hopf algebras could be constructed from Hopf monoids $\mathbf{h}$ in species. These are the Hopf algebras $\mathcal{K}_q(\mathbf{h})$ and $\mathcal{K}_q^\vee(\mathbf{h})$ when $p = 1$ and $q = \pm 1$. More generally, one may set $p = 1$ and let $q$ be arbitrary. In this situation, one obtains $q$-Hopf algebras starting with usual Hopf monoids. This shows that *every* Hopf algebra arising from a Hopf monoid in species admits a $q$-deformation. We encounter instances of this in Section 16.5.

We now turn our attention to the degenerate case. Letting either $p$ or $q$ to be 0 yields the following interesting corollaries.

**Corollary 16.8.** *The free Fock functor $\mathcal{K}_0$ sends $p$-Hopf monoids to 0-Hopf algebras.*

**Corollary 16.9.** *If $\mathbf{h}$ is a 0-Hopf monoid, then $\mathcal{K}_q(\mathbf{h})$ and $\mathcal{K}_q^\vee(\mathbf{h})$ are 0-Hopf algebras.*

Note that both the results allow us to construct 0-Hopf algebras. The first result for $p = 1$ is a special case of a result of Livernet [233, Theorem 4.1.2].

**Theorem 16.10.** *Let $\mathbf{h}$ be a finite-dimensional $p$-Hopf monoid. There is a natural isomorphism of $pq$-Hopf algebras*

$$\mathcal{K}_q^\vee(\mathbf{h}) \cong \mathcal{K}_q(\mathbf{h}^*)^*$$

*given by the canonical identification $\mathbf{h}[n] \cong (\mathbf{h}[n]^*)^*$.*

The above result follows from Propositions 16.3 and 3.32.

**16.1.7. Freeness and cofreeness results of Livernet.** Livernet explained how certain freeness and cofreeness results can be established for Hopf algebras which arise from Hopf monoids. We explain below how these ideas fit in our framework.

**Proposition 16.11.** *If $\mathbf{h}$ is a connected 0-bimonoid, then both $\mathcal{K}_q(\mathbf{h})$ and $\mathcal{K}_q^\vee(\mathbf{h})$ are free and cofree. If $\mathbf{h}$ is a connected $p$-bimonoid, then $\mathcal{K}_q(\mathbf{h})$ is free and $\mathcal{K}_q^\vee(\mathbf{h})$ is cofree.*

Proof. For the first part, we note that both $\mathcal{K}_q(\mathbf{h})$ and $\mathcal{K}_q^\vee(\mathbf{h})$ are connected 0-bialgebras. Hence the claim follows from Theorem 2.13.

For the second part, we know that $\mathcal{K}_0(\mathbf{h})$ is a connected 0-bialgebra and hence by Theorem 2.13, it is free. Now note that the product of $\mathcal{K}_q(\mathbf{h})$ is independent of $q$. Hence it follows that $\mathcal{K}_q(\mathbf{h})$ is also free. The cofreeness of $\mathcal{K}_q^\vee(\mathbf{h})$ is established in a similar manner. □

The second part of this result for $p = q = 1$ is due to Livernet [233, Theorem 4.2.2].

## 16.2. The deformed norm transformation

Recall that for $q = 1$, we had used the full Fock functors to construct the bosonic Fock functors by taking invariants and coinvariants. A major difference between that context and the present context is that invariants and coinvariants no longer lead to bilax monoidal functors. The right point of view is to deform the norm transformation (Section 15.4) and to look for its image (or coimage).

In this section, we construct this deformation. Further, we show that for generic values of $q$, it is an isomorphism. The image of the deformed norm transformation will be further explored in Section 16.3.

**16.2.1. Relating the structure constants of the deformed full Fock functors.** Let $\mathrm{Sh}(s,t)$ denote the set of $(s,t)$-shuffle permutations (2.21), and let $\mathrm{inv}(\zeta)$ denote the number of inversions of the permutation $\zeta$ (2.20).

**Lemma 16.12.** *The structure maps $\varphi$ and $\psi_q^\vee$, and $\psi_q$ and $\varphi^\vee$, are related by the formulas*

$$\psi_q^\vee(x\otimes y) = \sum_{\zeta\in \mathrm{Sh}\,(s,t)} q^{\mathrm{inv}(\zeta)}\,\zeta(\varphi(x\otimes y)) \qquad \text{for } x\in \mathbf{p}[s],\ y\in \mathbf{q}[t], \tag{16.4}$$

$$\psi_q(a\otimes b) = \sum_{\zeta\in \mathrm{Sh}\,(|S|,|T|)} q^{\mathrm{inv}(\zeta)}\,\varphi^\vee(\zeta^{-1}(a\otimes b)) \quad \text{for } a\in \mathbf{p}[S],\ b\in \mathbf{q}[T]. \tag{16.5}$$

The result follows from the proof of Lemma 15.18 complemented with (2.26).

### 16.2.2. The $q$-norm transformation from $\mathcal{K}_q$ to $\mathcal{K}_q^\vee$.

**Definition 16.13.** For any species $\mathbf{p}$, let $(\kappa_q)_{\mathbf{p}}\colon \mathcal{K}_q(\mathbf{p}) \to \mathcal{K}_q^\vee(\mathbf{p})$ be the map of graded vector spaces given by

$$\mathbf{p}[n] \to \mathbf{p}[n] \qquad (\kappa_q)_{\mathbf{p}}(z) = \sum_{\sigma\in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)}\,\sigma\cdot z, \tag{16.6}$$

for any $z\in \mathbf{p}[n]$. This defines a natural transformation $\kappa_q\colon \mathcal{K}_q \Rightarrow \mathcal{K}_q^\vee$ which we call the *$q$-norm.*

**Proposition 16.14.** *For finite-dimensional species, the $q$-norm is self-dual. In other words,*

$$(\kappa_q)^\vee = \kappa_q.$$

PROOF. One starts out by writing the norm map on the dual species $\mathbf{p}^*$. Thus

$$(\kappa_q)_{\mathbf{p}^*}\colon \bigoplus \mathbf{p}^*[n] \to \bigoplus \mathbf{p}^*[n] \qquad \alpha \mapsto \sum_{\sigma\in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)}\,\sigma\cdot\alpha$$

for $\alpha\in \mathbf{p}^*[n]$. Dualizing this map and identifying $(\mathbf{p}^*)^*$ with $\mathbf{p}$, we obtain:

$$(\kappa_q)_{\mathbf{p}}^\vee\colon \bigoplus \mathbf{p}[n] \to \bigoplus \mathbf{p}[n] \qquad z \mapsto \sum_{\sigma\in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)}\,\sigma^{-1}\cdot z$$

for any $z\in \mathbf{p}[n]$. The result now follows from (2.25). ☐

**Proposition 16.15.** *The $q$-norm is a morphism of bilax monoidal functors*

$$\kappa_q\colon \mathcal{K}_q \Rightarrow \mathcal{K}_q^\vee.$$

PROOF. We use the same notation and we argue as in the proof of Proposition 15.20. Using (16.5) we find

$$\begin{aligned}
&\big((\kappa_q)_{\mathbf{p}}\cdot(\kappa_q)_{\mathbf{q}}\big)(\psi_q)_{\mathbf{p},\mathbf{q}}(a\otimes b)\\
&\qquad\qquad = \sum_{\substack{\sigma\in \mathrm{S}_s\\ \tau\in \mathrm{S}_t}} \sum_{\zeta\in \mathrm{Sh}(s,t)} q^{\mathrm{inv}(\zeta)+\mathrm{inv}(\sigma)+\mathrm{inv}(\tau)}\,\varphi^\vee_{\mathbf{p},\mathbf{q}}\big((\sigma\times\tau)\cdot\zeta^{-1}\cdot(a\otimes b)\big).
\end{aligned}$$

Using (2.24) and (2.25) we deduce

$$\big((\kappa_q)_{\mathbf{p}}\cdot(\kappa_q)_{\mathbf{q}}\big)(\psi_q)_{\mathbf{p},\mathbf{q}}(a\otimes b) = \sum_{\rho\in \mathrm{S}_n} q^{\mathrm{inv}(\rho)}\,\varphi^\vee_{\mathbf{p},\mathbf{q}}(\rho\cdot(a\otimes b)) = \varphi^\vee_{\mathbf{p},\mathbf{q}}(\kappa_q)_{\mathbf{p}\cdot\mathbf{q}}(a\otimes b).$$

Thus

$$(\kappa_q\cdot\kappa_q)\psi_q = \varphi^\vee\kappa_q,$$

which proves that $\kappa_q$ is a morphism of colax functors.

The fact that it is a morphism of lax functors can be similarly verified. In the finite-dimensional setting, it follows from Proposition 16.14. ☐

**Proposition 16.16.** *The morphism of bilax monoidal functors obtained by precomposing $\kappa_q$ with the signature functor is isomorphic to $\kappa_{-q}$.*

PROOF. Consider the morphism of bilax monoidal functors obtained by precomposing $\kappa_q$ with the signature functor. Evaluated on the degree $n$ component of a species $\mathbf{p}$, one obtains a map

$$\mathbf{p}[n] \otimes \mathbf{E}^{-}[n] \to \mathbf{p}[n] \otimes \mathbf{E}^{-}[n]$$

given by

$$\begin{aligned} z \otimes (1 \wedge 2 \wedge \cdots \wedge n) &\mapsto \sum_{\sigma \in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)}\, \sigma \cdot (z \otimes (1 \wedge 2 \wedge \cdots \wedge n)) \\ &= \sum_{\sigma \in \mathrm{S}_n} (-q)^{\mathrm{inv}(\sigma)}\, (\sigma \cdot z) \otimes (1 \wedge 2 \wedge \cdots \wedge n). \end{aligned}$$

After making the identification (16.3), one sees that the above is the same as $\kappa_{-q}$ evaluated on the degree $n$ component of $\mathbf{p}$. $\square$

**16.2.3. The generic case for the norm map.** Up to this point, the relation between the $q$-deformed functors is analogous to that for the case $q = 1$ discussed in Section 15.4. A closer look at the transformation $\kappa_q$ reveals that the generic case is substantially different. Let us first study the behavior of $\kappa_q$ on the species $\mathbf{L}$.

**Example 16.17.** The map

$$\kappa_{\mathbf{L}} \colon \mathcal{K}(\mathbf{L}) \to \mathcal{K}^{\vee}(\mathbf{L})$$

has components

$$\mathbf{L}[n] \to \mathbf{L}[n], \qquad l \mapsto \sum_{\sigma \in \mathrm{S}_n} \sigma \cdot l = \sum_{l' \in \mathbf{L}[n]} l'.$$

The image of this map is one-dimensional, so $\kappa_{\mathbf{L}}$ is far from being an isomorphism. On the other hand, the map

$$(\kappa_q)_{\mathbf{L}} \colon \mathcal{K}_q(\mathbf{L}) \to \mathcal{K}_q^{\vee}(\mathbf{L})$$

has components

$$\mathbf{L}[n] \to \mathbf{L}[n], \qquad l \mapsto \sum_{\sigma \in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)}\, \sigma \cdot l.$$

In view of (10.27), the matrix of this map in the canonical basis of linear orders agrees with the matrix of the bilinear form on chambers (10.132). Its determinant is given by Zagier's formula (10.133). The only factors of this determinant are roots of unity. It follows that $(\kappa_q)_{\mathbf{L}}$ is an isomorphism if $q$ is not a root of unity.

The behavior of $\kappa_q$ on the species $\mathbf{L}$ has far-reaching implications.

**Theorem 16.18.** *Assume that $q$ is not a root of unity and the field characteristic is 0. Then the $q$-norm transformation*

$$\kappa_q \colon \mathcal{K}_q \Rightarrow \mathcal{K}_q^{\vee}$$

*is an isomorphism of bilax monoidal functors.*

PROOF. In view of Proposition 16.15, we only need to show that for any species $\mathbf{p}$, the components $\mathbf{p}[n] \to \mathbf{p}[n]$ of $(\kappa_q)_{\mathbf{p}}$ are bijective. Now, $\mathbf{p}[n]$ is an $\mathrm{S}_n$-module, and since the base field is of charactersitic 0, $\mathbf{p}[n]$ decomposes into a direct sum of irreducible $\mathrm{S}_n$-modules. By naturality, the map $(\kappa_q)_{\mathbf{p}}$ preserves this decomposition, so in order to conclude its invertibility we have to show that its restriction to each irreducible component is invertible. But every irreducible $\mathrm{S}_n$-module occurs as a direct summand of $\mathbf{L}[n]$, being this the regular representation. Again by naturality, the restrictions of $(\kappa_q)_{\mathbf{p}}$ and $(\kappa_q)_{\mathbf{L}}$ to a common irreducible submodule coincide. In turn, the map $(\kappa_q)_{\mathbf{L}}$ preserves the decomposition of $\mathbf{L}[n]$ into irreducibles, so the invertibility of each restriction follows from the invertibility of $(\kappa_q)_{\mathbf{L}}$ on $\mathbf{L}[n]$ discussed in Example 16.17. ☐

For $q = 0$, the functors $\mathcal{K}_0$ and $\mathcal{K}_0^\vee$ are identical as noted in (16.1). In this case, observe that the norm map $\kappa_0$ is the identity.

**16.2.4. The deformed full Fock functor and duality.** The above discussion shows that Zagier's formula is at the basis of a much more general result which we give below.

**Theorem 16.19.** *Let* $\mathbf{h}$ *be a p-Hopf monoid. There is a natural morphism of pq-Hopf algebras*

$$(\kappa_q)_{\mathbf{h}} \colon \mathcal{K}_q(\mathbf{h}) \to \mathcal{K}_q^\vee(\mathbf{h})$$

*which is an isomorphism if $q$ is not a root of unity and the field characteristic is* 0.

PROOF. The first claim follows from Proposition 3.32 and Proposition 16.15. The second claim follows from Theorem 16.18. ☐

Thus, in the generic case, the functor $\mathcal{K}_q$ is self-dual. Hence, it preserves self-dual Hopf monoids (Proposition 3.107). We state this below and repeat the argument for its proof.

**Corollary 16.20.** *If* $\mathbf{h}$ *is a finite-dimensional self-dual p-Hopf monoid, $q$ is not a root of unity, and the field characteristic is* 0, *then* $\mathcal{K}_q(\mathbf{h})$ *is a self-dual graded pq-Hopf algebra.*

PROOF. By combining the isomorphisms of Theorems 16.2 and 16.19 with the self-duality of $\mathbf{h}$, we obtain

$$\mathcal{K}_q(\mathbf{h})^* \cong \mathcal{K}_q^\vee(\mathbf{h}^*) \cong \mathcal{K}_q^\vee(\mathbf{h}) \cong \mathcal{K}_q(\mathbf{h}). \qquad \square$$

## 16.3. The fermionic and anyonic Fock bilax monoidal functors

So far in this chapter, we have discussed deformations of the full Fock functors. In this section, we introduce and study the fermionic partners of the bosonic Fock functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$. We explain how they can be viewed as the image of the decorated norm at $q = -1$. We conclude by considering anyonic Fock functors which are images of the decorated norm for any scalar $q$. Roughly speaking, these interpolate between bosonic and fermionic Fock functors.

**16.3.1. The fermionic Fock functors $\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^{\vee}_{-1}$.** We first note that the functors

$$\overline{\mathcal{K}}, \overline{\mathcal{K}}^{\vee} \colon (\mathsf{Sp}, \cdot, \beta_q) \to (\mathsf{gVec}, \cdot, \beta_q)$$

continue to be bilax with the same structure maps as before (15.3). The braiding and hence the parameter $q$ only enters in diagram (3.11). One readily sees that the presence of $q$ is innocuous and the diagram continues to commute. We now define the fermionic partner of these functors.

**Definition 16.21.** Let

$$\overline{\mathcal{K}}_{-1}, \overline{\mathcal{K}}^{\vee}_{-1} \colon (\mathsf{Sp}, \cdot, \beta_q) \to (\mathsf{gVec}, \cdot, \beta_{-q})$$

be the functors defined by

$$\overline{\mathcal{K}}_{-1}(\mathbf{q}) := \bigoplus_{n \geq 0} (\mathbf{q}[n] \otimes \mathbf{E}^{-}[n])_{\mathrm{S}_n} \qquad \text{and} \qquad \overline{\mathcal{K}}^{\vee}_{-1}(\mathbf{q}) := \bigoplus_{n \geq 0} (\mathbf{q}[n] \otimes \mathbf{E}^{-}[n])^{\mathrm{S}_n},$$

where $\mathbf{E}^{-}$ is the signed exponential species (Section 9.3).

Recall that $\mathbf{E}^{-}[n]$ is the sign representation of $\mathrm{S}_n$. Comparing and contrasting with the situation for bosonic Fock functors, in each degree $n$ component, one now takes $\mathrm{S}_n$ invariants and coinvariants of $\mathbf{q}[n]$ with respect to the usual action twisted by the sign representation of $\mathrm{S}_n$.

In terms of the signature functor (9.10),

$$\overline{\mathcal{K}}_{-1}(-) = \overline{\mathcal{K}}\big((-)^{-}\big) \qquad \text{and} \qquad \overline{\mathcal{K}}^{\vee}_{-1}(-) = \overline{\mathcal{K}}^{\vee}\big((-)^{-}\big). \tag{16.7}$$

Since the signature functor and the bosonic Fock functors are bistrong, it follows that their composites are also bistrong. This turns the fermionic Fock functors into bistrong functors. In particular they are braided bilax and preserve commutativity and cocommutativity.

The above definition enables us to quickly assert that many properties of the bosonic Fock functors go over to the fermionic case. We record the important ones below.

**16.3.2. Relation to the decorated norm at $q = -1$.** The self-duality of the signature functor (Proposition 9.11) along with Proposition 15.8 implies that, on finite-dimensional species, $\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^{\vee}_{-1}$ are contragredients of each other (as already suggested by the notation).

Precomposing (15.15) with the signature functor and using Proposition 16.16, we obtain the commutative diagram below.

$$\begin{tikzcd} \mathcal{K}_{-1} \arrow[r, Rightarrow, "\kappa_{-1}"] \arrow[d, Rightarrow] & \mathcal{K}^{\vee}_{-1} \\ \overline{\mathcal{K}}_{-1} \arrow[r, Rightarrow, "\overline{\kappa}_{-1}"'] & \overline{\mathcal{K}}^{\vee}_{-1} \arrow[u, Rightarrow] \end{tikzcd} \tag{16.8}$$

Further, it follows from Proposition 15.21 that, over a field of characteristic 0,

$$\overline{\kappa}_{-1} \colon \overline{\mathcal{K}}_{-1} \Rightarrow \overline{\mathcal{K}}^{\vee}_{-1}$$

is an isomorphism, and from Proposition 15.24 that, on finite-dimensional species, (16.8) is self-dual. It now follows that, in characteristic 0, the isomorphic functors

$\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^{\vee}_{-1}$ are self-dual, and are the coimage and image of the norm transformation $\kappa_{-1}$, in the sense of Section 3.11.

**16.3.3. Relation to the Hopf monoid of linear orders.** We showed in Proposition 16.6 that the deformations $\mathcal{K}_q$ and $\mathcal{K}^{\vee}_q$ can be expressed in terms of the bosonic Fock functors and the $q$-Hopf monoids $\mathbf{L}_q$ and $\mathbf{L}^*_q$. The same can be done with the fermionic Fock functors as follows.

**Proposition 16.22.** *We have isomorphisms of bilax functors*

$$\mathcal{K}_q(-) \cong \overline{\mathcal{K}}_{-1}(\mathbf{L}_{-q} \times (-)) \qquad \textit{and} \qquad \mathcal{K}^{\vee}_q(-) \cong \overline{\mathcal{K}}^{\vee}_{-1}(\mathbf{L}^*_{-q} \times (-))$$

*from* $(\mathsf{Sp}, \cdot, \beta_p)$ *to* $(\mathsf{Sp}, \cdot, \beta_{pq})$.

Proof. The first claim follows from the following sequence of isomorphisms.

$$\overline{\mathcal{K}}_{-1}(\mathbf{L}_{-q} \times (-)) \cong \overline{\mathcal{K}}\left(\big(\mathbf{L}_{-q} \times (-)\big)^{-}\right) \cong \overline{\mathcal{K}}\big((\mathbf{L}_{-q})^{-} \times (-)\big) \cong \overline{\mathcal{K}}(\mathbf{L}_q \times (-)) \cong \mathcal{K}_q(-).$$

The first isomorphism follows from definition, the second follows from Proposition 9.12, the third follows from (9.20), and the last follows from Proposition 16.6.

The second claim follows along similar lines. $\square$

**16.3.4. Evaluating the fermionic Fock functors.** As a consequence of (16.8) and the discussion after it, we obtain that for any $q$-Hopf monoid $\mathbf{h}$, there is a commutative diagram of $(-q)$-Hopf algebras

$$\begin{tikzcd}
\mathcal{K}_{-1}(\mathbf{h}) \arrow[r, "(\kappa_{-1})_{\mathbf{h}}"] \arrow[d, two heads] & \mathcal{K}^{\vee}_{-1}(\mathbf{h}) \\
\overline{\mathcal{K}}_{-1}(\mathbf{h}) \arrow[r, "(\overline{\kappa}_{-1})_{\mathbf{h}}"'] & \overline{\mathcal{K}}^{\vee}_{-1}(\mathbf{h}). \arrow[u, hook]
\end{tikzcd} \tag{16.9}$$

**Corollary 16.23.** *If the field characteristic is* $0$, *then*

$$(\overline{\kappa}_{-1})_{\mathbf{h}} \colon \overline{\mathcal{K}}_{-1}(\mathbf{h}) \to \overline{\mathcal{K}}^{\vee}_{-1}(\mathbf{h})$$

*is an isomorphism of* $(-q)$*-Hopf algebras.*

Since on finite-dimensional species the columns in (16.9) are contragredient of each other, the preceding result can be reformulated as follows. For any finite-dimensional $q$-Hopf monoid $\mathbf{h}$, the following are commutative diagrams of $(-q)$-Hopf algebras.

$$\begin{tikzcd}
\mathcal{K}_{-1}(\mathbf{h}^*) \arrow[r, "(\kappa_{-1})_{\mathbf{h}^*}"] \arrow[d, two heads] & \mathcal{K}_{-1}(\mathbf{h})^* \\
\overline{\mathcal{K}}_{-1}(\mathbf{h}^*) \arrow[r, "(\overline{\kappa}_{-1})_{\mathbf{h}^*}"'] & \overline{\mathcal{K}}_{-1}(\mathbf{h})^* \arrow[u, hook]
\end{tikzcd} \qquad \begin{tikzcd}
\mathcal{K}^{\vee}_{-1}(\mathbf{h})^* \arrow[r, "(\kappa_{-1})_{\mathbf{h}^*}"] \arrow[d, two heads] & \mathcal{K}^{\vee}_{-1}(\mathbf{h}^*) \\
\overline{\mathcal{K}}^{\vee}_{-1}(\mathbf{h})^* \arrow[r, "(\overline{\kappa}_{-1})_{\mathbf{h}^*}"'] & \overline{\mathcal{K}}^{\vee}_{-1}(\mathbf{h}^*) \arrow[u, hook]
\end{tikzcd}$$

(The two diagrams are the same.) Applying Corollary 16.23 to $\mathbf{h}^*$ we obtain the following result.

**Corollary 16.24.** *Let* $\mathbf{h}$ *be a finite-dimensional* $q$*-Hopf monoid. If the field characteristic is* 0, *then* $(\overline{\kappa}_{-1})_{\mathbf{h}^*}$ *is an isomorphism of* $(-q)$*-Hopf algebras. If in addition* $\mathbf{h}$ *is self-dual, then the* $(-q)$*-Hopf algebra* $\overline{\mathcal{K}}_{-1}(\mathbf{h})$ *is self-dual.*

**16.3.5. The anyonic Fock functor.** It is natural to ask whether one can deform the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^{\vee}$ using a parameter $q$, with $q = -1$ yielding the fermionic case discussed above.

We proceed along the lines of Section 15.4.3. By Proposition 16.15, the deformed norm transformation $\kappa_q \colon \mathcal{K}_q \Rightarrow \mathcal{K}_q^{\vee}$ is a morphism of bilax monoidal functors. Let $\Im_q$ denote its (co)image. According to the constructions of Section 3.11,

$$\Im_q \colon (\mathsf{Sp}, \cdot, \beta_p) \to (\mathsf{gVec}, \cdot, \beta_{pq})$$

is another bilax monoidal functor fitting in the commutative diagram below.

$$\begin{tikzcd} \mathcal{K}_q \arrow[rr, Rightarrow, "\kappa_q"] \arrow[dr, Rightarrow] & & \mathcal{K}_q^{\vee} \\ & \Im_q \arrow[ur, Rightarrow] & \end{tikzcd}$$

We call $\Im_q$ the *anyonic Fock functor*. It follows from Propositions 3.119 and 16.14 that on finite-dimensional species, it is self-dual (regardless of the characteristic). More generally, the above diagram is self-dual.

Theorem 16.18 says that if $\Bbbk$ is of characteristic 0 and $q$ is not a root of unity, then we do not obtain anything new:

$$\mathcal{K}_q \cong \Im_q \cong \mathcal{K}_q^{\vee}.$$

On the other hand, in characteristic zero we have

$$\overline{\mathcal{K}} \cong \Im_1 \cong \overline{\mathcal{K}}^{\vee} \qquad \text{and} \qquad \overline{\mathcal{K}}_{-1} \cong \Im_{-1} \cong \overline{\mathcal{K}}_{-1}^{\vee}.$$

For $q = 0$, the functors $\mathcal{K}_0$ and $\mathcal{K}_0^{\vee}$ are identical (16.1) and the above is the identity transformation. Thus, $\Im_0$ is the free Fock functor of Section 16.1.3.

Some information about particular values of the functor $\Im_q$ when $q$ is a root of unity is given in Section 16.5, but apart from this we do not pursue the study of this functor in this monograph. This appears to be an interesting and challenging topic.

## 16.4. The deformed full Fock functor and commutativity

We now concentrate on the functor $\mathcal{K}_q$ and study its behavior with respect to commutativity, as is done in Section 15.5 for the functor $\mathcal{K}$. In this section, we assume that $p$, $q$ and $r$ are nonzero scalars.

**16.4.1. Conjugating the functor $\mathcal{K}_q$ by the braiding.** The colax monoidal functor $(\mathcal{K}, \psi)$ is braided (Proposition 15.26), so we have ${}^{b}\psi = \psi$. The situation is slightly more delicate for the $q$-deformation. It proves convenient to state the result in terms of the various monoidal functors obtained from $\mathcal{K}_q$ by conjugation with the deformed braidings on species and graded vector spaces.

**Definition 16.25.** Following Definition 3.14, we define transformations:

$$\varphi^{b(p,r)} \colon \mathcal{K}(\mathbf{p}) \cdot \mathcal{K}(\mathbf{q}) \xrightarrow{\beta_r} \mathcal{K}(\mathbf{q}) \cdot \mathcal{K}(\mathbf{p}) \xrightarrow{\varphi_{\mathbf{q},\mathbf{p}}} \mathcal{K}(\mathbf{q} \cdot \mathbf{p}) \xrightarrow{\mathcal{K}(\beta_p^{-1})} \mathcal{K}(\mathbf{p} \cdot \mathbf{q}),$$

$$
\begin{aligned}
{}^{b(p,r)}\varphi\colon \mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}) &\xrightarrow{\beta_r^{-1}} \mathcal{K}(\mathbf{q})\cdot\mathcal{K}(\mathbf{p}) \xrightarrow{\varphi_{\mathbf{q},\mathbf{p}}} \mathcal{K}(\mathbf{q}\cdot\mathbf{p}) \xrightarrow{\mathcal{K}(\beta_p)} \mathcal{K}(\mathbf{p}\cdot\mathbf{q}),\\
\psi_q^{b(p,r)}\colon \mathcal{K}(\mathbf{p}\cdot\mathbf{q}) &\xrightarrow{\mathcal{K}(\beta_p)} \mathcal{K}(\mathbf{q}\cdot\mathbf{p}) \xrightarrow{(\psi_q)_{\mathbf{q},\mathbf{p}}} \mathcal{K}(\mathbf{q})\cdot\mathcal{K}(\mathbf{p}) \xrightarrow{\beta_r^{-1}} \mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}),\\
{}^{b(p,r)}\psi_q\colon \mathcal{K}(\mathbf{p}\cdot\mathbf{q}) &\xrightarrow{\mathcal{K}(\beta_p^{-1})} \mathcal{K}(\mathbf{q}\cdot\mathbf{p}) \xrightarrow{(\psi_q)_{\mathbf{q},\mathbf{p}}} \mathcal{K}(\mathbf{q})\cdot\mathcal{K}(\mathbf{p}) \xrightarrow{\beta_r} \mathcal{K}(\mathbf{p})\cdot\mathcal{K}(\mathbf{q}).
\end{aligned}
$$

The notation emphasizes the dependence on $p$, $q$ and $r$ and allows us to keep these constructions apart from the maps $\psi^b = {}^b\psi$ and $\varphi^b = {}^b\varphi$ of Section 15.5.

Since $\beta_q^{-1} = \beta_{q^{-1}}$, we have

$$(\psi_q)^{b(p,r)} = {}^{b(p^{-1},r^{-1})}(\psi_q) \qquad \text{and} \qquad \varphi^{b(p,r)} = {}^{b(p^{-1},r^{-1})}\varphi.$$

**Proposition 16.26.** *There is an equality*

$$(\mathcal{K}, \varphi, (\psi_q)^{b(p,pq)}) = (\mathcal{K}, \varphi, \psi_{q^{-1}})$$

*of bilax monoidal functors*

$$(\mathsf{Sp}, \cdot, \beta_{p^{-1}}) \to (\mathsf{gVec}, \cdot, \beta_{p^{-1}q^{-1}}).$$

The former is a conjugate of $\mathcal{K}_q$, as in Proposition 3.16, and the latter is the functor $\mathcal{K}_{q^{-1}}$.

Proof. We need to show that

$$(\psi_q)^{b(p,pq)} = \psi_{q^{-1}}.$$

As in the proof of Proposition 15.26, this entails that the diagram

$$
\begin{tikzcd}
\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[rr, "q^{\mathrm{sch}_n(S)}\,\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]"] & & \mathbf{p}[s]\otimes\mathbf{q}[t] \arrow[d, "\beta_{pq}^{-1}"] \\
\mathbf{q}[T]\otimes\mathbf{p}[S] \arrow[u, "\beta_p"] \arrow[rr, "q^{-\mathrm{sch}_n(T)}\,\mathbf{q}[\mathrm{cano}]\otimes\mathbf{p}[\mathrm{cano}]"'] & & \mathbf{q}[t]\otimes\mathbf{p}[s]
\end{tikzcd}
$$

commutes, where $S \sqcup T = [n]$ and $s = |S|$, $t = |T|$. We know from Proposition 15.26 that it commutes up to a power of $p$ and $q$. One can readily see that the powers of $p$ in question are both zero. The powers of $q$ in question are

$$\mathrm{sch}_n(S) - st \quad \text{and} \quad -\mathrm{sch}_n(T),$$

so the diagram commutes by (2.15). ∎

**16.4.2. The deformed half-twist transformation.** As in Definition 15.29, let $\omega_n$ be the longest permutation in $\mathrm{S}_n$. Define a natural transformation $\theta_q\colon \mathcal{K} \Rightarrow \mathcal{K}$ with components

$$\mathbf{p}[n] \to \mathbf{p}[n], \qquad x \mapsto q^{\binom{n}{2}}\,\mathbf{p}[\omega_n](x).$$

We call it the *deformed half-twist transformation.*

**Proposition 16.27.** *The transformation $\theta_q$ is an isomorphism of bilax monoidal functors*

$$(\mathcal{K}, {}^{b(p,pq)}\varphi, {}^{b(p,pq)}\psi_q) \Rightarrow (\mathcal{K}, \varphi, \psi_q).$$

PROOF. For the colax part, the diagram whose commutativity one needs to check is

$$\begin{tikzcd}
\mathbf{p}[S] \otimes \mathbf{q}[T] \arrow[r, "\beta_p^{-1}"] \arrow[d, "q^{\binom{n}{2}}(\mathbf{p}\cdot\mathbf{q})[\omega_n]"] & \mathbf{q}[T] \otimes \mathbf{p}[S] \arrow[r, "q^{\mathrm{sch}_n(T)}\mathbf{q}[\mathrm{cano}]\otimes\mathbf{p}[\mathrm{cano}]"] & \mathbf{q}[t] \otimes \mathbf{p}[s] \arrow[r, "\beta_{pq}"] & \mathbf{p}[s] \otimes \mathbf{q}[t] \arrow[d, "q^{\binom{s}{2}+\binom{t}{2}}\mathbf{p}[\omega_s]\otimes\mathbf{q}[\omega_t]"] \\
\mathbf{p}[S'] \otimes \mathbf{q}[T'] \arrow[rrr, "q^{\mathrm{sch}_n(S')}\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]"'] & & & \mathbf{p}[s] \otimes \mathbf{q}[t],
\end{tikzcd}$$

where $S \sqcup T = [n]$ and $s = |S|$, $t = |T|$. We know from Proposition 15.30 that it commutes up to a power of $p$ and $q$. It is clear that the powers of $p$ in question are both zero. The powers of $q$ in question are

$$\mathrm{sch}_n(T) + ts + \binom{s}{2} + \binom{t}{2} \quad \text{and} \quad \binom{n}{2} + \mathrm{sch}_n(S'),$$

so the diagram commutes by (2.16).

The proof of the lax part is similar; in fact, it is simpler since the Schubert statistic does not enter the calculation. □

**16.4.3. Commutativity of the Hopf monoids obtained by evaluating $\mathcal{K}_q$.** We now state the consequences for Hopf monoids of the properties of the functor $\mathcal{K}_q$ regarding commutativity. They will be expressed in terms of the op and cop constructions of Section 1.2.9. Recall that $p$ and $q$ are assumed to be nonzero scalars.

In contrast to Corollary 15.27, we have:

**Corollary 16.28.** *For any comonoid (Hopf monoid)* $\mathbf{h}$ *in* $(\mathsf{Sp}, \cdot, \beta_p)$,

$$\mathcal{K}_{q^{-1}}(\mathbf{h}^{\mathrm{cop}}) = \mathcal{K}_q(\mathbf{h})^{\mathrm{cop}}$$

*as comonoids (Hopf monoids) in* $(\mathsf{gVec}, \cdot, \beta_{(pq)^{-1}})$.

PROOF. We explain the Hopf monoid case; it contains the proof of the comonoid case. Since $\mathbf{h}$ is a $p$-Hopf monoid, $\mathbf{h}^{\mathrm{cop}}$ is a $p^{-1}$-Hopf monoid. Applying Proposition 16.26 to $\mathbf{h}^{\mathrm{cop}}$ yields

$$\mathcal{K}_{q^{-1}}(\mathbf{h}^{\mathrm{cop}}) = (\mathcal{K}, \varphi, (\psi_q)^{b(p,pq)})(\mathbf{h}^{\mathrm{cop}})$$

as $(pq)^{-1}$-Hopf algebras. Since the functor on the right is the conjugate of $\mathcal{K}_q$, the result follows by using the first part of Proposition 3.34. □

**Corollary 16.29.** *For any $p$-Hopf monoid* $\mathbf{h}$*, the map*

$$\mathcal{K}_q(\mathbf{h}^{\mathrm{op,cop}}) \to \mathcal{K}_q(\mathbf{h})^{\mathrm{op,cop}}$$

*whose degree $n$ component is* $q^{\binom{n}{2}}\mathbf{h}[\omega_n]$ *is a natural isomorphism of $pq$-Hopf algebras.*

PROOF. Follow the proof of Corollary 15.32, and use Proposition 16.27 instead of Proposition 15.30. □

## 16.5. Deformations of Hopf algebras arising from species

In this section, we apply the results of the preceding sections to the simplest Hopf monoids of Section 8.5. Throughout the present discussion, the value of the parameter $p$ is set equal to 1. Thus, $\mathcal{K}_q$ is viewed as a bilax functor

$$(\mathsf{Sp}, \cdot, \beta) \to (\mathsf{Sp}, \cdot, \beta_q),$$

and similarly for the other functors. Theorem 16.7 shows that $\mathcal{K}_q$ sends a Hopf monoid in species to a $q$-Hopf algebra. These are deformations of the Hopf algebras that arise from the undeformed functor $\mathcal{K}$. Thus, in this section, we will construct deformations of the Hopf algebras encountered in Section 15.3.

**Example 16.30.** We begin with the Hopf monoid $\mathbf{E}$ of Example 8.15. Its values under the undeformed Fock functors were calculated in Example 15.14. We now generalize that discussion. As before, $\mathcal{K}_q(\mathbf{E})$ can be identified with the polynomial algebra in the variable $x$. The coproduct

$$\mathcal{K}(\mathbf{E}) \xrightarrow{\mathcal{K}(\Delta)} \mathcal{K}(\mathbf{E} \cdot \mathbf{E}) \xrightarrow{\psi_q} \mathcal{K}(\mathbf{E}) \cdot \mathcal{K}(\mathbf{E})$$

can be calculated as follows.

$$x^n \mapsto \sum_{S \sqcup T = [n]} x^{|S|} \otimes x^{|T|} \mapsto \sum_{S \subseteq [n], |S| = s} q^{\mathrm{sch}_n(S)}\, x^s \otimes x^{n-s} = \sum_{s=0}^{n} \binom{n}{s}_q x^s \otimes x^{n-s}.$$

The last identity makes use of (2.27). Thus,

$$\mathcal{K}_q(\mathbf{E}) = \Bbbk_q[x],$$

the Eulerian $q$-Hopf algebra defined in Example 2.9. Similarly, or by duality, one finds that $\mathcal{K}_q^{\vee}(\mathbf{E}) = \Bbbk_q\{x\}$, the $q$-version of the divided power Hopf algebra.

Recall the fermionic Fock functors of Section 16.3. If the field characteristic is not 2, then it follows directly from Definition 16.21 that $\overline{\mathcal{K}}_{-1}(\mathbf{E})$ and $\overline{\mathcal{K}}_{-1}^{\vee}(\mathbf{E})$ are canonically isomorphic and equal to

$$\Bbbk_{-1}[x]/(x^2),$$

with the generator $x$ being primitive. This is the exterior algebra on one generator. This two-dimensional algebra is also called the algebra of *dual numbers*.

The norm transformation $\kappa_q$ yields the morphism of $q$-Hopf algebras (2.52). It follows that this map is an isomorphism if the field characteristic is zero and $q$ is not a root of unity.

Recall the functor $\Im_q$ of Section 16.3.5. We continue with the assumption that the field characteristic is zero. If $q$, with $q \neq 1$, is a root of unity of order $N$, then the kernel of this map is the ideal generated by $x^N$. This follows from (2.29). Thus,

$$\Im_q(\mathbf{E}) = \Bbbk_q[x]/(x^N). \tag{16.10}$$

In particular, for $q = -1$, one obtains the exterior algebra on one generator mentioned above.

We now turn to the commutativity issue. Let $q$ be nonzero. The Hopf monoid $\mathbf{E}$ is both commutative and cocommutative. So it stays the same under the op and cop constructions. Corollary 16.28 applied to $\mathbf{E}$ says that the cop construction applied to the Eulerian $q$-Hopf algebra yields the Eulerian $q^{-1}$-Hopf algebra. This fact was noted in (2.53) where it was checked directly. Corollary 16.29 yields the isomorphism $(\theta_q)_{\mathbf{E}}$ of $q$-Hopf algebras given in (2.54).

**Example 16.31.** We now unify Examples 15.16 and 16.30. Let $V$ be a vector space and $\mathbf{E}_V$ be the Hopf monoid of Example 8.18. Then

$$\mathcal{K}_q(\mathbf{E}_V) = \mathcal{T}_q(V), \ \overline{\mathcal{K}}_{-1}(\mathbf{E}_V) = \Lambda(V), \ \mathcal{K}_q^\vee(\mathbf{E}_V) = \mathcal{T}_q^\vee(V) \text{ and } \overline{\mathcal{K}}^\vee(\mathbf{E}_V) = \Lambda^\vee(V).$$

The objects in the right-hand sides include the deformed tensor Hopf algebra, the deformed shuffle Hopf algebra and the exterior Hopf algebra on a vector space. These were discussed in Section 2.6.2. The additional results presented there all follow from the various considerations in this chapter as we now explain.

Theorem 16.19 says that if $q$ is not a root of unity and the field characteristic is 0, then $\kappa_q$ yields an isomorphism of $q$-Hopf algebras

$$\mathcal{T}_q(V) \cong \mathcal{T}_q^\vee(V).$$

This is mentioned in (2.67). If $q = 0$, then the functors $\mathcal{K}_0$ and $\mathcal{K}_0^\vee$ are identical and the norm map $\kappa_0$ is the identity. It follows that $\mathcal{T}_0(V) = \mathcal{T}_0^\vee(V)$. If $q = -1$, then diagram (16.9) specializes to diagram (2.68).

If $q$ is a root of unity, then the image $\Im_q(\mathbf{E}_V)$ is a certain proper quotient of $\mathcal{T}_q(V)$. It is an example of a *Nichols algebra*. (We say more about quantum shuffle algebras and Nichols algebras in Sections 19.9 and 20.5.)

The self-duality results in Section 2.6.3 all follow from corresponding self-duality results for the functors $\mathcal{K}_q$, $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}_{-1}$, and the self-duality of the Hopf monoid $\mathbf{E}_V$. We point out that the latter requires a choice of an isomorphism between $V$ and $V^*$.

Let $q$ be nonzero. Since $\mathbf{E}_V$ is both commutative and cocommutative, it stays the same under the op and cop constructions. Corollary 16.28 applied to $\mathbf{E}_V$ yields

$$\mathcal{T}_q(V)^{\mathrm{cop}} = \mathcal{T}_{q^{-1}}(V)$$

as $q^{-1}$-Hopf algebras. Further, Corollary 16.29 yields the isomorphism

$$(\theta_q)_{\mathbf{E}_V} : \mathcal{T}_q(V) \to \mathcal{T}_q(V)^{\mathrm{op,cop}}, \qquad v_1 \otimes \cdots \otimes v_n \mapsto q^{\binom{n}{2}}\, v_n \otimes \cdots \otimes v_1$$

of $q$-Hopf algebras.

**Example 16.32.** We now turn to linear orders. Consider the Hopf monoid $\mathbf{L}^*$ of Example 8.24. In Example 15.17 we saw that

$$\mathcal{K}(\mathbf{L}^*) \cong \mathrm{S}\Lambda,$$

the graded Hopf algebra of permutations. Now applying $\mathcal{K}_q$ to $\mathbf{L}^*$ yields a $q$-version, which we denote by $\mathrm{S}\Lambda_q$. This object has been defined and studied by Foissy [130]. The product is the same as before, while the coproduct is

$$F_l \mapsto \sum_{s=0}^{n} q^{\mathrm{sch}_n(\{l^1,\dots,l^s\})}\, F_{\mathrm{std}(l^1|\cdots|l^s)} \otimes F_{\mathrm{std}(l^{s+1}|\cdots|l^n)},$$

where $l = l^1|\cdots|l^n \in \mathbf{L}[n]$ and $\mathrm{sch}_n(S)$ is the Schubert statistic.

This object can be arrived at in a number of different ways. From Proposition 16.6, we obtain

$$\mathcal{K}_q(\mathbf{L}^*) \cong \overline{\mathcal{K}}(\mathbf{L}_q \times \mathbf{L}^*) \cong \overline{\mathcal{K}}^\vee(\mathbf{L}_q \times \mathbf{L}^*) \cong \mathcal{K}^\vee(\mathbf{L}_q). \tag{16.11}$$

The first and last isomorphisms follow from Proposition 16.6, while the middle isomorphism follows from Corollary 15.22 and the fact that $\mathbf{L} \times \mathbf{L}^*$ consists of free (and hence flat) $\mathrm{S}_n$-modules. By duality or by proceeding directly as above, we obtain

$$\mathcal{K}_q^\vee(\mathbf{L}) \cong \overline{\mathcal{K}}^\vee(\mathbf{L} \times \mathbf{L}_q^*) \cong \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{L}_q^*) \cong \mathcal{K}(\mathbf{L}_q^*). \tag{16.12}$$

These objects are dual to those in (16.11). Further, if $q \neq 0$, then it follows from Corollary 12.11 that the objects in (16.11) are isomorphic to those in (16.12). In other words, $\mathrm{S}\Lambda_q$ is self-dual if $q \neq 0$.

Similarly, again by applying Proposition 16.6, we obtain

$$\mathcal{K}_q(\mathbf{L}) \cong \overline{\mathcal{K}}(\mathbf{L}_q \times \mathbf{L}) \cong \mathcal{K}(\mathbf{L}_q)$$

and

$$\mathcal{K}_q^\vee(\mathbf{L}^*) \cong \overline{\mathcal{K}}^\vee(\mathbf{L}^* \times \mathbf{L}_q^*) \cong \mathcal{K}^\vee(\mathbf{L}_q^*).$$

The former is a deformation of the Hopf algebra $\mathcal{K}(\mathbf{L})$ which was explicitly described in Example 15.17. The product is the same as before, while the coproduct is

$$\Delta(l) = \sum_{[n]=S \sqcup T} q^{\mathrm{sch}_n(S)} \, \mathrm{std}(l|_S) \otimes \mathrm{std}(l|_T),$$

where $l \in \mathbf{L}[n]$ and $\mathrm{sch}_n(S)$ is the Schubert statistic. The Hopf monoid $\mathbf{L}$ is cocommutative but the $q$-Hopf algebra $\mathcal{K}_q(\mathbf{L})$ is not. The correct statement to make about the latter is

$$\mathcal{K}_q(\mathbf{L})^{\mathrm{cop}} = \mathcal{K}_{q^{-1}}(\mathbf{L}).$$

This is a consequence of Corollary 16.28.

Applying the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ to the Hopf monoids $\mathbf{L}_q$ and $\mathbf{L}_q^*$ either yields the Eulerian $q$-Hopf algebra or its dual (Example 2.9) as follows.

$$\overline{\mathcal{K}}(\mathbf{L}_q) \cong \overline{\mathcal{K}}^\vee(\mathbf{L}_q) \cong \Bbbk_q[x] \qquad \text{and} \qquad \overline{\mathcal{K}}(\mathbf{L}_q^*) \cong \overline{\mathcal{K}}^\vee(\mathbf{L}_q^*) \cong \Bbbk_q\{x\}.$$

By applying (16.7) to the above, we obtain:

$$\overline{\mathcal{K}}_{-1}(\mathbf{L}_q) \cong \overline{\mathcal{K}}_{-1}^\vee(\mathbf{L}_q) \cong \Bbbk_{-q}[x] \qquad \text{and} \qquad \overline{\mathcal{K}}_{-1}(\mathbf{L}_q^*) \cong \overline{\mathcal{K}}_{-1}^\vee(\mathbf{L}_q^*) \cong \Bbbk_{-q}\{x\}.$$

The determination of $\Im_q(\mathbf{L})$ or $\Im_q(\mathbf{L}^*)$, when $q$ is a root of unity different from $\pm 1$, appears to be a difficult problem. Some information can be gleaned from the results of Hanlon and Stanley [159, Theorem 3.3] and of Denham [96, Theorem 3.1] on the kernel of $(\kappa_q)_{\mathbf{L}}$.

CHAPTER 17

# From Hopf Monoids to Hopf Algebras: Examples

In Chapters 12 and 13, we constructed many Hopf monoids using geometric and combinatorial ideas. Applying the Fock functors to these Hopf monoids yields Hopf algebras, many of which have appeared in the literature. An illustration of this on the Hopf monoids **E** and **L** was given in Section 15.3. The goal of this chapter is to explain these connections for other, more elaborate Hopf monoids.

**Hopf algebras from geometry.** The Hopf monoids of Chapter 12 fit into the commutative diagram (12.14). Applying the Fock functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ to this diagram, one obtains the commutative diagram of Hopf algebras considered in [12, Theorem 6.1.4]. In particular:

**Proposition 17.1.** *We have*

$$\mathcal{K}(\mathbf{\Sigma}^*) = \mathrm{P\Pi}, \quad \mathcal{K}(\overrightarrow{\mathbf{\Sigma}}^*) = \mathrm{Q\Pi}, \quad \mathcal{K}(\mathbb{L}^*) = \mathrm{S\Pi}, \quad \mathcal{K}(\mathbb{L}) = \mathrm{R\Pi}, \quad \mathcal{K}(\overrightarrow{\mathbf{\Sigma}}) = \mathrm{N\Pi},$$
$$\mathcal{K}(\mathbf{\Sigma}) = \mathrm{M\Pi}, \quad \mathcal{K}(\mathbf{\Pi}) = \Pi_{\mathrm{L}}, \quad \mathcal{K}(\overrightarrow{\mathbf{\Pi}}) = \Pi_{\mathrm{Z}}, \quad \mathcal{K}(\mathbf{\Pi}^*) = \Pi_{\mathrm{L}^*}, \quad \mathcal{K}(\overrightarrow{\mathbf{\Pi}}^*) = \Pi_{\mathrm{Z}^*}$$
$$\overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}^*) = \mathrm{Q\Lambda}, \quad \overline{\mathcal{K}}(\mathbb{L}^*) = \mathrm{S\Lambda}, \quad \overline{\mathcal{K}}(\mathbb{L}) = \mathrm{R\Lambda}, \quad \overline{\mathcal{K}}(\mathbf{\Sigma}) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}) = \mathrm{N\Lambda}$$
$$\overline{\mathcal{K}}(\mathbf{\Pi}) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}) = \Lambda_{\mathrm{L}}, \quad \overline{\mathcal{K}}(\mathbf{\Pi}^*) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}^*) = \Lambda_{\mathrm{L}^*}.$$

The Hopf algebras on the right-hand side have all appeared under various names in the literature. We have used the notation of [12], where all these objects are studied. In particular, $\mathrm{Q\Lambda}$ and $\mathrm{N\Lambda}$ are the Hopf algebras of quasi-symmetric and noncommutative symmetric functions, respectively. Also, $\Lambda_{\mathrm{L}} \cong \Lambda_{\mathrm{L}^*}$ is the Hopf algebra of symmetric functions.

One of our goals is to elaborate on this proposition and study its implications. For the claims involving the functor $\mathcal{K}$, we use the canonical identification between the degree $n$ parts of the Hopf monoid and the corresponding Hopf algebra. For the claim involving the functor $\overline{\mathcal{K}}$, we need to describe how to identify a coinvariant class of the degree $n$ part of the Hopf monoid with a degree $n$ element of the corresponding Hopf algebra. This is explained in subsequent sections as we go over each Hopf monoid individually. More details can be found in [12, Theorem 6.1.4, Propositions 7.1.1 and 8.1.1].

For a recent survey on topics related to (some of) the above Hopf algebras, see Hazewinkel [163, 164]. For applications to the representation theory of the symmetric group, see Blessenohl and Schocker [54]. For a survey on noncommutative symmetric functions and their applications, see Thibon [359].

**Hopf algebras from combinatorics.** The Hopf monoids of Chapter 13 give rise to another long list of Hopf algebras. Several of these Hopf algebras have received much attention in the recent literature; references are provided as each example is discussed in Section 17.5. We mention in particular work of Gessel and Malvenuto in connection to the Hopf algebras of posets (Section 17.5.1); of Ehrenborg in connection to the Hopf algebras of set-graded posets (Section 17.5.2); of Sagan, Schmitt, and Stanley in connection to the Hopf algebras of graphs (Section 17.5.3); of Connes–Kreimer and Grossman–Larson in connection to the Hopf algebras of forests (Section 17.5.4); of Crapo and Schmitt in connection to the Hopf algebras of matroids (Section 17.5.5).

We also discuss the combinatorial invariants that result from some of the morphisms of Hopf monoids of Chapter 13. They include Gessel's enumerator of poset partitions [144], the enumerator of descents of Bergeron and Sottile [42], Ehrenborg's flag function [112], the chromatic function of graphs of Ray and Wright [302] and of Stanley [339], a variant of this for labeled graphs due to Gebhard and Sagan [140], and a generating function for matroids constructed by Billera, Jia, and Reiner [48].

## 17.1. Shifting and standardization

Shifting and standardization are defined in Notation 2.5. Let $\mathbf{h}$ be a Hopf monoid in species. Recall from Section 15.2 that the coproduct of $\mathcal{K}(\mathbf{h})$ is constructed from the coproduct of $\mathbf{h}$ and the standardization maps. Similarly, the product of $\mathcal{K}(\mathbf{h})$ is constructed from the product of $\mathbf{h}$ and the shift map. This shows that the description of any Hopf algebra of the form $\mathcal{K}(\mathbf{h})$ involves the combinatorial procedures of shifting and standardization.

In this section, we look at the geometric aspect of these procedures. Recall the break and join maps, namely, $b_K$ and $j_K$ of Section 10.11. Composing these with the standardization and shift maps, yields maps which we denote by $b'_K$ and $j'_K$. These were defined in [12, Sections 6.3.3 and 6.6.3] and we recall them below. Unlike the break and join maps, their (co)domains only involve the Coxeter complexes $\Sigma[n]$ for varying $n$.

Recall that the Hopf monoids of Chapter 12 were defined geometrically using the break and join maps. The main point of this section is: If $\mathbf{h}$ is any of these Hopf monoids, then the corresponding Hopf algebra $\mathcal{K}(\mathbf{h})$ can be described in the same way by replacing the break and join maps by $b'_K$ and $j'_K$. We illustrate this on the Hopf monoid of linear orders.

**17.1.1. A variation on the break and join maps.** Let $K = S|T$ be a vertex of $\Sigma[n]$ of type $(s, t)$, thus, $s$ and $t$ are the cardinalities of $S$ and $T$ respectively. Define the map $b'_{S|T}$ as the composite

$$\begin{tikzcd} & \Sigma[S] \times \Sigma[T] \arrow[dr, "\Sigma[\mathrm{cano}]\times\Sigma[\mathrm{cano}]", "\cong"'] & \\ \mathrm{Star}(S|T) \arrow[ur, "b_{S|T}", "\cong"'] \arrow[rr, "b'_{S|T}"'] & & \Sigma[s] \times \Sigma[t]. \end{tikzcd}$$

The first isomorphism is the break map (10.57). The second isomorphism is induced by the order-preserving standardization maps $S \to [s]$ and $T \to [t]$. For example, for $K = 138|24567$,

$$b'_{S|T}\colon \mathrm{Star}(138|24567) \xrightarrow{\cong} \Sigma[3] \times \Sigma[5] \qquad\qquad 3|18|57|4|26 \mapsto (2|13, 35|2|14).$$

In other words, we first break the given set composition into two set compositions and then standardize each one of them. Note that this map relates the canonical linear orders $C_{(n)}$, $C_{(s)}$ and $C_{(t)}$ by

$$b'_K(KC_{(n)}) = (C_{(s)}, C_{(t)}).$$

Now let $(s,t)$ be a composition of $n$, and let $K$ be the vertex of $C_{(n)}$ of type $(s,t)$, that is, $K = S|T$ where $S = [s]$ and $T = [s+1, s+t]$. Consider the composite map

$$\begin{tikzcd} & \Sigma[S] \times \Sigma[T] \arrow[dr, "j_{S|T}", "\cong"'] & \\ \Sigma[s] \times \Sigma[t] \arrow[ur, "\Sigma[\mathrm{cano}] \times \Sigma[\mathrm{cano}]", "\cong"'] \arrow[rr, "j'_{S|T}"'] & & \mathrm{Star}(S|T). \end{tikzcd}$$

The second isomorphism is the join map (10.57). The first isomorphism is induced by the identity map $[s] \to S$ and the order-preserving shift map $[t] \to T$. For example, for the composition $(3,4)$, we have $K = 123|4567$ and

$$j'_K \colon \Sigma[3] \times \Sigma[4] \xrightarrow{\cong} \mathrm{Star}(123|4567) \qquad\qquad (2|13, 24|3|1) \mapsto 2|13|57|6|4.$$

In other words, we first shift up the indices in the second set composition and then concatenate it to the first set composition. Note that this map relates the canonical linear orders by

$$j_K(C_{(s)}, C_{(t)}) = C_{(n)}.$$

**17.1.2. The Hopf algebra of linear orders.** Recall from Definition 12.2 that the product and coproduct for the Hopf monoid on linear orders are as follows.

$$\begin{aligned} \mathbf{L}[S] \otimes \mathbf{L}[T] &\to \mathbf{L}[I] \\ C_1 \otimes C_2 &\mapsto j_{S|T}(C_1, C_2) \end{aligned} \qquad\qquad \begin{aligned} \mathbf{L}[I] &\to \mathbf{L}[S] \otimes \mathbf{L}[T] \\ C &\mapsto C_1 \otimes C_2, \end{aligned}$$

where $b_{S|T}(C) = (C_1, C_2)$.

It follows that the product and coproduct for the graded Hopf algebra $\mathcal{K}(\mathbf{L})$ can be described as follows.

$$\begin{aligned} \mathcal{K}(\mathbf{L})[s] \otimes \mathcal{K}(\mathbf{L})[t] &\to \mathcal{K}(\mathbf{L})[n] \\ C_1 \otimes C_2 &\mapsto j'_{S|T}(C_1, C_2) \end{aligned} \qquad\qquad \begin{aligned} \mathcal{K}(\mathbf{L})[n] &\to \mathcal{K}(\mathbf{L})[s] \otimes \mathcal{K}(\mathbf{L})[t] \\ C &\mapsto \sum_{\substack{S \sqcup T = [n] \\ |S|=s, |T|=t}} C_1 \otimes C_2, \end{aligned}$$

where for each summand $b'_{S|T}(C) = (C_1, C_2)$. This is a geometric reformulation of the description of $\mathcal{K}(\mathbf{L})$ given in Example 15.17.

Note the similarity between the description of the Hopf monoid and the corresponding Hopf algebra: the former uses the break and join maps whereas the latter uses the variants discussed above. It should be clear that this feature is displayed by all the Hopf monoids of Chapter 12.

We mention that for describing the Hopf algebras which correspond to the Hopf monoids of set partitions, one would need variants of the break and join maps for set partitions. These are defined in [12, Sections 6.5.10 and 6.8.9].

## 17.2. The Hopf algebra of permutations

In Section 12.3, we introduced and studied in detail two Hopf monoids indexed by pairs of linear orders. They were denoted $\mathbf{L}$ and $\mathbf{L}^*$ and were duals of each other. There is a $H$ and $K$ basis on $\mathbf{L}$, and a corresponding dual $M$ and $F$ basis on $\mathbf{L}^*$. We described the product, coproduct and antipode explicitly on all these bases. Further, we showed that $\mathbf{L}$ is self-dual, an explicit isomorphism $s\colon \mathbf{L} \to \mathbf{L}^*$ being given by the switch map

$$K_{(D,C)} \mapsto F_{(C,D)}.$$

We also constructed a one-parameter deformation of these Hopf monoids, denoted by $\mathbf{L}_q$ and $\mathbf{L}_q^*$.

**17.2.1. The associated Hopf algebras.** Now let us apply the Fock functors to these Hopf monoids. As mentioned in Proposition 17.1, we have

$$\mathcal{K}(\mathbf{L}^*) = \mathrm{S\Pi}, \quad \overline{\mathcal{K}}(\mathbf{L}^*) = \mathrm{S\Lambda}, \quad \mathcal{K}(\mathbf{L}) = \mathrm{R\Pi}, \quad \overline{\mathcal{K}}(\mathbf{L}) = \mathrm{R\Lambda}.$$

The object $\mathrm{S\Lambda}$ is the Hopf algebra of permutations of Malvenuto and Reutenauer [255, 256] that we encountered in Example 15.17 (where an explicit definition and more references are given). The object $\mathrm{S\Pi}$ is the Hopf algebra of pairs of permutations introduced in [12, Chapter 7]. Similar to the Hopf monoids, the Hopf algebras $\mathrm{R\Pi}$ and $\mathrm{R\Lambda}$ can be given a $H$ and $K$ basis, while $\mathrm{S\Pi}$ and $\mathrm{S\Lambda}$ can be given a $M$ and $F$ basis.

The identification $\overline{\mathcal{K}}(\mathbf{L}^*) = \mathrm{S\Lambda}$ is done as follows: the class of a pair of linear orders $F_{(C,D)} \in \mathbf{L}^*[I]$ identifies with the basis element $F_{d(C,D)}$, where $d(C,D)$ is the Weyl-valued distance between $C$ and $D$ given in (10.28). The identification $\overline{\mathcal{K}}(\mathbf{L}) = \mathrm{R\Lambda}$ is similar: the class $K_{(C,D)} \in \mathbf{L}[I]$ identifies with $K_{d(C,D)}$.

**17.2.2. Self-duality.** The switch map yields and isomorphism $\overline{\mathcal{K}}(s)\colon \mathrm{R\Lambda} \cong \mathrm{S\Lambda}$ of graded Hopf algebras. Using the above identifications we see that $\overline{\mathcal{K}}(s)$ sends $K_\sigma$ to $F_{\sigma^{-1}}$.

Since $\overline{\mathcal{K}}$ preserves duality, it also follows that $\mathrm{R\Lambda}$ and $\mathrm{S\Lambda}$ are duals of each other, and hence self-dual.

The switch map yields an isomorphism $\mathcal{K}(s)\colon \mathrm{R\Pi} \cong \mathrm{S\Pi}$ of graded Hopf algebras which sends $K_{(D,C)}$ to $F_{(C,D)}$. This does not imply that $\mathrm{R\Pi}$ (or $\mathrm{S\Pi}$) is self-dual (since unlike $\overline{\mathcal{K}}$, the functor $\mathcal{K}$ does not preserve duality). In fact, $\mathrm{R\Pi}$ is not self-dual: a calculation of primitive elements (on degree 3) suffices to distinguish the Hopf algebra $\mathrm{R\Pi}$ from its dual.

**17.2.3. Formulas on different bases.** The coproduct and product of the Hopf algebra $\mathrm{S\Pi}$ on the $F$ basis can be described in terms of the

$$\mathrm{Star}(K) \xrightarrow{b'_K} \Sigma[s] \times \Sigma[t] \quad \text{and} \quad \Sigma[s] \times \Sigma[t] \xrightarrow{j'_K} \mathrm{Star}(K)$$

of Section 17.1.1.

The positive part of the coproduct on $\mathrm{S\Pi}$ is given by

$$\Delta_+(F_{(C,D)}) = \sum_{K\colon \mathrm{rank}\, K=1\ K \leq D} F_{(C_1,D_1)} \otimes F_{(C_2,D_2)},$$

where $b'_K(D) = (D_1, D_2)$ and $b'_K(KC) = (C_1, C_2)$.

The product on SΠ is given by

$$F_{(C_1,D_1)} * F_{(C_2,D_2)} = \sum_{D:\, KD = j'_K(D_1,D_2)} F_{\left(j'_K(C_1,C_2),D\right)}.$$

Here $K \in \Sigma[n]$ is the unique vertex of $C_{(n)}$ of type $(s,t)$, where $C_1 \in \Sigma[s]$ and $C_2 \in \Sigma[t]$.

Observe that these are very similar to the formulas for $\mathbf{L}^*$ in Definition 12.7. As discussed in Section 17.1.2 for the Hopf algebra of linear orders, to obtain the formulas for the Hopf algebra SΠ one simply replaces each instance of the break and join maps $b_K$ and $j_K$ in the formulas for the Hopf monoid $\mathbf{L}^*$, by their variants $b'_K$ and $j'_K$.

The formula on the $M$ basis of SΠ given in [12, Theorems 7.3.1 and 7.3.4] can be derived in the same manner from the formula for $\mathbf{L}^*$ in Theorem 12.13. The term involving $q$ comes along for a ride. The story for the $K$ and $H$ basis of $\mathbf{L}$ is similar. The corresponding formulas for RΠ are given in [12, Definitions 6.5.6 and 6.8.6, Theorems 7.5.1 and 7.5.3].

We repeat that the $H$ basis of RΠ is not dual to the $M$ basis of SΠ. So the formulas on these bases, at the level of Hopf algebras, do not follow from each other; they have to be derived separately as done in [12]. On the other hand, if one works with species, then one can begin with the $M$ basis formula of $\mathbf{L}^*$, dualize to get the $H$ basis formula of $\mathbf{L}$, from which the corresponding formulas for both SΠ and RΠ follow.

Applying the functor $\overline{\mathcal{K}}$ yields formulas on the $M$ basis of SΛ. These were obtained in [14, Theorems 3.1 and 4.1]; for more information, see [12, Theorems 7.3.3 and 7.3.6]. The dual formulas on the $H$ basis of RΛ are written down in [12, Theorems 7.5.2 and 7.5.5].

**17.2.4. Antipode.** Since the functor $\overline{\mathcal{K}}$ is bistrong, it preserves antipodes. Thus Theorems 12.17 and 12.18 yield antipode formulas for SΛ on the $F$ and $M$ basis respectively. These were obtained in [14, Theorems 5.4 and 5.5]. We explain this in more detail below.

Let $v$ be a permutation on $n$ letters and $T$ be a subset of $[n-1]$. Choose chambers $C$ and $D$ such that $d(C,D) = v$. Define

$$v_T := d(HC, D),$$

where $H$ is the face of $D$ of type $T$. Since the Weyl distance and projections commute with the group action, it follows that $v_T$ only depends on $v$ and $T$ and not on the particular choice of $C$ and $D$.

Applying $\overline{\mathcal{K}}$ to formula (12.13), with $q = 1$, yields

$$\mathrm{s}(F_v) = \sum_w \Bigg( \sum_{T:\mathrm{Des}(w^{-1}v_T) \subseteq T} (-1)^{|T|+1} \Bigg) F_w.$$

The notations relate in the following manner:

$$d(C,D) = v, \quad d(C',D') = w, \quad d(C',D) = v_T, \quad \text{and} \quad \mathrm{type}(H) = T.$$

The condition $HC = C'$ is accounted for in the definition of $v_T$, while

$$HD' = D \iff \mathrm{Des}(D',D) \leq H \implies \mathrm{Des}(w^{-1}v_T) \subseteq T.$$

The equivalence follows from (10.42). For the implication, we apply the type map, and use the first diagram in (10.40), and

$$d(D', D) = d(D', C')\, d(C', D) = w^{-1} v_T.$$

Now we go back to the formula. Splitting the sum over $T$ into two parts depending on the parity of $|T|$ yields [14, Theorem 5.4].

The formula provided by Theorem 12.17 unlike formula (12.13) takes all cancellations into account. This can be expressed as follows. For fixed permutations $v$ and $w$, consider the map:

$$f\colon \{T \mid \mathrm{Des}(w^{-1}v_T) \subseteq T\} \to \mathrm{S}_n, \qquad T \mapsto w^{-1}v_T.$$

Then

$$\mathrm{s}(F_v) = \sum_w \sum (-1)^{|T|+1} F_w,$$

where the inside sum is over those subsets $T$ for which the inverse image $f^{-1}(w^{-1}v_T)$ is the singleton set $\{T\}$.

Applying $\overline{\mathcal{K}}$ to the formula provided by Theorem 12.18 for $q = 1$ yields:

$$\mathrm{s}(M_v) = (-1)^{\mathrm{gdes}(v)+1} \sum_w \kappa(v, w) M_w,$$

where $T = \mathrm{gDes}(v)$ is the set of global descents of $v$, and $\kappa(v, w)$ is the number of $T$-shuffle permutations satisfying the following conditions.

(i) $v_T\zeta^{-1} \leq w$,
(ii) if $v \leq v'$ and $v'_T\zeta^{-1} \leq w$, then $v = v'$, and
(iii) if $\mathrm{Des}(\zeta) \subseteq R \subseteq T$ and $v_R\zeta^{-1} \leq w$, then $R = T$.

Here $\leq$ denotes the weak left Bruhat order on permutations (Section 10.7.4). The notations relate in the following manner:

$$d(C, D) = v,\ \ d(C', D') = w,\ \ d(HC, D) = v_T,\ \ d(D', D) = \zeta,\ \text{ and }\ \mathrm{type}(H) = T.$$

The gallery condition $C' - D - D'$ yields (i), the condition $\overline{H}C' = C$ yields (ii), the condition $HD' = D$ says that $\zeta$ is a $T$-shuffle permutation, and the uniqueness of $H$ yields (iii). More details on how to effect these translations can be found in [12, Section 5.3.6].

**17.2.5. Primitive elements.** The primitive elements of $\mathbf{L}^*$ are determined in Section 12.3.5: the elements $M_{(C,D)}$ such that $D \wedge \overline{C} = \emptyset$ form a basis for this species.

Let $C$ and $D$ be linear orders and $d(C, D) = w$. The study of global descents in Section 10.7 shows that

$$D \wedge \overline{C} = \emptyset \iff w \text{ has no global descents.}$$

This implies that the elements $M_w$ where the permutation $w$ has no global descents form a basis of the space of primitive elements of $\mathrm{S}\Lambda$. A description for the coradical filtration of $\mathrm{S}\Lambda$ can be similarly deduced from the results in Section 12.3.5. This recovers the results in [14, Section 6]. Related results (for primitive elements) appeared earlier in the work of Poirier and Reutenauer [297] and Duchamp, Hivert and Thibon [107, Proposition 3.6].

The situation for $\mathrm{S}\Pi$ is much more complicated. From Proposition 15.36, we know that the primitive elements of $\mathbf{L}^*$ continue to be primitive for $\mathrm{S}\Pi$; also see the remark after [12, Theorem 7.3.2]. However, $\mathrm{S}\Pi$ has many more primitive elements, the dimensions of the components $n = 1, 2$ and 3 are 1, 3, and 29, whereas those for $\mathbb{L}^*$ are 1, 2 and 18. This provides an example of a Hopf monoid $\mathbf{h}$ for which the inclusion $\mathcal{K}\big(\mathcal{P}(\mathbf{h})\big) \subset \mathcal{P}\big(\mathcal{K}(\mathbf{h})\big)$ is strict.

The primitive elements, and more generally the coradical filtration, of $\mathrm{S\Pi}$ has been determined in [12, Theorem 7.4.3]. We recall that the strategy was to use a third basis, called the $S$ basis, on $\mathbb{L}^*[n]$. Just as the $F$ and $M$ basis are related using the partial order $\leq$, the $F$ and $S$ basis are related using a different partial order which was denoted $\preceq$. We remark that this partial order makes use of the canonical linear order on the set $[n]$; hence it cannot be defined on $\mathbb{L}^*[I]$. Therefore, it played no role in the study of $\mathbb{L}^*$.

**17.2.6. Interchanging the coordinates on the $M$ and $H$ basis.** It is interesting to apply the Fock functors to the map $t_1 \colon \mathbb{L}^* \to (\mathbb{L}^*)^{\mathrm{cop}}$ defined in (12.10). We recall that $\mathcal{K}$ is braided colax but not braided lax, while $\overline{\mathcal{K}}$ is both braided lax and colax. From Corollary 12.16 and Proposition 3.36, we obtain:

**Corollary 17.2** ([12, Corollaries 7.3.1 and 7.3.2])**.** *The map $\mathrm{S\Pi} \to \mathrm{S\Pi}^{\mathrm{cop}}$ that sends $M_{(C,D)}$ to $M_{(D,C)}$ and the map $\mathrm{S\Lambda} \to \mathrm{S\Lambda}^{\mathrm{cop}}$ that sends $M_w$ to $M_{w^{-1}}$ are isomorphisms of Hopf algebras.*

Since $\mathcal{K}$ is braided colax but not braided lax, Corollary 12.16 and Proposition 3.35 allow us to deduce the following result only.

**Corollary 17.3** ([12, Corollaries 7.5.1 and 7.5.2])**.** *The map $\mathrm{R\Pi}^{\mathrm{op}} \to \mathrm{R\Pi}$ that sends $H_{(C,D)}$ to $H_{(D,C)}$ is an isomorphism of coalgebras. The map $\mathrm{R\Lambda}^{\mathrm{op}} \to \mathrm{R\Lambda}$ that sends $H_w$ to $H_{w^{-1}}$ is an isomorphism of Hopf algebras.*

We see that at the level of Hopf algebras, interchanging coordinates on the $H$ basis is not so nice as that on the $M$ basis. The properties of the Fock functors are responsible for this fact.

**17.2.7. The deformation of Foissy.** Now consider the $q$-Hopf monoids $\mathbb{L}_q$ and $\mathbb{L}_q^*$. Note that applying $\overline{\mathcal{K}}$ yields $q$-Hopf algebras which deform $\mathrm{R\Lambda}$ and $\mathrm{S\Lambda}$ respectively. Let us denote these by $\mathrm{R\Lambda}_q$ and $\mathrm{S\Lambda}_q$. The former is the deformation considered by Foissy [130]; here, the coproduct is deformed while the product is as in the undeformed case. For the latter, the situation is reversed.

For a permutation $w$, let $\mathrm{inv}(w)$ be the number of inversions of $w$ as in (2.20). Using Proposition 12.12 and the relation between inversions and the gallery metric (10.27), one sees that the map

$$\mathrm{R\Lambda}_q \to \mathrm{S\Lambda}_q, \qquad\qquad K_w \mapsto q^{\mathrm{inv}(w)} F_{w^{-1}}$$

is an isomorphism if $q \neq 0$.

There is a different way of obtaining these $q$-Hopf algebras. It is explained in Example 16.32.

## 17.3. Quasi-symmetric and noncommutative symmetric functions

In Sections 12.4 and 12.5, we introduced and studied in detail Hopf monoids indexed by faces and directed faces. The ones indexed by faces were denoted $\mathbf{\Sigma}$ and $\mathbf{\Sigma}^*$. The canonical bases were denoted $H$ and $M$ respectively. The Hopf monoids indexed by directed faces were denoted $\overrightarrow{\mathbf{\Sigma}}$ and $\overrightarrow{\mathbf{\Sigma}}^*$. The former was equipped with the $H$ and $K$ bases and the latter with the dual $M$ and $F$ bases. We gave explicit descriptions for the products, coproducts and antipodes.

**17.3.1. The associated Hopf algebras.** Now let us apply the Fock functors to these Hopf monoids. As mentioned in Proposition 17.1, we have

$$\mathcal{K}(\mathbf{\Sigma}^*) = \mathrm{P}\Pi, \qquad \mathcal{K}(\overrightarrow{\mathbf{\Sigma}}^*) = \mathrm{Q}\Pi, \qquad \mathcal{K}(\mathbf{\Sigma}) = \mathrm{M}\Pi, \qquad \mathcal{K}(\overrightarrow{\mathbf{\Sigma}}) = \mathrm{N}\Pi$$
$$\overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}^*) = \mathrm{Q}\Lambda, \qquad \overline{\mathcal{K}}(\mathbf{\Sigma}) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}) = \mathrm{N}\Lambda.$$

The object $\mathrm{Q}\Lambda$ is the Hopf algebra of quasi-symmetric functions. It was introduced by Gessel [144] as a subalgebra of the algebra of formal power series in countably many variables (although with hindsight one can recognize this in work of Cartier [76]). It is discussed by Stanley [343, Section 7.19], Reutenauer [311, Section 9.4] and Bertet, Krob, Morvan, Novelli, Phan and Thibon [44]. The Hopf algebra structure of $\mathrm{Q}\Lambda$ was introduced by Malvenuto [255, Section 4.1]. The description of the product in some form or another can be found in works of Cartier [76, Formula (7)], Hoffman [169], Hazewinkel [161] and Ehrenborg [112, Lemma 3.3].

The object $\mathrm{N}\Lambda$ is the Hopf algebra of noncommutative symmetric functions introduced by Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon [142]. It can be viewed as a subalgebra of the algebra of formal power series in countably many noncommutative variables. More information can be found in [211, 108, 212, 213, 106, 107, 359].

The Hopf algebras $\mathrm{P}\Pi$, $\mathrm{Q}\Pi$, $\mathrm{M}\Pi$ and $\mathrm{N}\Pi$ are considered in [12, Section 6.2 and Chapter 8]. The objects $\mathrm{N}\Pi$ and $\mathrm{Q}\Pi$ have been studied under various names in [41, 43, 79, 166, 285, 289, 291, 292, 293].

Let $M$ and $F$ denote the bases of monomial and fundamental quasi-symmetric functions [144]. The latter are denoted $L$ in [343]. The identification $\overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \mathrm{Q}\Lambda$ is as follows:

$$\overline{M}_F \in \overline{\mathcal{K}}(\mathbf{\Sigma}^*) \;\longleftrightarrow\; M_{\mathrm{type}(F)} \in \mathrm{Q}\Lambda.$$

Here $\overline{M}_F$ denotes the class of $M_F \in \mathbf{\Sigma}^*[n]$ under the action of the symmetric group $\mathrm{S}_n$, and $\mathrm{type}(F)$ is the composition of $n$ underlying $F$.

Let $H$ and $K$ denote the bases of complete and ribbon noncommutative symmetric functions. They are denoted $S$ and $R$ in [359].

The identification $\overline{\mathcal{K}}(\mathbf{\Sigma}) = \mathrm{N}\Lambda$ involves a coefficient:

$$\overline{H}_G \in \overline{\mathcal{K}}(\mathbf{\Sigma}) \;\longleftrightarrow\; G!\, H_{\mathrm{type}(G)} \in \mathrm{N}\Lambda,$$

where $G!$ is as in (10.7).

The identifications $\overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}^*) = \mathrm{Q}\Lambda$ and $\overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}) = \mathrm{N}\Lambda$ are

$$\begin{aligned}
\overline{M}_{(F,C)} \in \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}^*) &\;\longleftrightarrow\; M_{\mathrm{type}(F)} \in \mathrm{Q}\Lambda\\
\overline{F}_{(F,C)} \in \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}^*) &\;\longleftrightarrow\; F_{\mathrm{type}(F)} \in \mathrm{Q}\Lambda\\
\overline{H}_{(G,D)} \in \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}) &\;\longleftrightarrow\; H_{\mathrm{type}(G)} \in \mathrm{N}\Lambda\\
\overline{K}_{(G,D)} \in \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}) &\;\longleftrightarrow\; K_{\mathrm{type}(G)} \in \mathrm{N}\Lambda.
\end{aligned}$$

**17.3.2. Formulas on different bases.** The Hopf monoid $\mathbf{\Sigma}^*$ gives rise to $\mathrm{P}\Pi$ and $\mathrm{Q}\Lambda$. The results in Section 12.4.1 yield the formulas on the $M$ basis of $\mathrm{P}\Pi$ given in [12, Section 8.3], and the familiar formulas on the $M$ basis of $\mathrm{Q}\Lambda$ (which we recall below).

The Hopf monoid $\overrightarrow{\mathbf{\Sigma}}^*$ gives rise to $\mathrm{Q}\Pi$ and $\mathrm{Q}\Lambda$. The formulas on the $F$ and $M$ bases of $\overrightarrow{\mathbf{\Sigma}}^*$ given in Sections 12.5.1 and 12.5.2 immediately imply those for the Hopf algebra $\mathrm{Q}\Pi$ given in [12, Section 8.2]. They also give rise to the familiar formulas on the $M$ and $F$ bases of $\mathrm{Q}\Lambda$. The latter are well-known and appear in most of the references cited above. We recall them next.

The product and coproduct on the $M$ basis of $\mathrm{Q}\Lambda$ are as follows.

$$\Delta(M_{(\alpha_1,\alpha_2,\dots,\alpha_k)}) = \sum_{i=0}^{k} M_{(\alpha_1,\dots,\alpha_i)} \otimes M_{(\alpha_{i+1},\dots,\alpha_k)},$$
$$M_\alpha * M_\beta = \sum_{\gamma} M_\gamma.$$

The second sum is over all quasi-shuffles $\gamma$ of $\alpha$ and $\beta$. (Quasi-shuffle of compositions is defined similarly to quasi-shuffle of set compositions, and is a special case of the construction of Section 2.6.6.)

Since the functor $\overline{\mathcal{K}}$ is bistrong, it preserves antipodes. Thus, either one of Theorems 12.21 or 12.34 yields the following antipode formula for $\mathrm{Q}\Lambda$ on the $M$ basis:

$$\mathrm{s}(M_\alpha) = (-1)^{\deg(\alpha)} \sum_{\beta:\beta\leq\overline{\alpha}} M_\beta,$$

where $\deg(\alpha)$ is the number of parts of $\alpha$ and $\overline{\alpha}$ is the composition $\alpha$ written in reverse order. This formula was obtained independently by Malvenuto [255, Corollaire 4.20] and Ehrenborg [112, Proposition 3.4].

We turn to the $F$ basis of $\mathrm{Q}\Lambda$. It is convenient to index the basis by binary words. A composition of $n$ is equivalent to a subset of $[n-1]$ and the latter to a word of length $n-1$ in the alphabet $\{+,-\}$. As an illustration,

$$(2,1,2,3) \quad \longleftrightarrow \quad \{2,3,5\} \quad \longleftrightarrow \quad -++-+--.$$

The coproduct of $\mathrm{Q}\Lambda$ is given by

$$\Delta(F_{\xi_1\xi_2\dots\xi_{n-1}}) = \sum_{i=0}^{n} F_{\xi_1\dots\xi_{i-1}} \otimes F_{\xi_{i+1}\dots\xi_{n-1}},$$

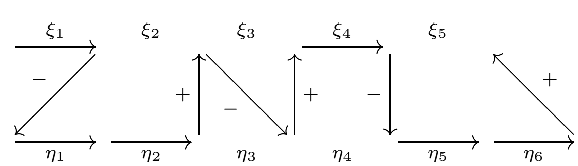
where each $\xi_i$ is either $+$ or $-$.

To describe the product, we need a notation. Let $S$ be a shuffle of the linear orders $1|2|\cdots|g_1$ and $1'|2'|\cdots|g_2'$.

This can be shown by a diagram as below, where we have taken $g_1 = 6$ and $g_2 = 7$ for illustration.

$$S = \begin{tikzcd}
1 \arrow[r] & 2 \arrow[dl] & 3 \arrow[dr] & 4 \arrow[r] & 5 \arrow[d] & 6 & \\
1' \arrow[r] & 2' \arrow[r] & 3' \arrow[u] & 4' \arrow[u] & 5' \arrow[r] & 6' \arrow[r] & 7' \arrow[ul]
\end{tikzcd}$$

Now suppose $\xi = \xi_1\xi_2\dots\xi_{g_1-1}$ and $\eta = \eta_1\eta_2\dots\eta_{g_2-1}$ are two sign sequences of lengths $g_1-1$ and $g_2-1$ respectively. Then, using the shuffle $S$, one can define a sign sequence $S(\xi,\eta)$ of length $g_1+g_2-1$ as illustrated below.

$S(\xi,\eta) =$ 

Namely, first draw the diagram for the shuffle $S$. Then put a $-$ sign on the arrows going down and a $+$ sign on the arrows going up. The horizontal arrows get labeled $\xi_i$ or $\eta_i$. In the example above, $S(\xi,\eta) = \xi_1 - \eta_1\eta_2 + - + \xi_4 - \eta_5\eta_6 +$.

Let $\xi$ and $\eta$ be sign sequences of length $g_1 - 1$ and $g_2 - 1$ respectively. The product on the $F$ basis is given by

$$F_\xi * F_\eta \;=\; \sum_S F_{S(\xi,\eta)},$$

where the sum if over all shuffles $S$ of $1|\cdots|g_1$ and $1'|\cdots|g_2'$. This description of the product of $\mathrm{Q}\Lambda$ appears in [12, Theorem 8.4.2]. It is a reformulation of well-known formulas such as [343, Exercise 7.93].

Let $\xi$ be a sign sequence of length $n-1$. The antipode on the $F$ basis is given by

$$\mathrm{s}(F_\xi) = (-1)^n\, F_{\overline{\xi}^c},$$

where $\overline{\xi}^c$ is the sign sequence obtained from $\xi$ by first replacing $+$ by $-$, and $-$ by $+$, and then writing the word backwards. For example,

$$-++-+-- \text{ first changes to } +--+-++ \text{ and then to } ++-+--+\,.$$

In the subset notation,

$$\mathrm{s}(F_T) \;=\; (-1)^n\, F_{\overline{T}^c},$$

where $\overline{T}^c$ is obtained from $T$ by first taking complement in $[n-1]$ and then replacing each entry by $n$ minus that entry. This formula is due to Malvenuto [255, Corollaire 4.20].

The antipode formula given by Theorem 12.31 is cancellation-free. However, the formula obtained by applying $\overline{\mathcal{K}}$ to it is not cancellation-free. The above formula has been written after taking these further cancellations into account. This will become clear from the $q$-case discussed below.

**17.3.3. The coradical filtrations.** Similar to the situation for $\mathbf{L}^*$, the $M$ basis of $\mathbf{\Sigma}^*$ and $\mathbf{\Sigma}^*$ can be used to determine their primitive elements. It follows from (15.19) applied to any of these two Hopf monoids that the space of primitive elements of $\mathrm{Q}\Lambda$ is spanned by compositions with one part.

The situation for $\mathrm{Q}\Pi$ and $\mathrm{P}\Pi$ is much more complicated. Similar to the story for $\mathrm{S}\Pi$, these have many more primitive elements than the corresponding Hopf monoids. The primitive elements, and more generally the coradical filtration, of $\mathrm{Q}\Pi$ has been determined in [12, Theorem 8.2.2]. There is a similar result for $\mathrm{P}\Pi$, which is not explicitly stated, but which follows from [12, Theorem 8.3.1]. The strategy again is to use the canonical linear order on the set $[n]$ to define a $S$ basis on the set of faces and the set of directed faces.

**17.3.4. Deformations.** Now consider the $q$-Hopf monoids $\overrightarrow{\mathbf{\Sigma}}_q$ and $\overrightarrow{\mathbf{\Sigma}}^*_q$. Note that applying $\overline{\mathcal{K}}$ yields $q$-Hopf algebras which deform $\mathrm{N}\Lambda$ and $\mathrm{Q}\Lambda$ respectively. Let us denote these by $\mathrm{N}\Lambda_q$ and $\mathrm{Q}\Lambda_q$. We make the latter explicit. The coproduct is as before, while the product is modified by

$$M_\alpha * M_\beta = \sum_\gamma q^{c(\alpha,\beta;\gamma)}\, M_\gamma$$

where the sum is over all quasi-shuffles of $\alpha$ and $\beta$, and $c(\alpha,\beta,\gamma)$ is the cost of unwinding the quasi-shuffle. We explain this by an example. Suppose

$$\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4), \quad \beta = (\beta_1,\beta_2,\beta_3), \quad \text{and} \quad \gamma = (\alpha_1,\alpha_2+\beta_1,\beta_2,\alpha_3+\beta_3,\alpha_4).$$

Then

$$c(\alpha,\beta;\gamma) = \beta_1\alpha_3 + \beta_1\alpha_4 + \beta_2\alpha_3 + \beta_2\alpha_4 + \beta_3\alpha_4.$$

The antipode of $\mathrm{Q}\Lambda_q$ is

$$\mathrm{s}(M_\alpha) = (-1)^{\deg(\alpha)} \sum_{\beta:\beta\leq\overline{\alpha}} q^{d(\beta)}\, M_\beta,$$

where for $\beta = (\beta_1,\dots,\beta_k)$,

$$d(\beta) := \sum_{1\leq i<j\leq k} \beta_i\beta_j.$$

This may be viewed as the cost of going from $\beta$ to its reverse composition.

In the $F$ basis, the coproduct is the same as before and the product is modified by multiplying each term by a power of $q$. The exponent is the cost of unwinding the shuffle $S(\xi,\eta)$. The antipode formula is as below. Let $T$ be a subset of $[n-1]$. Then

$$\mathrm{s}(F_T) \;=\; (-1)^n \sum_{V\,:\,\overline{T}^c\subseteq V} \left( \sum_{U\,:\,\overline{T}^c\subseteq U\subseteq V} (-1)^{|U|} q^{d(U)} \right) (-1)^{|V|} F_V, \tag{17.1}$$

where $\overline{T}^c$ and $d(U)$ are as defined above. If $q=1$, then the inner sum is zero unless $V=\overline{T}^c$, thus recovering the earlier formula. In general, in the inner sum, the unique smallest power of $q$ occurs when $U=\overline{T}^c$ and the unique largest power of $q$ occurs when $U=V$. Thus if $q$ is not an algebraic integer, then formula (17.1) is cancellation-free.

We showed in Proposition 12.38 that for $q$ not an algebraic integer, the $q$-Hopf monoids $\overrightarrow{\mathbf{\Sigma}}_q$ and $\overrightarrow{\mathbf{\Sigma}}^*_q$ are isomorphic. Applying the functor $\overline{\mathcal{K}}$, it follows that under the same hypothesis, there is an isomorphism

$$\mathrm{N}\Lambda_q \xrightarrow{\cong} \mathrm{Q}\Lambda_q$$

induced by the nondegenerate bilinear form (10.138). Since the product of $\mathrm{N}\Lambda_q$ is the same as that of $\mathrm{N}\Lambda$ (this being the case for the corresponding Hopf monoids), one obtains an algebra isomorphism from noncommutative symmetric functions to the $q$-version of quasi-symmetric functions. This result appears in [358, Section 3].

## 17.4. Symmetric functions

In Sections 12.6 and 12.7, we introduced and studied in detail Hopf monoids indexed by flats and directed flats. The ones indexed by flats were denoted $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$. The former was equipped with the $h$ and $q$ bases and the latter with the dual $m$ and $p$ bases. We showed that the Hopf monoids $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ are isomorphic and hence self-dual. The Hopf monoids indexed by directed flats were denoted $\overrightarrow{\mathbf{\Pi}}$ and $\overrightarrow{\mathbf{\Pi}}^*$. The canonical bases were denoted by $h$ and $m$ respectively. We gave explicit descriptions for the products, coproducts and antipodes on all these bases.

**17.4.1. The associated Hopf algebras.** Now let us apply the Fock functors to these Hopf monoids. As mentioned in Proposition 17.1, we have

$$\mathcal{K}(\mathbf{\Pi}) = \Pi_{\mathrm{L}}, \qquad \mathcal{K}(\overrightarrow{\mathbf{\Pi}}) = \Pi_{\mathrm{Z}}, \qquad \mathcal{K}(\mathbf{\Pi}^*) = \Pi_{\mathrm{L}^*}, \qquad \mathcal{K}(\overrightarrow{\mathbf{\Pi}}^*) = \Pi_{\mathrm{Z}^*},$$
$$\overline{\mathcal{K}}(\mathbf{\Pi}) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}) = \Lambda_{\mathrm{L}}, \qquad \overline{\mathcal{K}}(\mathbf{\Pi}^*) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}^*) = \Lambda_{\mathrm{L}^*}.$$

Since $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ are isomorphic, it follows that $\Lambda_{\mathrm{L}} \cong \Lambda_{\mathrm{L}^*}$ and $\Pi_{\mathrm{L}} \cong \Pi_{\mathrm{L}^*}$. Let us write $\Lambda$ for the former and $\Pi$ for the latter from now on.

The object $\Lambda$ is the Hopf algebra of symmetric functions. It is most often viewed as a subalgebra of the algebra of formal power series in countably many variables [138, 252, 319, 341]. The Hopf algebra viewpoint can be found in the works of Geissinger [141] and Zelevinsky [381]. The object $\Pi$ is the Hopf algebra of symmetric functions in noncommuting variables studied in [140, 315, 377], Even though the Hopf monoid $\mathbf{\Pi}$ is commutative, the Hopf algebra $\Pi$ is not. This example shows that $\mathcal{K}$ does not preserve commutativity.

The Hopf algebras $\Pi_{\mathrm{Z}}$ and $\Pi_{\mathrm{Z}^*}$ are described in [12, Sections 6.5.10 and 6.8.9]. The former is neither commutative nor cocommutative, while the latter is not commutative but cocommutative. Thus these Hopf algebras are neither isomorphic nor duals of each other. This example shows that $\mathcal{K}$ does not preserve duality.

We make use of the type and base maps of Section 10.1.5 in the following discussion. Let $h, m$ and $p$ denote the bases of complete, monomial and power sum symmetric functions respectively, as in [252, Section I.2] or [319, Section 4.3].

Suppose $X$ is a set partition into $k$ blocks, with $r_i$ blocks of size $i$, $i = 1, \dots, k$. Let $\lambda$ be the type of $X$, that is, the part $i$ occurs $r_i$ times in the partition $\lambda$. The identifications $\overline{\mathcal{K}}(\mathbf{\Pi}^*) = \overline{\mathcal{K}}(\mathbf{\Pi}) = \Lambda$ are as follows:

$$\begin{aligned}
\overline{m}_X \in \overline{\mathcal{K}}(\mathbf{\Pi}^*) &\longleftrightarrow r_1! \dots r_k!\, m_\lambda \in \Lambda;\\
\overline{p}_X \in \overline{\mathcal{K}}(\mathbf{\Pi}^*) &\longleftrightarrow p_\lambda \in \Lambda;\\
\overline{h}_X \in \overline{\mathcal{K}}(\mathbf{\Pi}) &\longleftrightarrow \lambda!\, h_\lambda \in \Lambda.
\end{aligned}$$

Here $\overline{m}_X$ denotes the class of $m_X$ under the action of the symmetric group and $\lambda!$ is defined in (10.8).

The Hopf algebra $\Pi$ arises similarly via the functor $\mathcal{K}$ and therefore maps canonically onto $\Lambda$. The above identifications are compatible with the description of this quotient map in [315, Theorem 2.1].

Suppose now that $L$ is a linear set partition into $k$ blocks, $X$ is its base, and $\lambda$ is the type of $X$. As above, let $X$ have $r_i$ blocks of size $i$, $i = 1, \dots, k$. The identifications $\overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}^*) = \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}) = \Lambda$ are as follows:

$$\begin{aligned}
\overline{m}_L \in \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}^*) &\longleftrightarrow r_1! \dots r_k!\, m_\lambda \in \Lambda;\\
\overline{h}_L \in \overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}) &\longleftrightarrow h_\lambda \in \Lambda.
\end{aligned}$$

The self-duality of the Hopf monoids $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ implies the self-duality of the Hopf algebra of symmetric functions, since $\overline{\mathcal{K}}$ preserves duality (Section 15.4.4).

**17.4.2. Formulas on different bases.** The product and coproduct formulas on the monomial, complete, and power sum bases of symmetric functions are well-known [12, Section 3.2.1]. These follow from the formulas on the corresponding bases of either $\mathbf{\Pi}$ or $\mathbf{\Pi}^*$. The same can be said for the antipode formulas.

## 17.5. Combinatorial Hopf algebras

We now consider some of the Hopf algebras arising from the Hopf monoids of Chapter 13, as well as some others that arise from the universal constructions of Chapter 11.

**17.5.1. Posets.** We begin with Hopf monoids of posets from Section 13.1. They were denoted $\mathbf{P}$ and $\mathbf{P}^*$ and were dual to each other. Applying the Fock functors yields various graded Hopf algebras, some of which have been considered in the literature. The Hopf algebra $\mathcal{K}(\mathbf{P})$ is implicit in the works of Gessel [144] and Malvenuto [255]. It has a basis consisting of posets on the sets $[n]$, $n \geq 0$. These are often called vertex-labeled posets in the combinatorics literature (the vertices are labeled by $[n]$). The Hopf algebra $\overline{\mathcal{K}}(\mathbf{P})$ has a basis consisting of isomorphism classes of finite posets (which may be called unlabeled posets). It has been explicitly described in [323, Section 16] and [10, Example 2.3].

Recall that $\mathcal{K}(\mathbf{L}^*) = \mathrm{S}\Lambda$ is the Hopf algebra of permutations (Example 15.17). Now consider the morphism

$$\hat{\omega}\colon \mathbf{P} \to \mathbf{L}^*$$

which arises from cofreeness of $\mathbf{L}^*$ (Section 13.1.3). Applying the functor $\mathcal{K}$ yields a morphism of Hopf algebras

$$\mathcal{K}(\hat{\omega})\colon \mathcal{K}(\mathbf{P}) \to \mathcal{K}(\mathbf{L}^*) = \mathrm{S}\Lambda.$$

This is familiar in the literature. Note that unless one works with species as above, this morphism cannot be constructed by universal properties, since in the passage from Hopf monoids to Hopf algebras via the functor $\mathcal{K}$ the universality of $\mathbf{L}^*$ is lost.

Recall that $\overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \mathrm{Q}\Lambda$ is the familiar Hopf algebra of quasi-symmetric functions (Section 17.3). Now consider the morphism

$$\hat{\zeta}\colon \mathbf{L} \times \mathbf{P} \to \mathbf{\Sigma}^*$$

which arises from cofreeness of the latter (Section 13.1.4). Applying the functor $\overline{\mathcal{K}}$ yields a morphism of Hopf algebras

$$\mathcal{K}(\mathbf{P}) = \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{P}) \xrightarrow{\overline{\mathcal{K}}(\hat{\zeta})} \overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \mathrm{Q}\Lambda.$$

This is Gessel's enumerator of poset partitions. The first equality follows from relation (15.6) between the functors $\mathcal{K}$ and $\overline{\mathcal{K}}$.

**17.5.2. Set graded posets.** One of the main Hopf algebras in combinatorics, and perhaps the first one to have arisen in the literature, is the graded Hopf algebra of graded posets of Joni and Rota [179, Section IV], which was further investigated by Ehrenborg [112] and Schmitt [323]. There is also a related Hopf algebra of *edge-labeled graded posets* defined by Bergeron and Sottile [42]. In Section 13.6.4, we constructed the Hopf monoid $\mathbf{sgP}$ based on the species of set-graded posets, in a manner which paralleled the definition of the Hopf algebra of (edge-labeled) graded posets.

The Hopf algebra $\mathcal{K}(\mathbf{sgP})$ has a basis consisting of edge-labeled posets for which the labels of any maximal chain form a permutation of $[n]$, $n \geq 0$. It is closely related to the Hopf algebra of edge-labeled graded posets of Bergeron and Sottile (their labelings are arbitrary). A basis for the Hopf algebra $\overline{\mathcal{K}}(\mathbf{sgP})$ appears to be difficult to describe.

Now consider the morphism

$$\hat{\eta}\colon \mathbf{sgP} \to \mathbf{\Sigma}^*$$

which arises from cofreeness of the latter (Section 13.6.5). Applying the functor $\overline{\mathcal{K}}$ yields a morphism of Hopf algebras

$$\overline{\mathcal{K}}(\hat{\eta})\colon \overline{\mathcal{K}}(\mathbf{sgP}) \to \overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \mathrm{Q}\Lambda.$$

This is Ehrenborg's *flag quasi-symmetric function* [112]. Indeed, starting from an arbitrary graded poset $P$ and choosing an arbitrary $[n]$-labeling $\lambda$, we have that

$$\overline{\mathcal{K}}(\hat{\eta})(P,\lambda) = \sum_{\alpha} f_\alpha(P)\, M_\alpha,$$

where the coefficient $f_\alpha(P)$ counts the number of chains $C$ in $P$ such that the underlying composition of $\lambda(C)$ is $\alpha$. This is precisely Ehrenborg's function.

Now consider the morphism

$$\hat{\zeta}\colon \mathbf{L} \times \mathbf{sgP} \to \mathbf{\Sigma}^*$$

which arises from cofreeness of the latter (Section 13.6.6). Applying the functor $\overline{\mathcal{K}}$ yields a morphism of Hopf algebras

$$\overline{\mathcal{K}}(\hat{\zeta})\colon \mathcal{K}(\mathbf{sgP}) = \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{sgP}) \to \overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \mathrm{Q}\Lambda.$$

This is closely related to the enumerator of descents defined by Bergeron and Sottile [42].

**17.5.3. Graphs.** We now consider Hopf monoids of graphs from Section 13.2. They were denoted $\mathbf{G}$ and $\mathbf{G}^*$ and were dual to each other. Applying the Fock functors yields various Hopf algebras of graphs. The Hopf algebra $\mathcal{K}(\mathbf{G})$ has a basis consisting of simple graphs with vertex set $[n]$, $n \geq 0$. The Hopf algebra $\overline{\mathcal{K}}(\mathbf{G})$ has a basis consisting of isomorphism classes of simple graphs. To distinguish between the two classes, one sometimes refers to the former as labeled graphs and to the latter as unlabeled.

The Hopf algebras $\mathcal{K}(\mathbf{G})$ and $\overline{\mathcal{K}}(\mathbf{G})$ were constructed by different means by Schmitt: they respectively agree with the Hopf algebras of Sections 13 and 12 in [323].

In Section 13.2.2, we constructed the morphism

$$\hat{\zeta}\colon \mathbf{G} \to \mathbf{\Pi}^*$$

from the cofreeness of the latter. Applying the Fock functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ to this morphism yields morphisms of Hopf algebras

$$\mathcal{K}(\mathbf{G}) \to \mathcal{K}(\mathbf{\Pi}^*) \quad \text{and} \quad \overline{\mathcal{K}}(\mathbf{G}) \to \overline{\mathcal{K}}(\mathbf{\Pi}^*).$$

Recall from Section 17.4 that $\mathcal{K}(\mathbf{\Pi}^*)$ is the algebra of symmetric functions in noncommuting variables and $\overline{\mathcal{K}}(\mathbf{\Pi}^*)$ is the familiar Hopf algebra of symmetric functions. Thus the first map associates a symmetric function in noncommuting variables to a labeled graph, and the second associates a symmetric function to an unlabeled graph. The latter is the chromatic symmetric function of Ray and Wright [302] and Stanley [339, Proposition 2.4]. The former is a variant introduced by Gebhard and Sagan [140].

Stanley constructs a more general version of the chromatic symmetric function that depends on a parameter $q \in \Bbbk$ and reduces to the previous version when

$q = 0$ [342]. It is easy to see that this generating function arises from a $q$-version of $\hat{\zeta}$ starting with (13.14).

**17.5.4. Forests.** In Section 13.3, we discussed two Hopf monoids based on rooted forests and on planar rooted forests. They were denoted $\mathbf{F}$ and $\overrightarrow{\mathbf{F}}$ respectively. We also discussed a $q$-deformation of the latter denoted $\overrightarrow{\mathbf{F}}_q$. Applying the functor $\overline{\mathcal{K}}$ to the Hopf monoid of rooted forests yields the Connes–Kreimer Hopf algebra. The set of unlabeled rooted forests is a linear basis of $\overline{\mathcal{K}}(\mathbf{F})$. Applying $\overline{\mathcal{K}}$ to the antipode formula of Theorem 13.5 yields Zimmermann's forest formula in quantum field theory [383].

This Hopf algebra appeared in early work of Grossman and Larson [155] and Schmitt [323, Example 16.1]. Kreimer [210] and Connes and Kreimer [83] brought it to the forefront by discussing connections with renormalization theory. It has been further studied in several works including [170, 278]. The fact that Zimmermann's formula can be seen as an antipode formula is due to Kreimer [210]; see also [83, p. 219], [126] and [149, Section 14.1].

Applying the functor $\overline{\mathcal{K}}$ to the Hopf monoid $\overrightarrow{\mathbf{F}}$ yields the Hopf algebra of planar rooted forests sometimes known as the the noncommutative Connes–Kreimer Hopf algebra. The set of unlabeled planar rooted forests is a linear basis of $\overline{\mathcal{K}}(\overrightarrow{\mathbf{F}})$. This Hopf algebra was introduced by Foissy [128, 129] and is further studied in several works including [171, 288] and [365, Section 7].

Foissy also considers a decorated version of this Hopf algebra. This arises from $\overrightarrow{\mathbf{F}}$ via the decorated Fock functor $\overline{\mathcal{K}}_V$ which will be studied in Chapter 19. Applying $\overline{\mathcal{K}}$ to the antipode formula of Theorem 13.4 (with $q = 1$) yields the formula of Foissy [128, Théorème 44] (in the undecorated setting).

Now consider the morphisms

$$\overrightarrow{\mathbf{F}} \xrightarrow{\vec{\phi}} \mathbf{L} \times \mathbf{P} \xrightarrow{\mathrm{id}\times\hat{\omega}} \mathbf{L} \times \mathbf{L}^* = \mathbf{I\!L}^*.$$

This is a portion of the diagram (13.18). The map $\vec{\phi}$ was defined in (13.17). The map $\hat{\omega}$ was defined in Section 13.1.3 and also considered in Section 17.5.1 while discussing Hopf algebras on posets. It is straightforward to check that the above composite is injective. Therefore, the noncommutative Connes–Kreimer Hopf algebra $\overline{\mathcal{K}}(\overrightarrow{\mathbf{F}})$ can be viewed as a Hopf subalgebra of the Hopf algebra of permutations $\mathrm{S}\Lambda = \overline{\mathcal{K}}(\mathbf{I\!L}^*)$. This gives rise to an isomorphism between the noncommutative Connes–Kreimer Hopf algebra and the Hopf algebra of unlabeled planar binary trees of Loday and Ronco [242, 243]. Another such isomorphism is discussed in [15, Section 8]. Applying the functor $\overline{\mathcal{K}}$ to the leftmost square in (13.18) yields the commutative diagram given in [15, Theorem 8.12], which relates Connes–Kreimer to symmetric functions and noncommutative symmetric functions.

The $q$-Hopf algebra $\overline{\mathcal{K}}(\overrightarrow{\mathbf{F}}_q)$ is considered by Foissy in [130, Section 1.1].

**17.5.5. Matroids.** Consider the Hopf monoid $\mathbf{M}$ of matroids from Section 13.8.2. The (commutative) Hopf algebra $\overline{\mathcal{K}}(\mathbf{M})$ has a basis consisting of isomorphism classes of finite matroids. It was introduced by Schmitt [323, Section 15] and further studied by Crapo and Schmitt [84, 85, 86].

Recall that we constructed a morphism of Hopf monoids

$$\hat{\zeta} \colon \mathbf{M} \to \mathbf{\Sigma}^*.$$

Applying $\overline{\mathcal{K}}$ yields a morphism of Hopf algebras

$$\overline{\mathcal{K}}(\hat{\zeta})\colon \overline{\mathcal{K}}(\mathbf{M}) \to \overline{\mathcal{K}}(\mathbf{\Sigma}^*) = \mathrm{Q}\Lambda,$$

where $\mathrm{Q}\Lambda$ is the Hopf algebra of quasi-symmetric functions. The above quasi-symmetric generating function was introduced by Billera, Jia, and Reiner in [48].

**17.5.6. Examples arising from universal constructions.** There are additional examples of Hopf algebras in the combinatorial literature that can be understood as follows. The universal constructions of Chapter 11 produce a large number of Hopf monoids, each with a certain universal property. Applying the Fock functors one then obtains Hopf algebras.

The Hopf algebras of Sections 17.3 and 17.4 arise in this manner. We discuss other examples next.

Hivert, Novelli, and Thibon define certain Hopf algebras of permutations and endofunctions in [167]. They arise from the free commutative Hopf monoid construction of Section 11.3, as follows.

Recall the Hopf monoid $\mathbf{b}$ of bijections from Example 11.16. We have $\mathbf{b} = \mathcal{S}(\mathbf{c})$ where $\mathbf{c}$ is the species of cycles. In other words, $\mathbf{b}$ is the free commutative Hopf monoid on the trivial positive monoid $\mathbf{c}$. Applying the functor $\mathcal{K}^{\vee}$ yields precisely the commutative Hopf algebra of permutations introduced by Hivert, Novelli, and Thibon in [167, Section 3].

The commutative Hopf algebra of endofunctions of [167, Section 2] can be obtained in a similar manner by replacing $\mathbf{c}$ for $\mathbf{c} \circ \mathbf{a}$, where $\mathbf{a}$ is the species of rooted trees (Section 13.3.1), using [40, §1.4, equation (1)] or [181, Exemple 12].

We turn to a Hopf algebra construction of Schmitt. Let Q be a species with restrictions (Section 8.7.8) and $\mathbf{q} = \Bbbk\mathrm{Q}$ its linearization. Then $\mathbf{q}$ carries the cocommutative comonoid structure described in Section 8.7.8. We may then consider the free commutative monoid $\mathcal{S}(\mathbf{q}_+)$, with the Hopf monoid structure of Section 11.3.2. The Hopf algebra $\mathcal{K}\big(\mathcal{S}(\mathbf{q}_+)\big)$ is the object constructed by Schmitt in [322, Section 3.3].

Schmitt also considers more general Hopf monoid structures on $\mathcal{S}(\mathbf{q}_+)$ that do not necessarily come from a comonoid structure on $\mathbf{q}$ [322, Example 3.3.3 and Section 4]. This allows him to obtain interesting examples such as the *Faà-di-Bruno* Hopf algebra. We do not deal with them in this monograph.

Note that in any of the above cases, we may apply the other universal constructions of Chapter 11 (the functors $\mathcal{T}$ or $\mathcal{T}^{\vee}$), combine them with any of the Fock functors and obtain in this manner many new Hopf algebras of a similar combinatorial flavor.

CHAPTER 18

# Adjoints of the Fock Functors

The Fock functors from species to graded vector spaces have been studied in detail in earlier chapters. In this chapter, we describe the adjoints of the Fock functors whenever they exist. Thus, important notions such as the free monoid in species on a graded algebra are discussed here. This study also clarifies various inter-relationships between these functors and the tensor and symmetric algebra functors on graded vector spaces and the corresponding functors $\mathcal{T}$ and $\mathcal{S}$ on species from Chapter 11. This further enriches our understanding of the interplay between species and graded vector spaces. The results of Section 3.9 on adjunctions play an important role in this discussion.

The adjoints are summarized in Tables 18.1 and 18.2. We use $\mathsf{Mon}(\mathsf{Sp})$ and $\mathsf{Comon}(\mathsf{Sp})$ for the categories of monoids and of comonoids in species, and $\mathsf{gAlg}$ and $\mathsf{gCoalg}$ for the categories of graded algebras and of graded coalgebras. The main goal of this chapter is to explain the functors in the second and third columns. The

Table 18.1. Adjoints of $\mathcal{K}$ and $\overline{\mathcal{K}}$.

| Functor | Left adjoint | Right adjoint |
|---|---|---|
| $\mathcal{K}\colon \mathsf{Sp} \to \mathsf{gVec}$ | $\mathcal{L}\colon \mathsf{gVec} \to \mathsf{Sp}$ | $\mathcal{R}\colon \mathsf{gVec} \to \mathsf{Sp}$ |
| $\mathcal{K}\colon \mathsf{Mon}(\mathsf{Sp}) \to \mathsf{gAlg}$ | $\mathcal{L}\colon \mathsf{gAlg} \to \mathsf{Mon}(\mathsf{Sp})$ | $\nexists$ |
| $\mathcal{K}\colon \mathsf{Comon}(\mathsf{Sp}) \to \mathsf{gCoalg}$ | $\nexists$ | $\mathcal{Q}\colon \mathsf{gCoalg} \to \mathsf{Comon}(\mathsf{Sp})$ |
| $\overline{\mathcal{K}}\colon \mathsf{Sp} \to \mathsf{gVec}$ | $(0)$ | $\overline{\mathcal{R}}\colon \mathsf{gVec} \to \mathsf{Sp}$ |
| $\overline{\mathcal{K}}\colon \mathsf{Mon}(\mathsf{Sp}) \to \mathsf{gAlg}$ | $(0)$ | $\overline{\mathcal{R}}\colon \mathsf{gAlg} \to \mathsf{Mon}(\mathsf{Sp})$ |
| $\overline{\mathcal{K}}\colon \mathsf{Comon}(\mathsf{Sp}) \to \mathsf{gCoalg}$ | $(0)$ | $\overline{\mathcal{Q}}\colon \mathsf{gCoalg} \to \mathsf{Comon}(\mathsf{Sp})$ |

Table 18.2. Adjoints of $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$.

| Functor | Left adjoint | Right adjoint |
|---|---|---|
| $\mathcal{K}^\vee\colon \mathsf{Sp} \to \mathsf{gVec}$ | $\mathcal{R}^\vee\colon \mathsf{gVec} \to \mathsf{Sp}$ | $\mathcal{L}^\vee\colon \mathsf{gVec} \to \mathsf{Sp}$ |
| $\mathcal{K}^\vee\colon \mathsf{Mon}(\mathsf{Sp}) \to \mathsf{gAlg}$ | $\mathcal{Q}^\vee\colon \mathsf{gAlg} \to \mathsf{Mon}(\mathsf{Sp})$ | $\nexists$ |
| $\mathcal{K}^\vee\colon \mathsf{Comon}(\mathsf{Sp}) \to \mathsf{gCoalg}$ | $\nexists$ | $\mathcal{L}^\vee\colon \mathsf{gCoalg} \to \mathsf{Comon}(\mathsf{Sp})$ |
| $\overline{\mathcal{K}}^\vee\colon \mathsf{Sp} \to \mathsf{gVec}$ | $\overline{\mathcal{R}}^\vee\colon \mathsf{gVec} \to \mathsf{Sp}$ | $(0)$ |
| $\overline{\mathcal{K}}^\vee\colon \mathsf{Mon}(\mathsf{Sp}) \to \mathsf{gAlg}$ | $\overline{\mathcal{Q}}^\vee\colon \mathsf{gAlg} \to \mathsf{Mon}(\mathsf{Sp})$ | $(0)$ |
| $\overline{\mathcal{K}}^\vee\colon \mathsf{Comon}(\mathsf{Sp}) \to \mathsf{gCoalg}$ | $\overline{\mathcal{R}}^\vee\colon \mathsf{gCoalg} \to \mathsf{Comon}(\mathsf{Sp})$ | $(0)$ |

symbol $\nexists$ indicates that the adjoint does not exist and the symbol (0) indicates that it exists only in characteristic zero.

Since $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$ are the contragredients of $\mathcal{K}$ and $\overline{\mathcal{K}}$ (Proposition 15.8), we know that the corresponding adjoints are related in a similar manner (Proposition 3.103). For this reason, for the most part, we concentrate on the adjoints of $\mathcal{K}$ and $\overline{\mathcal{K}}$. Further, in characteristic 0, the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ are isomorphic (Proposition 15.21), so the adjoint in this case can be told from either of the two tables.

## 18.1. The right adjoint of $\overline{\mathcal{K}}$

There is an obvious functor from graded vector spaces to species, where a graded vector space is viewed as a species with trivial actions of the symmetric groups. We begin with this functor since it plays an important role in the construction of many adjoints.

**18.1.1. The trivialization functor.** Define a functor

$$\mathsf{t}_{(-)}\colon \mathsf{gVec} \to \mathsf{Sp} \qquad \text{by} \qquad \mathsf{t}_V[I] := V_{|I|},$$

where $V = (V_n)_{n\geq 0}$ is a graded vector space. For any bijection $I \to J$, we let $\mathsf{t}_V[I] \to \mathsf{t}_V[J]$ be the identity map. Equivalently, $\mathsf{t}_V$ is the species given by the sequence of vector spaces $V_0,\ V_1,\ V_2,\ \ldots$ with trivial actions of the symmetric groups. A linear map $V \to W$ induces a morphism of species $\mathsf{t}_V \to \mathsf{t}_W$ in the obvious way. Hence $\mathsf{t}_{(-)}$ is a functor, which we refer to as the *trivialization functor*.

We now proceed to turn $\mathsf{t}_{(-)}$ into a braided lax monoidal functor with respect to the Cauchy product. Let

$$\varphi_{V,W}\colon \mathsf{t}_V \cdot \mathsf{t}_W \to \mathsf{t}_{V\cdot W}$$

be given by the maps

$$\bigoplus_{S\sqcup T=I} V_{|S|}\otimes W_{|T|} \to \bigoplus_{s+t=|I|} V_s \otimes W_t, \qquad v\otimes w \mapsto v\otimes w, \tag{18.1}$$

with the $(S,T)$-summand mapping to the $(s,t)$-summand for $|S| = s$ and $|T| = t$. Let the map $\varphi_0\colon \mathbf{1} \to \mathsf{t}_1$ be the obvious isomorphism, where we recall that 1 is the one-dimensional graded vector space concentrated in degree 0.

**Proposition 18.1.** *The functor* $(\mathsf{t}_{(-)}, \varphi)\colon (\mathsf{gVec}, \cdot) \to (\mathsf{Sp}, \cdot)$ *is lax monoidal.*

This is a straightforward check. The contragredient $(\mathsf{t}^\vee_{(-)}, \varphi^\vee)$ is then a colax monoidal functor by Proposition 3.102. We first observe that

$$\mathsf{t}_{(-)} \cong \mathsf{t}^\vee_{(-)}$$

as functors. The colax structure $\varphi^\vee$ is given by mapping the $(s,t)$-summand by the identity to all $(S,T)$-summands for which $|S| = s$ and $|T| = t$.

We know that a (co)lax monoidal functor induces a functor on the corresponding category of (co)monoids. As an example for the above functors, we note that

$$\mathsf{t}_{\Bbbk[x]} = \mathbf{E} \text{ as monoids} \qquad \text{and} \qquad \mathsf{t}^\vee_{\Bbbk\{x\}} = \mathbf{E} \text{ as comonoids,} \tag{18.2}$$

where $\mathbf{E}$ is the exponential species of Example 8.15.

We remark that the functor $(\mathsf{t}_{(-)}, \varphi, \varphi^\vee)$ is not bilax.

**18.1.2. A braided lax-lax adjunction.** We now show that the functor $\mathsf{t}_{(-)}$ is the right adjoint of $\overline{\mathcal{K}}$. To ensure uniformity of nomenclature for all the adjoints, we give another name to this functor and call it $\overline{\mathcal{R}}$.

**Lemma 18.2.** *Let $G$ be a group and $M$ and $N$ be $\Bbbk G$-modules with the action of $G$ on $N$ being trivial. Let $f\colon M \to N$ be a $\Bbbk$-linear map. Then $f\colon M \to N$ is a morphism of $G$-modules if and only if $f$ induces a $\Bbbk$-linear map $M_G \to N$.*

The above is a consequence of the definition of group coinvariants (Section 2.5.1). More formally, there is an adjunction

$$\begin{tikzcd} \mathsf{Mod}_G \arrow[r, bend left, "(-)_G"] & \mathsf{Vec}, \arrow[l, bend left] \end{tikzcd}$$

the right adjoint to $(-)_G$ being given by the functor which sends a vector space to itself with the trivial $G$-action. Applying this result to the family of symmetric groups yields:

**Proposition 18.3.** *The functor $\overline{\mathcal{R}}$ is the right adjoint to $\overline{\mathcal{K}}$. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{gVec}}(\overline{\mathcal{K}}(\mathbf{p}), V) \cong \mathrm{Hom}_{\mathsf{Sp}}\big(\mathbf{p}, \overline{\mathcal{R}}(V)\big) \tag{18.3}$$

*which are natural in $V$ and $\mathbf{p}$.*

**Proposition 18.4.** *The adjunction $(\overline{\mathcal{K}}, \overline{\mathcal{R}})$ in (18.3) is braided lax-lax, as well as braided colax-lax.*

Proof. The functor $\overline{\mathcal{K}}$ is braided strong. So by Proposition 3.95, there is a unique braided lax structure on $\overline{\mathcal{R}}$ such that the adjunction $(\overline{\mathcal{K}}, \overline{\mathcal{R}})$ is braided lax-lax, or equivalently, braided colax-lax. It is easy to verify that this lax structure matches the one that was defined in (18.1). □

**Proposition 18.5.** *The functor $\overline{\mathcal{R}}$ is the right adjoint to $\overline{\mathcal{K}}$ at the level of monoids. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{gAlg}}(\overline{\mathcal{K}}(\mathbf{p}), A) \cong \mathrm{Hom}_{\mathsf{Mon(Sp)}}\big(\mathbf{p}, \overline{\mathcal{R}}(A)\big), \tag{18.4}$$

*which are natural in $A$ and $\mathbf{p}$.*

*Similar statements hold for the commutative and Lie cases.*

Proof. This follows from Propositions 3.91, 3.92 and 18.4. □

**Remark 18.6.** In the associative and commutative case, one can say more. Namely, at the level of monoids and commutative monoids, the adjunction $(\overline{\mathcal{K}}, \overline{\mathcal{R}})$ continues to be braided lax-lax. This follows from the discussion in Example 7.48.

The functor $\overline{\mathcal{K}}$ does have a right adjoint at the level of comonoids. This is discussed in Section 18.5.

**18.1.3. Interaction with the functors $\mathcal{T}$ and $\mathcal{S}$.** The construction of $\overline{\mathcal{R}}$ has some interesting consequences which we briefly discuss. Consider the forgetful functors

$$\mathsf{gAlg} \to \mathsf{gVec}, \qquad \mathsf{gAlg}^{\mathsf{co}} \to \mathsf{gVec}, \qquad \text{and} \qquad \mathsf{gLie} \to \mathsf{gVec},$$

where the source categories are those of graded algebras, graded commutative algebras and graded Lie algebras. The left adjoints of these functors are given by the tensor algebra functor $\mathcal{T}$, the symmetric algebra functor $\mathcal{S}$ and the free Lie algebra functor $\mathcal{L}ie$. Analogues of these functors for species were discussed in Chapter 11. We now claim that the functor $\overline{\mathcal{K}}$ commutes with these functors.

**Proposition 18.7.** *The following diagrams commute.*

$$\begin{tikzcd} \mathsf{Mon(Sp)} \arrow[r, "\overline{\mathcal{K}}"] & \mathsf{gAlg} \\ \mathsf{Sp} \arrow[u, "\mathcal{T}"] \arrow[r, "\overline{\mathcal{K}}"'] & \mathsf{gVec} \arrow[u, "\mathcal{T}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Mon(Sp)} \arrow[d, "f\ell"'] & \mathsf{gAlg} \arrow[l, "\overline{\mathcal{R}}"'] \arrow[d, "f\ell"] \\ \mathsf{Sp} & \mathsf{gVec} \arrow[l, "\overline{\mathcal{R}}"] \end{tikzcd} \tag{18.5}$$

$$\begin{tikzcd} \mathsf{Mon^{co}(Sp)} \arrow[r, "\overline{\mathcal{K}}"] & \mathsf{gAlg^{co}} \\ \mathsf{Sp} \arrow[u, "\mathcal{S}"] \arrow[r, "\overline{\mathcal{K}}"'] & \mathsf{gVec} \arrow[u, "\mathcal{S}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Mon^{co}(Sp)} \arrow[d, "f\ell"'] & \mathsf{gAlg^{co}} \arrow[l, "\overline{\mathcal{R}}"'] \arrow[d, "f\ell"] \\ \mathsf{Sp} & \mathsf{gVec} \arrow[l, "\overline{\mathcal{R}}"] \end{tikzcd} \tag{18.6}$$

$$\begin{tikzcd} \mathsf{Lie(Sp)} \arrow[r, "\overline{\mathcal{K}}"] & \mathsf{gLie} \\ \mathsf{Sp} \arrow[u, "\mathcal{L}ie"] \arrow[r, "\overline{\mathcal{K}}"'] & \mathsf{gVec} \arrow[u, "\mathcal{L}ie"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Lie(Sp)} \arrow[d, "f\ell"'] & \mathsf{gLie} \arrow[l, "\overline{\mathcal{R}}"'] \arrow[d, "f\ell"] \\ \mathsf{Sp} & \mathsf{gVec} \arrow[l, "\overline{\mathcal{R}}"] \end{tikzcd} \tag{18.7}$$

*The functors in the diagram on the left are the left adjoints of the corresponding functors in the diagram on the right.*

Proof. The adjunction $(\overline{\mathcal{K}}, \overline{\mathcal{R}})$ was discussed in Proposition 18.5. The commutativity of the diagrams on the right is clear. The uniqueness of adjoints then implies the commutativity of the diagrams on the left. $\square$

**Remark 18.8.** Using Remark 18.6, one can say more regarding diagrams (18.5) and (18.6). Firstly, since $\overline{\mathcal{R}}$ continues to be braided lax at the level of monoids and commutative monoids, it follows that the diagrams on the right commute as lax functors. Secondly, the functors corresponding to each other in adjacent diagrams form a colax-lax adjunction. This implies that the diagrams on the left in (18.5) and (18.6) commute as colax functors.

Let $\odot$ refer to the modified Cauchy product on positive species (8.55). We claim that there are colax-lax adjunctions

$$\begin{tikzcd} (\mathsf{Sp}_+, \odot) \arrow[r, bend left, "\mathcal{T}"] & (\mathsf{Mon}(\mathsf{Sp}^{\circ}), \cdot) \arrow[l, bend left, "(-)_+"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} (\mathsf{Sp}_+, \odot) \arrow[r, bend left, "\mathcal{S}"] & (\mathsf{Mon^{co}}(\mathsf{Sp}^{\circ}), \cdot). \arrow[l, bend left, "(-)_+"] \end{tikzcd}$$

The first claim was shown in Lemma 11.6 and the second can be established in the same way. The difference with the above situation is that we now start with a positive species and end with a connected monoid. As a result, the image of a finite-dimensional species is again finite-dimensional. Interestingly, the discussion in Remark 18.8 remains valid in this setting as well. The result is stated below.

**Proposition 18.9.** *Consider the diagrams*

(18.8)
$$\begin{tikzcd}
(\mathsf{Mon}(\mathsf{Sp}^\circ), \cdot) \arrow[r, "\overline{\mathcal{K}}"] & (\mathsf{cgAlg}, \cdot) \\
(\mathsf{Sp}_+, \odot) \arrow[r, "\overline{\mathcal{K}}"'] \arrow[u, "\mathcal{T}"] & (\mathsf{gVec}_+, \odot) \arrow[u, "\mathcal{T}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathsf{Mon}(\mathsf{Sp}^\circ), \cdot) \arrow[d, "(-)_+"'] & (\mathsf{cgAlg}, \cdot) \arrow[l, "\overline{\mathcal{R}}"'] \arrow[d, "(-)_+"] \\
(\mathsf{Sp}_+, \odot) & (\mathsf{gVec}_+, \odot) \arrow[l, "\overline{\mathcal{R}}"]
\end{tikzcd}$$

(18.9)
$$\begin{tikzcd}
(\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}^\circ), \cdot) \arrow[r, "\overline{\mathcal{K}}"] & (\mathsf{cgAlg}^{\mathsf{co}}, \cdot) \\
(\mathsf{Sp}_+, \odot) \arrow[r, "\overline{\mathcal{K}}"'] \arrow[u, "\mathcal{S}"] & (\mathsf{gVec}_+, \odot) \arrow[u, "\mathcal{S}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}^\circ), \cdot) \arrow[d, "(-)_+"'] & (\mathsf{cgAlg}^{\mathsf{co}}, \cdot) \arrow[l, "\overline{\mathcal{R}}"'] \arrow[d, "(-)_+"] \\
(\mathsf{Sp}_+, \odot) & (\mathsf{gVec}_+, \odot) \arrow[l, "\overline{\mathcal{R}}"]
\end{tikzcd}$$

*where* $\mathsf{cg}$ *stands for connected graded. The diagrams on the left commute as colax functors, while the ones on the right commute as lax functors. Further, the functors in the diagrams on the left and the corresponding functors in the diagrams on the right form a colax-lax adjunction.*

Proof. One needs to establish the colax-lax adjunction $(\overline{\mathcal{K}}, \overline{\mathcal{R}})$ on the bottom horizontal arrows and check that the diagram on the right commutes as lax functors. The details are omitted. $\square$

Passing the diagrams on the left in (18.8) and (18.9) to the categories of comonoids, we obtain commutative diagrams:

(18.10)
$$\begin{tikzcd}
\mathsf{Hopf}(\mathsf{Sp}^\circ) \arrow[r, "\overline{\mathcal{K}}"] & \mathsf{cgHopf} \\
\mathsf{Comon}(\mathsf{Sp}_+) \arrow[r, "\overline{\mathcal{K}}"'] \arrow[u, "\mathcal{T}"] & \mathsf{gCoalg}_+ \arrow[u, "\mathcal{T}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Hopf}^{\mathsf{co}}(\mathsf{Sp}^\circ) \arrow[r, "\overline{\mathcal{K}}"] & \mathsf{cgHopf}^{\mathsf{co}} \\
\mathsf{Comon}(\mathsf{Sp}_+) \arrow[r, "\overline{\mathcal{K}}"'] \arrow[u, "\mathcal{S}"] & \mathsf{gCoalg}_+ \arrow[u, "\mathcal{S}"']
\end{tikzcd}$$

where $\mathsf{gCoalg}_+$ denotes the category of positively graded (noncounital) coalgebras.

**Example 18.10.** We illustrate the preceding results in the simplest situation. Consider the trivial positive comonoid $\mathbf{X}$. We have

$$\overline{\mathcal{K}}(\mathbf{X}) \cong X,$$

the 1-dimensional space with basis $\{x\}$. The preceding results will thus relate the free monoid on $\mathbf{X}$ to the free algebra on one generator.

Applying the left diagram in (18.10) to $\mathbf{X}$, one obtains

$$\overline{\mathcal{K}}(\mathbf{L}) \cong \overline{\mathcal{K}}\,(\mathcal{T}(\mathbf{X})) \cong \mathcal{T}\overline{\mathcal{K}}(\mathbf{X}) \cong \mathcal{T}(X) \cong \Bbbk[x].$$

The equalities are as graded Hopf algebras. The right diagram in (18.10) similarly yields

$$\overline{\mathcal{K}}(\mathbf{E}) \cong \Bbbk[x].$$

We saw this by an explicit computation in Examples 15.14 and 15.17.

The Lie case is similar. Recall from Section 11.9.1 that $\mathbf{Lie} \cong \mathcal{L}ie(\mathbf{X})$. Applying the diagram on the left in (18.7) to the species $\mathbf{X}$ one obtains that

$$\overline{\mathcal{K}}(\mathbf{Lie}) \cong \overline{\mathcal{K}}\big(\mathcal{L}ie(\mathbf{X})\big) \cong \mathcal{L}ie\big(\overline{\mathcal{K}}(\mathbf{X})\big) \cong \mathcal{L}ie(X),$$

the free Lie algebra on one generator. (It is one-dimensional except in characteristic 2.) By definition of $\overline{\mathcal{K}}$,

$$\overline{\mathcal{K}}(\mathbf{Lie}) = \bigoplus_n (\mathbf{Lie}[n])_{\mathrm{S}_n}$$

as graded vector spaces.

**Example 18.11.** We now illustrate the preceding results for the positive comonoids $\mathbf{L}^*_+$ and $\mathbf{E}^*_+$. We know from Examples 15.14 and 15.17 that

$$\overline{\mathcal{K}}(\mathbf{L}^*) \cong \overline{\mathcal{K}}(\mathbf{E}^*) \cong \Bbbk\{x\},$$

the divided power coalgebra. In characteristic 0, $\Bbbk\{x\} \cong \Bbbk[x]$ (Example 2.3).

The Hopf monoids $\overrightarrow{\mathbf{\Sigma}}$, $\mathbf{\Sigma}$, $\overrightarrow{\mathbf{\Pi}}$ and $\mathbf{\Pi}$ are studied in Chapter 12. According to Proposition 12.59, we have

$$\mathbf{\Sigma} \cong \mathcal{T}(\mathbf{E}^*_+), \quad \overrightarrow{\mathbf{\Sigma}} \cong \mathcal{T}(\mathbf{L}^*_+), \quad \mathbf{\Pi} \cong \mathcal{S}(\mathbf{E}^*_+) \quad \text{and} \quad \overrightarrow{\mathbf{\Pi}} \cong \mathcal{S}(\mathbf{L}^*_+).$$

Let $\mathrm{N}\Lambda$ and $\Lambda$ be the Hopf algebras of noncommutative symmetric functions and symmetric functions respectively. There are isomorphisms

$$\mathrm{N}\Lambda \cong \mathcal{T}(\Bbbk\{x\}_+), \quad H_{(a_1,\dots,a_k)} \mapsto x^{(a_1)} \otimes \cdots \otimes x^{(a_k)}$$

and

$$\Lambda \cong \mathcal{S}(\Bbbk\{x\}_+), \quad h_{(\lambda_1,\dots,\lambda_k)} \mapsto x^{(\lambda_1)} \cdots x^{(\lambda_k)}.$$

Here $H$ and $h$ denote the bases of complete (noncommutative) symmetric functions of $\mathrm{N}\Lambda$ and $\Lambda$ respectively, as in Sections 17.3 and 17.4.

Applying the diagrams in (18.10) to the noncounital comonoid $\mathbf{L}^*_+$, we obtain

$$\overline{\mathcal{K}}(\overrightarrow{\mathbf{\Sigma}}) \cong \overline{\mathcal{K}}\big(\mathcal{T}(\mathbf{L}^*_+)\big) \cong \mathcal{T}\big(\overline{\mathcal{K}}(\mathbf{L}^*_+)\big) \cong \mathcal{T}(\Bbbk\{x\}_+) \cong \mathrm{N}\Lambda,$$
$$\overline{\mathcal{K}}(\overrightarrow{\mathbf{\Pi}}) \cong \overline{\mathcal{K}}\big(\mathcal{S}(\mathbf{L}^*_+)\big) \cong \mathcal{S}\big(\overline{\mathcal{K}}(\mathbf{L}^*_+)\big) \cong \mathcal{S}(\Bbbk\{x\}_+) \cong \Lambda.$$

We emphasize that these result hold in any characteristic.

Similarly, applying the two diagrams to the noncounital comonoid $\mathbf{E}^*_+$, we obtain

$$\overline{\mathcal{K}}(\mathbf{\Sigma}) \cong \overline{\mathcal{K}}\big(\mathcal{T}(\mathbf{E}^*_+)\big) \cong \mathcal{T}\big(\overline{\mathcal{K}}(\mathbf{E}^*_+)\big) \cong \mathcal{T}(\Bbbk[x]_+) \overset{0}{\cong} \mathrm{N}\Lambda,$$
$$\overline{\mathcal{K}}(\mathbf{\Pi}) \cong \overline{\mathcal{K}}\big(\mathcal{S}(\mathbf{E}^*_+)\big) \cong \mathcal{S}\big(\overline{\mathcal{K}}(\mathbf{E}^*_+)\big) \cong \mathcal{S}(\Bbbk[x]_+) \overset{0}{\cong} \Lambda.$$

The symbol $\overset{0}{\cong}$ indicates that the isomorphism is only valid in characteristic zero. The rest of the computation is characteristic free.

## 18.2. The right adjoint of $\mathcal{K}$

In this section, we construct the right adjoint of $\mathcal{K}$ at the level of species and graded vector spaces, and study its monoidal properties. It is defined in terms of the trivialization functor $\mathtt{t}_{(-)}$ of Section 18.1 and the species $\mathbf{L}$ of linear orders (Example 8.3).

Recall from Section 10.2 that $\mathbf{L}$ is the linearization of the set species of chambers L. In this section as well as Section 18.5 we will make use of some elementary ideas and notations from Section 10.2.

Let $\mathcal{R}\colon \mathsf{gVec} \to \mathsf{Sp}$ be the lax functor defined by

$$\mathcal{R} := \mathbf{L}^* \times \mathsf{t}_{(-)}.$$

The lax structure of $\mathcal{R}$ is obtained by viewing it as the composite of the lax functors $\mathbf{L}^* \times (-)$ and $\mathsf{t}_{(-)}$. For the former, one views $\mathbf{L}^*$ as a monoid as in Example 8.24 and uses Proposition 8.66.

Now consider the following composite of colax-lax adjunctions.

$$\begin{tikzcd} \mathsf{Sp} \arrow[r, bend left, "\mathbf{L}\times(-)"] & \mathsf{Sp} \arrow[l, bend left, "\mathbf{L}^*\times(-)"] \arrow[r, bend left, "\overline{\mathcal{K}}"] & \mathsf{gVec}. \arrow[l, bend left, "\mathsf{t}_{(-)}"] \end{tikzcd}$$

The first adjunction is colax-lax by (8.81). The composite on the top is the colax functor $\mathcal{K}$ by (15.6). This leads to the following result.

**Proposition 18.12.** *The functor $\mathcal{R}$ is the right adjoint to $\mathcal{K}$. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{gVec}}(\mathcal{K}(\mathbf{p}), V) \cong \mathrm{Hom}_{\mathsf{Sp}}\big(\mathbf{p}, \mathcal{R}(V)\big)$$

*which are natural in $V$ and $\mathbf{p}$. Further, this adjunction is colax-lax.*

The unit of the adjunction

$$\mathbf{p} \xrightarrow{\eta(\mathbf{p})} \mathcal{R}\mathcal{K}(\mathbf{p})$$

is given by the maps

$$\mathbf{p}[I] \to \mathbf{L}[I]^* \otimes \mathbf{p}[|I|], \qquad z \mapsto \sum_{w \in \mathrm{Bij}(|I|, I)} (wC_{(n)})^* \otimes p[w^{-1}](z),$$

and the counit

$$\mathcal{K}\mathcal{R}(V) \xrightarrow{\xi(V)} V$$

is given by

$$\mathbf{L}[n]^* \otimes V_n \to V_n, \qquad l^* \otimes v \mapsto \begin{cases} v & \text{if } l = C_{(n)}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $C_{(n)}$ denotes the canonical linear order on the set $[n]$.

**Remark 18.13.** Since the functor $\mathcal{K}$ is not strong, Proposition 3.96 implies that there is no lax structure on $\mathcal{R}$ for which the adjunction $(\mathcal{K}, \mathcal{R})$ is lax-lax. In fact, we will see later that $\mathcal{K}$ does not have a right adjoint at the level of monoids (Section 18.4).

The situation for comonoids is different. The functor $\mathcal{K}$ does have a right adjoint at the level of comonoids; however it is more complicated to define than $\mathcal{R}$.

## 18.3. The left adjoint of $\mathcal{K}$

In this section, we construct the left adjoint of $\mathcal{K}$ at the level of species and graded vector spaces, as well as at the level of monoids and graded algebras. The discussion has similarities as well as differences with the discussion in Section 18.2.

**18.3.1. The strong functor $\mathcal{L}$.** Let $\mathcal{L}\colon \mathsf{gVec} \to \mathsf{Sp}$ be the lax functor defined by

$$\mathcal{L} := \mathbf{L} \times \mathsf{t}_{(-)},$$

where $\mathbf{L}$ is the monoid in Example 8.16. Let us denote the lax structure of $\mathcal{L}$ by $\gamma$. Explicitly, $\gamma_0\colon \mathbf{1} \to \mathcal{L}(\Bbbk)$ is the obvious isomorphism while

$$\gamma_{V,W}\colon \mathcal{L}(V) \cdot \mathcal{L}(W) \longrightarrow \mathcal{L}(V \cdot W) \tag{18.11}$$

is given by

$$\bigoplus_{S \sqcup T = I} \mathbf{L}[S] \otimes V_{|S|} \otimes \mathbf{L}[T] \otimes W_{|T|} \longrightarrow \bigoplus_{s+t=|I|} \mathbf{L}[I] \otimes V_s \otimes W_t$$
$$l \otimes v \otimes m \otimes w \longmapsto l \cdot m \otimes v \otimes w,$$

where $l \cdot m$ is the concatenation of the linear orders $l$ and $m$. The $(S,T)$-summand in the left-hand side maps to the $(s,t)$-summand in the right-hand side with $|S| = s$ and $|T| = t$. As a consequence:

**Proposition 18.14.** *The functor $(\mathcal{L}, \gamma)$ is strong.*

Let $\delta$ denote the inverse of $\gamma$. It yields a colax structure on $\mathcal{L}$. Explicitly, the map $\delta_0\colon \mathcal{L}(\Bbbk) \to \mathbf{1}$ is the obvious isomorphism, while

$$\delta_{U,V}\colon \mathcal{L}(U \cdot V) \longrightarrow \mathcal{L}(U) \cdot \mathcal{L}(V) \tag{18.12}$$

has components

$$\bigoplus_{s+t=|I|} \mathbf{L}[I] \otimes U_s \otimes V_t \longrightarrow \bigoplus_{S \sqcup T = I} \mathbf{L}[S] \otimes U_{|S|} \otimes \mathbf{L}[T] \otimes V_{|T|}$$
$$l \otimes u \otimes v \longmapsto l_1 \otimes u \otimes l_2 \otimes v$$

Here we write $l = wC_{(n)}$ where $w$ is a bijection from $[|I|]$ to $I$ and let $S = w([s])$ and $T = w([s+1, s+t])$. The linear orders $l_1$ and $l_2$ are the restrictions $l|_S$ and $l|_T$.

**Proposition 18.15.** *There is an isomorphism*

$$(\mathcal{L}, \delta) \cong (\mathcal{L}^\vee, \gamma^\vee)$$

*of costrong functors.*

Proof. First note that the contragredient of $\mathcal{L}$ is given by the formula

$$\mathcal{L}^\vee = \mathbf{L}^* \times \mathsf{t}^\vee_{(-)}.$$

Next observe that $\mathbf{L}$ and $\mathbf{L}^*$ are isomorphic as species and $\mathsf{t}_{(-)}$ and $\mathsf{t}^\vee_{(-)}$ are isomorphic as functors. It follows that $\mathcal{L}$ and $\mathcal{L}^\vee$ are also isomorphic as functors. It is then straightforward to check that the required costrong structures match. $\square$

It follows from the above discussion and (18.2) that

$$\mathcal{L}(\Bbbk[x]) = \mathbf{L} \text{ as monoids} \qquad \text{and} \qquad \mathcal{L}(\Bbbk\{x\}) = \mathbf{L}^* \text{ as comonoids.} \tag{18.13}$$

**Remark 18.16.** The definitions of $\mathcal{R}$ and $\mathcal{L}$ are quite similar; for the former one uses $\mathbf{L}^*$ while for the latter one uses $\mathbf{L}$. So they are isomorphic as functors; however they differ as lax functors. The functor $\mathcal{L}$ is strong, while $\mathcal{R}$ is not.

**18.3.2. The left adjoint of $\mathcal{K}$.** One can discover the above lax and colax structures of $\mathcal{L}$ by following a different route as follows.

**Proposition 18.17.** *The functor $\mathcal{L}$ is the left adjoint to $\mathcal{K}$. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Sp}}(\mathcal{L}(V), \mathbf{p}) \cong \mathrm{Hom}_{\mathsf{gVec}}\big(V, \mathcal{K}(\mathbf{p})\big), \tag{18.14}$$

*which are natural in $V$ and $\mathbf{p}$.*

Proof. We construct the unit and counit of the adjunction. Using (10.10), we define

$$\begin{aligned} V &\xrightarrow{\eta(V)} \mathcal{K}\mathcal{L}(V) \\ V_n &\longrightarrow \mathbf{L}[n] \otimes V_n \\ v &\longmapsto C_{(n)} \otimes v, \end{aligned} \qquad \begin{aligned} \mathcal{L}\mathcal{K}(\mathbf{p}) &\xrightarrow{\xi(\mathbf{p})} \mathbf{p} \\ \mathbf{L}[I] \otimes \mathbf{p}[|I|] &\longrightarrow \mathbf{p}[I] \\ wC_{(n)} \otimes z &\longmapsto \mathbf{p}[w](z), \end{aligned}$$

where $w$ is a bijection from $[|I|]$ to $I$. Conditions (A.3) may be checked without difficulty. $\square$

Since $\mathcal{K}$ is a lax monoidal functor, Proposition 3.84 guarantees the existence of a unique colax monoidal functor structure on $\mathcal{L}$ such that the adjunction $(\mathcal{L}, \mathcal{K})$ is colax-lax. One finds that this colax structure coincides with $\delta$. We already know that $\delta$ is invertible with inverse $\gamma$. Since the adjunction between $(\mathcal{L}, \delta)$ and $(\mathcal{K}, \varphi)$ is colax-lax, the adjunction between $(\mathcal{L}, \gamma)$ and $(\mathcal{K}, \varphi)$ is lax-lax, by Proposition 3.93. To summarize:

**Proposition 18.18.** *With respect to the lax structure $\varphi$ of $\mathcal{K}$, the lax structure $\gamma$ of $\mathcal{L}$ in (18.11) and the colax structure $\delta$ of $\mathcal{L}$ in (18.12), the adjunction $(\mathcal{L}, \mathcal{K})$ in (18.14) is lax-lax and colax-lax.*

The fact that the adjunction $(\mathcal{L}, \mathcal{K})$ is lax-lax can be further exploited: Proposition 3.91 tells us that the adjunction restricts to the categories of monoids.

**Proposition 18.19.** *The functor $\mathcal{L}$ is the left adjoint to $\mathcal{K}$ at the level of monoids. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Mon(Sp)}}(\mathcal{L}(A), \mathbf{p}) \cong \mathrm{Hom}_{\mathsf{gAlg}}\big(A, \mathcal{K}(\mathbf{p})\big), \tag{18.15}$$

*which are natural in $A$ and $\mathbf{p}$.*

**18.3.3. The free twisted algebra of Barratt.** The monoid $\mathcal{L}(A)$, in the special case when $A$ is a free algebra, appears in the work of Barratt [33, Definition 3], under the name *free twisted algebra*. A context for this object is given by the result below.

**Proposition 18.20.** *The following diagrams commute.*

$$\begin{tikzcd} \mathsf{gAlg} \arrow[r, "\mathcal{L}"] & \mathsf{Mon(Sp)} \\ \mathsf{gVec} \arrow[u, "\mathcal{T}"] \arrow[r, "\mathcal{L}"'] & \mathsf{Sp} \arrow[u, "\mathcal{T}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{gAlg} \arrow[d, "f\ell"'] & \mathsf{Mon(Sp)} \arrow[l, "\mathcal{K}"'] \arrow[d, "f\ell"] \\ \mathsf{gVec} & \mathsf{Sp} \arrow[l, "\mathcal{K}"] \end{tikzcd} \tag{18.16}$$

*The functors in the left diagram are the left adjoints of the corresponding functors in the right diagram.*

PROOF. The adjunction $(\mathcal{T}, f\ell)$ was discussed in the proof of Theorem 11.3, while the adjunction $(\mathcal{L}, \mathcal{K})$ was discussed in Proposition 18.19. The commutativity of the diagram on the right is clear. The uniqueness of adjoints then implies the commutativity of the diagram on the left. □

As an example, applying the diagram on the left in (18.16) to a graded vector space concentrated in degree 1 (denote the component in degree 1 by $V$), we obtain:

$$\mathcal{L}\big(\mathcal{T}(V)\big) = \mathcal{T}\big(\mathcal{L}(V)\big) = \mathbf{L} \circ \mathbf{X}_V = \mathbf{L} \times \mathbf{E}_V \quad \text{as monoids.}$$

In the terminology of Barratt, the object on the left is the free twisted algebra on $V$. The object on the right was considered in Example 11.11. In the special case when $V$ is one-dimensional, we have:

$$\mathcal{L}(\Bbbk[x]) = \mathbf{L} \circ \mathbf{X} = \mathbf{L} \quad \text{as monoids.}$$

We have seen this result in (18.13) where it was arrived at by different means.

**18.3.4. Remarks in the negative.** Since the functor $\mathcal{L}$ is lax and colax, it sends graded algebras to monoids in species and graded coalgebras to comonoids in species, by Proposition 3.29. However, $(\mathcal{L}, \gamma, \delta)$ is not bilax: diagram (3.11) does not commute. Alternatively, $(\mathcal{L}, \gamma)$ is clearly not braided lax.

One may wonder about the behavior of the adjunction $(\mathcal{L}, \mathcal{K})$ with respect to the colax structure of $\mathcal{K}$. The answer is that there is no colax structure on $\mathcal{L}$ for which the adjunction $(\mathcal{L}, \mathcal{K})$ is colax-colax. If this were the case, then by Proposition 3.96 the functor $\mathcal{K}$ would be strong, which is not true.

Since the adjunction $(\mathcal{L}, \mathcal{K})$ is not colax-colax, we cannot conclude the existence of a left adjoint to $\mathcal{K}$ on the categories of comonoids. In fact, such a left adjoint does not exist (Section 18.4).

## 18.4. Nonexistence of certain adjoints

We establish the nonexistence of certain adjoints of the Fock functors.

**18.4.1. Categorical products and coproducts.** Let us first consider the category of graded connected algebras. The (categorical) product of two objects $A$ and $B$ is given by the direct sum $A \oplus B$, with component-wise product. The coproduct is a little more complicated. The component of positive degree $n$ of the coproduct of $A$ and $B$ is the direct sum of the spaces

$$(E_1)_{s_1} \otimes \cdots \otimes (E_k)_{s_k}$$

over all decompositions $n = s_1 + \cdots + s_k$ into positive integers, where $E_i$ is either $A$ or $B$ and $E_i \neq E_{i+1}$.

Now consider the category of connected monoids in species. The product of $\mathbf{p}$ and $\mathbf{q}$ is given by the sum $\mathbf{p} + \mathbf{q}$ as defined in (8.2) while the coproduct is the species whose $I$-component is the direct sum of the spaces

$$\mathbf{r}_1[S_1] \otimes \cdots \otimes \mathbf{r}_k[S_k]$$

over all decompositions $I = S_1 \sqcup \cdots \sqcup S_k$ into nonempty subsets, where $\mathbf{r}_i$ is either $\mathbf{p}$ or $\mathbf{q}$ and $\mathbf{r}_i \neq \mathbf{r}_{i+1}$.

For the categories of graded connected coalgebras or connected comonoids in species, the descriptions of the products and coproducts get switched.

**18.4.2. The functor $\mathcal{K}$.** From the above descriptions, it follows that at the level of monoids, the functor $\mathcal{K}$ preserves products but not coproducts (compare the dimensions). Similarly, at the level of comonoids, the functor $\mathcal{K}$ preserves coproducts but not products. Applying Proposition A.10, item (ii), we have

$$\mathsf{Mon(Sp)} \underset{\not\leftarrow}{\overset{\mathcal{K}}{\rightleftarrows}} \mathsf{gAlg} \quad \text{and} \quad \mathsf{gCoalg} \underset{\mathcal{K}}{\overset{\not\rightarrow}{\leftrightarrows}} \mathsf{Comon(Sp)}, \tag{18.17}$$

which says that at the level of monoids (comonoids), the functor $\mathcal{K}$ does not have a right (left) adjoint.

**18.4.3. The functor $\overline{\mathcal{K}}$.** Now we turn our attention to the functor $\overline{\mathcal{K}}$. We note that $\Bbbk$ viewed as a trivial module over $\Bbbk\mathrm{S}_n$ is flat for all $n$ only if the characteristic of $\Bbbk$ is zero. (Indeed, suppose $\Bbbk$ is flat as $\Bbbk\mathrm{S}_n$-module. Then, as mentioned in Section 2.5.2, it is also projective. Hence the augmentation $\epsilon\colon \Bbbk\mathrm{S}_n \to \Bbbk$ admits a section $\iota\colon \Bbbk \to \Bbbk\mathrm{S}_n$ of $\Bbbk\mathrm{S}_n$-modules. This implies that $1 = a \cdot n!$ in $\Bbbk$ where $\iota(1) = a \cdot \sum_{\sigma\in\mathrm{S}_n} \sigma$.) Since

$$M_{\Bbbk\mathrm{S}_n} = M \otimes_{\Bbbk\mathrm{S}_n} \Bbbk,$$

it follows that the functor $\overline{\mathcal{K}}\colon \mathsf{Sp} \to \mathsf{gVec}$ (defined using coinvariants) is not left exact in positive characteristic. Hence, it does not preserve kernels and therefore by Proposition A.10, item (ii), it cannot have a left adjoint.

The nonexistence of a left adjoint for $\overline{\mathcal{K}}$ in positive characteristic at the level of (co)monoids can be shown similarly as follows. Let $p$ be the field characteristic. Fix $n \geq p$. Choose an injective morphism $f\colon M \to N$ of $\Bbbk\mathrm{S}_n$-modules such that $M_{\mathrm{S}_n} \to N_{\mathrm{S}_n}$ is not injective (take, for example, $\iota\colon \Bbbk \to \Bbbk\mathrm{S}_n$ as defined above). Let $\mathbf{p}$ and $\mathbf{q}$ be the positive species concentrated in degree $n$, whose degree $n$ parts are $M$ and $N$ respectively. View them as non(co)unital (co)monoids with the trivial (co)product. It is clear that $f$ induces a map $\mathbf{p} \to \mathbf{q}$ of non(co)unital (co)monoids. Now adjoin (co)units to obtain an injective map $\mathbf{p}^\circ \to \mathbf{q}^\circ$ of (co)monoids. However, the image of this map under $\overline{\mathcal{K}}$ is not injective. Thus, $\overline{\mathcal{K}}$ does not preserve kernels and therefore by Proposition A.10, item (ii), it cannot have a left adjoint.

## 18.5. The right adjoints of $\mathcal{K}$ and $\overline{\mathcal{K}}$ on comonoids

The content of this section is summarized in the following four adjunctions. In other words, both $\mathcal{K}$ and $\overline{\mathcal{K}}$ have right adjoints at the level of (cocommutative) comonoids.

$$\mathsf{Comon(Sp)} \underset{\mathcal{Q}}{\overset{\mathcal{K}}{\rightleftarrows}} \mathsf{gCoalg}, \qquad \mathsf{Comon(Sp)} \underset{\overline{\mathcal{Q}}}{\overset{\overline{\mathcal{K}}}{\rightleftarrows}} \mathsf{gCoalg}.$$

$$^{\mathrm{co}}\mathsf{Comon(Sp)} \underset{c\mathcal{Q}}{\overset{\mathcal{K}}{\rightleftarrows}} {}^{\mathrm{co}}\mathsf{gCoalg}, \qquad {}^{\mathrm{co}}\mathsf{Comon(Sp)} \underset{c\overline{\mathcal{Q}}}{\overset{\overline{\mathcal{K}}}{\rightleftarrows}} {}^{\mathrm{co}}\mathsf{gCoalg}.$$

Most of the work lies in establishing the adjunctions in the top line. The remaining two then follow by general considerations.

Instead of working with this setup, we will equivalently describe the left adjoints of $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$. The geometric language of projection maps will be used for this discussion; adequate background is provided in Chapter 10, particularly Sections 10.4 and 10.5.

**18.5.1. The functor $\mathcal{Q}^\vee$ from algebras to monoids.** We now define a functor

$$\mathcal{Q}^\vee \colon \mathsf{gAlg} \to \mathsf{Mon}(\mathsf{Sp}).$$

For a graded algebra $A$, $\mathcal{Q}^\vee(A)$ is defined to be the quotient of the monoid $\mathcal{L}\big(\mathcal{T}(A)\big)$ as below.

$$\mathcal{Q}^\vee(A)[I] := \frac{\mathcal{L}\big(\mathcal{T}(A)\big)[I]}{Rel} = \frac{\mathbf{L}[I] \otimes \big( \bigoplus_{r_1+\cdots+r_k=|I|} A_{r_1} \otimes \cdots \otimes A_{r_k} \big)}{Rel},$$

where $Rel$ is the ideal generated by the relations:

For $C \in \mathbf{L}[I]$, $a \in A_s$, $b \in A_t$, $s+t=|I|$,

$$(18.18)\qquad \begin{aligned} C \otimes ab &= \sum_{\substack{K \text{ a vertex in } \Sigma[I]:\\ \mathrm{type}(K)=(s,t)}} KC \otimes a \otimes b && \text{if } s \neq 0,\, t \neq 0.\\ C \otimes ab &= C \otimes a \otimes b && \text{if } s = 0 \text{ or } t = 0. \end{aligned}$$

The product of the monoid $\mathcal{L}\big(\mathcal{T}(A)\big)$ is given by concatenating the two linear orders and the two sets of tensor factors from $A$, and induces the product on the quotient $\mathcal{Q}^\vee(A)$.

Observe that

$$\mathcal{Q}^\vee(A)[\emptyset] = A_0.$$

As an example, for the set $I = \{g, o, p, i\}$, and $a, b \in A_2$, we have the relation:

$$(18.19)\quad g|o|p|i \otimes ab = (g|o|p|i + g|p|o|i + g|i|o|p + o|p|g|i + o|i|g|p + p|i|g|o) \otimes a \otimes b.$$

The first term corresponds to $K = go|pi$, the second corresponds to $K = gp|oi$, and so forth. Applying the bijection $I \to [4]$, which sends $g$, $o$, $p$, $i$ to 1, 2, 3, 4 respectively yields a relation where the terms on the right are indexed by $(2,2)$-shuffle permutations. More generally, we have the relation:

$$(18.20)\qquad C_{(n)} \otimes ab = \sum_{\zeta \in \mathrm{Sh}\,(s,t)} \zeta C_{(n)} \otimes a \otimes b.$$

This is a consequence of Proposition 10.6.

**18.5.2. An alternative description of $\mathcal{Q}^\vee$.** We have $\mathcal{L}\mathcal{T} = \mathcal{T}\mathcal{L}$ from (18.16). From here one can give an alternative description of $\mathcal{Q}^\vee(A)$ as a quotient of the monoid $\mathcal{T}\big(\mathcal{L}(A)\big)$ as follows.

$$\mathcal{Q}^\vee(A)[I] = \frac{\mathcal{T}\big(\mathcal{L}(A)\big)[I]}{Rel} = \frac{\bigoplus_{I_1 \sqcup \cdots \sqcup I_k = I} (\mathbf{L}[I_1] \otimes A_{|I_1|}) \otimes \cdots \otimes (\mathbf{L}[I_k] \otimes A_{|I_k|})}{Rel}$$

where $Rel$ is the ideal generated by the relations below.

For $C \in \mathbf{L}[I]$, $a \in A_s$, $b \in A_t$, $s+t=|I|$,

$$(18.21)\qquad \begin{aligned} C \otimes ab &= \bigoplus_{\substack{K \text{ a vertex in } \Sigma[I]:\\ \mathrm{type}(K)=(s,t)}} (C_1 \otimes a) \otimes (C_2 \otimes b) && \text{if } s \neq 0,\, t \neq 0.\\ C \otimes ab &= (C \otimes a) \otimes (C_{(0)} \otimes b) && \text{if } t = 0.\\ C \otimes ab &= (C_{(0)} \otimes a) \otimes (C \otimes b) && \text{if } s = 0. \end{aligned}$$

The linear orders $C_1$ and $C_2$ are defined by $b_K(KC) = (C_1, C_2)$, where $b_K$ is the break map (10.57), and $C_{(0)}$ denotes the basis element of $\mathbf{L}[\emptyset]$.

Returning to the previous example, relation (18.19) takes the form:

$$g|o|p|i \otimes ab = (g|o \otimes a) \otimes (p|i \otimes b) + (g|p \otimes a) \otimes (o|i \otimes b) + (g|i \otimes a) \otimes (o|p \otimes b) \\ + (o|p \otimes a) \otimes (g|i \otimes b) + (o|i \otimes a) \otimes (g|p \otimes b) + (p|i \otimes a) \otimes (g|o \otimes b).$$

Each term corresponds to a decomposition $S \sqcup T = I$ where $|S| = |T| = 2$. This reflects the general fact that vertices $K$ of $\Sigma[I]$ of type $(s, t)$ correspond to decompositions $S \sqcup T = I$ where $|S| = s$ and $|T| = t$.

**18.5.3. The ideal of relations.** The ideal generated by the second relation in (18.18) is linearly spanned by the following relations:

For $a_k \in A_0$ or $a_{k+1} \in A_0$,

$$C \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n = C \otimes a_1 \otimes \cdots \otimes a_k \otimes a_{k+1} \otimes \cdots \otimes a_n.$$

First consider the quotient of the monoid $\mathcal{L}\big(\mathcal{T}(A)\big)$ by this ideal. Its $I$-component for $I$ nonempty is given by

$$\mathbf{L}[I] \otimes \Bigg( \bigoplus_{(r_1,\ldots,r_k) \models |I|} A_{r_1} \otimes_{A_0} A_{r_2} \otimes_{A_0} \cdots \otimes_{A_0} A_{r_k} \Bigg),$$

where the tensor products are taken over $A_0$. Hence $\mathcal{Q}^\vee(A)$ can be viewed as the quotient of this space by the ideal generated by the first relation in (18.18).

**Proposition 18.21.** *For $C \in \mathbf{L}[I]$, a composition $\alpha = (\alpha_1, \ldots, \alpha_l)$ of $|I|$, and $a_i \in A_{\alpha_i}$,*

$$C \otimes a_1 a_2 \cdots a_l = \sum_{\substack{G \text{ a face of } \Sigma[I]: \\ \mathrm{type}(G) = \alpha}} GC \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_l.$$

Proof. For $l = 1$, there is only one term in the summation given by the empty face $G = \emptyset$; hence the result is a tautology. For $l = 2$, the result is the same as the first relation in (18.18). We provide details for $l = 3$, the general case is along the same lines.

$$\begin{aligned} C \otimes abc &= \sum_{\mathrm{type}(K) = (s+t, u)} KC \otimes ab \otimes c \\ &= \sum_{\mathrm{type}(K) = (s+t, u)} \; \sum_{\mathrm{type}(K') = (s,t)} j(K', \emptyset)(KC) \otimes a \otimes b \otimes c \\ &= \sum_{\mathrm{type}(G) = (s,t,u)} GC \otimes a \otimes b \otimes c. \end{aligned}$$

The first two steps are obtained by applying the first relation in (18.18) twice. For the last step, we use the fact that a face $G$ of type $(s, t, u)$ can be uniquely written in the form $j(K', \emptyset)K$ where $K$ is a vertex of type $(s + t, u)$ and $K'$ is a vertex of type $(s, t)$, and where $j(K', \emptyset)$ is the join map (10.57) applied to $(K', \emptyset)$. This is straightforward from the definitions; a related discussion is given in [12, Proposition 6.4.1]. ∎

**18.5.4. The adjunction $(\mathcal{Q}^\vee, \mathcal{K}^\vee)$.** We now come to the main computation in this section.

**Proposition 18.22.** *The functor $\mathcal{Q}^\vee$ is the left adjoint to $\mathcal{K}^\vee$ at the level of monoids. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Mon(Sp)}}(\mathcal{Q}^\vee(A), \mathbf{p}) \cong \mathrm{Hom}_{\mathsf{gAlg}}\big(A, \mathcal{K}^\vee(\mathbf{p})\big), \tag{18.22}$$

*which are natural in $A$ and $\mathbf{p}$.*

Proof. Let $\mathbf{p} = (\mathbf{p}, \mu, \iota)$ be a monoid in species and

$$\mathcal{K}(\mathbf{p}) = (\mathcal{K}(\mathbf{p}), \mu\varphi, \iota\varphi_0) \qquad \text{and} \qquad \mathcal{K}^\vee(\mathbf{p}) = (\mathcal{K}^\vee(\mathbf{p}), \mu\psi^\vee, \iota\psi_0^\vee)$$

be the corresponding algebras; for the notation see Definition 3.28. Below we construct the bijection $g \longleftrightarrow f$ required by (18.22).

*The map from right to left.* Given a morphism of algebras $f\colon A \to \mathcal{K}^\vee(\mathbf{p})$, define

$$g\colon \mathcal{L}\big(\mathcal{T}(A)\big) \to \mathbf{p}$$

as follows. We first view $f$ as a linear map $f\colon A \to \mathcal{K}(\mathbf{p})$. Now there are two equivalent ways to proceed. One way is to extend $f$ to a morphism of algebras $\mathcal{T}(A) \to \mathcal{K}(\mathbf{p})$ and then use the adjunction $(\mathcal{L}, \mathcal{K})$ in (18.15) to obtain $g$. Another way is to first use the adjunction $(\mathcal{L}, \mathcal{K})$ in (18.14) to obtain a morphism of species $\mathcal{L}(A) \to \mathbf{p}$ and then extend it to a morphism of monoids $\mathcal{T}\big(\mathcal{L}(A)\big) \to \mathbf{p}$. Both descriptions make it clear that $g$ is a morphism of monoids. An explicit formula for $g$ is given below.

$$g(C, a_1 \otimes \cdots \otimes a_k) := \mathbf{p}[w]\, \mu\varphi\big(f(a_1), \ldots, f(a_k)\big),$$

where $C = wC_{(n)}$ as in (10.10).

We next show that $f$ being a morphism of algebras implies that $g$ respects the relations in (18.18).

We start by checking that $g$ respects the first relation. Let $C = wC_{(n)}$ be a linear order on $I$, $a \in A_s$, $b \in A_t$ and $T = (s,t)$ with both $s$ and $t$ not zero.

$$\begin{aligned}
g(C, ab) &= \mathbf{p}[w]\, f(ab) \\
&= \mathbf{p}[w]\, \mu\psi^\vee\big(f(a), f(b)\big) \\
&= \sum_{\zeta \in \mathrm{Sh}(T)} \mathbf{p}[w]\mathbf{p}[\zeta]\, \mu\varphi\big(f(a), f(b)\big) \\
&= \sum_{\zeta \in \mathrm{Sh}(T)} \mathbf{p}[w\zeta]\, \mu\varphi\big(f(a), f(b)\big) \\
&= \sum_{\zeta \in \mathrm{Sh}(T)} g(w\zeta C_{(n)}, a \otimes b) \\
&= \sum_{\substack{F \text{ a vertex in } \Sigma[n]: \\ \mathrm{type}(F) = T}} g(w(FC_{(n)}), a \otimes b) \\
&= \sum_{\substack{K \text{ a vertex in } \Sigma[I]: \\ \mathrm{type}(K) = T}} g(KC, a \otimes b)
\end{aligned}$$

The first and fifth equality follows from the definition of $g$. The second equality uses the fact that $f$ is a morphism of algebras. The third equality uses the relation between $\varphi$ and $\psi^\vee$ involving shuffles given in (15.12). The sixth equality follows from the bijection between $T$-shuffle permutations and faces of type $T$, see Proposition 10.6. The last equality uses (10.15) with $J = [n]$, which says that $w$ is a type and product preserving map between $\Sigma[n]$ and $\Sigma[I]$.

The fact that $g$ respects the second relation in (18.18) is verified next. Let $a \in A_s$, $b \in A_t$ with either $s$ or $t$ equal to zero.

$$\begin{aligned} g(wC_{(n)}, ab) &= \mathbf{p}[w]\, f(ab) \\ &= \mathbf{p}[w]\, \mu\psi^\vee\big(f(a), f(b)\big) \\ &= \mathbf{p}[w]\, \mu\varphi\big(f(a), f(b)\big) \\ &= g(wC_{(n)}, a \otimes b). \end{aligned}$$

The main point is to note that $\psi^\vee\big(f(a), f(b)\big) = \varphi\big(f(a), f(b)\big)$ if either $a$ or $b$ is of degree zero. The remaining steps follow from the definitions.

*The map from left to right.* Given a morphism of monoids $g\colon \mathcal{Q}^\vee(A) \to \mathbf{p}$, define

$$f\colon A \to \mathcal{K}^\vee(\mathbf{p})$$

by

$$f(a) := g(C_{(n)}, a) \in \mathbf{p}[n],$$

for $a \in A_n$, where $C_{(n)}$ is the canonical linear order on $[n]$. The two computations below show that $f$ is a morphism of algebras. In a way, they also show the origin of the relations in (18.18).

Let $a$ and $b$ both have nonzero degree.

$$\begin{aligned} \mu\psi^\vee\big(f(a), f(b)\big) &= \mu\psi^\vee\big(g(C_{(s)}, a), g(C_{(t)}, b)\big) \\ &= \mu\Big(\sum_{\zeta \in Sh(T)} \zeta\varphi\big(g(C_{(s)}, a), g(C_{(t)}, b)\big)\Big) \\ &= \mu\Big(\sum_{\zeta \in Sh(T)} \zeta\big(g(C_{(s)}, a), g(C_{(s+1,s+t)}, b)\big)\Big) \\ &= \sum_{\zeta \in Sh(T)} \zeta\mu\big(g(C_{(s)}, a), g(C_{(s+1,s+t)}, b)\big) \\ &= \sum_{\zeta \in Sh(T)} \zeta g(C_{(s+t)}, a \otimes b) \\ &= g\Big(\sum_{\zeta \in Sh(T)} (\zeta C_{(s+t)}, a \otimes b)\Big) \\ &= g(C_{(s+t)}, ab) \\ &= f(ab). \end{aligned}$$

The first and last equality follows from the definition of $f$. For the second equality, we use the relation between $\varphi$ and $\psi^\vee$ given in (15.12). For the third equality, we use the definition of $\varphi$ and the fact that $g$ is a morphism of species. Here $C_{(s+1,s+t)}$ refers to the canonical linear order on $[s+1, s+t]$. For the fourth equality, we use that $\mu$ is a morphism of species and hence commutes with $\zeta$. For the fifth equality, we use that $g$ is a morphism of monoids. For the seventh equality, we use relation (18.20).

Let the degree of either $a$ or $b$ be zero.

$$\begin{aligned}
\mu\psi^{\vee}\big(f(a), f(b)\big) &= \mu\varphi\big(f(a), f(b)\big) \\
&= \mu\varphi\big(g(C_{(s)}, a), g(C_{(t)}, b)\big) \\
&= \mu\big(g(C_{(s)}, a), g(C_{(s+1,s+t)}, b)\big) \\
&= g(C_{(s+t)}, a \otimes b) \\
&= g(C_{(s+t)}, ab) \\
&= f(ab).
\end{aligned}$$

In the first equality, we use $\psi^{\vee}\big(f(a), f(b)\big) = \varphi\big(f(a), f(b)\big)$ which is implied by our hypothesis. In the fourth equality, we use the second relation in (18.18). The remaining steps follow from the definitions.

□

**18.5.5. The functor $\overline{\mathcal{Q}}^{\vee}$ from algebras to monoids.** We now define a functor

$$\overline{\mathcal{Q}}^{\vee} : \mathsf{gAlg} \to \mathsf{Mon}(\mathsf{Sp}).$$

For a graded algebra $A$, $\overline{\mathcal{Q}}^{\vee}(A)$ is defined to be the quotient of the monoid $\mathcal{T}(\mathsf{t}_A)$ as below.

$$\overline{\mathcal{Q}}^{\vee}(A)[I] := \frac{\mathcal{T}(\mathsf{t}_A)[I]}{Rel} = \frac{\displaystyle\bigoplus_{I_1 \sqcup \cdots \sqcup I_k = I} A_{|I_1|} \otimes \cdots \otimes A_{|I_k|}}{Rel}$$

where $Rel$ is the ideal generated by the relations below.

For $a \in A_s$, $b \in A_t$, $s + t = |I|$,

$$ab = \bigoplus_{\substack{S \sqcup T = I: \\ |S| = s, |T| = t}} a \otimes b. \tag{18.23}$$

This is similar to the definition of $\mathcal{Q}^{\vee}(A)$ as a quotient of $\mathcal{T}\big(\mathcal{L}(A)\big)$, with the above relation being the analogue of (18.21). Comparing the two definitions, it is clear that

$$\overline{\mathcal{Q}}^{\vee}(A)[I] = \frac{\mathcal{Q}^{\vee}(A)[I]}{\langle C \otimes a = D \otimes a \text{ for } C, D \in \mathbf{L}[I],\ a \in A\rangle}.$$

An argument along the lines of Proposition 18.22 shows that:

**Proposition 18.23.** *The functor $\overline{\mathcal{Q}}^{\vee}$ is the left adjoint to $\overline{\mathcal{K}}^{\vee}$ at the level of monoids. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Mon}(\mathsf{Sp})}(\overline{\mathcal{Q}}^{\vee}(A), \mathbf{p}) \cong \mathrm{Hom}_{\mathsf{gAlg}}\big(A, \overline{\mathcal{K}}^{\vee}(\mathbf{p})\big), \tag{18.24}$$

*which are natural in $A$ and $\mathbf{p}$.*

**18.5.6. The commutative version.** Let $c\mathcal{Q}^{\vee}$ and $c\overline{\mathcal{Q}}^{\vee}$ be the functors respectively defined as the composites

$$\mathsf{gAlg}^{\mathsf{co}} \to \mathsf{gAlg} \xrightarrow{\mathcal{Q}^{\vee}} \mathsf{Mon}(\mathsf{Sp}) \xrightarrow{(-)_c} \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp})$$

$$\mathsf{gAlg}^{\mathsf{co}} \to \mathsf{gAlg} \xrightarrow{\overline{\mathcal{Q}}^{\vee}} \mathsf{Mon}(\mathsf{Sp}) \xrightarrow{(-)_c} \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp})$$

where the functor $(-)_c$ sends a monoid to its abelianization. The following are commutative analogues of Propositions 18.22 and 18.23.

**Proposition 18.24.** *The functor $c\mathcal{Q}^\vee$ is the left adjoint to $\mathcal{K}^\vee$ at the level of commutative monoids. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp})}(c\mathcal{Q}^\vee(A), \mathbf{p}) \cong \mathrm{Hom}_{\mathsf{gAlg}^{\mathsf{co}}}\big(A, \mathcal{K}^\vee(\mathbf{p})\big), \tag{18.25}$$

*which are natural in $A$ and $\mathbf{p}$.*

Proof. Observe that the following is an adjunction

$$\begin{tikzcd} \mathsf{Mon}(\mathsf{Sp}) \arrow[r, bend left, "(-)_c"] & \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}). \arrow[l, bend left, "inc"] \end{tikzcd}$$

By composing it with the adjunction $(\mathcal{Q}^\vee, \mathcal{K}^\vee)$, we obtain an adjunction between $\mathsf{gAlg}$ and $\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp})$. Note that $\mathsf{gAlg}^{\mathsf{co}}$ is a full subcategory of $\mathsf{gAlg}$, and that $\mathcal{K}^\vee$ preserves commutativity. Now the result follows by using Proposition A.5. ☐

By a similar argument:

**Proposition 18.25.** *The functor $c\overline{\mathcal{Q}}^\vee$ is the left adjoint to $\overline{\mathcal{K}}^\vee$ at the level of commutative monoids. In other words, we have isomorphisms*

$$\mathrm{Hom}_{\mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp})}(c\overline{\mathcal{Q}}^\vee(A), \mathbf{p}) \cong \mathrm{Hom}_{\mathsf{gAlg}^{\mathsf{co}}}\big(A, \overline{\mathcal{K}}^\vee(\mathbf{p})\big), \tag{18.26}$$

*which are natural in $A$ and $\mathbf{p}$.*

**18.5.7. Interaction with the functors $\mathcal{T}$ and $\mathcal{S}$.** For completeness, we record some commutative diagrams involving the $\mathcal{Q}$ functors. Similar results for the functors $\overline{\mathcal{R}}$ and $\mathcal{L}$ have been discussed earlier in this chapter.

**Proposition 18.26.** *The following diagrams commute. The functors in the left diagrams are the left adjoints of the corresponding functors in the right diagrams.*

$$\begin{tikzcd} \mathsf{gAlg} \arrow[r, "\mathcal{Q}^\vee"] & \mathsf{Mon}(\mathsf{Sp}) \\ \mathsf{gVec} \arrow[u, "\mathcal{T}"] \arrow[r, "\mathcal{R}^\vee"'] & \mathsf{Sp} \arrow[u, "\mathcal{T}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{gAlg} \arrow[d, "f\ell"'] & \mathsf{Mon}(\mathsf{Sp}) \arrow[l, "\mathcal{K}^\vee"'] \arrow[d, "f\ell"] \\ \mathsf{gVec} & \mathsf{Sp} \arrow[l, "\mathcal{K}^\vee"] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{gAlg} \arrow[r, "\overline{\mathcal{Q}}^\vee"] & \mathsf{Mon}(\mathsf{Sp}) \\ \mathsf{gVec} \arrow[u, "\mathcal{T}"] \arrow[r, "\overline{\mathcal{R}}^\vee"'] & \mathsf{Sp} \arrow[u, "\mathcal{T}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{gAlg} \arrow[d, "f\ell"'] & \mathsf{Mon}(\mathsf{Sp}) \arrow[l, "\overline{\mathcal{K}}^\vee"'] \arrow[d, "f\ell"] \\ \mathsf{gVec} & \mathsf{Sp} \arrow[l, "\overline{\mathcal{K}}^\vee"] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{gAlg}^{\mathsf{co}} \arrow[r, "c\mathcal{Q}^\vee"] & \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}) \\ \mathsf{gVec} \arrow[u, "\mathcal{S}"] \arrow[r, "\mathcal{R}^\vee"'] & \mathsf{Sp} \arrow[u, "\mathcal{S}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{gAlg}^{\mathsf{co}} \arrow[d, "f\ell"'] & \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}) \arrow[l, "\mathcal{K}^\vee"'] \arrow[d, "f\ell"] \\ \mathsf{gVec} & \mathsf{Sp} \arrow[l, "\mathcal{K}^\vee"] \end{tikzcd}$$

$$
\begin{tikzcd}
\mathsf{gAlg}^{\mathsf{co}} \arrow[r, "c\overline{\mathcal{Q}}^{\vee}"] & \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}) \\
\mathsf{gVec} \arrow[u, "\mathcal{S}"] \arrow[r, "\overline{\mathcal{R}}^{\vee}"'] & \mathsf{Sp} \arrow[u, "\mathcal{S}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{gAlg}^{\mathsf{co}} \arrow[d, "f\ell"'] & \mathsf{Mon}^{\mathsf{co}}(\mathsf{Sp}) \arrow[l, "\overline{\mathcal{K}}^{\vee}"'] \arrow[d, "f\ell"] \\
\mathsf{gVec} & \mathsf{Sp} \arrow[l, "\overline{\mathcal{K}}^{\vee}"]
\end{tikzcd}
$$

As an example, applying the diagrams on the left to a graded vector space concentrated in degree 1 (denote the component in degree 1 by $V$), we obtain:

$$
\mathcal{Q}^{\vee}\big(\mathcal{T}(V)\big) = \overline{\mathcal{Q}}^{\vee}\big(\mathcal{T}(V)\big) = \mathbf{L} \circ \mathbf{X}_V = \mathbf{L} \times \mathbf{E}_V
$$

$$
c\mathcal{Q}^{\vee}\big(\mathcal{S}(V)\big) = c\overline{\mathcal{Q}}^{\vee}\big(\mathcal{S}(V)\big) = \mathbf{E} \circ \mathbf{X}_V = \mathbf{E}_V.
$$

The first object was considered in Example 11.11. It is the free twisted algebra of Barratt (Section 18.3.3). It is the free object for the forgetful functor from twisted algebras to vector spaces. Thus the functors $\mathcal{L}$ and $\mathcal{Q}^{\vee}$ though different yield the same monoid when evaluated on the free algebra. The second object was considered in Example 8.18. It is the free commutative twisted algebra [291, Proposition 10]. It is the free object for the forgetful functor from commutative twisted algebras to vector spaces.

In the special case when $V$ is one-dimensional, we have:

$$
\mathcal{Q}^{\vee}(\Bbbk[x]) = \overline{\mathcal{Q}}^{\vee}(\Bbbk[x]) = \mathbf{L} \qquad\qquad c\mathcal{Q}^{\vee}(\Bbbk[x]) = c\overline{\mathcal{Q}}^{\vee}(\Bbbk[x]) = \mathbf{E}.
$$

This is consistent with the fact that $\mathbf{E}$ is the commutative quotient of $\mathbf{L}$.

CHAPTER 19

# Decorated Fock Functors and Creation-Annihilation

In Chapters 15 and 16, we defined and studied various Fock functors and their deformations. We recall that these are bilax functors from species to graded vector spaces. In this chapter, we consider generalizations of these functors which depend on a vector space (the space of *decorations*). They are summarized in Table 19.1. When the vector space is the base field $\Bbbk$, we recover the earlier Fock functors. The earlier theory generalizes in a straightforward way to this more general setting.

In a sense, one may view the result of applying the functor $\mathcal{K}_V$ (or its relatives) to a species $\mathbf{p}$ as a version of the graded vector space $\mathcal{K}(\mathbf{p})$ (or its relatives) in which the given combinatorial structure determined by the species $\mathbf{p}$ has been decorated with elements of the vector space $V$.

We begin by defining the decorated full Fock functors and the decorated bosonic Fock functors in Section 19.1 and show that they are bilax. We also explain how they can be constructed from their undecorated counterparts using the decorated exponential species. The discussion is continued in Section 19.2 where interrelationships between these functors are understood via the decorated norm transformation.

The values of the various decorated Fock functors on the exponential species are the tensor Hopf algebra, the shuffle Hopf algebra, the symmetric and exterior Hopf algebras, and their deformations (Section 2.6). The underlying vector spaces of these Hopf algebras are known as Fock spaces; the standard terminology of these spaces is summarized in Table 19.2. This constitutes our motivation for the terminology "Fock functors".

We now turn to a feature which is new to this chapter. Graded vector spaces with creation-annihilation operators were discussed in Section 2.8. Fock spaces are examples of such spaces. Further, the creation-annihilation operators that they carry satisfy canonical commutation relations. The point of view of this chapter is

TABLE 19.1. Decorated Fock functors.

| Fock functor | Name |
|---|---|
| $\mathcal{K}_V$, $\mathcal{K}_V^\vee$ | Decorated full Fock functor |
| $\mathcal{K}_{V,q}$, $\mathcal{K}_{V,q}^\vee$ | Deformed decorated full Fock functor |
| $\Im_{V,q}$ | Decorated anyonic Fock functor |
| $\overline{\mathcal{K}}_V$, $\Im_V$, $\overline{\mathcal{K}}_V^\vee$ | Decorated bosonic Fock functor |
| $\overline{\mathcal{K}}_{V,-1}$, $\Im_{V,-1}$, $\overline{\mathcal{K}}_{V,-1}^\vee$ | Decorated fermionic Fock functor |
| $\mathcal{K}_{V,0}$, $\Im_{V,0}$, $\mathcal{K}_{V,0}^\vee$ | Decorated free Fock functor |

TABLE 19.2. Fock spaces.

| Fock spaces | Name |
|---|---|
| $\mathcal{K}_V(\mathbf{E})$, $\mathcal{K}_V^\vee(\mathbf{E})$, $\mathcal{K}_{V,q}(\mathbf{E})$, $\mathcal{K}_{V,q}^\vee(\mathbf{E})$ | Full Fock space |
| $\Im_{V,q}(\mathbf{E})$ | Anyonic Fock space |
| $\overline{\mathcal{K}}_V(\mathbf{E})$, $\Im_V(\mathbf{E})$, $\overline{\mathcal{K}}_V^\vee(\mathbf{E})$ | Bosonic Fock space |
| $\overline{\mathcal{K}}_{V,-1}(\mathbf{E})$, $\Im_{V,-1}(\mathbf{E})$, $\overline{\mathcal{K}}_{V,-1}^\vee(\mathbf{E})$ | Fermionic Fock space |
| $\mathcal{K}_{V,0}(\mathbf{E})$, $\Im_{V,0}(\mathbf{E})$, $\mathcal{K}_{V,0}^\vee(\mathbf{E})$ | Free Fock space |

as follows. Species with up-down operators were discussed in Section 8.12, the basic example being that of the exponential species (Example 8.55). Now Fock functors convert up-down operators to creation-annihilation operators. This provides an explanation for the existence of such operators on Fock spaces. Further, we may now apply the Fock functors to other species which carry up-down operators leading to more general Fock spaces equipped with creation-annihilation operators. These ideas are due to Guţă and Maassen [158] and Bożejko and Guţă [64], and are explained in Sections 19.3 and 19.4.

Sections 19.5 and 19.6 deal with commutation relations. We introduce the notion of a species with balanced operators. This is a species with up-down operators where the up and down operators need to satisfy some compatibility relations. The exponential species is the basic example of a species with balanced operators. The main result here is that the Fock functors convert a species with balanced operators to a graded vector space with creation-annihilation operators which satisfy the canonical commutation relations.

The rest of the chapter deals with deformations. Deformations of the decorated full Fock functors, along with the fermionic and anyonic cases are treated in Section 19.7. The $q$-commutation relation is treated in Section 19.8. In Section 19.9, we consider a general situation in which the decorated Fock functors are deformed using a Yang–Baxter operator on $V$. The anyonic Fock space in this case is the Nichols algebra associated to $V$ (also called the quantum symmetric algebra).

There is another approach to combinatorial models for Fock spaces due to Baez and Dolan [29, Section 5]. It involves a generalization of the notion of species called *stuff type*. We also point the reader to the related works [30] and [280]. We do not pursue the connections between this interesting approach and the ideas presented here.

We thank Roland Speicher for making us aware of [64, 158].

## 19.1. Decorated Fock functors

In this section, we define the decorated Fock functors along with their bilax structures. We explain how the decorated and undecorated Fock functors determine each other. We also address the behavior of the functors with respect to duality and the contragredient construction of Section 3.10. The connection to Schur functors is also explained.

**19.1.1. Decorated Fock functors.** Let $V$ be a vector space. For each $n \geq 0$, there is a left action of the symmetric group $\mathrm{S}_n$ on $V^{\otimes n}$ given by

$$\sigma \cdot (v_1 \cdots v_n) := v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}.$$

For simplicity, we omit the tensor symbols between the $v_i$'s.

Let $\mathbf{p}$ be a species. Then $\mathrm{S}_n$ acts diagonally on $\mathbf{p}[n] \otimes V^{\otimes n}$,

$$\sigma \cdot (x \otimes v_1 \cdots v_n) = \mathbf{p}[\sigma](x) \otimes \sigma \cdot (v_1 \cdots v_n). \tag{19.1}$$

The spaces of invariants and of coinvariants for this action are respectively denoted by

$$\mathbf{p}[n] \otimes^{\mathrm{S}_n} V^{\otimes n} := \big(\mathbf{p}[n] \otimes V^{\otimes n}\big)^{\mathrm{S}_n}$$

and

$$\mathbf{p}[n] \otimes_{\mathrm{S}_n} V^{\otimes n} := \big(\mathbf{p}[n] \otimes V^{\otimes n}\big)_{\mathrm{S}_n}.$$

In other words, $\mathbf{p}[n] \otimes^{\mathrm{S}_n} V^{\otimes n}$ is the subspace of $\mathbf{p}[n] \otimes V^{\otimes n}$ consisting of those tensors $\sum_i x_i \otimes v_1^i \cdots v_n^i$ such that

$$\sum_i \mathbf{p}[\sigma](x_i) \otimes \sigma \cdot (v_1^i \cdots v_n^i) = \sum_i x_i \otimes v_1^i \cdots v_n^i$$

for all $\sigma \in \mathrm{S}_n$, and $\mathbf{p}[n] \otimes_{\mathrm{S}_n} V^{\otimes n}$ is the quotient of $\mathbf{p}[n] \otimes V^{\otimes n}$ in which

$$\overline{x \otimes v_1 \cdots v_n} = \overline{\mathbf{p}[\sigma](x) \otimes \sigma \cdot (v_1 \cdots v_n)}$$

for all $\sigma \in \mathrm{S}_n$.

**Definition 19.1.** The decorated Fock functors

$$\mathcal{K}_V, \mathcal{K}_V^{\vee}, \overline{\mathcal{K}}_V, \overline{\mathcal{K}}_V^{\vee} \colon \mathsf{Sp} \to \mathsf{gVec}$$

are defined by

$$\mathcal{K}_V(\mathbf{p}) := \mathcal{K}_V^{\vee}(\mathbf{p}) := \bigoplus_{n\geq 0} \mathbf{p}[n] \otimes V^{\otimes n},$$

$$\overline{\mathcal{K}}_V(\mathbf{p}) := \bigoplus_{n\geq 0} \mathbf{p}[n] \otimes_{\mathrm{S}_n} V^{\otimes n},$$

$$\overline{\mathcal{K}}_V^{\vee}(\mathbf{p}) := \bigoplus_{n\geq 0} \mathbf{p}[n] \otimes^{\mathrm{S}_n} V^{\otimes n}.$$

The quotient maps $\mathcal{K}_V(\mathbf{p}) \twoheadrightarrow \overline{\mathcal{K}}_V(\mathbf{p})$ and the inclusions $\overline{\mathcal{K}}_V^{\vee}(\mathbf{p}) \hookrightarrow \mathcal{K}_V^{\vee}(\mathbf{p})$ define natural transformations

$$\mathcal{K}_V \Rightarrow \overline{\mathcal{K}}_V \qquad \text{and} \qquad \overline{\mathcal{K}}_V^{\vee} \Rightarrow \mathcal{K}_V^{\vee}.$$

We refer to $\mathcal{K}_V$ and $\mathcal{K}_V^{\vee}$ as the *decorated full Fock functors* and to $\overline{\mathcal{K}}_V$ and $\overline{\mathcal{K}}_V^{\vee}$ as the *decorated bosonic Fock functors*.

We refer to $V$ as the space of *decorations*. Setting $V = \Bbbk$ recovers the (undecorated) Fock functors of Definitions 15.1 and 15.5. The first thing we do below is to extend the bilax monoidal structure of these functors to the decorated context. It is this structure that distinguishes between $\mathcal{K}_V$ and $\mathcal{K}_V^{\vee}$ (as in the undecorated context). In the next section, we will see via the decorated norm transformation that the functors $\overline{\mathcal{K}}_V$ and $\overline{\mathcal{K}}_V^{\vee}$ are isomorphic in characteristic 0.

**19.1.2. Schur functors.** Let $\mathbf{p}$ be a fixed species. The *Schur functor* associated to $\mathbf{p}$ is [260, Definition 1.24]

$$\mathcal{S}_{\mathbf{p}} \colon \mathsf{Vec} \to \mathsf{Vec}, \qquad \mathcal{S}_{\mathbf{p}}(V) := \bigoplus_{n \geq 0} \mathbf{p}[n] \otimes_{\mathrm{S}_n} V^{\otimes n}.$$

The functor $\mathcal{S}_{\mathbf{p}}$ is *analytic*. An intrinsic characterization of analytic functors is given by Joyal in [182, Théorème 1, Appendice]. Together with the results of [182, §2.0 and §4.1], this implies that if $\Bbbk$ is a field of characteristic 0, any analytic functor on $\mathsf{Vec}$ is the Schur functor of a unique species $\mathbf{p}$. More precisely, there is an equivalence between the category of analytic functors on $\mathsf{Vec}$ and that of species. A related result is given by Fresse [137, Proposition 1.2.5].

When dealing with the decorated Fock functors $\mathsf{Sp} \to \mathsf{gVec}$, the vector space $V$ is fixed and the species $\mathbf{p}$ is varying. Thus

$$\overline{\mathcal{K}}_V(\mathbf{p}) = \mathcal{S}_{\mathbf{p}}(V).$$

The same perspective can be adopted for the functor $\overline{\mathcal{K}}_V^{\vee}$. This leads to the *divided power functor* $\Gamma_{\mathbf{p}} \colon \mathsf{Vec} \to \mathsf{Vec}$ defined by

$$\Gamma_{\mathbf{p}}(V) := \overline{\mathcal{K}}_V^{\vee}(\mathbf{p}).$$

This functor is studied by Fresse in [136, Section 1] and [137, Section 1.2.12].

Additional information on Schur functors can be found in [137, Section 1.2].

**19.1.3. Bilax structure of the decorated Fock functors.** Let $V$ be a fixed vector space. We proceed to endow the $V$-decorated Fock functors with a bilax monoidal structure.

Given species $\mathbf{p}$ and $\mathbf{q}$, we define morphisms of graded vector spaces

$$\mathcal{K}_V(\mathbf{p}) \cdot \mathcal{K}_V(\mathbf{q}) \underset{\psi_{\mathbf{p},\mathbf{q}}}{\overset{\varphi_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} \mathcal{K}_V(\mathbf{p} \cdot \mathbf{q})$$

as follows. The degree $n$ components of these maps

$$\bigoplus_{s+t=n} \big(\mathbf{p}[s] \otimes V^{\otimes s}\big) \otimes \big(\mathbf{q}[t] \otimes V^{\otimes t}\big) \underset{\psi_{\mathbf{p},\mathbf{q}}}{\overset{\varphi_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} \Big(\bigoplus_{S \sqcup T = [n]} \mathbf{p}[S] \otimes \mathbf{q}[T]\Big) \otimes V^{\otimes n}$$

are the direct sum of the following maps:

$$\begin{gathered} \mathbf{p}[s] \otimes V^{\otimes s} \otimes \mathbf{q}[t] \otimes V^{\otimes t} \xrightarrow{\varphi_{\mathbf{p},\mathbf{q}}} \mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t] \otimes V^{\otimes n} \\ x \otimes v_1 \cdots v_s \otimes y \otimes w_1 \cdots w_t \longmapsto x \otimes \mathbf{q}[\mathrm{cano}](y) \otimes v_1 \cdots v_s w_1 \cdots w_t, \end{gathered}$$

$$\begin{gathered} \mathbf{p}[S] \otimes \mathbf{q}[T] \otimes V^{\otimes n} \xrightarrow{\psi_{\mathbf{p},\mathbf{q}}} \mathbf{p}[s] \otimes V^{\otimes s} \otimes \mathbf{q}[t] \otimes V^{\otimes t} \\ x \otimes y \otimes v_1 \cdots v_n \longmapsto \mathbf{p}[\mathrm{cano}](x) \otimes v_{i_1} \cdots v_{i_s} \otimes \mathbf{q}[\mathrm{cano}](y) \otimes v_{j_1} \cdots v_{j_t}, \end{gathered}$$

where we have written $S = \{i_1 < \cdots < i_s\}$ and $T = \{j_1 < \cdots < j_t\}$ and cano denotes the canonical order-preserving maps, as in Notation 2.5. Thus $\varphi$ and $\psi$ act on the species part as in the undecorated case (Section 15.1.1), while on tensors $\varphi$ concatenates and $\psi$ deshuffles. Note that the composite $\psi_{\mathbf{p},\mathbf{q}}\varphi_{\mathbf{p},\mathbf{q}}$ is the identity, but in general these maps are not invertible on the degree $n$ component, as before.

Note that $\mathcal{K}_V(\mathbf{1}) = 1$, the graded vector space of (2.7). We let $\varphi_0$ and $\psi_0$ be the identity maps

$$1 \underset{\psi_0}{\overset{\varphi_0}{\rightleftarrows}} \mathcal{K}_V(\mathbf{1}).$$

We proceed similarly for the functor $\mathcal{K}_V^\vee$. The maps

$$\bigoplus_{s+t=n} \big(\mathbf{p}[s] \otimes V^{\otimes s}\big) \otimes \big(\mathbf{q}[t] \otimes V^{\otimes t}\big) \underset{\varphi^\vee_{\mathbf{p},\mathbf{q}}}{\overset{\psi^\vee_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} \Big(\bigoplus_{S \sqcup T=[n]} \mathbf{p}[S] \otimes \mathbf{q}[T]\Big) \otimes V^{\otimes n}$$

as follows. The lax structure map $\psi^\vee_{\mathbf{p},\mathbf{q}}$ is the direct sum of the following maps, one for each $s, t$ and each summand in the target with $|S| = s$ and $|T| = t$:

$$\begin{aligned} \mathbf{p}[s] \otimes V^{\otimes s} \otimes \mathbf{q}[t] \otimes V^{\otimes t} &\longrightarrow \mathbf{p}[S] \otimes \mathbf{q}[T] \otimes V^{\otimes n} \\ x \otimes v_1 \cdots v_s \otimes y \otimes w_1 \cdots w_t &\longmapsto \mathbf{p}[\mathrm{cano}](x) \otimes \mathbf{q}[\mathrm{cano}](y) \otimes u_1 \cdots u_n. \end{aligned}$$

Here, we write $S = \{i_1 < \cdots < i_s\}$ and $T = \{j_1 < \cdots < j_t\}$ and define

$$u_h := \begin{cases} v_k & \text{if } h = i_k \in S, \\ w_k & \text{if } h = j_k \in T. \end{cases}$$

In other words, the tensor $u_1 \cdots u_n$ is the result of shuffling the tensors $v_1 \cdots v_s$ and $w_1 \cdots w_t$ according to the shuffle determined by $S$ and $T$.

The colax structure map $\varphi^\vee_{\mathbf{p},\mathbf{q}}$ is the direct sum of the following maps:

$$\begin{aligned} \mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t] \otimes V^{\otimes n} &\longrightarrow \mathbf{p}[s] \otimes V^{\otimes s} \otimes \mathbf{q}[t] \otimes V^{\otimes t} \\ x \otimes y \otimes v_1 \cdots v_n &\longmapsto x \otimes v_1 \cdots v_s \otimes \mathbf{q}[\mathrm{cano}](y) \otimes v_{s+1} \cdots v_{s+t}. \end{aligned}$$

On the components for which $S \neq [s]$ (and hence $T \neq [s+1, s+t]$), the map $\varphi^\vee_{\mathbf{p},\mathbf{q}}$ is zero.

We let $\varphi^\vee_0$ and $\psi^\vee_0$ be the identity maps of $\mathcal{K}^\vee_V(\mathbf{1}) = 1$.

The structure maps of $\mathcal{K}_V$ descend to coinvariants and those of $\mathcal{K}^\vee_V$ restrict to invariants, as indicated below.

$$\begin{tikzcd} \mathcal{K}_V(\mathbf{p}) \cdot \mathcal{K}_V(\mathbf{q}) \arrow[r, shift left, "\varphi_{\mathbf{p},\mathbf{q}}"] \arrow[d, two heads] & \mathcal{K}_V(\mathbf{p} \cdot \mathbf{q}) \arrow[l, shift left, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, two heads] \\ \overline{\mathcal{K}}_V(\mathbf{p}) \cdot \overline{\mathcal{K}}_V(\mathbf{q}) \arrow[r, shift left, dashed, "\overline{\varphi}_{\mathbf{p},\mathbf{q}}"] & \overline{\mathcal{K}}_V(\mathbf{p} \cdot \mathbf{q}) \arrow[l, shift left, dashed, "\overline{\psi}_{\mathbf{p},\mathbf{q}}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}^\vee_V(\mathbf{p}) \cdot \mathcal{K}^\vee_V(\mathbf{q}) \arrow[r, shift left, "\psi^\vee_{\mathbf{p},\mathbf{q}}"] & \mathcal{K}^\vee_V(\mathbf{p} \cdot \mathbf{q}) \arrow[l, shift left, "\varphi^\vee_{\mathbf{p},\mathbf{q}}"] \\ \overline{\mathcal{K}}^\vee_V(\mathbf{p}) \cdot \overline{\mathcal{K}}^\vee_V(\mathbf{q}) \arrow[u, hook] \arrow[r, shift left, dashed, "\overline{\psi}^\vee_{\mathbf{p},\mathbf{q}}"] & \overline{\mathcal{K}}^\vee_V(\mathbf{p} \cdot \mathbf{q}) \arrow[u, hook] \arrow[l, shift left, dashed, "\overline{\varphi}^\vee_{\mathbf{p},\mathbf{q}}"] \end{tikzcd}$$

The proofs of these statements are similar to those in the undecorated case; see the proof of Proposition 15.2 for the coinvariant case.

We now state the main result of this section.

**Theorem 19.2.** *The functors*

$$(\mathcal{K}_V, \varphi, \psi),\ (\mathcal{K}^\vee_V, \psi^\vee, \varphi^\vee) \colon (\mathsf{Sp}, \cdot, \beta_q) \to (\mathsf{gVec}, \cdot, \beta_q)$$

*are bilax monoidal. The functors*

$$(\overline{\mathcal{K}}_V, \overline{\varphi}, \overline{\psi}),\ (\overline{\mathcal{K}}^\vee_V, \overline{\psi}^\vee, \overline{\varphi}^\vee) \colon (\mathsf{Sp}, \cdot, \beta_q) \to (\mathsf{gVec}, \cdot, \beta_q)$$

*are bistrong monoidal. The natural transformations* $\mathcal{K}_V \Rightarrow \overline{\mathcal{K}}_V$ *and* $\overline{\mathcal{K}}_V^\vee \Rightarrow \mathcal{K}_V^\vee$ *are morphisms of bilax monoidal functors.*

The proofs are again similar to those in the undecorated case; see Theorems 15.3 and 15.6. We recall here that $\beta_q$ are the deformed braidings on species (9.1) and vector spaces (2.50). Due to the similarity in their definitions, the parameter $q$ plays a passive role in the proof.

Alternatively, the above result can be deduced from the following result (whose proof is straightforward) used in conjunction with Theorem 3.22.

**Proposition 19.3.** *The functor* $\mathcal{K}_V$ *is the following composite of bilax functors*:

$$(\mathsf{Sp}, \cdot, \beta_q) \xrightarrow{(-)\times \mathbf{E}_V} (\mathsf{Sp}, \cdot, \beta_q) \xrightarrow{\mathcal{K}} (\mathsf{gVec}, \cdot, \beta_q). \tag{19.2}$$

*The same result holds for the other decorated Fock functors as well*; *they are obtained by precomposing their undecorated counterparts with* $(-) \times \mathbf{E}_V$*.*

Here, $\mathbf{E}_V$ is the bimonoid of the decorated exponential species discussed in Example 8.18, and $(-)\times\mathbf{E}_V$ is the bilax functor associated to it as in Proposition 8.66. This functor is in fact bistrong. Strictly speaking, the latter functor was studied for the case $q = 1$, but the same can be done for a general $q$ by using Proposition 9.5.

To summarize, the undecorated and decorated Fock functors determine each other. The former is the special case of the latter in which $V = \Bbbk$, while the latter can be obtained from the former by precomposing with $(-) \times \mathbf{E}_V$.

**Example 19.4.** We apply the decorated Fock functors to the Hopf monoid $\mathbf{E}$. In view of Proposition 19.3 and Example 15.16, we obtain

$$\mathcal{K}_V(\mathbf{E}) = \mathcal{K}(\mathbf{E}_V) = \mathcal{T}(V) \qquad \text{and} \qquad \overline{\mathcal{K}}_V(\mathbf{E}) = \overline{\mathcal{K}}(\mathbf{E}_V) = \mathcal{S}(V),$$

the tensor and symmetric Hopf algebras on $V$ (the elements of $V$ have degree one). Similarly,

$$\mathcal{K}_V^\vee(\mathbf{E}) = \mathcal{T}^\vee(V) \qquad \text{and} \qquad \overline{\mathcal{K}}_V^\vee(\mathbf{E}) = \mathcal{S}^\vee(V),$$

the shuffle Hopf algebra and its symmetric Hopf subalgebra. These examples have been considered by Fresse [136, Section 1.2.11].

Thus, one may view the result of applying the functor $\mathcal{K}_V$ (or its relatives) to a Hopf monoid $\mathbf{p}$ in species as a decorated version of the Hopf algebra $\mathcal{K}(\mathbf{p})$ (or its relatives). Hence, every Hopf algebra of the form $\mathcal{K}(\mathbf{p})$ admits a decorated version in this sense.

**19.1.4. Duality between decorated Fock functors.** Let $V$ be a finite-dimensional vector space. This implies a natural isomorphism of $\mathrm{S}_n$-modules

$$\mathbf{p}[n]^* \otimes (V^*)^{\otimes n} \cong (\mathbf{p}[n] \otimes V^{\otimes n})^*$$

for any finite-dimensional species $\mathbf{p}$ and $n \geq 0$, and hence an isomorphism of functors

$$\mathcal{K}_{V^*}^\vee(\mathbf{p}^*) \cong \mathcal{K}_V(\mathbf{p})^*.$$

If $\mathbf{p}$ and $\mathbf{q}$ are finite-dimensional species, then the bilax structures of $\mathcal{K}_V$ and $\mathcal{K}^\vee_{V^*}$ are related through duality as expressed by the following commutative diagrams.

$$\begin{array}{ccc} \mathcal{K}^\vee_{V^*}(\mathbf{p}^*)\cdot\mathcal{K}^\vee_{V^*}(\mathbf{q}^*) & \xrightarrow{\psi^\vee} & \mathcal{K}^\vee_{V^*}(\mathbf{p}^*\cdot\mathbf{q}^*) \\ \| & & \| \\ \mathcal{K}_V(\mathbf{p})^*\cdot\mathcal{K}_V(\mathbf{q})^* & & \mathcal{K}^\vee_{V^*}\big((\mathbf{p}\cdot\mathbf{q})^*\big) \\ \| & & \| \\ \big(\mathcal{K}_V(\mathbf{p})\cdot\mathcal{K}_V(\mathbf{q})\big)^* & \xrightarrow[\psi^*]{} & \mathcal{K}_V(\mathbf{p}\cdot\mathbf{q})^* \end{array} \qquad \begin{array}{ccc} \mathcal{K}^\vee_{V^*}(\mathbf{p}^*)\cdot\mathcal{K}^\vee_{V^*}(\mathbf{q}^*) & \xleftarrow{\varphi^\vee} & \mathcal{K}^\vee_{V^*}(\mathbf{p}^*\cdot\mathbf{q}^*) \\ \| & & \| \\ \mathcal{K}_V(\mathbf{p})^*\cdot\mathcal{K}_V(\mathbf{q})^* & & \mathcal{K}^\vee_{V^*}\big((\mathbf{p}\cdot\mathbf{q})^*\big) \\ \| & & \| \\ \Big(\mathcal{K}_V(\mathbf{p})\cdot\mathcal{K}_V(\mathbf{q})\Big)^* & \xleftarrow[\varphi^*]{} & \mathcal{K}_V(\mathbf{p}\cdot\mathbf{q})^* \end{array}$$

This means that the decorated full Fock functors $\mathcal{K}_V$ and $\mathcal{K}^\vee_{V^*}$ are contragredient, in the sense of Section 3.10. A similar statement holds for the decorated bosonic Fock functors. The above discussion can be summarized as follows.

**Proposition 19.5.** *Let $V$ be a finite-dimensional vector space. On finite-dimensional species, the bilax functors $(\mathcal{K}^\vee_{V^*}, \psi^\vee, \varphi^\vee)$ and $(\overline{\mathcal{K}}^\vee_{V^*}, \overline{\psi}^\vee, \overline{\varphi}^\vee)$ are respectively isomorphic to the contragredients of $(\mathcal{K}_V, \varphi, \psi)$ and $(\overline{\mathcal{K}}_V, \overline{\varphi}, \overline{\psi})$.*

This result can also be viewed more conceptually as a consequence of earlier results of a similar nature. For example, the fact that the contragredient of $\mathcal{K}_V$ is $\mathcal{K}^\vee_{V^*}$ can be deduced as follows.

$$(\mathcal{K}_V)^\vee(-) \cong \mathcal{K}^\vee\Big(\big((-)\times\mathbf{E}_V\big)^\vee\Big) \cong \mathcal{K}^\vee\big((-)\times\mathbf{E}_{V^*}\big) \cong \mathcal{K}^\vee_{V^*}.$$

The first isomorphism follows by applying the contragredient construction to (19.2), and noting that the contragredient of $\mathcal{K}$ is $\mathcal{K}^\vee$ (Proposition 15.8). The middle isomorphism follows from the self-duality of the Hadamard functor (Proposition 8.60), and noting that the dual of $\mathbf{E}_V$ is $\mathbf{E}_{V^*}$ (Example 8.23). The last isomorphism follows from Proposition 19.3 applied to $\mathcal{K}^\vee_{V^*}$.

## 19.2. The decorated norm transformation

Let $V$ be a vector space. The norm transformation (Definition 15.19) can be extended to the decorated context.

**Definition 19.6.** For any species $\mathbf{p}$, let $\kappa_{\mathbf{p}}\colon \mathcal{K}_V(\mathbf{p}) \to \mathcal{K}^\vee_V(\mathbf{p})$ be the map of graded vector spaces given by

$$\kappa_{\mathbf{p}}(x\otimes v_1\cdots v_n) := \sum_{\sigma\in \mathrm{S}_n} \sigma\cdot(x\otimes v_1\cdots v_n),$$

for any $x\in\mathbf{p}[n]$, $v_i\in V$.

The action of $\mathrm{S}_n$ on $\mathbf{p}[n]\otimes V^{\otimes n}$ is as in (19.1). Each homogeneous component of $\kappa_{\mathbf{p}}$ is an instance of the norm map of Section 2.5. It follows that $\kappa\colon \mathcal{K}_V \Rightarrow \mathcal{K}^\vee_V$ is a natural transformation, which we call the *decorated norm*. Note that the dependence of $\kappa$ on $V$ is not manifest in the notation.

**Proposition 19.7.** *The decorated norm is a morphism of bilax monoidal functors*

$$\kappa\colon \mathcal{K}_V \Rightarrow \mathcal{K}^\vee_V.$$

Proof. This can be proved directly in the same way as Proposition 15.20. Alternatively, it may also be deduced from it, using Proposition 19.3 and noting that the decorated norm $\kappa\colon \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee$ is the composition of the undecorated norm $\kappa\colon \mathcal{K} \Rightarrow \mathcal{K}^\vee$ with the bilax functor $(-)\times \mathbf{E}_V$. ☐

The decorated norm map $\kappa_{\mathbf{p}}\colon \mathcal{K}_V(\mathbf{p}) \to \mathcal{K}_V^\vee(\mathbf{p})$ factors through coinvariants and its image consists of invariant elements (see Section 2.5). It therefore gives rise to a morphism of bilax monoidal functors

$$\bar{\kappa}\colon \overline{\mathcal{K}}_V \Rightarrow \overline{\mathcal{K}}_V^\vee$$

fitting in the commutative diagram below.

$$\begin{tikzcd} \mathcal{K}_V \arrow[r, Rightarrow, "\kappa"] \arrow[d, Rightarrow] & \mathcal{K}_V^\vee \\ \overline{\mathcal{K}}_V \arrow[r, Rightarrow, "\bar{\kappa}"'] & \overline{\mathcal{K}}_V^\vee \arrow[u, Rightarrow] \end{tikzcd} \tag{19.3}$$

**Proposition 19.8.** *If $\Bbbk$ is a field of characteristic 0, then the morphism of bistrong monoidal functors*

$$\bar{\kappa}\colon \overline{\mathcal{K}}_V \Rightarrow \overline{\mathcal{K}}_V^\vee$$

*is an isomorphism. More generally, for any commutative ring $\Bbbk$, if the species $\mathbf{p}$ consists of flat $\Bbbk\mathrm{S}_n$-modules $\mathbf{p}[n]$ and $V$ is a flat $\Bbbk$-module, then*

$$\bar{\kappa}_{\mathbf{p}}\colon \overline{\mathcal{K}}_V(\mathbf{p}) \to \overline{\mathcal{K}}_V^\vee(\mathbf{p})$$

*is invertible.*

Proof. If $V$ is flat as a $\Bbbk$-module, then so is $V^{\otimes n}$. This and the flatness of $\mathbf{p}[n]$ as a $\Bbbk\mathrm{S}_n$-module imply that $\mathbf{p}[n]\otimes V^{\otimes n}$ is flat as a $\Bbbk\mathrm{S}_n$-module, according to [69, Exercise III.0.1]. The result then follows from Lemma 2.20. ☐

**19.2.1. The image of the decorated norm.** Let $\Im_V$ denote the (co)image of the decorated norm transformation $\kappa\colon \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee$, in the sense of Section 3.11. It follows from Proposition 19.8 that, in characteristic 0,

$$\overline{\mathcal{K}}_V, \quad \Im_V, \quad \text{and} \quad \overline{\mathcal{K}}_V^\vee$$

are isomorphic bistrong monoidal functors. In general, these are three distinct bistrong monoidal functors related by morphisms of bistrong functors

$$\overline{\mathcal{K}}_V \Rightarrow \Im_V \Rightarrow \overline{\mathcal{K}}_V^\vee.$$

The connection between all five functors is as in the following diagram.

$$\begin{tikzcd} \mathcal{K}_V \arrow[rr, Rightarrow, "\kappa"] \arrow[d, Rightarrow] & & \mathcal{K}_V^\vee \\ \overline{\mathcal{K}}_V \arrow[r, Rightarrow] & \Im_V \arrow[r, Rightarrow] & \overline{\mathcal{K}}_V^\vee \arrow[u, Rightarrow] \end{tikzcd}$$

Suppose now that $V$ and $\mathbf{p}$ are finite-dimensional. It follows from Proposition 19.5 that $\kappa$ is related to its dual as follows.

$$\begin{tikzcd}
\mathcal{K}_V^\vee(\mathbf{p})^* \arrow[r, "(\kappa_{\mathbf{p}})^*"] \arrow[d, equal] & \mathcal{K}_V(\mathbf{p})^* \arrow[d, equal] \\
\mathcal{K}_{V^*}(\mathbf{p}^*) \arrow[r, "\kappa_{\mathbf{p}^*}"'] & \mathcal{K}_{V^*}^\vee(\mathbf{p}^*)
\end{tikzcd}$$

This means that the contragredient (3.47) of the $V$-decorated norm $\kappa\colon \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee$ is the $V^*$-decorated norm $\kappa\colon \mathcal{K}_{V^*} \Rightarrow \mathcal{K}_{V^*}^\vee$. A similar relation holds for $\bar{\kappa}\colon \overline{\mathcal{K}}_V \Rightarrow \overline{\mathcal{K}}_V^\vee$ and $\bar{\kappa}\colon \overline{\mathcal{K}}_{V^*} \Rightarrow \overline{\mathcal{K}}_{V^*}^\vee$. More generally, Lemma 2.22 yields:

**Proposition 19.9.** *Let $V$ be a finite-dimensional vector space. On finite-dimensional species, the contragredient of diagram (19.3) for $V$ is diagram (19.3) for $V^*$.*

We also note from Proposition 3.119 and the discussion preceding it that

$$(\Im_V)^\vee = \Im_{V^*},$$

and hence the image $\Im_V$ is self-dual (regardless of the characteristic).

**Remark 19.10.** Recall the connection between decorated Fock functors and Schur functors from Section 19.1.2. The norm transformation has been considered by Fresse in the context of Schur functors in [136, Section 1.1.14] and [137, Section 1.2.12].

## 19.3. Classical creation-annihilation operators

The mathematical context for creation-annihilation operators is that of graded vector spaces. This was discussed in Section 2.8. We now briefly review the main motivating example, namely that of creation-annihilation operators on Fock spaces. The terminology that we are using in this chapter is borrowed from this example.

Bosons and fermions are commonly used terms in particle physics; very roughly, they stand for classes of particles which behave like $+1$ and $-1$ respectively. They are named after the physicists Enrico Fermi and Satyendra Bose. Fock spaces are used in quantum mechanics to describe quantum states with a variable number of particles. They are named after the physicist V. A. Fock. The terms bosonic Fock space and fermionic Fock space are used depending on whether the particles are bosons or fermions. A creation operator acts on bosonic or fermionic Fock space by increasing the number of particles by 1. Similarly, an annihilation operator decreases the number of particles by 1. A discussion of these ideas can be found in the books by Merzbacher [272, Chapter 20] and Landau and Lifshitz [220, Chapter IX]. For the original work of Fock, see [127, paper 32-2].

To relate to the notation below, $V$ stands for the quantum states of a single particle. It is customary in physics textbooks to choose a basis for $V$ and proceed from there; however this is not necessary for our purposes. The canonical commutation relations between creation and annihilation operators are stated here without proof. They can be checked directly and will also follow from the generalities of subsequent sections.

**19.3.1. Classical Fock spaces.** Let $V$ be a vector space. The *algebraic Fock spaces* associated to $V$ are the underlying spaces of the tensor and symmetric algebras of $V$ [368, Examples 1.3.3]. More precisely, *full Fock space* is

$$\mathcal{T}(V) = \bigoplus_{n \geq 0} V^{\otimes n},$$

the underlying space of the tensor algebra, and *bosonic Fock space* is

$$\mathcal{S}(V) = \bigoplus_{n \geq 0} (V^{\otimes n})_{\mathrm{S}_n},$$

the underlying space of the symmetric algebra.

Suppose there is given a bilinear form $\langle\ ,\ \rangle$ on $V$. We may extend it to full Fock space by

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_m \rangle := \begin{cases} \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

In this context, the following operators on full Fock space are of interest [368, Example 1.5.3]: the (left) *annihilation operator* associated to $v \in V$,

$$a(v)\colon \mathcal{T}(V) \to \mathcal{T}(V), \qquad a(v)(1) = 0, \quad a(v)(v_1 \otimes \cdots \otimes v_n) = \langle v, v_1 \rangle \otimes v_2 \cdots \otimes v_n,$$

and the (left) *creation operator* associated to $v$,

$$c(v)\colon \mathcal{T}(V) \to \mathcal{T}(V), \qquad c(v)(1) = v, \quad c(v)(v_1 \otimes \cdots \otimes v_n) = v \otimes v_1 \otimes \cdots \otimes v_n.$$

The operators $a(v)$ and $c(v)$ are adjoint with respect to the above bilinear form on $\mathcal{T}(V)$, in the sense that

$$\langle a(v)(\xi), \eta \rangle = \langle \xi, c(v)(\eta) \rangle$$

for every $\xi, \eta \in \mathcal{T}(V)$.

Note that full Fock space is also the underlying space of the shuffle algebra $\mathcal{T}^\vee(V)$. Thus one may view the creation-annihilation operators as acting on either the tensor algebra or the shuffle algebra. It turns out that, from an algebraic point of view, it is more natural to let creation act on $\mathcal{T}(V)$ and annihilation on $\mathcal{T}^\vee(V)$. Indeed, it is easy to see that each annihilation operator is a derivation for the product of $\mathcal{T}^\vee(V)$ (shuffle), and each creation operator is a coderivation for the coproduct of $\mathcal{T}(V)$ (deshuffle). In addition, the creation operators descend to coinvariants and give rise to well-defined operators on bosonic Fock space. Dually, the annihilation operators restrict to invariants and give rise to well-defined operators on $\mathcal{S}^\vee(V)$. This is shown below.

$$\begin{tikzcd} \mathcal{T}(V) \arrow[r, "c(v)"] \arrow[d, two heads] & \mathcal{T}(V) \arrow[d, two heads] \\ \mathcal{S}(V) \arrow[r, dashed, "\bar{c}(v)"'] & \mathcal{S}(V) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}^\vee(V) \arrow[r, "a(v)"] & \mathcal{T}^\vee(V) \\ \mathcal{S}^\vee(V) \arrow[r, dashed, "\bar{a}(v)"'] \arrow[u, hook] & \mathcal{S}^\vee(V) \arrow[u, hook] \end{tikzcd}$$

If the characteristic of the base field is 0, bosonic Fock space may be identified with the underlying space of $\mathcal{S}^\vee(V)$ by means of the transformation $\bar{\kappa}$. In this situation, creation-annihilation operators act on bosonic Fock space and can therefore be composed. The following commutation relations hold on bosonic Fock space.

$$\begin{aligned}\tilde{c}(w)\tilde{c}(v) &= \tilde{c}(v)\tilde{c}(w),\\ \bar{a}(w)\bar{a}(v) &= \bar{a}(v)\bar{a}(w),\\ \bar{a}(w)\tilde{c}(v) - \tilde{c}(v)\bar{a}(w) &= \langle v, w\rangle \,\mathrm{id}\end{aligned}$$

These are identities of operators on $\mathcal{S}^\vee(V)$. To keep the notation straight, we have written $\tilde{c}(v)$ for the operator corresponding to $\bar{c}(v)$.

We will be working with a more general formulation of these relations given in (19.4).

**19.3.2. The one-dimensional case.** Let $V = \Bbbk$, the base field, equipped with the canonical inner product $\langle 1, 1\rangle = 1$. In this case, full Fock space and bosonic Fock space coincide and equal the space of polynomials in one variable:

$$\mathcal{T}(V) = \mathcal{S}(V) = \Bbbk[x] \qquad \text{and} \qquad \mathcal{T}^\vee(V) = \mathcal{S}^\vee(V) = \Bbbk\{x\}.$$

These are the polynomial and divided power Hopf algebras of Example 2.3. Since $V$ is one-dimensional, up to a scalar there is only one creation and one annihilation operator (corresponding to $v = 1$). These are given by

$$x\colon \Bbbk[x] \to \Bbbk[x] \qquad x^n \mapsto x^{n+1} \qquad \text{and} \qquad \Bbbk\{x\} \to \Bbbk\{x\} \qquad x^{(n)} \mapsto x^{(n-1)},$$

with the convention that $x^{(-1)} = 0$. The former is a coderivation, while the latter is a derivation.

In characteristic 0, the norm map $x^n \mapsto n! x^{(n)}$ provides an isomorphism of Hopf algebras from $\Bbbk[x]$ to $\Bbbk\{x\}$. The annihilation operator when viewed as an operator on $\Bbbk[x]$ via this isomorphism is the derivative operator

$$\frac{d}{dx}\colon \Bbbk[x] \to \Bbbk[x] \qquad x^n \mapsto nx^{n-1}.$$

It is well-known or one verifies directly that the creation-annihilation operators satisfy:

$$\frac{d}{dx}x - x\frac{d}{dx} = 1.$$

This is the simplest instance of the commutation relation on bosonic Fock space mentioned above.

**19.3.3. Generalized Fock spaces.** Guţă and Maassen [158] and Bożejko and Guţă [64] work with the assumption that $V$ is a Hilbert space. We limit our attention to the algebraic aspects of their constructions. This allows us to work in a slightly more general setting in which a bilinear form on $V$ is not required. In this setting, there is a creation operator for each $v \in V$ (as in the classical setting), and an annihilation operator for each functional $f \in V^*$ (rather than for each $v \in V$). The commutation relations on bosonic Fock space then take the following form.

$$\begin{aligned}\tilde{c}(w)\tilde{c}(v) &= \tilde{c}(v)\tilde{c}(w),\\ \bar{a}(g)\bar{a}(f) &= \bar{a}(f)\bar{a}(g),\\ \bar{a}(f)\tilde{c}(v) - \tilde{c}(v)\bar{a}(f) &= f(v)\,\mathrm{id}\end{aligned} \tag{19.4}$$

These are also called canonical commutation relations (usually abbreviated C.C.R).

As noted in Example 19.4, the classical Fock spaces are the values of the decorated Fock functors on the exponential species: $\mathcal{K}_V(\mathbf{E}) = \mathcal{T}(V)$ and $\overline{\mathcal{K}}_V(\mathbf{E}) = \mathcal{S}(V)$. In the next sections of this chapter, following [158, 64] we present a generalization of these constructions in which the classical Fock spaces are replaced by the values of the decorated Fock functors on species with up-down operators. We refer to these as generalized Fock spaces. We show that if the operators are balanced, then the commutation relations (19.4) continue to hold on generalized bosonic Fock spaces (Proposition 19.27).

**19.3.4. Fermionic Fock spaces.** The emphasis of this chapter is on bosonic Fock spaces. However, in the final sections, we do touch upon fermionic, and more generally, anyonic Fock spaces. The fermionic Fock space is

$$\Lambda(V) = \bigoplus_{n\geq 0} (V^{\otimes n})_{\mathrm{S}_n},$$

the underlying space of the exterior algebra; the invariants are taken with respect to the signed action of $\mathrm{S}_n$:

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto (-1)^{\operatorname{inv}\sigma}\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \tag{19.5}$$

where $\operatorname{inv}\sigma$ denotes the number of inversions of $\sigma$ (2.20). This is the usual action tensored with the sign representation.

The following commutation relations hold on fermionic Fock space.

$$\begin{aligned} \tilde{c}(w)\tilde{c}(v) &= -\tilde{c}(v)\tilde{c}(w), \\ \bar{a}(g)\bar{a}(f) &= -\bar{a}(f)\bar{a}(g), \\ \bar{a}(f)\tilde{c}(v) + \tilde{c}(v)\bar{a}(f) &= f(v)\,\mathrm{id}\,. \end{aligned} \tag{19.6}$$

We show that these relations continue to hold on generalized fermionic Fock spaces that arise from species with balanced operators (Proposition 19.39).

## 19.4. The generalized Fock spaces of Guţă and Maassen

We present a construction of generalized Fock spaces; these are termed *combinatorial Fock spaces* by Guţă and Maassen [158]. We formulate it in functorial terms, in agreement with the main ideas in this monograph. More precisely, generalized Fock spaces are the values of the decorated Fock functors on species with up-down operators. In particular, we add to their constructions by paying attention to the algebraic properties of the functors and of the resulting Fock spaces. Our setting is closer to that of Bożejko and Guţă [64, Section 2], but we work with arbitrary vector spaces rather than Hilbert spaces.

We point out that the entire theory applies to $V = \Bbbk$. In this case, there are canonical choices for $v$ and $f$. This yields results for undecorated Fock functors which go beyond those discussed in Chapter 15. We do not make them explicit.

TABLE 19.3. Categories with $+1$ and $-1$ operators.

| Categories | Description |
|---|---|
| $\mathsf{Sp}^{\mathsf{u}}$ | Species with up operators |
| $\mathsf{Sp}_{\mathsf{d}}$ | Species with down operators |
| $\mathsf{gVec}^{\mathsf{c}}$ | Graded vector spaces with creation operators |
| $\mathsf{gVec}_{\mathsf{a}}$ | Graded vector spaces with annihilation operators |

We will freely use the set up of Sections 2.8, 8.11 and 8.12. The notations for the categories are reviewed in Table 19.3. Throughout this section, $V$ is a fixed vector space, $v \in V$ is a vector, and $f \in V^*$ is a functional.

**19.4.1. Constructions of Guţă and Maassen.** We extend the decorated full Fock functor

$$\mathcal{K}_V : \mathsf{Sp} \to \mathsf{gVec}$$

to the category of species with up operators, and its companion

$$\mathcal{K}_V^\vee : \mathsf{Sp} \to \mathsf{gVec}$$

to the category of species with down operators. The choices of up operators for $\mathcal{K}$ and down operators for $\mathcal{K}^\vee$ are not arbitrary; they are justified below (see also Section 19.3).

**Definition 19.11.** We define a functor

$$\mathcal{K}_{V,v} : \mathsf{Sp}^{\mathsf{u}} \to \mathsf{gVec}^{\mathsf{c}}$$

by

$$\mathcal{K}_{V,v}(\mathbf{p}, u) := \big(\mathcal{K}_V(\mathbf{p}), c(v)\big),$$

where $\mathcal{K}_V : \mathsf{Sp} \to \mathsf{gVec}$ is the decorated full Fock functor, and the homogeneous map

$$c(v) : \mathcal{K}_V(\mathbf{p}) \to \mathcal{K}_V(\mathbf{p})$$

of degree 1 has components

$$\mathbf{p}[n] \otimes V^{\otimes n} \to \mathbf{p}[n+1] \otimes V^{\otimes (n+1)}$$

defined by

$$c(v)(x_0 \otimes 1) := u(x_0) \otimes v, \qquad c(v)(x_n \otimes v_1 \cdots v_n) := u(x_n) \otimes v v_1 \cdots v_n,$$

for $x_n \in \mathbf{p}[n]$, $v_i \in V$.

Here we make use of Convention 8.49 in order to have $u : \mathbf{p}[n] \to \mathbf{p}'[n] = \mathbf{p}[n+1]$. Note that the dependence of $c(v)$ on $u$ is not manifest in the notation.

A morphism in $\mathsf{Sp}^{\mathsf{u}}$ intertwines the up operators and hence its image under $\mathcal{K}_V$ intertwines the creation operators. Thus, $\mathcal{K}_{V,v}$ is a functor as stated.

**Definition 19.12.** We define a functor

$$\mathcal{K}_{V,f}^\vee : \mathsf{Sp}_{\mathsf{d}} \to \mathsf{gVec}_{\mathsf{a}}$$

by

$$\mathcal{K}_{V,f}^\vee(\mathbf{p}, d) := \big(\mathcal{K}_V^\vee(\mathbf{p}), a(f)\big),$$

where $\mathcal{K}_V^\vee : \mathsf{Sp} \to \mathsf{gVec}$ is the decorated full Fock functor, and the homogeneous map

$$a(f) : \mathcal{K}_V^\vee(\mathbf{p}) \to \mathcal{K}_V^\vee(\mathbf{p})$$

of degree $-1$ has components

$$\mathbf{p}[n] \otimes V^{\otimes n} \to \mathbf{p}[n-1] \otimes V^{\otimes (n-1)}$$

defined by

$$a(f)(x_0 \otimes 1) := 0, \qquad a(f)(x_n \otimes v_1 \cdots v_n) := d(x_n) \otimes f(v_1) v_2 \cdots v_n,$$

for $x_n \in \mathbf{p}[n]$, $v_i \in V$.

**19.4.2. Duality between creation and annihilation.** Let $V$ be a finite-dimensional vector space and $V^*$ be its dual. Recall that the full Fock functors $\mathcal{K}_V$ and $\mathcal{K}_V^\vee$ are related by

$$\mathcal{K}^\vee_{V^*}(\mathbf{p}^*) \cong \mathcal{K}_V(\mathbf{p})^*,$$

for any species $\mathbf{p}$ (Section 19.1.4). The functors $\mathcal{K}_{V,v}$ and $\mathcal{K}^\vee_{V,f}$ considered above are related in a similar manner. The precise result is given below. The proof is straightforward.

**Proposition 19.13.** *Let $(\mathbf{p}, u)$ be a species with up operators and $(\mathbf{q}, d)$ be a species with down operators. Let $(\mathbf{p}^*, u^*)$ and $(\mathbf{q}^*, d^*)$ be the dual species with down and up operators, respectively. Then, the following diagrams commute.*

$$\begin{tikzcd}
\mathcal{K}^\vee_{V^*,v}(\mathbf{p}^*, u^*) \arrow[r, "a(v)"] \arrow[d, equal] & \mathcal{K}^\vee_{V^*,v}(\mathbf{p}^*, u^*) \arrow[d, equal] \\
\mathcal{K}_{V,v}(\mathbf{p}, u)^* \arrow[r, "c(v)^*"'] & \mathcal{K}_{V,v}(\mathbf{p}, u)^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}^\vee_{V^*,f}(\mathbf{q}^*, d^*) \arrow[r, "c(f)"] \arrow[d, equal] & \mathcal{K}^\vee_{V^*,f}(\mathbf{q}^*, d^*) \arrow[d, equal] \\
\mathcal{K}_{V,f}(\mathbf{q}, d)^* \arrow[r, "a(f)^*"'] & \mathcal{K}_{V,f}(\mathbf{q}, d)^*
\end{tikzcd}$$

In other words, on finite-dimensional species, the functors $\mathcal{K}_{V,v}$ and $\mathcal{K}_{V,f}$ are contragredient to the functors $\mathcal{K}^\vee_{V^*,v}$ and $\mathcal{K}^\vee_{V^*,f}$ respectively.

### 19.4.3. From up-down to creation-annihilation.

**Notation 19.14.** Recall from Convention 8.49 that we view $[n]^+ = [1+n]$ with 1 being the distinguished element of $[n]^+$. We now extend this convention to sets of positive integers. Thus, if $S = \{i_1 < \cdots < i_s\}$ is such a set, we let

$$1 + S = \{1 + i_1 < \cdots < 1 + i_s\} \quad \text{and} \quad S^+ = \{1 < 1 + i_1 < \cdots < 1 + i_s\}. \tag{19.7}$$

Note that if $[n] = S \sqcup T$, then

$$[n]^+ = S^+ \sqcup (1 + T) = (1 + S) \sqcup T^+. \tag{19.8}$$

Here is a first result that explains why creation goes with $\mathcal{K}_V$ and annihilation with $\mathcal{K}_V^\vee$. Another reason is given later, in Proposition 19.19.

**Proposition 19.15.** *Let $(\mathbf{p}, u)$ and $(\mathbf{q}, w)$ be species with up operators. The creation operator $c(v)$ and the colax structure of $\mathcal{K}_V$ are related by the following commutative diagrams.*

$$\begin{tikzcd}
\mathcal{K}_{V,v}\big((\mathbf{p}, u)\cdot(\mathbf{q}, w)\big) \arrow[r, "\psi"] \arrow[d, "c(v)"'] & \mathcal{K}_{V,v}(\mathbf{p}, u)\cdot\mathcal{K}_{V,v}(\mathbf{q}, w) \arrow[d, "c(v)\cdot\mathrm{id}+\mathrm{id}\cdot c(v)"] \\
\mathcal{K}_{V,v}\big((\mathbf{p}, u)\cdot(\mathbf{q}, w)\big) \arrow[r, "\psi"'] & \mathcal{K}_{V,v}(\mathbf{p}, u)\cdot\mathcal{K}_{V,v}(\mathbf{q}, w)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}_{V,v}(\mathbf{1}) \arrow[d, "c(v)"'] \arrow[dr, "0"] & \\
\mathcal{K}_{V,v}(\mathbf{1}) \arrow[r, "\psi_0"'] & 1
\end{tikzcd} \tag{19.9}$$

*Let* $(\mathbf{p}, d)$ *and* $(\mathbf{q}, e)$ *be species with down operators. The annihilation operator* $a(f)$ *and the lax structure of* $\mathcal{K}_V^\vee$ *are related by the following commutative diagrams.*

$$\begin{tikzcd}
\mathcal{K}^\vee_{V,f}(\mathbf{p},d)\cdot\mathcal{K}^\vee_{V,f}(\mathbf{q},e) \arrow[r, "\psi^\vee"] \arrow[d, "a(f)\cdot\mathrm{id}+\mathrm{id}\cdot a(f)"'] & \mathcal{K}^\vee_{V,f}\big((\mathbf{p},d)\cdot(\mathbf{q},e)\big) \arrow[d, "a(f)"] \\
\mathcal{K}^\vee_{V,f}(\mathbf{p},d)\cdot\mathcal{K}^\vee_{V,f}(\mathbf{q},e) \arrow[r, "\psi^\vee"'] & \mathcal{K}^\vee_{V,f}\big((\mathbf{p},d)\cdot(\mathbf{q},e)\big)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\psi_0^\vee"] \arrow[dr, "0"'] & \mathcal{K}^\vee_{V,f}(\mathbf{1}) \arrow[d, "a(f)"] \\
& \mathcal{K}^\vee_{V,f}(\mathbf{1})
\end{tikzcd}
\tag{19.10}$$

In the finite-dimensional setting, the two statements are duals of each other. The same remark applies to the subsequent corollaries.

Proof. Consider the first diagram in (19.9). Start from an element

$$x \otimes y \otimes v_1 \cdots v_n \in \mathbf{p}[S] \otimes \mathbf{q}[T] \otimes V^{\otimes n}$$

in the component of degree $n$ of $\mathcal{K}_V\big((\mathbf{p},u)\cdot(\mathbf{q},w)\big)$ where $S \sqcup T = [n]$. Applying $\psi$ takes us to

$$\mathbf{p}[\mathrm{cano}](x) \otimes v_{i_1} \cdots v_{i_s} \otimes \mathbf{q}[\mathrm{cano}](y) \otimes v_{j_1} \cdots v_{j_t} \in \mathbf{p}[s] \otimes V^{\otimes s} \otimes \mathbf{q}[t] \otimes V^{\otimes t},$$

where $S = \{i_1 < \cdots < i_s\}$ and $T = \{j_1 < \cdots < j_t\}$. Applying now $c(v)\cdot\mathrm{id}+\mathrm{id}\cdot c(v)$ we obtain

$$u\big(\mathbf{p}[\mathrm{cano}](x)\big) \otimes vv_{i_1} \cdots v_{i_s} \otimes \mathbf{q}[\mathrm{cano}](y) \otimes v_{j_1} \cdots v_{j_t} \\ + \mathbf{p}[\mathrm{cano}](x) \otimes v_{i_1} \cdots v_{i_s} \otimes w\big(\mathbf{q}[\mathrm{cano}](y)\big) \otimes vv_{j_1} \cdots v_{j_t}.$$

On the other hand, applying $c(v)$ to $x \otimes y \otimes v_1 \cdots v_n$ we get, in view of (8.68) and (19.8),

$$u(x) \otimes y \otimes vv_1 \cdots v_n + x \otimes w(y) \otimes vv_1 \cdots v_n \\ \in \Big(\mathbf{p}[S^+] \otimes \mathbf{q}[1+T] \otimes V^{\otimes(n+1)}\Big) \oplus \Big(\mathbf{p}[1+S] \otimes \mathbf{q}[T^+] \otimes V^{\otimes(n+1)}\Big).$$

Therefore, applying $\psi$ we obtain, in view of (19.7),

$$\mathbf{p}[\mathrm{cano}]\big(u(x)\big) \otimes vv_{i_1} \cdots v_{i_s} \otimes \mathbf{q}[\mathrm{cano}](y) \otimes v_{j_1} \cdots v_{j_t} \\ + \mathbf{p}[\mathrm{cano}](x) \otimes v_{i_1} \cdots v_{i_s} \otimes \mathbf{q}[\mathrm{cano}]\big(w(y)\big) \otimes vv_{j_1} \cdots v_{j_t}.$$

This coincides with the expression obtained above since the up operators $u$ and $w$ are morphisms of species. Thus the first diagram in (19.9) commutes.

The commutativity of the second diagram in (19.9) follows from that of the first plus unitality of the lax structure (3.6) (or can be easily checked directly).

The proofs for (19.10) are similar. □

Recalling the definition of the monoidal structure on $\mathsf{gVec}^{\mathsf{c}}$ (2.75) we see that diagrams (19.9) say that $\psi$ and $\psi_0$ are morphisms in $\mathsf{gVec}^{\mathsf{c}}$. A similar remark applies to (19.10). In conjunction with Theorem 19.2 which says that $\mathcal{K}_V$ and $\mathcal{K}_V^\vee$ are bilax monoidal functors, we deduce at once the following result.

**Proposition 19.16.** *The functor*

$$(\mathcal{K}_{V,v}, \psi)\colon (\mathsf{Sp}^{\mathsf{u}}, \cdot) \to (\mathsf{gVec}^{\mathsf{c}}, \cdot)$$

*is colax monoidal. The functor*

$$(\mathcal{K}^\vee_{V,f}, \psi^\vee)\colon (\mathsf{Sp}_{\mathsf{d}}, \cdot) \to (\mathsf{gVec}_{\mathsf{a}}, \cdot)$$

*is lax monoidal.*

Recall that (co)lax monoidal functors preserve (co)monoids (Proposition 3.29). Further, recall that (co)monoids in categories with up, down, creation or annihilation operators are usual (co)monoids equipped with (co)derivations (Sections 8.12.5 and 2.8.3). This yields the following.

**Corollary 19.17.** *The functor $\mathcal{K}_{V,v}$ preserves up coderivations while the functor $\mathcal{K}^{\vee}_{V,f}$ preserves down derivations. Explicitly:*

*Let $\mathbf{p}$ be a comonoid in species equipped with an up coderivation $u$. Then the creation operator*

$$c(v)\colon \mathcal{K}_{V,v}(\mathbf{p}, u) \to \mathcal{K}_{V,v}(\mathbf{p}, u)$$

*is a coderivation of the coalgebra $\mathcal{K}_V(\mathbf{p})$ of degree $+1$.*

*Similarly, if $\mathbf{p}$ is a monoid in species equipped with a down derivation $d$, then the annihilation operator*

$$a(f)\colon \mathcal{K}^{\vee}_{V,f}(\mathbf{p}, d) \to \mathcal{K}^{\vee}_{V,f}(\mathbf{p}, d)$$

*is a derivation of the algebra $\mathcal{K}^{\vee}_V(\mathbf{p})$ of degree $-1$.*

**Example 19.18.** Recall from Example 8.55 that the exponential species is equipped with a down derivation and an up coderivation. We have

$$\mathcal{K}_V(\mathbf{E}) = \mathcal{T}(V) \qquad \text{and} \qquad \mathcal{K}^{\vee}_V(\mathbf{E}) = \mathcal{T}^{\vee}(V),$$

the tensor and shuffle algebras, respectively. The underlying space is classical full Fock space, in both cases. The creation-annihilation operators of Definitions 19.11 and 19.12 coincide with the classical creation-annihilation operators of Section 19.3. Corollary 19.17 recovers the facts, noted in Section 19.3, that the classical creation operator is a coderivation for the coproduct of the tensor algebra (deshuffle) and the classical annihilation operator is a derivation for the shuffle product.

More examples are discussed in Section 19.6.

**19.4.4. Creation-annihilation on bosonic Fock functors.** Let $V$ be a vector space. Recall (Theorem 19.2) that the full Fock functors are related to the bosonic Fock functors by means of transformations (morphisms of bilax functors)

$$\mathcal{K}_V \Rightarrow \overline{\mathcal{K}}_V, \qquad \overline{\mathcal{K}}^{\vee}_V \Rightarrow \mathcal{K}^{\vee}_V.$$

The creation-annihilation operators induce homogeneous maps on the bosonic Fock functors as follows.

**Proposition 19.19.** *For any species with up operators $(\mathbf{p}, u)$ and $v \in V$, the creation operator $c(v)$ descends to coinvariants*

$$\begin{tikzcd}
\mathcal{K}_V(\mathbf{p}, u) \arrow[r, "c(v)"] \arrow[d, two heads] & \mathcal{K}_V(\mathbf{p}, u) \arrow[d, two heads] \\
\overline{\mathcal{K}}_V(\mathbf{p}, u) \arrow[r, dashed, "\bar{c}(v)"'] & \overline{\mathcal{K}}_V(\mathbf{p}, u)
\end{tikzcd}$$

*yielding a homogeneous map* $\bar{c}(v)\colon \overline{\mathcal{K}}_V(\mathbf{p},u) \to \overline{\mathcal{K}}_V(\mathbf{p},u)$ *of degree* $+1$*. Dually, for any species with down operators* $(\mathbf{p},d)$ *and* $f \in V^*$*, the annihilation operator* $a(f)$ *restricts to invariants*

$$\begin{tikzcd} \mathcal{K}^{\vee}_V(\mathbf{p},d) \arrow[r, "a(f)"] & \mathcal{K}^{\vee}_V(\mathbf{p},d) \\ \overline{\mathcal{K}}^{\vee}_V(\mathbf{p},d) \arrow[u, hook] \arrow[r, dashed, "\bar{a}(f)"'] & \overline{\mathcal{K}}^{\vee}_V(\mathbf{p},d) \arrow[u, hook] \end{tikzcd}$$

*yielding a homogeneous map* $\bar{a}(f)\colon \overline{\mathcal{K}}^{\vee}_V(\mathbf{p},d) \to \overline{\mathcal{K}}^{\vee}_V(\mathbf{p},d)$ *of degree* $-1$*.*

Proof. We check the first assertion. Let $\sigma \in \mathrm{S}_n$ and consider the elements

$$x \otimes v_1 \cdots v_n \qquad \text{and} \qquad \mathbf{p}[\sigma](x) \otimes v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}$$

in $\mathbf{p}[n] \otimes V^{\otimes n}$, the component of degree $n$ of $\mathcal{K}_V(\mathbf{p},u)$. Note that the second is obtained from the first by acting by $\sigma$. Now applying $c(v)$ to both of them yields

$$u(x) \otimes vv_1 \cdots v_n$$

and

$$u\big(\mathbf{p}[\sigma](x)\big) \otimes vv_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)} = \mathbf{p}[\sigma^+]\big(u(x)\big) \otimes vv_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}.$$

The equality holds by naturality of $u$ and the definition of $\sigma^+$ (8.65). Observe that acting by $\sigma^+$ on the first element gives the second element above; so they yield the same element in the space of coinvariants. ☐

Proposition 19.19 allows us to give the following definition.

**Definition 19.20.** We define functors

$$\overline{\mathcal{K}}_{V,v}\colon \mathsf{Sp}^{\mathsf{u}} \to \mathsf{gVec}^{\mathsf{c}} \qquad \text{and} \qquad \overline{\mathcal{K}}^{\vee}_{V,f}\colon \mathsf{Sp}_{\mathsf{d}} \to \mathsf{gVec}_{\mathsf{a}}$$

by

$$\overline{\mathcal{K}}_{V,v}(\mathbf{p},u) := \big(\overline{\mathcal{K}}_V(\mathbf{p}), \bar{c}(v)\big) \qquad \text{and} \qquad \overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d) := \big(\overline{\mathcal{K}}^{\vee}_V(\mathbf{p}), \bar{a}(f)\big)$$

where $\bar{c}(v)$ and $\bar{a}(f)$ are the maps of Proposition 19.19.

Theorem 19.2 and Proposition 19.16 yield:

**Proposition 19.21.** *The functor*

$$(\overline{\mathcal{K}}_{V,v}, \overline{\psi})\colon (\mathsf{Sp}^{\mathsf{u}}, \cdot) \to (\mathsf{gVec}^{\mathsf{c}}, \cdot)$$

*is bistrong monoidal and* $\mathcal{K}_{V,v} \Rightarrow \overline{\mathcal{K}}_{V,v}$ *is a morphism of colax monoidal functors. The functor*

$$(\overline{\mathcal{K}}^{\vee}_{V,f}, \overline{\psi}^{\vee})\colon (\mathsf{Sp}_{\mathsf{d}}, \cdot) \to (\mathsf{gVec}_{\mathsf{a}}, \cdot)$$

*is bistrong monoidal and* $\mathcal{K}^{\vee}_{V,f} \Rightarrow \overline{\mathcal{K}}^{\vee}_{V,f}$ *is a morphism of lax monoidal functors.*

By arguing as for Corollary 19.17, we obtain the following consequence.

**Corollary 19.22.** *The functor* $\overline{\mathcal{K}}_{V,v}$ *preserves up derivations and up coderivations while the functor* $\overline{\mathcal{K}}^{\vee}_{V,f}$ *preserves down derivations and down coderivations.*

In view of the above observations, it is natural to consider the functor $\Im_V$ of Section 19.2.1. Indeed, there is an induced functor

$$\Im_{V,v,f} \colon \mathsf{Sp}^{\mathsf{u}}_{\mathsf{d}} \to \mathsf{gVec}^{\mathsf{c}}_{\mathsf{a}}$$

from species with up-down operators to graded vector spaces with creation-annihilation operators. This point of view will be taken up later in Section 19.8.4.

## 19.5. Creation-annihilation on generalized bosonic Fock spaces

Classically, creation and annihilation are viewed as operators on the same (Fock) space, and therefore can be composed. Our presentation leads to operators acting on different spaces ($\mathcal{K}_V$ and $\overline{\mathcal{K}}_V$ for creation, $\mathcal{K}^\vee_V$ and $\overline{\mathcal{K}}^\vee_V$ for annihilation). However, over a field of characteristic 0, the bosonic functors $\overline{\mathcal{K}}_V$ and $\overline{\mathcal{K}}^\vee_V$ are naturally isomorphic (Proposition 19.8). This identification allows us to compose creation and annihilation operators at the bosonic level.

These operators do not commute. A result of Guţă and Maassen [158, Lemmas 6 and 7] describes the situation explicitly. This is recalled in Proposition 19.25. Since our setting is slightly more general and the notation is different from theirs, we provide a proof. The main result of this section is Proposition 19.24 which is a variant of this result. It is in fact easier to derive and more useful for later purposes.

We assume throughout this section that $\Bbbk$ is a field of characteristic 0. We also continue to assume that $V$ is a fixed vector space, $v \in V$ is a vector, and $f \in V^*$ is a functional.

**19.5.1. The commutation setup.** Let $(\mathbf{p}, u, d)$ be a species with up-down operators. Consider the following (noncommutative) diagram

(19.11)
$$\begin{tikzcd}
 & \mathcal{K}_{V,v}(\mathbf{p},u) \arrow[r, "c(v)"] \arrow[dl, "\kappa"'] & \mathcal{K}_{V,v}(\mathbf{p},u) \arrow[dr, "\kappa"] & \\
\mathcal{K}^\vee_{V,f}(\mathbf{p},d) \arrow[d, "a(f)"'] & & & \mathcal{K}^\vee_{V,f}(\mathbf{p},d) \arrow[d, "a(f)"] \\
\mathcal{K}^\vee_{V,f}(\mathbf{p},d) & & & \mathcal{K}^\vee_{V,f}(\mathbf{p},d) \\
 & \mathcal{K}_{V,v}(\mathbf{p},u) \arrow[r, "c(v)"'] \arrow[ul, "\kappa"] & \mathcal{K}_{V,v}(\mathbf{p},u) \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

in which $\kappa$ is the decorated norm transformation (Definition 19.6). Using the invertibility of $\kappa$ at the bosonic level, this diagram yields two new noncommutative diagrams as follows.

$$
\begin{tikzcd}
\overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d) \arrow[r, "\tilde{c}(v)"] \arrow[d, "\bar{a}(f)"'] & \overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d) \arrow[d, "\bar{a}(f)"] \\
\overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d) \arrow[r, "\tilde{c}(v)"'] & \overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathcal{K}}_{V,v}(\mathbf{p},u) \arrow[r, "\bar{c}(v)"] \arrow[d, "\tilde{a}(f)"'] & \overline{\mathcal{K}}_{V,v}(\mathbf{p},u) \arrow[d, "\tilde{a}(f)"] \\
\overline{\mathcal{K}}_{V,v}(\mathbf{p},u) \arrow[r, "\bar{c}(v)"'] & \overline{\mathcal{K}}_{V,v}(\mathbf{p},u)
\end{tikzcd}
\tag{19.12}
$$

Here $\tilde{c}(v)$ denotes the conjugate of $\bar{c}(v)$ by $\bar{\kappa}$,

$$
\overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d) \xrightarrow{\bar{\kappa}^{-1}} \overline{\mathcal{K}}_{V,v}(\mathbf{p},u) \xrightarrow{\bar{c}(v)} \overline{\mathcal{K}}_{V,v}(\mathbf{p},u) \xrightarrow{\bar{\kappa}} \overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},u),
\tag{19.13}
$$

and $\tilde{a}(f)$ denotes the conjugate of $\bar{a}(f)$ by $\bar{\kappa}^{-1}$,

$$
\overline{\mathcal{K}}_{V,v}(\mathbf{p},u) \xrightarrow{\bar{\kappa}} \overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d) \xrightarrow{\bar{a}(f)} \overline{\mathcal{K}}^{\vee}_{V,f}(\mathbf{p},d) \xrightarrow{\bar{\kappa}^{-1}} \overline{\mathcal{K}}_{V,v}(\mathbf{p},u).
\tag{19.14}
$$

The lack of commutativity of diagrams (19.12) is of interest; it is systematically studied in the rest of this section. We will use the first diagram in (19.12) for working purposes.

**Notation 19.23.** For each $1 \leq i, j \leq n$, we let $(i,j) \in \mathrm{S}_n$ denote the transposition that switches $i$ with $j$. For simplicity, we use $\sigma \cdot x$ instead of $\mathbf{p}[\sigma](x)$ to denote the action of $\sigma \in \mathrm{S}_n$ on $x \in \mathbf{p}[n]$.

We let $(k, \dots, 1)$ be the permutation which sends $k$ to $k-1$, $k-1$ to $k-2$, and so on, and finally 1 to $k$, while fixing the elements greater than $k$.

**19.5.2. Composition formulas: first version.** We begin by deriving an explicit formula for the creation operator $\tilde{c}(v)$. We have emphasized that creation goes with $\mathcal{K}_V$; however there is a sensible creation operator on $\mathcal{K}^{\vee}_V$ given by

$$
c(v)\colon x \otimes v_1 \cdots v_n \mapsto \sum_{k=0}^{n} (k+1, \dots, 1) \cdot u(x) \otimes v_1 \cdots v_k v \cdots v_n.
\tag{19.15}
$$

With this definition, the following diagram commutes.

$$
\begin{tikzcd}
\mathcal{K}_V(\mathbf{p},u) \arrow[r, "c(v)"] \arrow[d, "\kappa"'] & \mathcal{K}_V(\mathbf{p},u) \arrow[d, "\kappa"] \\
\mathcal{K}^{\vee}_V(\mathbf{p},u) \arrow[r, "c(v)"'] & \mathcal{K}^{\vee}_V(\mathbf{p},u)
\end{tikzcd}
$$

To see this, note that

$$\begin{aligned}
&\kappa\big(c(v)(x \otimes v_1 \cdots v_n)\big) \\
&\qquad = \sum_{\tau \in \mathrm{S}_{n+1}} \tau \cdot \big(u(x)\big) \otimes \tau \cdot (v v_1 \cdots v_n) \\
&\qquad = \sum_{k=0}^{n} \sum_{\sigma \in \mathrm{S}_n} (k+1, \dots, 1)\sigma^{+} \cdot u(x) \otimes v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(k)} v \cdots v_{\sigma^{-1}(n)} \\
&\qquad = \sum_{k=0}^{n} \sum_{\sigma \in \mathrm{S}_n} (k+1, \dots, 1) \cdot u(\sigma \cdot x) \otimes (k+1, \dots, 1) \cdot (v v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}) \\
&\qquad = c(v)\big(\kappa(x \otimes v_1 \cdots v_n)\big).
\end{aligned}$$

Here $\sigma^{+}$ is as in (8.65). For the equalities, note that any permutation $\tau$ of $vv_1 \cdots v_n$ is the composite of a permutation $\sigma^{+}$ of $vv_1 \cdots v_n$ which fixes $v$ in the first position followed by a permutation $(k+1, \dots, 1)$ which inserts $v$ in the $(k+1)$-st position.

Since $\overline{\mathcal{K}}^{\vee}_{V}(\mathbf{p})$ is a subspace of $\mathcal{K}^{\vee}_{V}(\mathbf{p})$, it follows that the creation operator $\tilde{c}(v)$ is given by the same formula (19.15). Using this, we obtain:

**Proposition 19.24.** *Let* $(\mathbf{p}, u, d)$ *be a species with up-down operators and*

$$\sum_i x_i \otimes v^i_1 \cdots v^i_n \in \mathbf{p}[n] \otimes^{\mathrm{S}_n} V^{\otimes n}$$

*an element of degree* $n$ *in* $\overline{\mathcal{K}}^{\vee}_{V}(\mathbf{p}, u, d)$. *We have*

$$\begin{aligned}
\bar{a}(f)\tilde{c}(v)\Big(\sum_i x_i \otimes v^i_1 \cdots v^i_n\Big)
&= \sum_i du(x_i) \otimes f(v)\, v^i_1 \cdots v^i_n \\
&\quad + \sum_i \sum_{k=1}^{n} d\big((k+1, \dots, 1) \cdot u(x_i)\big) \otimes f(v^i_1)\, v^i_2 \cdots v^i_k v \cdots v^i_n
\end{aligned} \tag{19.16}$$

*and*

$$\tilde{c}(v)\bar{a}(f)\Big(\sum_i x_i \otimes v^i_1 \cdots v^i_n\Big) = \sum_i \sum_{k=1}^{n} (k, \dots, 1) \cdot ud(x_i) \otimes f(v^i_1)\, v^i_2 \cdots v^i_k v \cdots v^i_n. \tag{19.17}$$

A $q$-deformation of this result is given later in Proposition 19.40.

**19.5.3. Composition formulas: second version.** We now present variants of the above formulas, mainly for completeness. We begin with a slight variant of (19.15) which is as follows.

$$\begin{aligned}
\tilde{c}(v)\Big(\sum_i x_i \otimes v^i_1 \cdots v^i_n\Big) &= \sum_i u(x_i) \otimes v v^i_1 \cdots v^i_n \\
&\quad + \sum_i \sum_{k=1}^{n} (k+1, 1) \cdot u(x_i) \otimes v^i_k v^i_1 \cdots v \cdots v^i_n,
\end{aligned} \tag{19.18}$$

where $v$ appears at position $k+1$.

Proof. To prove this, we note that by $\mathrm{S}_n$-invariance, for $1 \leq k \leq n$,

$$\sum_i x_i \otimes v_1^i \cdots v_n^i = \sum_i (k, \dots, 1)^{-1} \cdot x_i \otimes v_k^i v_1^i \cdots v_{k-1}^i v_{k+1}^i \cdots v_n^i.$$

Now apply the operator which acts by the up operator $u$ followed by the permutation $(k+1, \dots, 1)$ on the first factor and inserts $v$ in position $k+1$ in the second factor. Now summing over $1 \leq k \leq n$ and adding the term

$$\sum_i u(x_i) \otimes v v_1^i \cdots v_n^i$$

to both sides, one obtains (19.18). Here we made use of (19.15), the observation that

$$(k+1, \dots, 1) = (k+1, 1)(k+1, \dots, 2) \qquad \text{for } 1 \leq k \leq n,$$

and the $\mathrm{S}_n$-invariance of $u$. $\square$

**Proposition 19.25 (Guţă and Maassen).** *Let $(\mathbf{p}, u, d)$ be a species with up-down operators and*

$$\sum_i x_i \otimes v_1^i \cdots v_n^i \in \mathbf{p}[n] \otimes^{\mathrm{S}_n} V^{\otimes n}$$

*an element of degree $n$ in $\overline{\mathcal{K}}_V^{\vee}(\mathbf{p}, u, d)$. We have*

$$\begin{aligned}(19.19)\quad \bar{a}(f)\tilde{c}(v)\Big(\sum_i x_i \otimes v_1^i \cdots v_n^i\Big) &= \sum_i du(x_i) \otimes f(v)\, v_1^i \cdots v_n^i \\ &\quad + \sum_i \sum_{k=1}^{n} d\big((k+1, 1) \cdot u(x_i)\big) \otimes f(v_k^i)\, v_1^i \cdots v \cdots v_n^i\end{aligned}$$

*and*

$$\begin{aligned}(19.20)\quad \tilde{c}(v)\bar{a}(f)\Big(\sum_i x_i \otimes v_1^i \cdots v_n^i\Big) &= \sum_i \sum_{k=1}^{n} (k, 1) \cdot ud\big((k, 1) \cdot x_i\big) \otimes f(v_k^i)\, v_1^i \cdots v \cdots v_n^i.\end{aligned}$$

*In both summations over $k$, $v$ appears at position $k$.*

Proof. The first part follows directly from (19.18). For the second part, we apply the transposition $(k, 1)$ to the $\mathrm{S}_n$-invariant element and then apply $\bar{a}(f)$ to obtain

$$\bar{a}(f)\Big(\sum_i x_i \otimes v_1^i \cdots v_n^i\Big) = \sum_i (k, 1) \cdot x_i \otimes f(v_k^i)\, v_2^i \cdots v_1^i \cdots v_n^i,$$

where $v_1^i$ is in position $k$. This holds for $1 \leq k \leq n$ and further we have the same $\mathrm{S}_{n-1}$-invariant element written in $n$ different ways. Note that (19.18) expresses $\tilde{c}(v)$ acting on the degree $n$ part as a sum of $n+1$ operators. Here $\tilde{c}(v)$ is acting on the degree $n-1$ part since we are first applying the annihilation operator. So it is a sum of $n$ operators. By letting the $k$-th operator act on the above formula, and summing over all $1 \leq k \leq n$, the result follows. $\square$

For deriving the formulas in the second version, we crucially used the fact that we were dealing with an $\mathrm{S}_n$-invariant element. Hence, unlike for the first version, these formulas do not generalize to the $q$-setting. In that scenario, one has to deal with the generalized anyonic Fock spaces $\Im_{V,q}(\mathbf{p})$ elements of which cannot be interpreted as invariants.

## 19.6. Species with balanced operators

So far, we have dealt with species with up-down operators but never specified any relations between these operators. The discussion in Section 19.5 motivates the following definition.

**Definition 19.26.** A species *with balanced operators* is a species $(\mathbf{p}, u, d)$ with up-down operators such that all relations (19.21)–(19.23c) below hold. We use Notation 19.23.

**The up-up and down-down relations.** For $n = 0, 1, 2, \ldots,$

$$(1,2) \cdot u^2(-) = u^2(-), \tag{19.21}$$

$$d^2\big((1,2) \cdot (-)\big) = d^2(-) \tag{19.22}$$

as maps

$$\mathbf{p}[n] \to \mathbf{p}[n+2] \qquad \text{and} \qquad \mathbf{p}[n+2] \to \mathbf{p}[n]$$

respectively.

**The up-down relations.** For $n = 0, 1, 2, \ldots$ and $1 \leq k \leq n$,

$$du = \lambda_n \,\mathrm{id}, \tag{19.23a}$$

$$d\big((k+1, \ldots, 1) \cdot u(-)\big) = (k, \ldots, 1) \cdot u\big(d(-)\big), \tag{19.23b}$$

$$d\big((k+1, 1) \cdot u(-)\big) = (k, 1) \cdot u(d((k,1) \cdot (-)), \tag{19.23c}$$

where $\lambda_n$ is an arbitrary scalar. In the left-hand sides above,

$$u\colon \mathbf{p}[n] \to \mathbf{p}[n+1] \qquad \text{and} \qquad d\colon \mathbf{p}[n+1] \to \mathbf{p}[n],$$

while in the right-hand sides,

$$u\colon \mathbf{p}[n-1] \to \mathbf{p}[n] \qquad \text{and} \qquad d\colon \mathbf{p}[n] \to \mathbf{p}[n-1].$$

By using invariance under the appropriate symmetric groups, one sees that relations (19.23b) and (19.23c) imply each other; in other words, they are equivalent.

**Proposition 19.27.** *Let $(\mathbf{p}, u, d)$ be a species with up-down operators. Let $v, w \in V$ and $f, g \in V^*$, and let $\tilde{c}(v)$ and $\bar{a}(f)$ be as in (19.12).*

(i) *If (19.21) holds, then $\tilde{c}(w)\tilde{c}(v) = \tilde{c}(v)\tilde{c}(w)$.*
(ii) *If (19.22) holds, then $\bar{a}(g)\bar{a}(f) = \bar{a}(f)\bar{a}(g)$.*
(iii) *If (19.23) holds, then $\bar{a}(f)\tilde{c}(v) - \tilde{c}(v)\bar{a}(f) = \lambda_n f(v)\,\mathrm{id}$.*

*In the third statement, (19.23) refers to all three relations (19.23a)–(19.23c).*

In particular, the generalized bosonic Fock space of a species with balanced operators satisfies the usual bosonic commutation relations (19.4). Conjugating by $\bar{\kappa}$ or its inverse and using (19.13) and (19.14), the above result is equivalent to:

(i) If (19.21) holds, then $\bar{c}(w)\bar{c}(v) = \bar{c}(v)\bar{c}(w)$.
(ii) If (19.22) holds, then $\tilde{a}(g)\tilde{a}(f) = \tilde{a}(f)\tilde{a}(g)$.
(iii) If (19.23) holds, then $\tilde{a}(f)\bar{c}(v) - \bar{c}(v)\tilde{a}(f) = \lambda_n f(v)\,\mathrm{id}$.

Proof. Consider (i), the case of up operators. It is convenient here to work with $\bar{c}$ rather than $\tilde{c}$. Applying the operators $c(w)c(v)$ and $c(v)c(w)$ to $x \otimes v_1 v_2 \cdots v_n$ yields

$$u^2(x) \otimes w v v_1 v_2 \cdots v_n \qquad \text{and} \qquad u^2(x) \otimes v w v_1 v_2 \cdots v_n$$

respectively. If (19.21) holds, then applying the transposition $(1,2)$ to one yields the other. So they represent the same element in the space of coinvariants, thus $\bar{c}(w)\bar{c}(v) = \bar{c}(v)\bar{c}(w)$, proving (i).

Case (ii) is similar and omitted. Case (iii) follows by applying either Propositions 19.24 or 19.25. □

We illustrate this result with some interesting examples. Examples 19.31 and 19.32 are due to Guţă and Maassen [158, Section 4.1].

**Example 19.28.** Let $(\mathbf{E}, u, d)$ be the exponential species with up-down operators, as in Example 8.55. In this case $u$ and $d$ are inverse and the symmetric group action is trivial. Thus $(\mathbf{E}, u, d)$ is a species with balanced operators with $\lambda_n = 1$. By applying Proposition 19.27, one recovers the bosonic commutation relations (19.4).

**Example 19.29.** Let $\mathbf{e}$ be the species of elements defined in Section 8.13.7. Thus, $\mathbf{e}[I] = \Bbbk I$, the vector space with basis the elements of $I$. We first note that $\mathbf{e}$ carries up-down operators, the up operator being the natural inclusion and the down operator being given by

$$\mathbf{e}[I^+] \to \mathbf{e}[I] \qquad\qquad i \mapsto \begin{cases} i & i \in I, \\ 0 & i = *_I. \end{cases}$$

It is straightforward to check that $(\mathbf{e}, u, d)$ is a species with balanced operators with $\lambda_n = 1$. By applying Proposition 19.27, one sees that the same commutation relations (19.4) hold on the generalized bosonic space of the species of elements.

This can be understood more explicitly as follows. The generalized bosonic space of the species of elements can be identified with

$$\overline{\mathcal{K}}_V(\mathbf{e}) \xrightarrow{\cong} \Bbbk \oplus V \otimes S(V), \qquad \overline{i \otimes v_1 \cdots v_n} \mapsto v_i \otimes v_1 \cdots v_{i-1} v_{i+1} \cdots v_n.$$

Under this identification, the creation operator $\bar{c}(v)$ and annihilation operator $\tilde{a}(f)$ send $v_0 \otimes v_1 \cdots v_n$ to

$$v_0 \otimes v v_1 \cdots v_n \qquad \text{and} \qquad \sum_{i=1}^{n} v_0 \otimes v_1 \cdots f(v_i) \cdots v_n$$

respectively. With these descriptions, the commutation relations may also be checked directly.

**Example 19.30.** Let $\mathbf{E}^{\cdot 2} = \mathbf{E} \cdot \mathbf{E}$ be the species of subsets (Example 8.17). We may use the up-down operators of $\mathbf{E}$ on either factor to define

$$u_1, u_2 \colon \mathbf{E}^{\cdot 2}[I] \to \mathbf{E}^{\cdot 2}[I^+] \qquad \text{and} \qquad d_1, d_2 \colon \mathbf{E}^{\cdot 2}[I^+] \to \mathbf{E}^{\cdot 2}[I]$$

by

$$u_1(S) = S, \qquad\qquad u_2(S) = S \cup \{*_I\},$$

and

$$d_1(S) = \begin{cases} S & *_I \notin S, \\ 0 & *_I \in S, \end{cases} \qquad\qquad d_2(S) = \begin{cases} 0 & *_I \notin S, \\ S \setminus \{*_I\} & *_I \in S. \end{cases}$$

It is straightforward to check that $(\mathbf{E}^{\cdot 2}, u_1, d_1)$ and $(\mathbf{E}^{\cdot 2}, u_2, d_2)$ are species with balanced operators with $\lambda_n = 1$, while $(\mathbf{E}^{\cdot 2}, u_1, d_2)$ and $(\mathbf{E}^{\cdot 2}, u_2, d_1)$ are species with balanced operators with $\lambda_n = 0$.

Proceeding more directly, generalized bosonic space of the subset species can be identified with

$$\overline{\mathcal{K}}_V(\mathbf{E}^{\cdot 2}) \xrightarrow{\cong} S(V) \otimes S(V), \qquad \overline{T \otimes v_1 \cdots v_n} \mapsto v_{i_1} \cdots v_{i_t} \otimes v_{j_1} \cdots v_{j_s},$$

where $T = \{i_1, \ldots, i_t\}$ and its complement in $[n]$ is $\{j_1, \ldots, j_s\}$. Under this identification, the creation operators $\bar{c}_1(v)$ and $\bar{c}_2(v)$ send $u_1 \cdots u_m \otimes v_1 \cdots v_n$ to

$$vu_1 \cdots u_m \otimes v_1 \cdots v_n \qquad \text{and} \qquad u_1 \cdots u_m \otimes vv_1 \cdots v_n$$

respectively, and the annihilation operators $\tilde{a}_1(f)$ and $\tilde{a}_2(f)$ send it to

$$\sum_{i=1}^{m} u_1 \cdots f(u_i) \cdots u_m \otimes v_1 \cdots v_n \qquad \text{and} \qquad \sum_{i=1}^{n} u_1 \cdots u_m \otimes v_1 \cdots f(v_i) \cdots v_n$$

respectively. With these descriptions, the various commutation relations asserted above can be checked directly.

**Example 19.31.** Let $(\mathbf{L}, u, d)$ be the species of linear orders with up-down operators, as in Example 8.56. Interestingly, $(\mathbf{L}, u, d)$ is *not* a species with balanced operators; among the required relations only (19.23a) holds. So Proposition 19.27 does not apply.

Let us write down the creation and annihilation operators explicitly. First observe that

$$\overline{\mathcal{K}}_V(\mathbf{L}) \xrightarrow{\cong} \mathcal{T}(V), \qquad \overline{1|\cdots|n \otimes v_1 \cdots v_n} \mapsto v_1 \cdots v_n,$$

which is classical full Fock space. Under this identification, the creation operator $\bar{c}(v)$ and the annihilation operator $\tilde{a}(f)$ send $v_1 \cdots v_n$ to

$$vv_1 \cdots v_n \qquad \text{and} \qquad f(v_1)v_2 \cdots v_n$$

respectively. These are the classical creation and annihilation operators on full Fock space of Section 19.3. It follows that

$$\tilde{a}(f)\bar{c}(v) = \bar{a}(f)\tilde{c}(v) = f(v)\,\mathrm{id}\,.$$

This can also be verified from equation (19.16) as follows.

For any $l \in \mathbf{L}[n]$ we have

$$du(l) = l \qquad \text{and} \qquad d\big((k+1, \ldots, 1) \cdot u(l)\big) = 0 \quad \text{for every } k = 1, \ldots, n,$$

since the minimum element of $(k+1, \ldots, 1) \cdot u(l^1|\cdots|l^n)$ is not 1.

The relation noted above does not look like a commutation relation. However, it is indeed the case $q = 0$ of a $q$-commutation relation. This point will be clarified later in Example 19.43.

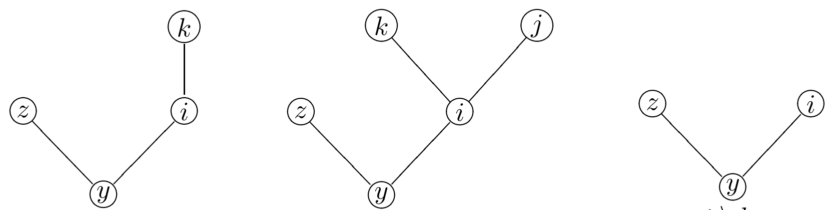
**Example 19.32.** Consider the species $\mathbf{a}$ of rooted trees (Section 13.3.1): $\mathbf{a}[I]$ is the space with basis consisting of all rooted trees with vertex set $I$. Given a rooted tree $t \in \mathbf{a}[I]$, a vertex $i \in I$, and a new element $j \notin I$, let

$$t_i^j \in \mathbf{a}[I \sqcup \{j\}]$$

be the rooted tree obtained by attaching a new leaf with label $j$ to vertex $i$. In addition, given a leaf $k$ of $t$, let

$$t \setminus k \in \mathbf{a}[I \setminus \{k\}]$$

be the rooted tree obtained by removing leaf $k$ from $t$. These constructions are illustrated below, for $I = \{i, k, y, z\}$.



$t \qquad t_i^j \qquad t \setminus k$

The maps

$$\mathbf{a}[I] \underset{d}{\overset{u}{\rightleftarrows}} \mathbf{a}[I^+]$$

given by

$$u(t) := \sum_{i \in I} t_i^{*_I} \quad \text{for } t \in \mathbf{a}[I]$$

and

$$d(t) := \begin{cases} t \setminus *_I & \text{if } *_I \text{ is a leaf of } t, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } t \in \mathbf{a}[I^+],$$

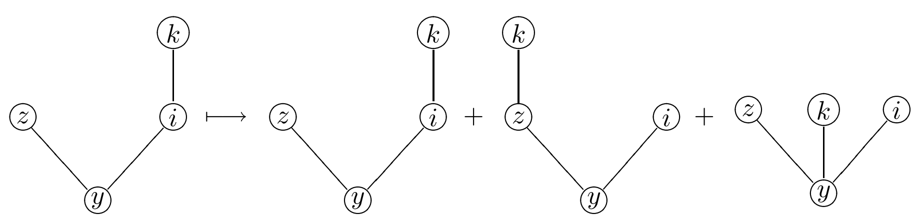
turn $\mathbf{a}$ into a species with up-down operators. However, $(\mathbf{a}, u, d)$ is *not* a species with balanced operators; relations (19.21) and (19.22) do not hold. However, relation (19.23) holds with $\lambda_n = n$. Details are as follows. For any $t \in \mathbf{a}[n]$ we have

$$du(t) = n\, t$$

and for any $k = 1, \ldots, n$

$$\begin{aligned} d\big((k+1,1) \cdot u(t)\big) &= \begin{cases} \sum\limits_{i \in [n] \setminus \{k\}} (t \setminus k)_i^k & \text{if } k \text{ is a leaf of } t \\ 0 & \text{otherwise} \end{cases} \\ &= (k,1) \cdot ud\big((k,1) \cdot t\big). \end{aligned}$$

The summation above consists of all trees obtained from $t$ by removing leaf $k$ and reattaching it to a vertex of the remaining tree. An example follows.

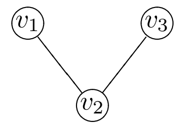


It follows from Proposition 19.27 that

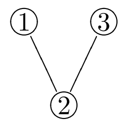
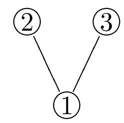
$$\big(\bar{a}(f)\tilde{c}(v) - \tilde{c}(v)\bar{a}(f)\big)(x) = n\, f(v)\, x \tag{19.24}$$

for any unlabeled, $V$-decorated rooted tree $x$ with $n$ vertices.

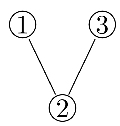
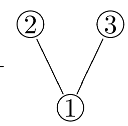
Proceeding more directly, the generalized bosonic Fock space of rooted trees can be identified with the space of *unlabeled* rooted trees with $n$ vertices, each vertex decorated by an element of $V$. For instance, the unlabeled $V$-decorated tree below

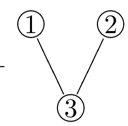


if viewed as a coinvariant, is the image of

 $\otimes\, v_1 v_2 v_3$ or, say,  $\otimes\, v_2 v_1 v_3$

and, if viewed as an invariant, is the following element of $\mathbf{a}[3] \otimes^{S_3} V^{\otimes 3}$.

 $\otimes\,(v_1v_2v_3 + v_3v_2v_1)$ $+$  $\otimes\,(v_2v_1v_3 + v_2v_3v_1)$

$+$  $\otimes\,(v_1v_3v_2 + v_3v_1v_2)$

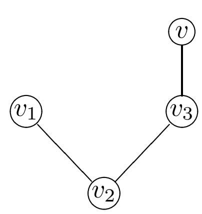
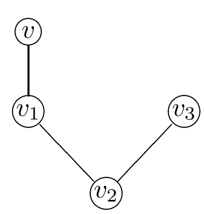
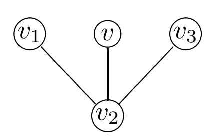
In keeping with the previous notation, we make the following definitions. For an unlabeled $V$-decorated tree $t$, vertex $n$ and $v \in V$, we let $t_n^v$ be the tree obtained by attaching a new leaf with label $v$ to vertex $n$. In addition, for a leaf $l$, we let $t \setminus l$ be the tree obtained by removing leaf $l$ from $t$.

Under the above identification (say with invariants), the creation and annihilation operators are:

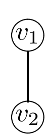
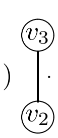
$$\tilde{c}(v)(t) = \sum_{n:\text{vertex of } t} t_n^v \qquad \text{and} \qquad \bar{a}(f)(t) = \sum_{l:\text{leaf of } t} f(v_l)\, t \setminus l,$$

where $v_l \in V$ is the label of the leaf $l$.

The creation operator $\tilde{c}(v)$ applied to the above example yields

 $+$  $+$ 

while the annihilation operator $\bar{a}(f)$ yields

$f(v_3)$  $+$ $f(v_1)$  .

One can now check directly that (19.24) holds. Further, one sees that neither the creation operators nor the annihilation operators commute. This is consistent with the conclusion drawn from Proposition 19.27.

## 19.7. Deformations of decorated Fock functors

The theory of undecorated Fock functors developed in Chapter 15 admits several generalizations, one of them being the one-parameter deformations of Chapter 16. In this section we briefly consider this type of deformation in the decorated case.

**19.7.1. The decorated fermionic Fock functor.** It is natural to ask for a decorated version of the fermionic Fock functor $\overline{\mathcal{K}}_{-1}$. Recall that the bosonic and fermionic Fock functors determine each other by precomposing with the signature functor. Following this idea, we define the decorated fermionic Fock functor, denoted $\overline{\mathcal{K}}_{V,-1}$, as the composite:

$$(\mathsf{Sp}, \cdot, \beta_p) \xrightarrow{(-)^-} (\mathsf{Sp}, \cdot, \beta_{-p}) \xrightarrow{\overline{\mathcal{K}}_V} (\mathsf{gVec}, \cdot, \beta_{-p}),$$

where we recall that $(-)^-$ is the signature functor (9.10). As a functor, $\overline{\mathcal{K}}_{V,-1}(\mathbf{p})$ is given by the same formula as given for $\overline{\mathcal{K}}_V(\mathbf{p})$ in Definition 19.1, with the understanding that the coinvariants are taken with respect to the signed action of $\mathrm{S}_n$ on $V^{\otimes n}$ as in (19.5). Since the composite of bilax functors is bilax, the above formulation defines $\overline{\mathcal{K}}_{V,-1}$ not just as a functor but as a bilax functor. Since $\overline{\mathcal{K}}_V$ itself can be viewed as a composite (Proposition 19.3), there are alternative ways of viewing $\overline{\mathcal{K}}_{V,-1}$ as a composite. For example, the composites

$$\begin{gathered}(\mathsf{Sp}, \cdot, \beta_p) \xrightarrow{(-)\times(\mathbf{E}_V)^-} (\mathsf{Sp}, \cdot, \beta_{-p}) \xrightarrow{\overline{\mathcal{K}}} (\mathsf{gVec}, \cdot, \beta_{-p}), \\ (\mathsf{Sp}, \cdot, \beta_p) \xrightarrow{(-)\times\mathbf{E}_V} (\mathsf{Sp}, \cdot, \beta_p) \xrightarrow{\overline{\mathcal{K}}_{-1}} (\mathsf{gVec}, \cdot, \beta_{-p})\end{gathered} \tag{19.25}$$

both yield $\overline{\mathcal{K}}_{V,-1}$. Here $(\mathbf{E}_V)^-$ is the signed partner of $\mathbf{E}_V$ which is the same as the signature functor applied to $\mathbf{E}_V$.

The functor $\overline{\mathcal{K}}^{\vee}_{V,-1}$ can be defined similarly as the composite:

$$(\mathsf{Sp}, \cdot, \beta_p) \xrightarrow{(-)^-} (\mathsf{Sp}, \cdot, \beta_{-p}) \xrightarrow{\overline{\mathcal{K}}^{\vee}_V} (\mathsf{gVec}, \cdot, \beta_{-p}),$$

and described in alternative ways by using $\overline{\mathcal{K}}^{\vee}$ and $\overline{\mathcal{K}}^{\vee}_{-1}$ in the above discussion. It follows from Proposition 19.5 that

$$\overline{\mathcal{K}}^{\vee}_{V^*,-1} \cong (\overline{\mathcal{K}}_{V,-1})^{\vee}.$$

**19.7.2. One-parameter deformations.** Recall that the full Fock functors can be deformed to yield functors $\mathcal{K}_q$ and $\mathcal{K}^{\vee}_q$ which depend on a scalar $q$. This was explained in Section 16.1. In a similar manner, the decorated full Fock functors can be deformed to yield functors $\mathcal{K}_{V,q}$ and $\mathcal{K}^{\vee}_{V,q}$. We call them the deformed decorated full Fock functors. We now comment on their bilax structures.

Recall the structure maps $\varphi$ and $\psi$ of $\mathcal{K}_V$ from Section 19.1.3. For the functor $\mathcal{K}_{V,q}$, the lax structure $\varphi$ remains unchanged while the colax structure $\psi$ is deformed to $\psi_q$ by the coefficient $q^{\mathrm{sch}_n(S)}$, the exponent being the Schubert statistic. The structure maps of $\mathcal{K}^{\vee}_{V,q}$, denoted $\psi^{\vee}_q$ and $\varphi^{\vee}$, can be made explicit in the same manner.

**Theorem 19.33.** *The functors*

$$(\mathcal{K}_{V,q}, \varphi, \psi_q),\ (\mathcal{K}^{\vee}_{V,q}, \psi^{\vee}_q, \varphi^{\vee})\colon (\mathsf{Sp}, \cdot, \beta_p) \to (\mathsf{gVec}, \cdot, \beta_{pq}).$$

*are bilax monoidal.*

**Proposition 19.34.** *The functors $\mathcal{K}_{V,q}$ and $\mathcal{K}^{\vee}_{V,q}$ are the following composites of bilax functors*

$$(\mathsf{Sp},\cdot,\beta_p) \xrightarrow{(-)\times\mathbf{E}_V} (\mathsf{Sp},\cdot,\beta_p) \xrightarrow{\mathcal{K}_q} (\mathsf{gVec},\cdot,\beta_{pq})$$
$$(\mathsf{Sp},\cdot,\beta_p) \xrightarrow{(-)\times\mathbf{E}_V} (\mathsf{Sp},\cdot,\beta_p) \xrightarrow{\mathcal{K}^{\vee}_q} (\mathsf{gVec},\cdot,\beta_{pq})$$

*respectively.*

Decorated versions of results in Chapter 16 can be obtained as a consequence of this result. We state some of them below.

**Proposition 19.35.** *There are isomorphisms of bilax functors*

$$\mathcal{K}_{V,q}(-) \cong \overline{\mathcal{K}}_V\big(\mathbf{L}_q \times (-)\big) \qquad \textit{and} \qquad \mathcal{K}^{\vee}_{V,q}(-) \cong \overline{\mathcal{K}}^{\vee}_V\big(\mathbf{L}^*_q \times (-)\big)$$

*from* $(\mathsf{Sp},\cdot,\beta_p)$ *to* $(\mathsf{gVec},\cdot,\beta_{pq})$.

Proof. We give the argument for the first part.

$$\mathcal{K}_{V,q}(-) \cong \mathcal{K}_q\big((-)\times\mathbf{E}_V\big) \cong \overline{\mathcal{K}}(\mathbf{L}_q \times (-) \times \mathbf{E}_V) \cong \overline{\mathcal{K}}_V\big(\mathbf{L}_q \times (-)\big).$$

The first isomorphism follows from Proposition 19.34, the second follows from Proposition 16.6, and the third follows from Proposition 19.3. □

We now discuss a deformation of the decorated norm transformation of Definition 19.6. Let

$$\kappa_q \colon \mathcal{K}_{V,q} \Rightarrow \mathcal{K}^{\vee}_{V,q}$$

be defined as follows. For any species $\mathbf{p}$, let

$$(\kappa_q)_{\mathbf{p}}(x \otimes v_1 \cdots v_n) := \sum_{\sigma\in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)}\sigma\cdot(x\otimes v_1\cdots v_n), \tag{19.26}$$

for any $x \in \mathbf{p}[n]$, $v_i \in V$. This is the decorated $q$-norm. The dependence of $\kappa_q$ on $V$ is not manifest in the notation.

It follows directly from the definition that the decorated $q$-norm is the result of precomposing the undecorated $q$-norm $\kappa_q$ of Definition 16.13 with the functor $(-)\times\mathbf{E}_V$. It then follows from Proposition 16.15 that the decorated $q$-norm is a morphism of bilax functors. Similarly, one can deduce from Proposition 16.14 that the contragredient for the decorated $q$-norm for $V$ is the decorated $q$-norm for $V^*$.

Let $\Im_{V,q}$ be the image of the decorated $q$-norm. We call it the *decorated anyonic Fock functor*. It fits into the following commutative diagram.

$$\begin{tikzcd} \mathcal{K}_{V,q} \arrow[rr, Rightarrow, "\kappa_q"] \arrow[dr, Rightarrow] & & \mathcal{K}^{\vee}_{V,q} \\ & \Im_{V,q}. \arrow[ur, Rightarrow] & \end{tikzcd} \tag{19.27}$$

On finite-dimensional species, the dual of this diagram is the same diagram with $V$ replaced by $V^*$.

It follows that $\Im_{V,q}$ is the composite:

$$(\mathsf{Sp},\cdot,\beta_p) \xrightarrow{(-)\times\mathbf{E}_V} (\mathsf{Sp},\cdot,\beta_p) \xrightarrow{\Im_q} (\mathsf{gVec},\cdot,\beta_{pq}),$$

where $\Im_q$ is the anyonic Fock functor (Section 16.3.5). We define *anyonic Fock space* to be the value of the decorated anyonic Fock functor on the exponential species. We note that

$$\Im_{V,1} = \Im_V,$$

the functor considered in Section 19.2.1. This is the case $q = 1$. For $q = -1$, in characteristic 0, we have

$$\Im_{V,-1} \cong \overline{\mathcal{K}}_{V,-1} \cong \overline{\mathcal{K}}^{\vee}_{V,-1},$$

the decorated fermionic Fock functors. For $q = 0$, the decorated 0-norm is the identity, so

$$\mathcal{K}_{V,0} = \Im_{V,0} = \mathcal{K}^{\vee}_{V,0}. \tag{19.28}$$

We call this the *decorated free Fock functor*. In this situation, the result of Proposition 19.35 says that

$$\Im_{V,0}(-) \cong \Im_V\big(\mathbf{L}_0 \times (-)\big). \tag{19.29}$$

This is an isomorphism of bilax functors from $(\mathsf{Sp}, \cdot, \beta_p)$ to $(\mathsf{gVec}, \cdot, \beta_0)$.

## 19.8. Deformations related to up-down and creation-annihilation

In the previous section, we looked at deformations of decorated Fock functors. We now look at the behavior of these functors on species with up-down operators. The main result is Proposition 19.41 which says that if the up-down operators are balanced (in a weaker sense) then the resulting creation-annihilation operators satisfy a $q$-commutation relation.

**19.8.1. The Hadamard and signature functors.** We know from Proposition 8.58 that the Hadamard functor $(\times, \varphi, \psi)$ on species is bilax with respect to the Cauchy product. We would like to upgrade this result to species with up or down operators. This is not possible. The best one can say is the following.

**Proposition 19.36.** *The functor*

$$(\times, \psi)\colon (\mathsf{Sp}^{\mathsf{u}} \times \mathsf{Sp}^{\mathsf{u}}, \cdot_p \times \cdot_q) \to (\mathsf{Sp}^{\mathsf{u}}, \cdot_{pq})$$

*is colax. Dually, the functor*

$$(\times, \varphi)\colon (\mathsf{Sp}_{\mathsf{d}} \times \mathsf{Sp}_{\mathsf{d}}, \cdot_p \times \cdot_q) \to (\mathsf{Sp}_{\mathsf{d}}, \cdot_{pq})$$

*is lax.*

Proof. We give the argument for the first part. First, define the Hadamard product of two species with up operators $(\mathbf{p}, u)$ and $(\mathbf{q}, v)$ to be $(\mathbf{p} \cdot \mathbf{q}, w)$ where

$$w\colon \mathbf{p} \times \mathbf{q} \xrightarrow{u \times v} \mathbf{p}' \times \mathbf{q}' = (\mathbf{p} \times \mathbf{q})'. \tag{19.30}$$

Let $(\mathbf{p}_1, u_1)$, $(\mathbf{p}_2, u_2)$, $(\mathbf{q}_1, v_1)$ and $(\mathbf{q}_2, v_2)$ be species with up operators. We need to check that the following diagram commutes, with $\tau_q$ as in (8.69).

$$\begin{tikzcd}
(\mathbf{p}_1 \cdot \mathbf{q}_1) \times (\mathbf{p}_2 \cdot \mathbf{q}_2) \arrow[r, "\psi"] \arrow[d, "(u_1 \cdot \mathrm{id} + \tau_p \cdot v_1) \times (u_2 \cdot \mathrm{id} + \tau_q \cdot v_2)"'] & (\mathbf{p}_1 \times \mathbf{p}_2) \cdot (\mathbf{q}_1 \times \mathbf{q}_2) \arrow[d, "(u_1 \times u_2) \cdot \mathrm{id} + \tau_{pq} \cdot (v_1 \times v_2)"] \\
(\mathbf{p}_1 \cdot \mathbf{q}_1)' \times (\mathbf{p}_2 \cdot \mathbf{q}_2)' \arrow[r, "\psi"'] & \big((\mathbf{p}_1 \times \mathbf{p}_2) \cdot (\mathbf{q}_1 \times \mathbf{q}_2)\big)'
\end{tikzcd}$$

One verifies this on each component. The first observation is that the diagram commutes trivially unless one starts in a component of the form

$$(\mathbf{p}_1[S] \otimes \mathbf{q}_1[T]) \otimes (\mathbf{p}_2[S] \otimes \mathbf{q}_2[T]).$$

On this component, the check is straightforward. One needs to use $\tau_p \tau_q = \tau_{pq}$. $\square$

Recall the signature functor $(-)^-$ which sends a species $\mathbf{p}$ to its signed partner $\mathbf{p}^- = \mathbf{p} \times \mathbf{E}^-$. The above result along with the observation of Section 9.3 that $\mathbf{E}^-$ is a comonoid in $(\mathsf{Sp}^{\mathsf{u}}, \cdot_{-1})$ and a monoid in $(\mathsf{Sp}_{\mathsf{d}}, \cdot_{-1})$ implies the following.

**Proposition 19.37.** *The functor*

$$(-)^- \colon (\mathsf{Sp}^{\mathsf{u}}, \cdot_q) \to (\mathsf{Sp}^{\mathsf{u}}, \cdot_{-q})$$

*is colax. Dually, the functor*

$$(-)^- \colon (\mathsf{Sp}_{\mathsf{d}}, \cdot_q) \to (\mathsf{Sp}_{\mathsf{d}}, \cdot_{-q})$$

*is lax.*

**19.8.2. The up-down properties of the deformed Fock functors.** Let $v \in V$ and $f \in V^*$ be fixed. Consider the functors

$$\mathcal{K}_{V,q,v} \colon \mathsf{Sp}^{\mathsf{u}} \to \mathsf{gVec}^{\mathsf{c}} \qquad \text{and} \qquad \mathcal{K}^{\vee}_{V,q,f} \colon \mathsf{Sp}_{\mathsf{d}} \to \mathsf{gVec}_{\mathsf{a}},$$

with the creation operator on the former and the annihilation operator on the latter defined in the same way as before. So far, there is no dependence on $q$. The dependence comes when one considers the monoidal properties of these functors. Keeping in mind the undeformed case, we do not expect these functors to be bilax; rather, we expect the former to be colax, and dually the latter to be lax. Accordingly:

**Proposition 19.38.** *The functor*

$$(\mathcal{K}_{V,q,v}, \psi_q) \colon (\mathsf{Sp}^{\mathsf{u}}, \cdot_p) \to (\mathsf{gVec}^{\mathsf{c}}, \cdot_{pq})$$

*is colax monoidal. The functor*

$$(\mathcal{K}^{\vee}_{V,q,f}, \psi^{\vee}_q) \colon (\mathsf{Sp}_{\mathsf{d}}, \cdot_p) \to (\mathsf{gVec}_{\mathsf{a}}, \cdot_{pq})$$

*is lax monoidal.*

The proof is straightforward and omitted.

**19.8.3. Creation-annihilation on generalized fermionic Fock spaces.** The entire discussion in Section 19.5 can be carried out for the decorated fermionic Fock functors of Section 19.7.1. The starting point is to take diagram (19.11) and replace $\mathcal{K}_V$, $\mathcal{K}^{\vee}_V$ and $\kappa$ with $\mathcal{K}_{V,-1}$, $\mathcal{K}^{\vee}_{V,-1}$ and $\kappa_{-1}$ respectively. Up to isomorphism, this is equivalent to precomposing (19.11) with the signature functor. Using the invertibility of $\kappa_{-1}$ at the fermionic level, this yields two new noncommutative diagrams as follows.

$$\begin{tikzcd} \overline{\mathcal{K}}^{\vee}_{V,-1,f}(\mathbf{p},d) \arrow[r, "\bar{c}(v)"] \arrow[d, "\bar{a}(f)"'] & \overline{\mathcal{K}}^{\vee}_{V,-1,f}(\mathbf{p},d) \arrow[d, "\bar{a}(f)"] \\ \overline{\mathcal{K}}^{\vee}_{V,-1,f}(\mathbf{p},d) \arrow[r, "\bar{c}(v)"'] & \overline{\mathcal{K}}^{\vee}_{V,-1,f}(\mathbf{p},d) \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathcal{K}}_{V,-1,v}(\mathbf{p},u) \arrow[r, "\bar{c}(v)"] \arrow[d, "\bar{a}(f)"'] & \overline{\mathcal{K}}_{V,-1,v}(\mathbf{p},u) \arrow[d, "\bar{a}(f)"] \\ \overline{\mathcal{K}}_{V,-1,v}(\mathbf{p},u) \arrow[r, "\bar{c}(v)"'] & \overline{\mathcal{K}}_{V,-1,v}(\mathbf{p},u) \end{tikzcd} \tag{19.31}$$

Formulas (19.16) and (19.17) hold with the coefficients $(-1)^k$ and $(-1)^{k-1}$ respectively inserted inside the double summations. More details are given in the $q$-version below. The same is true for the formulas (19.19) and (19.20). This leads to the following fermionic version of Proposition 19.27.

**Proposition 19.39.** *Let $(\mathbf{p}, u, d)$ be a species with up-down operators. Let $v, w \in V$ and $f, g \in V^*$, and let $\tilde{c}(v)$ and $\bar{a}(f)$ be as in (19.31).*

(i) *If (19.21) holds, then $\tilde{c}(w)\tilde{c}(v) = -\tilde{c}(v)\tilde{c}(w)$.*
(ii) *If (19.22) holds, then $\bar{a}(g)\bar{a}(f) = -\bar{a}(f)\bar{a}(g)$.*
(iii) *If (19.23) holds, then $\bar{a}(f)\tilde{c}(v) + \tilde{c}(v)\bar{a}(f) = \lambda_n f(v)\,\mathrm{id}$.*

*In the third statement, (19.23) refers to all three relations (19.23a)–(19.23c).*

In particular, the generalized fermionic Fock space of a species with balanced operators satisfies the usual fermionic commutation relations (19.6). Conjugating by $\bar{\kappa}_{-1}$ or its inverse, the same relations hold with $\tilde{c}$ and $\bar{a}$ replaced by $\bar{c}$ and $\tilde{a}$.

In the examples of Section 19.6, we had derived various bosonic commutation relations. In light of the above result, we see that they have corresponding fermionic analogues. We content ourselves by mentioning that for the exponential species, the above result recovers the commutation relations on fermionic Fock space (19.6).

**19.8.4. Creation-annihilation on generalized anyonic Fock spaces.** We would like to unify the bosonic and fermionic settings. For that purpose, we consider the decorated anyonic Fock functor $\Im_{V,q}$. It turns out that there is a induced functor

$$\Im_{V,q,v,f} \colon \mathsf{Sp}^{\mathsf{u}}_{\mathsf{d}} \to \mathsf{gVec}^{\mathsf{c}}_{\mathsf{a}},$$

from species with up-down operators to graded vector spaces with creation-annihilation operators. This means that

$$\mathcal{K}_{V,q,v} \Rightarrow \Im_{V,q,v,f} \qquad \text{and} \qquad \Im_{V,q,v,f} \Rightarrow \mathcal{K}^{\vee}_{V,q,f}$$

are natural transformations, the former for up and creation, and the latter for down and annihilation.

From now on, we simply write $\Im_{V,q}$, suppressing the dependence on $v$ and $f$. The value of this functor on a species with up-down operators is a generalized anyonic Fock space. It carries both creation and annihilation operators, which we denote simply by $c(v)$ and $a(f)$ without bar or tilde, keeping in view their unbiased nature.

Explicitly, the creation operator is given by

$$c(v) \colon x \otimes v_1 \cdots v_n \mapsto \sum_{k=0}^{n} q^k (k+1, \ldots, 1) \cdot u(x) \otimes v_1 \cdots v_k v \cdots v_n. \tag{19.32}$$

Note that this is (19.15) with the coefficient $q^k$ inserted. This formula can be derived in the same manner using the decorated $q$-norm (19.26). One sees that the exponent of $q$ must be the number of inversions of $(k+1, \ldots, 1)$ which is $k$.

The annihilation operator is given by

$$a(f) \colon x \otimes v_1 v_2 \cdots v_n \mapsto d(x) \otimes f(v_1)\, v_2 \cdots v_n, \tag{19.33}$$

as before with no dependence on $q$.

These formulas imply the following $q$-analogue of Proposition 19.24:

**Proposition 19.40.** *Let* $(\mathbf{p}, u, d)$ *be a species with up-down operators and*

$$\sum_i x_i \otimes v_1^i \cdots v_n^i$$

*an element of degree* $n$ *in* $\Im_{V,q}(\mathbf{p})$. *Then* (19.16) *and* (19.17) *hold with coefficients* $q^k$ *and* $q^{k-1}$ *respectively inserted inside the double summations.*

We point out that there is no similar statement for (19.19) and (19.20). As a further implication, we obtain the following $q$-analogue which simultaneously generalizes part (iii) of Propositions 19.27 and 19.39. Note that no claim is being made about the commutativity of creation operators or of annihilation operators.

**Proposition 19.41.** *Let* $(\mathbf{p}, u, d)$ *be a species with up-down operators. If* (19.23) *holds, then*

$$a(f)c(v) - q\, c(v)a(f) = \lambda_n f(v)\,\mathrm{id}$$

*where the operators are acting on the degree* $n$ *component of* $\Im_{V,q}(\mathbf{p})$.

These type of deformed creation-annihilation operators and $q$-commutation relations for anyonic Fock space $\Im_{V,q}(\mathbf{E})$ have been considered in the literature, starting with the work of Bożejko and Speicher [65, Section 2]; for additional work and more recent references see also Anshelevich [26].

**19.8.5. Relating the decorated anyonic Fock functors.** Consider the decorated free Fock functor

$$\Im_{V,0} \colon \mathsf{Sp}_{\mathsf{d}}^{\mathsf{u}} \to \mathsf{gVec}_{\mathsf{a}}^{\mathsf{c}}.$$

One can see from (19.32) and (19.33), or using (19.28) that creation and annihilation on generalized free Fock space are given by

$$c(v) \colon x \otimes v_1 \cdots v_n \mapsto u(x) \otimes v v_1 \cdots v_n,$$
$$a(f) \colon x \otimes v_1 \cdots v_n \mapsto d(x) \otimes f(v_1)\, v_2 \cdots v_n.$$

They verify Proposition 19.41 for $q = 0$.

Now recall from Proposition 19.36 and its proof that if $\mathbf{p}$ and $\mathbf{q}$ are species with up-down operators, then so is their Hadamard product $\mathbf{p} \times \mathbf{q}$. Also let $\mathbf{L}$ be the linear order species with up-down operators as defined in Example 8.56.

**Proposition 19.42.** *The following is an isomorphism of functors*

$$\Im_{V,0}(-) \cong \Im_V\big(\mathbf{L} \times (-)\big) \tag{19.34}$$

*from* $\mathsf{Sp}_{\mathsf{d}}^{\mathsf{u}}$ *to* $\mathsf{gVec}_{\mathsf{a}}^{\mathsf{c}}$.

This is a straightforward check which we omit.

It is worth comparing the claim made above with (19.29). We point out that in the present situation we are not making any claims about the monoidal properties of the functors; so it does not matter whether we write $\mathbf{L}$ or $\mathbf{L}_0$.

**Example 19.43.** Applying (19.34) to the exponential species, we obtain

$$\Im_{V,0}(\mathbf{E}) \;\cong\; \Im_V(\mathbf{L}).$$

The first space is the free Fock space. Creation-annihilation operators on this space satisfy the 0-commutation relation, that is, the relation of Proposition 19.41 with $q = 0$. The second space is the generalized bosonic Fock space of the linear order species. However, $\mathbf{L}$ is not a species with balanced operators; so we do not expect the bosonic commutation relations to hold on this space. Rather, we are seeing that the 0-commutation relation should hold which is exactly what was noted in Example 19.31.

**Example 19.44.** Applying (19.34) to the species of elements (Example 19.29), we obtain

$$\Im_{V,0}(\mathbf{e}) \;\cong\; \Im_V(\mathbf{L} \times \mathbf{e}).$$

The first space is the generalized free Fock space of the species of elements. Since $\mathbf{e}$ is a species with balanced operators, the creation-annihilation operators acting on it satisfy the 0-commutation relation.

The pointing of the species of linear orders $\mathbf{L}^{\bullet} = \mathbf{L} \times \mathbf{e}$ carries up-down operators; however, they do not turn $\mathbf{L}^{\bullet}$ into a species with balanced operators. Hence we do not expect the creation-annihilation operators acting on generalized bosonic Fock space of $\mathbf{L}^{\bullet}$ to satisfy the usual commutation relation. Instead, the above isomorphism shows that they satisfy the 0-commutation relation.

## 19.9. Yang–Baxter deformations of decorated Fock functors

In Section 19.7, we discussed one-parameter deformations of the decorated Fock functors. We now sketch a more general framework for deformations in the decorated setting. To express the monoidal properties of these deformed functors, one needs to generalize the notion of a bilax functor to the context where the source category is braided but the target category is only partially braided. We explain the main idea behind this notion, and then discuss the examples of interest to us.

**19.9.1. Bilax functors in the context of a Yang–Baxter operator.** Let $(\mathsf{D}, \bullet)$ be a monoidal category. Recall that a *Yang–Baxter operator* on a functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ consists of a natural isomorphism

$$\nu \colon \mathcal{F}(A) \bullet \mathcal{F}(B) \to \mathcal{F}(B) \bullet \mathcal{F}(A) \tag{19.35}$$

satisfying the dodecagon axiom [184, Definition 2.4]. If $\mathsf{C}$ has only one arrow, then we recover the more common notion of a Yang–Baxter operator [191, Definition XIII.3.1].

Now suppose that $\mathsf{C}$ is a braided monoidal category and let $(\mathcal{F}, \nu)$ be as above. Note that we do not require $\mathsf{D}$ to be braided. Even then we can make sense of when $(\mathcal{F}, \nu)$ is bilax: in the braiding axiom (3.11) use $\nu$ instead of $\beta$. This idea can be used to give an abstract definition of a bilax functor in this setting. In the recent paper [265], McCurdy and Street have discussed this notion (independently from our work). However, we point out that in addition to the usual axioms one would also need compatibilities of $\nu$ with the lax and colax structures of $\mathcal{F}$. In the usual setting, these compatibilities follow from properties of the braiding. One reason for this can be seen from the requirement: If $\mathsf{C}$ has only one object, then a bilax functor should specialize to a braided bialgebra [356, Definition 5.1].

The results of Takeuchi [356, Section 5] relating braided bialgebras to bialgebras in certain braided monoidal categories can be extended to results relating bilax monoidal functors in the context of a Yang–Baxter operator to bilax monoidal functors between braided monoidal categories.

**19.9.2. Yang–Baxter deformations of decorated full Fock functors.** Start with a Yang–Baxter operator $R$ on $V$. This implies that the tensor power $V^{\otimes n}$ carries an action of the braid group $\mathrm{B}_n$. Recall that there is a canonical section

$$s\colon \mathrm{S}_n \to \mathrm{B}_n \tag{19.36}$$

which sends generators to generators in the usual presentations of these groups [246, Section 2.1.2]. This section is not a group homomorphism.

For any decomposition $S \sqcup T = [n]$, consider the permutation $\zeta : [n] \to [n]$ whose restrictions to $S$ and $T$ are the order-preserving maps $\mathrm{cano} : S \to [s]$ and $\mathrm{cano} : T \to [s+1, s+t]$, where $s = |S|$ and $t = |T|$. Then there is an induced map

$$V^{\otimes n} \to V^{\otimes s} \otimes V^{\otimes t} \tag{19.37}$$

given by the action of the element $s(\zeta) \in \mathrm{B}_n$. More explicitly, we repeatedly apply the Yang–Baxter operator so that the $V$'s which lie in the positions specified by $S$ move to the first $s$ positions.

Now for $s + t = n$, consider the permutation $\zeta : [n] \to [n]$ whose restrictions to $[s]$ and $[s+1, s+t]$ are the order-preserving maps $\mathrm{cano} : [s] \to [t+1, t+s]$ and $\mathrm{cano} : [s+1, s+t] \to [t]$. Then there is an induced map

$$V^{\otimes s} \otimes V^{\otimes t} \to V^{\otimes t} \otimes V^{\otimes s} \tag{19.38}$$

given by the action of the element $s(\zeta) \in \mathrm{B}_n$.

We now explain how these ideas can be used to construct a bilax functor

$$\mathcal{K}_{V,R}\colon (\mathsf{Sp}, \cdot, \beta) \to (\mathsf{gVec}, \cdot).$$

To start with, the functor is defined by:

$$\mathcal{K}_{V,R}(\mathbf{p}) := \mathbf{p}[n] \otimes V^{\otimes n}.$$

The lax structure is the same as for $\mathcal{K}_V$ while the colax structure is defined using (19.37). The structure map $\nu$ of (19.35) is defined using (19.38). This turns $\mathcal{K}_{V,R}$ into a bilax functor.

The functor $\mathcal{K}^{\vee}_{V,R}$ is constructed along similar lines.

For any species $\mathbf{p}$, let $\kappa_{\mathbf{p}}\colon \mathcal{K}_{V,R}(\mathbf{p}) \to \mathcal{K}^{\vee}_{V,R}(\mathbf{p})$ be the map of graded vector spaces given by

$$\kappa_{\mathbf{p}}(x \otimes v_1 \cdots v_n) := \sum_{\sigma \in \mathrm{S}_n} \sigma \cdot x \otimes s(\sigma) \cdot (v_1 \cdots v_n),$$

for any $x \in \mathbf{p}[n]$, $v_i \in V$. This defines the norm transformation

$$\kappa\colon \mathcal{K}_{V,R} \Rightarrow \mathcal{K}^{\vee}_{V,R}.$$

It is a morphism of bilax functors. The image gives rise to the bilax functor $\Im_{V,R}$.

**Example 19.45.** Consider the flip operator on $V$ which interchanges the two tensor factors of $V \otimes V$. This, as well as any scalar multiple $q$ of it, is a Yang–Baxter operator $R_q$ on $V$. This gives the representation of $\mathrm{B}_n$ in which the action of the standard generators is by multiplication by $q$. In this case, the Yang–Baxter operator $\nu$ on $\mathcal{K}_{V,R_q}$ extends in fact to the braiding $\beta_q$ on $\mathsf{gVec}$. The functors we obtain in this situation are

$$\mathcal{K}_{V,R_q} = \mathcal{K}_{V,q}, \quad \mathcal{K}^{\vee}_{V,R_q} = \mathcal{K}^{\vee}_{V,q}, \quad \text{and} \quad \Im_{V,R_q} = \Im_{V,q},$$

the functors of diagram (19.27). These are bilax in the usual sense.

**Example 19.46.** We now generalize the previous example. Let $Q$ be a square matrix of size $r$, where $r$ is the dimension of $V$. Fix a basis $x_1, x_2, \ldots, x_r$ of $V$, and consider the Yang–Baxter $R_Q$ operator on $V$:

$$V \otimes V \to V \otimes V, \qquad x_i \otimes x_j \mapsto q_{ji}\, x_j \otimes x_i$$

where $i$ and $j$ vary between 1 and $r$, and $q_{ji}$ denotes the entries of the matrix $Q$. (We recover the previous example if all entries are equal.) The operator $R_Q$ is an involution precisely if $Q$ is log-antisymmetric. Let us denote the resulting functors by

$$\mathcal{K}_{V,R_Q},\ \mathcal{K}^{\vee}_{V,R_Q} \quad \text{and} \quad \Im_{V,R_Q}.$$

They are not bilax in the usual sense in general.

In Chapter 20, we construct these functors using colored species and multigraded vector spaces. We highlight an important result. If the field characteristic is 0 and $Q$ is such that no monomial in the $q_{ij}$'s equals 1 then the norm transformation is an isomorphism and the three functors above are isomorphic (Theorem 20.11).

**Example 19.47.** We continue the discussion in the preceding example. Applying the above deformed functors to a bimonoid in species yields a braided bialgebra rather than a usual bialgebra.

Recall the bimonoid $\mathbf{E}$ associated to the exponential species. The object $\mathcal{K}_{V,R_Q}(\mathbf{E})$ is the free algebra on $r$ generators $\Bbbk\langle x_1, \ldots, x_r\rangle$ of Example 2.14. The object $\mathcal{K}^{\vee}_{V,R_Q}(\mathbf{E})$ is the quantum shuffle algebra as defined by Green [152] and Rosso [316, Proposition 9]. It has the same underlying space as the free algebra but the structure maps are different: the product is a deformation of the shuffle product and the coproduct is deconcatenation. The object $\Im_{V,R_Q}(\mathbf{E})$ is Rosso's *quantum symmetric algebra* associated to the matrix $Q$. This is also called the *Nichols algebra of diagonal type* associated to $Q$ [23, Proposition 2.11]. These objects appear in the classification of pointed Hopf algebras with abelian coradical [20, 22, 23, 24, 25]. Sections 3.2 and 4 of the survey by Andruskiewitsch and Schneider [23] contain results on Nichols algebras of diagonal type. More information can be found in the lecture notes by Heckenberger [165].

A more detailed discussion of this example is given later in Example 20.21.

Other Yang–Baxter operators would lead to more general deformations of the decorated Fock functors. The resulting braided Hopf algebras after applying the functor $\Im_{V,R}$ would include, for the special case of the exponential species, the Nichols algebra (also called quantum symmetric algebra) associated to the Yang–Baxter operator $R$. For information on Nichols algebras, see [23] and [165]. They are named after Warren Nichols who considered them in [284].

**19.9.3. Up-down and creation-annihilation.** The construction in Section 19.8 of creation-annihilation operators from species with up-down operators can also be extended to the setting of Yang–Baxter operators. The creation-annihilation operators act on usual anyonic Fock space but the action is deformed by the Yang–Baxter operator. We explain this briefly.

Fix $v \in V$ and $f \in V^*$. Then the value of the functor $\Im_{V,R}$ on a species with up-down operators carries both creation and annihilation operators which are as follows.

The creation operator is given by

$$c(v)\colon x \otimes v_1 \cdots v_n \mapsto \sum_{k=0}^{n} (k+1, \dots, 1) \cdot u(x) \otimes s(k+1, \dots, 1) \cdot (vv_1 \cdots v_n), \tag{19.39}$$

where $s$ is the canonical section of (19.36). The Yang–Baxter operator $R$ appears in this map via the action on the tensors.

The annihilation operator is given by

$$a(f)\colon x \otimes v_1 v_2 \cdots v_n \mapsto d(x) \otimes f(v_1)\, v_2 \cdots v_n, \tag{19.40}$$

as before with no dependence on $R$

To get the commutation relations, pick a basis $x_1, x_2, \dots, x_r$ of $V$, and write

$$R(x_a \otimes x_b) = \sum_{c,d} R_{ab}^{cd}\, x_c \otimes x_d$$

for suitable coefficients $R_{ab}^{cd}$. One may check that:

**Proposition 19.48.** *Let $(\mathbf{p}, u, d)$ be a species with up-down operators, and let $R$ be a Yang–Baxter operator. If (19.23) holds, then for any $i$ and $j$ between $1$ and $r$,*

$$a(x_i^*)c(x_j) - \sum_{k,l} R_{jl}^{ik}\, c(x_k)a(x_l^*) = \lambda_n \delta_{ij} \operatorname{id}$$

*where the operators are acting on the degree $n$ component of $\Im_{V,q}(\mathbf{p})$.*

If $R = R_Q$ as in Example 19.46, then the above commutation relation becomes:

$$a(x_i^*)c(x_j) - q_{ij}\, c(x_j)a(x_i^*) = \lambda_n \delta_{ij} \operatorname{id}.$$

If all the $q_{ij}$'s are equal, then one recovers the $q$-commutation relation of Proposition 19.41.

The special case of the exponential species recovers the creation-annihilation operators introduced by Bożejko and Speicher [66]. The commutation relation of Proposition 19.48 is given on [66, p. 109].

CHAPTER 20

# Colored Fock Functors

The main goal of this chapter is the construction of multivariate versions of the Fock functors of Chapters 15 and 16. This is done by replacing species by colored species and graded vector spaces by multigraded vector spaces. One of the advantages of the multivariate setting is the existence of a variety of braidings on these categories, as we saw in Sections 2.4 and 14.1. In particular, for each integer square matrix $Q$ of size $r$ there is a braiding on the category of $r$-colored species as well as on the category of $\mathbb{N}^r$-graded vector spaces. When the matrix $Q$ is related to a matrix $A$ of Cartan type via (2.33), such braidings are used to define the deformations of simple Lie algebras usually known as quantum groups. They also play a key role in the theory of (abstract) pointed Hopf algebras. Remarkably, it is possible to construct monoidal functors from $r$-colored species to $\mathbb{N}^r$-graded vector spaces that are bilax with respect to these braidings. For $r = 1$, this recovers the Fock functors and their $q$-deformations.

This opens the doors to largely unexplored territory in the universe of combinatorial Hopf algebras and connects it with the world of quantum groups and the classification theory of abstract Hopf algebras.

In this chapter we content ourselves with the main details of these constructions, leaving further study for future work, and hopefully, other interested authors. We begin with the construction of the colored Fock functors in Section 20.1. We also introduce, for any log-antisymmetric matrix $Q$, a colored version which simultaneously generalizes the bosonic and fermionic Fock functors. In Section 20.2, we construct the colored norm transformation between the colored full Fock functors. Its image yields the colored anyonic Fock functor. In particular, we recover the bosonic and fermionic Fock functors when $Q = [1]$ and $Q = [-1]$ respectively. In Section 20.3, we study the behavior of the colored full Fock functor $\mathcal{K}_Q$ with respect to commutativity. The functor $\mathcal{K}_Q$ is not braided colax in general. We show that conjugating the colax structure with the braidings yields the functor $\mathcal{K}_{Q^{-t}}$. We also construct a $q$-analogue of the half-twist transformation. In Section 20.4, we relate these colored Fock functors to the deformations of the decorated Fock functors constructed in Section 19.9. We conclude with Section 20.5 which shows

Table 20.1. Colored Fock functors.

| Fock functor | Name |
|---|---|
| $\mathcal{K}_Q$, $\mathcal{K}_Q^\vee$ | Colored full Fock functor |
| $\overline{\mathcal{K}}_Q$, $\overline{\mathcal{K}}_Q^\vee$ | Colored bosonic-fermionic Fock functor |
| $\Im_Q$ | Colored anyonic Fock functor |

how nontrivial quantum groups arise by applying the colored Fock functors to even the simplest $Q$-Hopf monoids.

## 20.1. The colored Fock functors

In this section, we construct the colored or $Q$-deformed version of the full Fock functors. We refer to these as the $Q$-Fock functors. We obtain the free Fock functor as a special case. We then explain how invariants and coinvariants can be used to obtain colored generalizations of the bosonic and fermionic Fock functors.

**20.1.1. The colored full Fock functors.** Define a functor $\mathcal{K}^{(r)}$ by

$$\mathcal{K}^{(r)}(\mathbf{q}) := \bigoplus_{n\geq 0} \bigoplus_{f\colon [n]\to[r]} \mathbf{q}[n,f]. \tag{20.1}$$

This space is graded over the set of all functions $f\colon [n] \to [r]$, with $n \geq 0$ (the free monoid on $r$ generators). Hence it is also graded over $\mathbb{N}^r$ (the free commutative monoid on $r$ generators). Explicitly, the degree of an element in $\mathbf{q}[n,f]$ is $\mathsf{d}(f)$ as defined in (2.38). We may write

$$\mathcal{K}^{(r)}(\mathbf{q}) = \bigoplus_{\mathsf{d}\in\mathbb{N}^r} \bigoplus_{\substack{f\colon [n]\to[r]\\ \mathsf{d}(f)=\mathsf{d}}} \mathbf{q}[n,f].$$

We now proceed to turn this into a bilax monoidal functor

$$(\mathcal{K}^{(r)}, \varphi^{(r)}, \psi_Q^{(r)})\colon (\mathsf{Sp}^{(r)}, \cdot, \beta_P) \to (\mathsf{gVec}^{(r)}, \cdot, \beta_{P\times Q}),$$

where $\beta_P$ is the braiding on $r$-colored species defined in (14.3), $\beta_{P\times Q}$ is the braiding on $\mathbb{N}^r$-graded vector spaces defined in (2.60), and $P\times Q$ is the Hadamard product of matrices (Section 2.2.5). For the bilax structure, we need to define transformations

$$\mathcal{K}^{(r)}(\mathbf{p})\cdot\mathcal{K}^{(r)}(\mathbf{q}) \underset{(\psi_Q^{(r)})_{\mathbf{p},\mathbf{q}}}{\overset{\varphi^{(r)}_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} \mathcal{K}^{(r)}(\mathbf{p}\cdot\mathbf{q}).$$

Considering the component of degree $n$ of both spaces, and using the definitions of the Cauchy products (2.59) and (14.2), we need to define maps

$$\bigoplus_{s+t=n} \bigoplus_{\substack{f_1\colon [s]\to[r]\\ f_2\colon [t]\to[r]}} \mathbf{p}[s,f_1]\otimes\mathbf{q}[t,f_2] \underset{(\psi_Q^{(r)})_{\mathbf{p},\mathbf{q}}}{\overset{\varphi^{(r)}_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} \bigoplus_{S\sqcup T=[n]} \bigoplus_{f\colon [n]\to[r]} \mathbf{p}[S,f|_S]\otimes\mathbf{q}[T,f|_T].$$

The natural transformation $\varphi^{(r)}_{\mathbf{p},\mathbf{q}}$ is given by

$$\mathbf{p}[s,f_1]\otimes\mathbf{q}[t,f_2] \xrightarrow{\mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}]} \mathbf{p}[S,g]\otimes\mathbf{q}[T,h]$$

where

$$S=[s],\quad g=f_1,\quad T=[s+1,s+t],\quad h=f_2\,\mathrm{cano}^{-1},$$

with

$$\mathrm{cano}\colon [t]\to[s+1,s+t].$$

being the canonical order-preserving map. We are following Notation 2.5. Thus $\varphi^{(r)}$ simply embeds the domain into certain distinguished components of the codomain.

Before going to the colax structure, we review some terminology. Recall the multiplicative weighted Schubert statistic (2.35). It can be equivalently formulated as a weighted cocycle (10.102). Further, it is closely related to the multiplicative weighted distance function on chambers (10.75). The notations for these notions and the relations between them are summarized below.

$$\mathrm{sch}^Q_n(S,f) = \mathrm{sch}^Q_{S,T,f}(C_{(n)}) = \mathrm{dist}^Q_f(C_{(n)}, KC_{(n)}),$$

where $K$ is the vertex $S|T$ and $C_{(n)}$ is the canonical linear order on $[n]$. On the left is the statistic, in the middle is the cocycle, and on the right is the distance function. The equalities hold by (10.103) and the multiplicative version of (10.100).

We proceed. The natural transformation $(\psi^{(r)}_Q)_{\mathbf{p},\mathbf{q}}$ is given by

$$\begin{gathered}\mathbf{p}[S, f|_S] \otimes \mathbf{q}[T, f|_T] \to \mathbf{p}[s, f_1] \otimes \mathbf{q}[t, f_2] \\ x \otimes y \mapsto \mathrm{sch}^Q_n(S,f)\ \overline{x} \otimes \overline{y},\end{gathered} \tag{20.2}$$

or equivalently,

$$x \otimes y \mapsto \mathrm{dist}^Q_f(C_{(n)}, KC_{(n)})\ \overline{x} \otimes \overline{y}. \tag{20.3}$$

where

$$s = |S|, \quad t = |T|, \quad f_1 = f|_S\,\mathrm{cano}^{-1}, \quad f_2 = f|_T\,\mathrm{cano}^{-1},$$

with

$$\mathrm{cano}\colon S \to [s] \quad \text{and} \quad \mathrm{cano}\colon T \to [t].$$

being the canonical order-preserving maps,

$$\overline{x} = \mathbf{p}[\mathrm{cano}](x) \quad \text{and} \quad \overline{y} = \mathbf{q}[\mathrm{cano}](y).$$

Note that several components of the domain map onto one same component of the codomain under $\psi^{(r)}_Q$.

Finally, we let $\varphi^{(r)}_0$ and $(\psi^{(r)}_Q)_0$ be the identity maps

$$\Bbbk \xrightarrow{\varphi^{(r)}_0} \mathcal{K}^{(r)}(\mathbf{1}_{(r)}) \xrightarrow{(\psi^{(r)}_Q)_0} \Bbbk.$$

We use $\mathcal{K}_Q$ to shorten $(\mathcal{K}^{(r)}, \varphi^{(r)}, \psi^{(r)}_Q)$.

**Theorem 20.1.** *The functor*

$$\mathcal{K}_Q\colon (\mathsf{Sp}^{(r)}, \cdot, \beta_P) \to (\mathsf{gVec}^{(r)}, \cdot, \beta_{P\times Q})$$

*is bilax monoidal.*

Proof. The proof reduces to checking that certain products of the entries taken from the matrices $P$ and $Q$ match. This is a generalization of the proof of Theorem 16.1 where one checked that certain powers of $p$ and $q$ match (recall that in that case $P = [p]$ and $Q = [q]$). These checks boiled down to properties of the Schubert statistic. Now we need to use the corresponding properties of the weighted Schubert statistic. For example, coassociativity (dual of diagram (3.5)) follows from (2.42). Similarly, the braiding axiom (3.11) follows from (2.43) (also see the results on braids in [6] in this regard). □

It follows from the definitions that for $r = 1$ and $Q = [q]$,

$$\mathcal{K}_Q = \mathcal{K}_q,$$

the bilax monoidal functor $\mathcal{K}_q$ of Section 9.4. For this reason, $\mathcal{K}_Q$ is a colored version of $\mathcal{K}_q$, and hence also a deformation of the full Fock functor $\mathcal{K}$. Another special case worth mentioning is when $Q = 1_{r,r}$. The multiplicative distance is identically 1 in this case, so it essentially drops out of the definition of the colax structure.

We now proceed to define another bilax functor

$$\mathcal{K}_Q^\vee = ((\mathcal{K}^{(r)})^\vee, (\psi_{Q^t}^{(r)})^\vee, (\varphi^{(r)})^\vee).$$

The reason for this terminology will be clear shortly. The functor

$$(\mathcal{K}^{(r)})^\vee := \mathcal{K}^{(r)}$$

and is given by (20.1). Define a natural transformation

$$(\psi_{Q^t}^{(r)})^\vee_{\mathbf{p},\mathbf{q}} \colon (\mathcal{K}^{(r)})^\vee(\mathbf{p}) \cdot (\mathcal{K}^{(r)})^\vee(\mathbf{q}) \to (\mathcal{K}^{(r)})^\vee(\mathbf{p} \cdot \mathbf{q})$$

with components

$$\mathbf{p}[s, f_1] \otimes \mathbf{q}[t, f_2] \to \bigoplus_{S \sqcup T = [n]} \mathbf{p}[S, g] \otimes \mathbf{q}[T, h]$$

which map

$$x \otimes y \mapsto \sum_{\substack{S \sqcup T = [n] \\ |S|=s, |T|=t}} \mathrm{sch}_n^Q(S, f)\, \mathbf{p}[\mathrm{cano}](x) \otimes \mathbf{q}[\mathrm{cano}](y), \tag{20.4}$$

where the canonical maps in question are, for each term in the sum,

$$\mathrm{cano} \colon [s] \to S \qquad \text{and} \qquad \mathrm{cano} \colon [t] \to T,$$

and $f$ is defined such that its restriction to $S$ is $f_1 \,\mathrm{cano}^{-1}$ and its restriction to $T$ is $f_2 \,\mathrm{cano}^{-1}$. Note that $g = f_1 \,\mathrm{cano}^{-1}$ and $h = f_2 \,\mathrm{cano}^{-1}$ necessarily.

The natural transformation

$$(\varphi^{(r)})^\vee_{\mathbf{p},\mathbf{q}} \colon (\mathcal{K}^{(r)})^\vee(\mathbf{p} \cdot \mathbf{q}) \to (\mathcal{K}^{(r)})^\vee(\mathbf{p}) \cdot (\mathcal{K}^{(r)})^\vee(\mathbf{q})$$

is the direct sum of the following maps:

$$\mathbf{p}[s, g] \otimes \mathbf{q}[[s+1, s+t], h] \xrightarrow{\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}]} \mathbf{p}[s, f_1] \otimes \mathbf{q}[t, f_2]$$

where $f_1 = g$ and $f_2 = h \,\mathrm{cano}^{-1}$ for

$$\mathrm{cano} \colon [s+1, s+t] \to [t].$$

On the components for which $S \neq [s]$ (and $T \neq [s+1, s+t]$), the map $(\varphi^{(r)})^\vee_{\mathbf{p},\mathbf{q}}$ is zero.

We let $(\psi_{Q^t}^{(r)})^\vee_0$ and $(\varphi^{(r)})^\vee_0$ be the identity maps

$$\Bbbk \xrightarrow{(\psi_{Q^t}^{(r)})^\vee_0} (\mathcal{K}^{(r)})^\vee(\mathbf{1}_{(r)}) \xrightarrow{(\varphi^{(r)})^\vee_0} \Bbbk.$$

It is straightforward to show that:

**Theorem 20.2.** *The functor*

$$\mathcal{K}_Q^\vee \colon (\mathsf{Sp}^{(r)}, \cdot, \beta_P) \to (\mathsf{gVec}^{(r)}, \cdot, \beta_{P \times Q})$$

*is bilax monoidal.*

Recall that duality on colored species (14.6) changes the braiding $\beta_P$ to $\beta_{P^t}$. Similarly, duality on multigraded vector spaces (2.63) changes $\beta_{P\times Q}$ to $\beta_{P^t\times Q^t}$. Thus, applying the contragredient construction of Section 3.10 to $\mathcal{K}_Q$ yields a new bilax monoidal functor which is related to $\mathcal{K}_Q^\vee$ (in the finite-dimensional setting) via

$$\mathcal{K}_Q^\vee = (\mathcal{K}_{Q^t})^\vee.$$

This explains why we write $Q^t$ in the notation for the colax structure of $\mathcal{K}_Q^\vee$.

We refer to the functors $\mathcal{K}_Q$ and $\mathcal{K}_Q^\vee$ collectively as the *colored full Fock functors*.

**20.1.2. The colored free Fock functor.** Let $0_{r,r}$ denote the matrix of size $r$, all of whose entries are zero. We claim that

$$\mathcal{K}_{0_{r,r}} = \mathcal{K}_{0_{r,r}}^\vee. \tag{20.5}$$

We check the colax part; the lax part follows similarly or by applying the contragredient. For the colax structure of $\mathcal{K}_{0_{r,r}}$, put $Q = 0_{r,r}$ in (20.3). Observe that this map is nonzero only if $C_{(n)} = KC_{(n)}$, that is, if $K = S|T$ is a face of $C_{(n)}$, or equivalently, if $S = [s]$ and $T = [s+1, s+t]$. In addition, whenever this occurs, the coefficient (multiplicative distance) is 1. So it clearly agrees with the colax structure of $\mathcal{K}_{0_{r,r}}^\vee$.

The bilax functor constructed in (20.5) is the colored version of the free Fock functor of (16.1). We refer to it as the *colored free Fock functor*.

**20.1.3. The colored bosonic-fermionic Fock functors.** We now introduce a colored version which simultaneously generalizes the bosonic and fermionic Fock functors. More precisely, for any log-antisymmetric matrix $Q$, we define bistrong functors $\overline{\mathcal{K}}_Q$ and $\overline{\mathcal{K}}_Q^\vee$ (which will be isomorphic in characteristic 0) such that

$$\overline{\mathcal{K}}_{[1]} = \overline{\mathcal{K}}, \qquad \overline{\mathcal{K}}_{[-1]} = \overline{\mathcal{K}}_{-1}, \qquad \overline{\mathcal{K}}_{[1]}^\vee = \overline{\mathcal{K}}^\vee \qquad \text{and} \qquad \overline{\mathcal{K}}_{[-1]}^\vee = \overline{\mathcal{K}}_{-1}^\vee.$$

In other words, the matrices $[1]$ and $[-1]$ yield the bosonic and fermionic functors respectively.

For simplicity, let us first discuss the case when $Q = 1_{r,r}$, that is, all matrix entries are 1. Let $\mathsf{d} \in \mathbb{N}^r$ be fixed. The starting point is the observation that for any $r$-colored species, the symmetric group $\mathrm{S}_n$ acts on the space

$$\bigoplus_{f\colon [n]\to[r],\, \mathsf{d}(f)=\mathsf{d}} \mathbf{q}[n,f]. \tag{20.6}$$

The action is as follows. Any element $\sigma \in \mathrm{S}_n$ defines a map

$$[n,f] \to [n, f\sigma^{-1}]$$

of colored sets which by functoriality induces a map of vector spaces

$$\mathbf{q}[\sigma]\colon \mathbf{q}[n,f] \to \mathbf{q}[n,f\sigma^{-1}] \qquad \text{denoted} \qquad z \mapsto \sigma\cdot z. \tag{20.7}$$

Now take direct sum of these maps to obtain the action. Observe that the $\Bbbk\mathrm{S}_n$-module (20.6) is the same as the induced module

$$\bigoplus_{f\colon [n]\to[r],\, \mathsf{d}(f)=\mathsf{d}} \mathbf{q}[n,f] = \Bbbk\mathrm{S}_n \otimes_{\Bbbk\mathrm{S}_{\mathsf{d}}} \mathbf{q}[n_{\mathsf{d}}, f_{\mathsf{d}}], \tag{20.8}$$

with notations as in Remark 14.2.

Now let

$$\overline{\mathcal{K}}_{1_{r,r}}(\mathbf{q}) := \bigoplus_{\mathsf{d}\in\mathbb{N}^r}\Bigg(\bigoplus_{f\colon [n]\to[r],\, \mathsf{d}(f)=\mathsf{d}} \mathbf{q}[n,f]\Bigg)_{\mathrm{S}_n} = \bigoplus_{\mathsf{d}\in\mathbb{N}^r}\big(\mathbf{q}[n_{\mathsf{d}}, f_{\mathsf{d}}]\big)_{\mathrm{S}_{\mathsf{d}}}. \tag{20.9}$$

The functor $\overline{\mathcal{K}}^{\vee}_{1_{r,r}}$ is defined similarly by taking invariants instead of coinvariants. Then, by definition, there are natural transformations

$$\mathcal{K}_{1_{r,r}} \Rightarrow \overline{\mathcal{K}}_{1_{r,r}} \qquad \text{and} \qquad \overline{\mathcal{K}}^{\vee}_{1_{r,r}} \Rightarrow \mathcal{K}^{\vee}_{1_{r,r}}.$$

Further, by arguing as in the proof of Proposition 15.2, one can show that these natural transformations induce bistrong structures on $\overline{\mathcal{K}}_{1_{r,r}}$ and $\overline{\mathcal{K}}^{\vee}_{1_{r,r}}$. It follows that these functors are contragredients of each other.

Now we go to the general case. For $Q$ a log-antisymmetric matrix, the functors $\overline{\mathcal{K}}_Q$ and $\overline{\mathcal{K}}^{\vee}_Q$ are defined as composites of bistrong functors:

$$\overline{\mathcal{K}}_Q(-) := \overline{\mathcal{K}}_{1_{r,r}}\big((-)_Q\big) \qquad \text{and} \qquad \overline{\mathcal{K}}^{\vee}_Q(-) := \overline{\mathcal{K}}^{\vee}_{1_{r,r}}\big((-)_Q\big) \tag{20.10}$$

where $(-)_Q$ is the colored signature functor (see Proposition 14.10). It follows that

$$\overline{\mathcal{K}}^{\vee}_Q = (\overline{\mathcal{K}}_{Q^t})^{\vee}.$$

Let us now understand the functor $\overline{\mathcal{K}}_Q$ in more explicit terms. The same discussion can be carried out for $\overline{\mathcal{K}}^{\vee}_Q$ by replacing coinvariants with invariants. Since the signature functor is defined as the Hadamard product with the colored exponential species $\mathbf{E}_Q$, it follows that

$$\overline{\mathcal{K}}_Q(\mathbf{q}) := \bigoplus_{\mathsf{d}\in\mathbb{N}^r}\Bigg(\bigoplus_{f\colon [n]\to[r],\, \mathsf{d}(f)=\mathsf{d}} \mathbf{q}[n,f]\otimes \mathbf{E}_Q[n,f]\Bigg)_{\mathrm{S}_n}. \tag{20.11}$$

Since $\mathbf{E}_Q$ is one-dimensional on each colored set, there is a canonical isomorphism

$$\mathbf{q}[n,f]\otimes \mathbf{E}_Q[n,f] \to \mathbf{q}[n,f] \qquad x\otimes(1\wedge\cdots\wedge n)\mapsto x.$$

Further, note that

$$\begin{aligned}\sigma\cdot\big(x\otimes(1\wedge\cdots\wedge n)\big) &= \sigma\cdot x\otimes\big(\sigma(1)\wedge\cdots\wedge\sigma(n)\big)\\ &= \mathrm{dist}^{Q}_{f\sigma^{-1}}(\sigma C_{(n)}, C_{(n)})\,\sigma\cdot x\otimes(1\wedge\cdots\wedge n).\end{aligned}$$

An example will serve to clarify this further. Using blue for color 1 and red for color 2,

$$(312)\cdot\big(x\otimes(1\wedge 2\wedge 3)\big) = (312)\cdot x\otimes(3\wedge 1\wedge 2) = q_{21}q_{11}\,(312)\cdot x\otimes(1\wedge 2\wedge 3).$$

The above coefficient can be written alternatively in terms of the weighted multiplicative inversion statistic (2.45) as follows.

$$\mathrm{dist}^{Q}_{f\sigma^{-1}}(\sigma C_{(n)}, C_{(n)}) = \mathrm{dist}^{Q}_{f}(C_{(n)}, \sigma^{-1}C_{(n)}) = \mathrm{inv}^{Q}_{f}(\sigma),$$

The first equality holds by (10.78) and the second equality holds by (10.116). This shows that:

**Proposition 20.3.** *If $Q$ is log-antisymmetric, then*

$$z \mapsto \sigma * z, \qquad \text{where } \sigma * z := \mathrm{inv}_f^Q(\sigma)\, \sigma \cdot z \tag{20.12}$$

*defines an action of* $\mathrm{S}_n$ *on the space* (20.6).

The above action is a twisted version of (20.7). Thus, (20.11) can be rewritten as

$$\overline{\mathcal{K}}_Q(\mathbf{q}) = \bigoplus_{\mathsf{d}\in\mathbb{N}^r} \Bigl( \bigoplus_{f\colon [n]\to[r],\, \mathsf{d}(f)=\mathsf{d}} \mathbf{q}[n,f] \Bigr)_{\mathrm{S}_n},$$

with the understanding that the coinvariants are taken with respect to the twisted action of $\mathrm{S}_n$ given in (20.12).

**20.1.4. Relations to the signature functor and the linear order species.** Recall the signature functor $(-)_Q$ on colored species from Section 14.4.3. It was used in the construction of the colored bosonic-fermionic functors. We now record its relation with the colored full Fock functor. This is a colored generalization of Proposition 16.5 and can be proved in the same manner.

**Proposition 20.4.** *Let $P, Q$, and $R$ be square matrices of the same size and let $Q$ be log-antisymmetric. The following diagram commutes* (*up to isomorphism*) *as bilax monoidal functors.*

$$\begin{tikzcd} (\mathsf{Sp}^{(r)}, \cdot, \beta_P) \arrow[rr, "(-)_Q"] \arrow[dr, "\mathcal{K}_{Q\times R}"'] & & (\mathsf{Sp}^{(r)}, \cdot, \beta_{P\times Q}) \arrow[dl, "\mathcal{K}_R"] \\ & (\mathsf{gVec}^{(r)}, \cdot, \beta_{P\times Q\times R}) & \end{tikzcd} \tag{20.13}$$

*The same result holds with $\mathcal{K}_Q$ replaced by $\mathcal{K}_Q^\vee$.*

We now show how the $Q$-Hopf monoids $\mathbf{L}_Q$ and $\mathbf{L}_Q^*$ of colored linear orders studied in Section 14.5 can be used to construct the colored full Fock functors from the colored bosonic-fermionic functors. This generalizes Propositions 16.6 and 16.22.

**Proposition 20.5.** *Let $P, Q$, and $R$ be square matrices of the same size and let $Q$ be log-antisymmetric. There are isomorphisms of bilax functors*

$$\mathcal{K}_{P\times Q}(-) \cong \overline{\mathcal{K}}_Q\bigl(\mathbf{L}_P \times (-)\bigr) \qquad \textit{and} \qquad \mathcal{K}^\vee_{P\times Q}(-) \cong \overline{\mathcal{K}}^\vee_Q\bigl(\mathbf{L}^*_P \times (-)\bigr)$$

*from* $(\mathsf{Sp}^{(r)}, \cdot, \beta_R)$ *to* $(\mathsf{gVec}^{(r)}, \cdot, \beta_{R\times P\times Q})$.

Proof. We start with the special case when all entries of $Q$ are 1. Given a colored species $\mathbf{p}$, define a map of multigraded vector spaces

$$\mathcal{K}_P(\mathbf{p}) \cong \overline{\mathcal{K}}_{1_{r,r}}(\mathbf{L}_P \times \mathbf{p})$$

with components

$$\mathbf{p}[n,f] \to \Bigl( \bigoplus_{f\colon [n]\to[r]} \mathbf{L}[n,f] \times \mathbf{q}[n,f] \Bigr)_{\mathrm{S}_n}, \qquad x \mapsto \overline{C_{(n)} \otimes x},$$

where $C_{(n)} = 1|\cdots|n$ is the canonical linear order on $[n]$ and the overline denotes the projection to coinvariants. By arguing as in the proofs of Propositions 15.9 and 16.6, one can show that this is an isomorphism of bilax functors.

The general case can be derived from this one as follows.

$$\begin{aligned}\mathcal{K}_{P\times Q}(-) &\cong \overline{\mathcal{K}}_{1_{r,r}}\big(\mathbf{L}_{P\times Q}\times(-)\big) \cong \overline{\mathcal{K}}_{1_{r,r}}\big((\mathbf{L}_P)_Q\times(-)\big)\\ &\cong \overline{\mathcal{K}}_{1_{r,r}}\Big(\big(\mathbf{L}_P\times(-)\big)_Q\Big) \cong \overline{\mathcal{K}}_Q\big(\mathbf{L}_P\times(-)\big)\end{aligned}$$

The second isomorphism follows from (14.24), the third follows from Proposition 14.11, and the last follows from (20.10). □

## 20.2. The colored norm transformation and the anyonic Fock functor

We generalize the $q$-norm transformation of Section 16.2 to the colored setting and study its implications. In particular, we consider the colored version of the anyonic Fock functor introduced in Section 16.3.5. Further, we show that the colored norm transformation is an isomorphism if $Q$ is symmetric and has generic entries.

### 20.2.1. Relating the structure constants of the colored full Fock functors.

Let $\mathrm{Sh}(s,t)$ denote the set of $(s,t)$-shuffle permutations (2.21).

**Lemma 20.6.** *The structure maps $\varphi^{(r)}$ and $(\psi^{(r)}_{Q^t})^\vee$, and $\psi^{(r)}_Q$ and $(\varphi^{(r)})^\vee$, are related by the formulas*

$$(\psi^{(r)}_{Q^t})^\vee(x\otimes y) = \sum_{\zeta\in\mathrm{Sh}\,(s,t)} \mathrm{inv}^Q_f(\zeta^{-1})\,\zeta\big(\varphi^{(r)}(x\otimes y)\big) \tag{20.14}$$

*for $x\in\mathbf{p}[s,f_1]$, $y\in\mathbf{q}[t,f_2]$, and $f$ as in (20.4),*

$$\psi^{(r)}_Q(a\otimes b) = \sum_{\zeta\in\mathrm{Sh}\,(|S|,|T|)} \mathrm{inv}^Q_f(\zeta^{-1})\,(\varphi^{(r)})^\vee\big(\zeta^{-1}(a\otimes b)\big) \tag{20.15}$$

*for $a\in\mathbf{p}[S,g]$, $b\in\mathbf{q}[T,h]$, and $f$ such that its restriction to $S$ is $g$ and to $T$ is $h$.*

The result follows along the lines of the proof of Lemma 15.18 complemented with (2.46).

### 20.2.2. The colored norm transformation.

**Definition 20.7.** For any colored species $\mathbf{p}$, let

$$(\kappa_Q)_{\mathbf{p}}\colon \mathcal{K}_Q(\mathbf{p})\to\mathcal{K}^\vee_Q(\mathbf{p})$$

be the map which on component (20.6) is given by

$$z\mapsto\sum_{\sigma\in\mathrm{S}_n}\mathrm{inv}^Q_f(\sigma)\,\sigma\cdot z,\qquad\text{or equivalently,}\qquad z\mapsto\sum_{\sigma\in\mathrm{S}_n}\sigma * z \tag{20.16}$$

for any $z\in\mathbf{p}[n,f]$, with $\sigma\cdot z$ as in (20.7), $\mathrm{inv}^Q_f$ as in (2.45), and $\sigma * z$ as in (20.12).

This defines a natural transformation $\kappa_Q\colon\mathcal{K}_Q\Rightarrow\mathcal{K}^\vee_Q$ which we call the *$Q$-norm*, or, *colored norm.*

**Proposition 20.8.** *For finite-dimensional colored species, for a symmetric matrix $Q$, the $Q$-norm is self-dual. More generally, we have*

$$(\kappa_Q)^\vee = \kappa_{Q^t}.$$

Proof. We proceed along the lines of the proof of Proposition 16.14. One starts out by writing the norm map on the dual species $\mathbf{p}^*$. Thus

$$(\kappa_Q)_{\mathbf{p}^*} \colon \bigoplus \mathbf{p}^*[n,f] \to \bigoplus \mathbf{p}^*[n,f] \qquad \alpha \mapsto \sum_{\sigma \in \mathrm{S}_n} \mathrm{inv}_f^Q(\sigma)\, \sigma \cdot \alpha$$

for $\alpha \in \mathbf{p}^*[n,f]$. Dualizing this map and identifying $(\mathbf{p}^*)^*$ with $\mathbf{p}$, we obtain:

$$(\kappa_Q)_{\mathbf{p}}^{\vee} \colon \bigoplus \mathbf{p}[n,g] \to \bigoplus \mathbf{p}[n,g] \qquad z \mapsto \sum_{\sigma \in \mathrm{S}_n} \mathrm{inv}_{g\sigma}^Q(\sigma)\, \sigma^{-1} \cdot z$$

for any $z \in \mathbf{p}[n,g]$. The result now follows from (2.47). $\square$

**Proposition 20.9.** *The $Q$-norm is a morphism of bilax monoidal functors*

$$\kappa_Q \colon \mathcal{K}_Q \Rightarrow \mathcal{K}_Q^{\vee}.$$

Proof. We proceed along the lines of the proof of Proposition 16.15. Let $a \otimes b \in \mathbf{p}[S,g] \otimes \mathbf{q}[T,h]$. Using (20.15) we find

$$\big((\kappa_Q)_{\mathbf{p}} \cdot (\kappa_Q)_{\mathbf{q}}\big)(\psi_Q^{(r)})_{\mathbf{p},\mathbf{q}}(a \otimes b)$$
$$= \sum_{\substack{\sigma \in \mathrm{S}_s \\ \tau \in \mathrm{S}_t}} \sum_{\zeta \in \mathrm{Sh}(s,t)} \mathrm{inv}_f^Q(\zeta^{-1})\, \mathrm{inv}_{\overline{g}}^Q(\sigma)\, \mathrm{inv}_{\overline{h}}^Q(\tau)\, (\varphi^{(r)})_{\mathbf{p},\mathbf{q}}^{\vee}\big((\sigma \times \tau) \cdot \zeta^{-1} \cdot (a \otimes b)\big),$$

where $\overline{g}$ and $\overline{h}$ are defined using (2.48)

$$= \sum_{\rho \in \mathrm{S}_n} \mathrm{inv}_f^Q(\rho)\, (\varphi^{(r)})_{\mathbf{p},\mathbf{q}}^{\vee}\big(\rho \cdot (a \otimes b)\big)$$

where $f$ is such that its restriction to $S$ is $g$ and to $T$ is $h$

$$= (\varphi^{(r)})_{\mathbf{p},\mathbf{q}}^{\vee}(\kappa_Q)_{\mathbf{p}\cdot\mathbf{q}}(a \otimes b).$$

The second to last equality requires argument. We need to show that

$$\mathrm{inv}_f^Q(\rho) = \mathrm{inv}_f^Q(\zeta^{-1})\, \mathrm{inv}_{\overline{g}}^Q(\sigma)\, \mathrm{inv}_{\overline{h}}^Q(\tau), \tag{20.17}$$

where $\rho = (\sigma \times \tau)\zeta^{-1}$. This follows from (2.22) and (2.49). Equivalently, in geometric terms, using (10.116), we need to show that

$$\mathrm{dist}_f^Q(C_{(n)}, \rho^{-1}C_{(n)}) = \mathrm{dist}_f^Q(C_{(n)}, \zeta C_{(n)})\, \mathrm{dist}_{\overline{g}}^Q(C_{(s)}, \sigma^{-1}C_{(s)})\, \mathrm{dist}_{\overline{h}}^Q(C_{(t)}, \tau^{-1}C_{(t)}).$$

This follows from (2.22) and (10.117). We emphasize that the gate property is at the root of this identity. To conclude:

$$(\kappa_Q \cdot \kappa_Q)\psi_Q^{(r)} = (\varphi^{(r)})^{\vee}\kappa_Q$$

and $\kappa_Q$ is a morphism of colax functors. The proof can be summarized in the following commutative diagram

$$\begin{tikzcd}
\mathbf{p}[S,g] \otimes \mathbf{q}[T,h] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"] \arrow[d, "{\mathbf{p}[\rho|_S]\otimes\mathbf{q}[\rho|_T]}"'] & \mathbf{p}[s,\overline{g}] \otimes \mathbf{q}[t,\overline{h}] \arrow[d, "{\mathbf{p}[\sigma]\otimes\mathbf{q}[\tau]}"] \\
\mathbf{p}[\rho(S), g\rho^{-1}|_{\rho(S)}] \otimes \mathbf{q}[\rho(T), h\rho^{-1}|_{\rho(T)}] \arrow[r, "{\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]}"'] & \mathbf{p}[s,\overline{g}\sigma^{-1}] \otimes \mathbf{q}[t,\overline{h}\tau^{-1}]
\end{tikzcd}$$

and (20.17), where $\rho = (\sigma \times \tau) \cdot \zeta^{-1}$ and $\zeta$ is the unique $(s,t)$-shuffle permutation which sends $[s]$ to $S$ and $[s+1, s+t]$ to $T$.

The fact that $\kappa_Q$ is a morphism of lax functors can be similarly verified. In the finite-dimensional setting, it follows from Proposition 20.8. $\square$

**20.2.3. The colored anyonic Fock functor.** Proposition 20.9 allows us to introduce a new bilax monoidal functor $\Im_Q$ as the image of $\kappa_Q$. It fits in a commutative diagram:

$$\begin{tikzcd} \mathcal{K}_Q \arrow[rr, Rightarrow, "\kappa_Q"] \arrow[dr, Rightarrow] & & \mathcal{K}_Q^\vee \\ & \Im_Q. \arrow[ur, Rightarrow] & \end{tikzcd}$$

For $Q = [q]$, the above discussion specializes to the one in Section 16.3.5; thus $\Im_Q$ is the colored version of the anyonic Fock functor. It follows from Propositions 3.119 and 20.8 that if $Q$ is symmetric, then $\Im_Q$ is self-dual (regardless of the characteristic). More generally, the dual of the above diagram is the same diagram with $Q$ replaced by $Q^t$.

If $Q$ is log-antisymmetric, then we have an expanded commutative diagram:

$$\begin{tikzcd} \mathcal{K}_Q \arrow[rr, Rightarrow, "\kappa_Q"] \arrow[d, Rightarrow] & & \mathcal{K}_Q^\vee \\ \overline{\mathcal{K}}_Q \arrow[r, Rightarrow] & \Im_Q \arrow[r, Rightarrow] & \overline{\mathcal{K}}_Q^\vee. \arrow[u, Rightarrow] \end{tikzcd}$$

Further, we claim that in characteristic 0,

$$\overline{\mathcal{K}}_Q \cong \Im_Q \cong \overline{\mathcal{K}}_Q^\vee.$$

In this situation, Proposition 20.3 shows that $\sigma * z$ defines an action of the symmetric group; hence, using Lemma 2.20, one may view the image as (co)invariants of this action. The claim follows.

**20.2.4. The generic case for the norm map.** Our goal now is to show that $\kappa_Q$ is an isomorphism if $Q$ is generic. We begin by studying the behavior of $\kappa_Q$ on the colored linear order species.

**Example 20.10.** Consider the colored linear order species $\mathbf{L}_{(r)}$ given by

$$\mathbf{L}_{(r)}[n, f] := \mathbf{L}[n].$$

We will denote an element of this component by $(C, f)$, where $C$ is a linear order on $n$.

The norm map $\kappa_Q$ on $\mathbf{L}_{(r)}$ can be written as

$$(C, g) \mapsto \sum_{\pi \in \mathrm{S}_n} \mathrm{dist}_g^Q(C_{(n)}, \pi^{-1} C_{(n)})\, (\pi C, g\pi^{-1}).$$

An illustrative example, with blue denoting color 1 and red denoting color 2, is given below.

$$3|1|2 \mapsto 3|1|2 + q_{12}\, 3|2|1 + q_{11}\, 2|1|3 + q_{11}q_{12}\, 1|2|3 + q_{12}^2\, 2|3|1 + q_{12}^2 q_{11}\, 1|3|2.$$

Observe that the labels change but the colors stay in the same position. Such a set of elements can be expressed in general as

$$\{(\sigma C_{(n)}, f\sigma^{-1}) \mid \sigma \in \mathrm{S}_n\}, \tag{20.18}$$

where $f\colon [n] \to [r]$ is fixed. In the above example, $(C_{(n)}, f) = 1|2|3$.

The norm map splits as a direct sum of $r^n$ maps, each map defined on the space spanned by the above set. Explicitly, using (10.78), it can be written as

$$(\sigma C_{(n)}, f\sigma^{-1}) \mapsto \sum_{\pi \in \mathrm{S}_n} \operatorname{dist}_f^Q(\sigma^{-1}C_{(n)}, (\pi\sigma)^{-1}C_{(n)})\,(\pi\sigma C_{(n)}, f(\pi\sigma)^{-1}).$$

This is induced by the following bilinear form on the space spanned by (20.18):

$$\big\langle(\sigma C_{(n)}, f\sigma^{-1}), (\pi\sigma C_{(n)}, f(\pi\sigma)^{-1})\big\rangle = \operatorname{dist}_f^Q(\sigma^{-1}C_{(n)}, (\pi\sigma)^{-1}C_{(n)}).$$

After precomposing with the bijection

$$\sigma^{-1}C_{(n)} \mapsto (\sigma C_{(n)}, f\sigma^{-1}),$$

between the set of linear orders on $[n]$ and the set in (20.18), the above bilinear form coincides with the bilinear form on $\mathbf{L}[n]$ given by (10.134). It follows from Lemma 10.33 that $(\kappa_Q)_{\mathbf{L}_{(r)}}$ is an isomorphism if no monomial in the $q_{ij}$'s equals 1.

**Theorem 20.11.** *Assume that for the matrix $Q$, no monomial in the $q_{ij}$'s equals 1 and the field characteristic is 0. Then the $Q$-norm transformation*

$$\kappa_Q\colon \mathcal{K}_Q \Rightarrow \mathcal{K}_Q^\vee$$

*is an isomorphism of bilax monoidal functors.*

Proof. Let $\mathbf{p}$ be a $r$-colored species and let $\mathsf{d} \in \mathbb{N}^r$. Consider the $\mathrm{S}_n$-module given in (20.6). Let us call this $M_{\mathsf{d}}$. We know from (20.8) that

$$M_{\mathsf{d}} = \Bbbk\mathrm{S}_n \otimes_{\Bbbk\mathrm{S}_{\mathsf{d}}} \mathbf{q}[n_{\mathsf{d}}, f_{\mathsf{d}}],$$

which is a representation induced from a parabolic subgroup.

We need to show that the norm map $\kappa_Q$ on $M_{\mathsf{d}}$ is an isomorphism. For this, we show that it is an isomorphism on every irreducible component of $M_{\mathsf{d}}$. Accordingly, let $I$ be an irreducible in $M_{\mathsf{d}}$. Then there is an irreducible, say $I_1 \otimes \cdots \otimes I_r$, in $\mathbf{q}[n_{\mathsf{d}}, f_{\mathsf{d}}]$ whose induction to $\mathrm{S}_n$ contains $I$.

Now, note that $\mathbf{L}_{(r)}[n_{\mathsf{d}}, f_{\mathsf{d}}]$ is the tensor product of the regular representations of the $\mathrm{S}_{d_i}$'s. So $I_1 \otimes \cdots \otimes I_r$ can be viewed as a submodule of this tensor product. It follows by inducing up that $I$ can be viewed as a submodule of

$$\bigoplus_{f\colon [n]\to[r],\, \mathsf{d}(f)=\mathsf{d}} \mathbf{L}_{(r)}[n, f].$$

The norm map $\kappa_Q$ is an isomorphism on this module by the analysis of Example 20.10. The result follows. $\square$

**20.2.5. From colored Hopf monoids to multigraded Hopf algebras.** Using the above discussion along with the proofs of Theorem 15.12, Theorem 15.13, and Corollary 15.22, one can immediately derive the following results.

**Theorem 20.12.** *If $\mathbf{h}$ is a $P$-Hopf monoid, then $\mathcal{K}_Q(\mathbf{h}), \mathcal{K}_Q^\vee(\mathbf{h})$ and $\Im_Q(\mathbf{h})$ are $(P \times Q)$-Hopf algebras. If $\mathbf{h}$ is finite-dimensional, there are natural isomorphisms of $(P \times Q)$-Hopf algebras*

$$\mathcal{K}_Q^\vee(\mathbf{h}) \cong \mathcal{K}_Q(\mathbf{h}^*)^*$$

*given by the canonical identification* $\mathbf{h}[n,f] \cong (\mathbf{h}[n,f]^*)^*$. *In addition, there is a commutative diagram of natural morphisms of* $(P\times Q)$*-Hopf algebras*

$$\begin{tikzcd} \mathcal{K}_Q(\mathbf{h}) \arrow[rr, "(\kappa_Q)_{\mathbf{h}}"] \arrow[dr, two heads] & & \mathcal{K}_Q^\vee(\mathbf{h}) \\ & \Im_Q(\mathbf{h}). \arrow[ur] & \end{tikzcd}$$

If $Q$ is log-antisymmetric, then in addition to all of the above, one can conclude:

**Theorem 20.13.** *Let* $Q$ *be a log-antisymmetric matrix. If* $\mathbf{h}$ *is a* $P$*-Hopf monoid, then* $\overline{\mathcal{K}}_Q(\mathbf{h})$ *and* $\overline{\mathcal{K}}_Q^\vee(\mathbf{h})$ *are* $(P\times Q)$*-Hopf algebras. If* $\mathbf{h}$ *is finite-dimensional, there are natural isomorphisms of* $(P\times Q)$*-Hopf algebras*

$$\overline{\mathcal{K}}_Q^\vee(\mathbf{h}) \cong \overline{\mathcal{K}}_Q(\mathbf{h}^*)^*$$

*given by the canonical identification* $\mathbf{h}[n] \cong (\mathbf{h}[n]^*)^*$. *In addition, there is a commutative diagram of natural morphisms of* $(P\times Q)$*-Hopf algebras*

$$\begin{tikzcd} \mathcal{K}_Q(\mathbf{h}) \arrow[rr, "(\kappa_Q)_{\mathbf{h}}"] \arrow[d, two heads] & & \mathcal{K}_Q^\vee(\mathbf{h}) \\ \overline{\mathcal{K}}_Q(\mathbf{h}) \arrow[r] & \Im_Q(\mathbf{h}) \arrow[r] & \overline{\mathcal{K}}_Q^\vee(\mathbf{h}). \arrow[u, hook] \end{tikzcd}$$

*If the field characteristic is* $0$*, then the maps on the bottom horizontal line are isomorphisms of* $(P\times Q)$*-Hopf algebras.*

If $Q$ is generic, then there is only one object to consider. More precisely, using Theorem 20.11, one can conclude:

**Theorem 20.14.** *Let* $Q$ *be such that no monomial in the* $q_{ij}$*'s equals* $1$*, and let the field characteristic be* $0$*. If* $\mathbf{h}$ *is a* $P$*-Hopf monoid, then*

$$\mathcal{K}_Q(\mathbf{h}) \cong \Im_Q(\mathbf{h}) \cong \mathcal{K}_Q^\vee(\mathbf{h})$$

*as* $(P\times Q)$*-Hopf algebras.*

## 20.3. The colored full Fock functor and commutativity

We now concentrate on the functor $\mathcal{K}_Q$ and study its behavior with respect to commutativity, as is done in Section 16.4 for the functor $\mathcal{K}_q$. In this section, we assume that $P$, $Q$ and $R$ are matrices of size $r$ all of whose entries are nonzero.

**20.3.1. Conjugating the functor $\mathcal{K}_Q$ by the braiding.** We have seen that the functor $\mathcal{K}_q$ is not braided colax in general. However, one can understand the situation by conjugating the colax structure by the braiding. We now do the same for the functor $\mathcal{K}_Q$. We use notations similar to the one-dimensional case. Accordingly, following Definition 16.25, we define four transformations

$$(\varphi^{(r)})^{b(P,R)}, \quad {}^{b(P,R)}(\varphi^{(r)}), \quad (\psi_Q^{(r)})^{b(P,R)} \quad \text{and} \quad {}^{b(P,R)}(\psi_Q^{(r)})$$

as the following composites:

$$\mathcal{K}^{(r)}(\mathbf{p}) \cdot \mathcal{K}^{(r)}(\mathbf{q}) \xrightarrow{\beta_R} \mathcal{K}^{(r)}(\mathbf{q}) \cdot \mathcal{K}^{(r)}(\mathbf{p}) \xrightarrow{\varphi^{(r)}_{\mathbf{q},\mathbf{p}}} \mathcal{K}^{(r)}(\mathbf{q} \cdot \mathbf{p}) \xrightarrow{\mathcal{K}^{(r)}(\beta_P^{-1})} \mathcal{K}^{(r)}(\mathbf{p} \cdot \mathbf{q}),$$

$$\mathcal{K}^{(r)}(\mathbf{p}) \cdot \mathcal{K}^{(r)}(\mathbf{q}) \xrightarrow{\beta_R^{-1}} \mathcal{K}^{(r)}(\mathbf{q}) \cdot \mathcal{K}^{(r)}(\mathbf{p}) \xrightarrow{\varphi^{(r)}_{\mathbf{q},\mathbf{p}}} \mathcal{K}^{(r)}(\mathbf{q} \cdot \mathbf{p}) \xrightarrow{\mathcal{K}^{(r)}(\beta_P)} \mathcal{K}^{(r)}(\mathbf{p} \cdot \mathbf{q}),$$

$$\mathcal{K}^{(r)}(\mathbf{p} \cdot \mathbf{q}) \xrightarrow{\mathcal{K}^{(r)}(\beta_P)} \mathcal{K}^{(r)}(\mathbf{q} \cdot \mathbf{p}) \xrightarrow{(\psi^{(r)}_Q)_{\mathbf{q},\mathbf{p}}} \mathcal{K}^{(r)}(\mathbf{q}) \cdot \mathcal{K}^{(r)}(\mathbf{p}) \xrightarrow{\beta_R^{-1}} \mathcal{K}^{(r)}(\mathbf{p}) \cdot \mathcal{K}^{(r)}(\mathbf{q}),$$

$$\mathcal{K}^{(r)}(\mathbf{p} \cdot \mathbf{q}) \xrightarrow{\mathcal{K}^{(r)}(\beta_P^{-1})} \mathcal{K}^{(r)}(\mathbf{q} \cdot \mathbf{p}) \xrightarrow{(\psi^{(r)}_Q)_{\mathbf{q},\mathbf{p}}} \mathcal{K}^{(r)}(\mathbf{q}) \cdot \mathcal{K}^{(r)}(\mathbf{p}) \xrightarrow{\beta_R} \mathcal{K}^{(r)}(\mathbf{p}) \cdot \mathcal{K}^{(r)}(\mathbf{q}).$$

Let $Q^{-t}$ denote the matrix obtained from $Q$ by taking transpose and inverting all the entries as in Section 2.2.5. Since

$$(\beta_Q)^{-1} = \beta_{Q^{-t}}$$

both for multigraded vector spaces (2.62) and for colored species (14.4), it follows that

$$(\psi^{(r)}_Q)^{b(P,R)} = {}^{b(P^{-t},R^{-t})}(\psi^{(r)}_Q) \qquad \text{and} \qquad (\varphi^{(r)})^{b(R,P)} = {}^{b(R^{-t},P^{-t})}(\varphi^{(r)}).$$

We have the following generalization of Proposition 16.26.

**Proposition 20.15.** *There is an equality*

$$(\mathcal{K}^{(r)}, \varphi^{(r)}, (\psi^{(r)}_Q)^{b(P, P\times Q)}) = (\mathcal{K}^{(r)}, \varphi^{(r)}, \psi^{(r)}_{Q^{-t}})$$

*of bilax monoidal functors*

$$(\mathsf{Sp}, \cdot, \beta_{P^{-t}}) \to (\mathsf{gVec}, \cdot, \beta_{P^{-t}\times Q^{-t}}).$$

*The former is a conjugate of* $\mathcal{K}_Q$, *as in Proposition* 3.16, *and the latter is the functor* $\mathcal{K}_{Q^{-t}}$.

Proof. The proof is similar to that of Proposition 16.26; we provide a geometric version of the previous proof. We need to check that the diagram

$$\begin{tikzcd}
\mathbf{p}[S,g] \otimes \mathbf{q}[T,h] \arrow[rr, "\psi^{(r)}_Q"] & & \mathbf{p}[s,\overline{g}] \otimes \mathbf{q}[t,\overline{h}] \arrow[d, "\beta^{-1}_{P\times Q}"] \\
\mathbf{q}[T,h] \otimes \mathbf{p}[S,g] \arrow[u, "\beta_P"] \arrow[rr, "\psi^{(r)}_{Q^{-t}}"'] & & \mathbf{q}[t,\overline{h}] \otimes \mathbf{p}[s,\overline{g}]
\end{tikzcd}$$

commutes, where $S \sqcup T = [n]$ and $s = |S|$, $t = |T|$, and $g$, $h$, $\overline{g}$ and $\overline{h}$ are related by (2.48).

For this, one essentially has the check that the factors coming from $P$ and $Q$, by following the two directions, match. The factors coming from $P$ clearly cancel, so it drops out of the calculation. We now consider the factors coming from $Q$. Let $f\colon [n] \to [r]$ be the function whose restriction to $S$ is $g$ and whose restriction to $T$ is $h$. We need to show that

$$\operatorname{sch}^Q_n(S,f)\,\operatorname{brd}^{Q^{-t}}_{\mathsf{d}(\overline{g}),\mathsf{d}(\overline{h})} = \operatorname{sch}^{Q^{-t}}_n(T,f).$$

The terms involving the statistic come from the colax structure by using (20.2), while the term involving the braid coefficient comes from the braiding (2.60). The usage of $Q^{-t}$ in the braid coefficient is due to (2.62). By transferring the $Q^{-t}$-terms to the other side (which gets rid of the inverse), one sees that the above identity reduces to (2.40). This completes the proof.

It is worth mentioning that the above identity, using (10.100) and (10.106), is equivalent to

$$\mathrm{sch}^{Q}_{S,T,f}(C_{(n)})\,\mathrm{brd}^{Q^{-t}}_{S,T,f} = \mathrm{sch}^{Q^{-t}}_{T,S,f}(C_{(n)}).$$

This identity follows from (10.109). It is further equivalent to

$$\mathrm{dist}^{Q}_{f}(C_{(n)}, KC_{(n)})\,\mathrm{dist}^{Q^{-t}}_{f}(KC_{(n)}, \overline{K}C_{(n)}) = \mathrm{dist}^{Q^{-t}}_{f}(C_{(n)}, \overline{K}C_{(n)})$$

where $K$ is the vertex $S|T$. This identity follows from (10.110). $\square$

**20.3.2. The colored half-twist transformation.** Define a natural transformation $\theta_Q \colon \mathcal{K}^{(r)} \Rightarrow \mathcal{K}^{(r)}$ with components

$$\mathbf{p}[n,f] \to \mathbf{p}[n, f\omega_n], \qquad x \mapsto \mathrm{inv}^{Q}_{f}(\omega_n)\mathbf{p}[\omega_n](x),$$

where $\omega_n$ is the longest permutation in $\mathrm{S}_n$ and

$$\mathrm{inv}^{Q}_{f}(\omega_n) = \mathrm{dist}^{Q}_{f}(C_{(n)}, \overline{C}_{(n)}) = \prod_{i<j} q_{f(j)f(i)}. \tag{20.19}$$

The left-hand side is the multiplicative inversion statistic, as defined in (2.45), applied to $\omega_n$. The first equality holds since $\omega_n$ applied to $C_{(n)}$ is its opposite chamber $\overline{C}_{(n)}$. Thus,

$$\mathrm{inv}^{Q}_{f}(\omega_n)$$

is the product of the weights of *all* half-spaces which contain $C_{(n)}$.

We call $\theta_Q$ the *colored half-twist transformation*.

**Proposition 20.16.** *The transformation $\theta_Q$ is an isomorphism of bilax monoidal functors*

$$\big(\mathcal{K}^{(r)}, {}^{b(P,P\times Q)}(\varphi^{(r)}), {}^{b(P,P\times Q)}(\psi^{(r)}_{Q})\big) \Rightarrow (\mathcal{K}^{(r)}, \varphi^{(r)}, \psi^{(r)}_{Q}).$$

Proof. We need to generalize the proof of Proposition 16.27. Since the details become quite intricate, we provide full details. We present the arguments in geometric language since it makes the computations more transparent.

For the lax part, the diagram whose commutativity one needs to check is

$$\begin{tikzcd}
\mathbf{p}[s,g]\otimes\mathbf{q}[t,h] \arrow[r, dotted, "\beta^{-1}_{P\times Q}"] \arrow[d, dotted, "(\theta_Q)_{\mathbf{p}}\cdot(\theta_Q)_{\mathbf{q}}"] & \mathbf{q}[t,h]\otimes\mathbf{p}[s,f] \arrow[r, "\varphi^{(r)}"] & \mathbf{q}[t,h]\otimes\mathbf{p}[\tilde{s},\tilde{g}] \arrow[r, "\beta_P"] & \mathbf{p}[\tilde{s},\tilde{g}]\otimes\mathbf{q}[t,h] \arrow[d, dotted, "(\theta_Q)_{\mathbf{p}\cdot\mathbf{q}}"'] \\
\mathbf{p}[s,g\omega_s]\otimes\mathbf{q}[t,h\omega_t] \arrow[rrr, "\varphi^{(r)}"'] & & & \mathbf{p}[s,g\omega_s]\otimes\mathbf{q}[\tilde{t},\widetilde{h\omega_t}]
\end{tikzcd}$$

where $\tilde{s} = [t+1, t+s]$ and $\tilde{t} = [s+1, s+t]$, with $\tilde{g}$ and $\widetilde{h\omega_t}$ are defined by

$$\begin{tikzcd}
{[s]} \arrow[rr, "\mathrm{cano}"] \arrow[dr, "g"'] & & {[t+1,t+s]} \arrow[dl, "\tilde{g}"] \\
& {[r]} &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[t]} \arrow[rr, "\mathrm{cano}"] \arrow[dr, "h\omega_t"'] & & {[s+1,s+t]} \arrow[dl, "\widetilde{h\omega_t}"] \\
& {[r].} &
\end{tikzcd}$$

If we let $P$ and $Q$ be the matrices with all entries 1, then the diagram clearly commutes. So the main step is to check that the factors coming from $P$ and $Q$ by following the two directions match. The matrix $P$ is only involved in two top horizontal arrows and it is clear that these contributions cancel each other. The interesting case is that of the matrix $Q$; it is involved in the three arrows, which are shown dotted. We need to show that

$$\mathrm{brd}^{Q^{-t}}_{\mathsf{d}(g),\mathsf{d}(h)} \mathrm{inv}^Q_f(\omega_n) = \mathrm{inv}^Q_g(\omega_s)\,\mathrm{inv}^Q_h(\omega_t).$$

The first term comes from the braiding (2.60), the second term comes from $(\theta_Q)\mathbf{p}\cdot\mathbf{q}$, while the term in the right-hand side come from $(\theta_Q)\mathbf{p}$ and $(\theta_Q)\mathbf{q}$. By transferring the $Q^{-t}$-term to the right (which gets rid of the inverse), the above is equivalent to

$$\begin{aligned}\mathrm{inv}^Q_f(\omega_n) &= \mathrm{brd}^Q_{\mathsf{d}(h),\mathsf{d}(g)} \mathrm{inv}^Q_g(\omega_s)\,\mathrm{inv}^Q_h(\omega_t),\\ &= \mathrm{brd}^Q_{[t],[t+1,t+s],f} \mathrm{inv}^Q_g(\omega_s)\,\mathrm{inv}^Q_h(\omega_t),\end{aligned}$$

where $f\colon [n]\to[r]$ is the function whose restriction to $[t]$ is $h$ and whose restriction to $[t+1,t+s]$ is $\tilde{g}$. The first equality follows from (2.37), while the second follows from (10.106). The above is a higher dimensional generalization of the identity

$$\binom{n}{2} = st + \binom{s}{2} + \binom{t}{2}.$$

We give a geometric proof. Let $F$ be the vertex of $C_{(n)}$ defined by

$$F := [t]|[t+1,t+s].$$

Then by using (20.19), (10.107), and the compatibility of the gallery metric with joins (10.82), we are reduced to showing that

$$\mathrm{dist}^Q_f(C_{(n)},\overline{C}_{(n)}) = \mathrm{dist}^Q_f(C,\overline{F}C)\,\mathrm{dist}^Q_f(\overline{F}C_{(n)},\overline{C}_{(n)}).$$

The first term on the right is independent of which chamber $C$ is chosen. By choosing $C = C_{(n)}$ and noting from Proposition 10.4 that there is a minimum gallery

$$C_{(n)} - \overline{F}C_{(n)} - \overline{C}_{(n)},$$

the identity follows from (10.79).

For the colax part, the diagram whose commutativity one needs to check is

$$\begin{tikzcd}
\mathbf{p}[S,g]\otimes\mathbf{q}[T,h] \arrow[r, "\beta_P^{-1}"] \arrow[d, dotted, "(\theta_Q)_{\mathbf{p}\cdot\mathbf{q}}"] & \mathbf{q}[T,h]\otimes\mathbf{p}[S,g] \arrow[r, dotted, "\psi_Q^{(r)}"] & \mathbf{q}[t,\overline{h}]\otimes\mathbf{p}[s,\overline{g}] \arrow[r, dotted, "\beta_{P\times Q}"] & \mathbf{p}[s,\overline{g}]\otimes\mathbf{q}[t,\overline{h}] \arrow[d, dotted, "(\theta_Q)_{\mathbf{p}}\cdot(\theta_Q)_{\mathbf{q}}"'] \\
\mathbf{p}[\omega_n(S),g\omega_n]\otimes\mathbf{q}[\omega_n(T),h\omega_n] \arrow[rrr, dotted, "\psi_Q^{(r)}"'] & & & \mathbf{p}[s,\overline{g}\omega_s]\otimes\mathbf{q}[t,\overline{h}\omega_t],
\end{tikzcd}$$

where $S\sqcup T = [n]$ and $s = |S|$, $t = |T|$, and where $g$, $h$, $\overline{g}$ and $\overline{h}$ are related by (2.48).

As in the lax case, the interesting part is to check that the factors coming from $Q$ by following the two directions match. The arrows that contribute are shown dotted. We need to show that

$$\mathrm{sch}^Q_n(T,f)\,\mathrm{brd}^Q_{\mathsf{d}(h),\mathsf{d}(g)} \mathrm{inv}^Q_{\overline{g}}(\omega_s)\,\mathrm{inv}^Q_{\overline{h}}(\omega_t) = \mathrm{inv}^Q_f(\omega_n)\,\mathrm{sch}^Q_n(\omega_n(S),f\omega_n),$$

where $f\colon [n] \to [r]$ is the function whose restriction to $S$ is $g$ and whose restriction to $T$ is $h$.

In the left-hand side, the first term comes from $\psi_Q^{(r)}$ using (20.2), the second comes from the braiding using (2.60), and the last two come from $(\theta_Q)_{\mathbf{p}}$ and $(\theta_Q)_{\mathbf{q}}$. Similarly, in the right-hand side, the first term comes from $(\theta_Q)_{\mathbf{p}\cdot\mathbf{q}}$ and the second comes from $\psi_Q^{(r)}$.

The complication in the above identity comes from the fact that it cannot be easily split into two. For example, the terms involving the Schubert statistic do not cancel: (2.41) does not apply because we are not assuming $Q$ to be symmetric. This complication was not present in the one-dimensional case.

The above identity can be rewritten in geometric language as below. The translations are made as in the lax case, with $K$ being the vertex $S|T$. The two terms involving the inversion statistic have been combined into one.

$$\begin{aligned}\operatorname{dist}_f^Q(C_{(n)}, \overline{K}C_{(n)}) \operatorname{dist}_f^Q(\overline{K}C, KC) \operatorname{dist}_f^Q(\overline{K}C_{(n)}, \overline{K}\overline{C}_{(n)}) \\ = \operatorname{dist}_f^Q(C_{(n)}, \overline{C}_{(n)}) \operatorname{dist}_{f\omega_n}^Q(C_{(n)}, \omega_n(K)C_{(n)}),\end{aligned}$$

where $C$ is any chamber. Let us prove this. There are no direct cancellations as mentioned above; so one has to cancel in steps.

Let us first rewrite the last term on the right using (10.78) as

$$\begin{aligned}\operatorname{dist}_{f\omega_n}^Q(C_{(n)}, \omega_n(K)C_{(n)}) &= \operatorname{dist}_{f\omega_n}^Q\big(\omega_n(\overline{C}_{(n)}), \omega_n(KC_{(n)})\big) \\ &= \operatorname{dist}_f^Q(\overline{C}_{(n)}, K\overline{C}_{(n)}).\end{aligned}$$

As a first step, using the minimum gallery

$$C_{(n)} - \overline{K}C_{(n)} - \overline{K}\overline{C}_{(n)} - \overline{C}_{(n)}$$

(existence follows from Proposition 10.4) and applying (10.79), we cancel off the first and third terms in the left-hand side from the first term in the right-hand side. So the identity to be proved reduces to

$$\operatorname{dist}_f^Q(\overline{K}C, KC) = \operatorname{dist}_f^Q(\overline{K}\overline{C}_{(n)}, \overline{C}_{(n)}) \operatorname{dist}_f^Q(\overline{C}_{(n)}, K\overline{C}_{(n)})$$

In the left-hand side, we put $C = \overline{C}_{(n)}$ and observe that there is a minimum gallery

$$\overline{K}\overline{C}_{(n)} - \overline{C}_{(n)} - K\overline{C}_{(n)}.$$

The identity follows from (10.79). $\square$

**20.3.3. Commutativity of the Hopf monoids obtained by evaluating $\mathcal{K}_Q$.** We now state the consequences for Hopf monoids of the properties of the functor $\mathcal{K}_Q$ regarding commutativity. They will be expressed in terms of the op and cop constructions of Section 1.2.9. These are the colored analogues of Corollaries 16.28 and 16.29. They can be proved in the same manner using Propositions 20.15 and 20.16. Recall that in this context $P$ and $Q$ are matrices with nonzero entries.

**Corollary 20.17.** *For any comonoid (Hopf monoid) $\mathbf{h}$ in $(\mathsf{Sp}^{(r)}, \cdot, \beta_P)$,*

$$\mathcal{K}_{Q^{-t}}(\mathbf{h}^{\mathrm{cop}}) = \mathcal{K}_Q(\mathbf{h})^{\mathrm{cop}}$$

*as comonoids (Hopf monoids) in $(\mathsf{gVec}^{(r)}, \cdot, \beta_{P^{-t}\times Q^{-t}})$.*

**Corollary 20.18.** *For any $P$-Hopf monoid $\mathbf{h}$, the map*

$$\mathcal{K}_Q(\mathbf{h}^{\mathrm{op,cop}}) \to \mathcal{K}_Q(\mathbf{h})^{\mathrm{op,cop}}$$

*whose degree $(n, f)$ component is $\mathrm{inv}_f^Q(\omega_n)\mathbf{h}[\omega_n]$ is a natural isomorphism of $(P \times Q)$-Hopf algebras.*

## 20.4. Colors and decorations

The colored Fock functors $\mathcal{K}_Q$, $\mathcal{K}_Q^\vee$ and $\Im_Q$ studied in this chapter are closely related to the deformed decorated Fock functors $\mathcal{K}_{V,R_Q}$, $\mathcal{K}_{V,R_Q}^\vee$ and $\Im_{V,R_Q}$ of Example 19.46. In this section, we explain the connection between the two situations.

**20.4.1. From multigraded vector spaces to graded vector spaces.** An $\mathbb{N}^r$-graded vector space has an underlying $\mathbb{N}$-grading for which an element of multidegree $(d^1, \dots, d^r)$ has degree $d^1 + \cdots + d^r$. More formally, there is a functor

$$\mathcal{F}\colon \mathsf{gVec}^{(r)} \to \mathsf{gVec}.$$

This functor is strong with respect to the Cauchy product. Further, viewed as a functor

$$(\mathsf{gVec}^{(r)}, \cdot, \beta_{q1_{r,r}}) \to (\mathsf{gVec}, \cdot, \beta_q),$$

it is braided.

It is natural to ask what happens if one uses the braiding $\beta_Q$ on multigraded vector spaces. A satisfactory answer can be given using the notion of a bilax functor with a Yang–Baxter operator (Section 19.9.1). More precisely, for multigraded vector spaces $V$ and $W$, define the structure map $\nu$ of (19.35) by

$$\begin{tikzcd}
\mathcal{F}(V) \cdot \mathcal{F}(W) \arrow[r, dashed, "\nu"] \arrow[d, equal] & \mathcal{F}(W) \cdot \mathcal{F}(V) \arrow[d, equal] \\
\mathcal{F}(V \cdot W) \arrow[r, "\mathcal{F}(\beta_Q)"'] & \mathcal{F}(W \cdot V)
\end{tikzcd}$$

where the vertical identifications use the fact that $\mathcal{F}$ is strong. Then

$$(\mathcal{F}, \nu)\colon (\mathsf{gVec}^{(r)}, \cdot, \beta_Q) \to (\mathsf{gVec}, \cdot)$$

is bilax in the sense of Section 19.9.1.

**20.4.2. Colors and decorations.** Let $V$ be a vector space with dimension $r$. Fix a basis of $V$, say $x_1, x_2, \dots, x_r$. The main observation is the following. The diagram

$$\begin{tikzcd}
\mathsf{Sp} \arrow[r, "\mathcal{K}_{V,R_Q}"] \arrow[d, "(-)_{(r)}"'] & \mathsf{gVec} \\
\mathsf{Sp}^{(r)} \arrow[r, "\mathcal{K}_Q"'] & \mathsf{gVec}^{(r)} \arrow[u, "\mathcal{F}"']
\end{tikzcd} \tag{20.20}$$

commutes up to isomorphism via the identification

$$\bigoplus_{f\colon [n]\to[r]} \mathbf{p}[n] \longleftrightarrow \mathbf{p}[n] \otimes V^{\otimes n}$$

which sends $z \in \mathbf{p}[n]$ in the component $f\colon [n] \to [r]$ to

$$z \otimes x_{f(1)} \otimes x_{f(2)} \otimes \cdots \otimes x_{f(n)}.$$

It is straightforward to check that:

**Theorem 20.19.** *The following diagram of bilax functors with Yang–Baxter operators*

$$\begin{tikzcd}
(\mathsf{Sp}, \cdot, \beta) \arrow[r, "\mathcal{K}_{V,R_Q}"] \arrow[d, "(-)_{(r)}"'] & (\mathsf{gVec}, \cdot) \\
(\mathsf{Sp}^{(r)}, \cdot, \beta_{1_{r,r}}) \arrow[r, "\mathcal{K}_Q"'] & (\mathsf{gVec}^{(r)}, \cdot, \beta_Q) \arrow[u, "(\mathcal{F},\nu)"']
\end{tikzcd} \tag{20.21}$$

*commutes up to isomorphism.*

The result holds with $\mathcal{K}_{V,R_Q}$ and $\mathcal{K}_Q$ replaced by $\mathcal{K}^\vee_{V,R_Q}$ and $\mathcal{K}^\vee_Q$. Further, these isomorphisms are compatible with the norm transformations, so the result also holds with $\mathcal{K}_{V,R_Q}$ and $\mathcal{K}_Q$ replaced by $\Im_{V,R_Q}$ and $\Im_Q$.

**Corollary 20.20.** *Let* $\mathbf{p}$ *be a bimonoid in species. Then*

$$\mathcal{K}_{V,R_Q}(\mathbf{p}) \cong \mathcal{K}_Q(\mathbf{p}_{(r)}), \quad \mathcal{K}^\vee_{V,R_Q}(\mathbf{p}) \cong \mathcal{K}^\vee_Q(\mathbf{p}_{(r)}), \quad \Im_{V,R_Q}(\mathbf{p}) \cong \Im_Q(\mathbf{p}_{(r)})$$

*as braided bialgebras.*

The objects on the right are multigraded vector spaces; one views them as graded vector spaces using the functor $\mathcal{F}$ which has been suppressed in the result.

## 20.5. Quantum objects

We now apply the results of the preceding sections to some of the simplest colored Hopf monoids. They will lead to some well-known $Q$-Hopf algebras, such as the quantum linear space, or more generally, the quantum symmetric algebra. The entries of the matrix $Q$ enter into the definitions of the product and coproduct of these objects making them both noncommutative and noncocommutative for generic $Q$. This is one justification for the quantum terminology. The quantum objects may be viewed as deformations of classical objects. The latter are either commutative or cocommutative and correspond to those $Q$ whose entries are more special such as 1, $-1$ or 0.

**Example 20.21.** We start from the simplest nontrivial Hopf monoid, the exponential species $\mathbf{E}$. Its Hopf monoid structure has been discussed throughout Chapter 8. When applied to $\mathbf{E}$, the bistrong functors of Proposition 14.4 yield the same result:

$$\mathbf{E}_{(r)} = \mathbf{E}^{(r)}.$$

This was discussed in Section 14.3. Let us write $\mathbf{E}_{(r)}$ from now on. We now apply the colored Fock functors to this object. In view of Corollary 20.20, this is equivalent to applying the deformed decorated Fock functors of Example 19.46 to $\mathbf{E}$. As a result, the present discussion parallels that in Example 19.47.

We have that

$$\mathcal{K}_Q(\mathbf{E}_{(r)}) \xrightarrow{\cong} \Bbbk\langle x_1, \dots, x_r\rangle,$$

the free algebra on $r$ generators of Example 2.14, under

$$*_{([n],f)} \leftrightarrow x_f = x_{f(1)} \cdots x_{f(n)}.$$

Corollary 20.17 says that

$$\mathcal{K}_Q(\mathbf{E}_{(r)})^{\mathrm{cop}} = \mathcal{K}_{Q^{-t}}(\mathbf{E}_{(r)})$$

and Corollary 20.18 says that the map

$$(\theta_Q)_{\mathbf{E}_{(r)}} \colon \mathcal{K}_Q(\mathbf{E}_{(r)}) \to \mathcal{K}_Q(\mathbf{E}_{(r)})^{\mathrm{op,cop}}, \qquad x_f \mapsto \Big(\prod_{i<j} q_{f(j)f(i)}\Big) x_{f\omega_n}$$

is an isomorphism of $Q$-Hopf algebras.

Similarly, $\mathcal{K}_Q^\vee(\mathbf{E}_{(r)})$ is the space $\Bbbk\langle x_1, \dots, x_r\rangle$ equipped with a deformation of the shuffle product and with the usual deconcatenation coproduct. This is the same as the quantum shuffle algebra as defined by Green [152] and Rosso [316, Proposition 9].

The $Q$-norm transformation is the map

$$(\kappa_Q)_{\mathbf{E}_{(r)}} \colon \mathcal{K}_Q(\mathbf{E}_{(r)}) \to \mathcal{K}_Q^\vee(\mathbf{E}_{(r)})$$
$$x_f \mapsto \sum_{\sigma\in \mathrm{S}_n} \Big(\prod_{(i,j)\in \mathrm{Inv}(\sigma)} q_{f(j)f(i)}\Big) x_{f\sigma^{-1}}.$$

Now let us consider the functor $\mathfrak{S}_Q$. The object $\mathfrak{S}_Q(\mathbf{E}_{(r)})$ is Rosso's quantum symmetric algebra, also called the Nichols algebra of diagonal type. We now consider some important special cases in which $\mathfrak{S}_Q(\mathbf{E}_{(r)})$ can be explicitly described.

- If $Q$ is such that no monomial in the $q_{ij}$'s equals 1, then Theorem 20.14 implies that $\mathfrak{S}_Q(\mathbf{E}_{(r)})$ is the free algebra on $r$ generators. For related work, see [268, Theorem 1.9.2].
- Let $Q$ be a log-antisymmetric matrix. In characteristic zero [316, Example 1, p. 409], we have

  $$\mathfrak{S}_Q(\mathbf{E}_{(r)}) = \overline{\mathcal{K}}_Q(\mathbf{E}_{(r)}) \xrightarrow{\cong} \Bbbk\langle x_1, \dots, x_r\rangle/(x_i x_j - q_{ji} x_j x_i),$$

  which is the quantum linear space of Example 2.15. Using (20.11) for the left-hand side, it is straightforward to see that the map

  $$i_1 \wedge \cdots \wedge i_n \mapsto x_{f(i_1)} \dots x_{f(i_n)}$$

  is an isomorphism. As an explicit example, with blue denoting color 1 and red denoting color 2,

  $$2 \wedge 3 \wedge 4 \wedge 1 \mapsto x_2 x_1 x_2 x_1.$$

  This situation can be generalized slightly: Let $Q$ satisfy $q_{ij}q_{ji} = 1$ for $i \neq j$. Then the Nichols algebra (still assuming characteristic zero) is given by the relations:

  $$x_i x_j = q_{ij} x_j x_i, \qquad x_i^{N_i} = 0,$$

  where $N_i$ is the order of $q_{ii}$ if $q_{ii}$ is a root of unity different from 1. The reason for the latter relation is the same as in the one-dimensional case (16.10). In characteristic $p$, there is an additional relation $x_i^p = 0$ whenever $q_{ii} = 1$.

  We thank Matías Graña for this input.

- Let $Q$ be of the form (2.33), $q$ is not a root of unity, and $A$ is the symmetrization of a symmetrizable generalized Cartan matrix $C$. Then a theorem of Lusztig [246, Corollary 33.1.5] and of Rosso [316, Theorem 15.1] states that
$$\Im_Q(\mathbf{E}_{(r)}) = U_q^+(C),$$
the nilpotent part of the quantum enveloping algebra associated with $C$.

  In general, this will be a proper quotient of the free algebra on $r$ generators. This does not contradict Theorem 20.14 because, in general, the matrix $A$ will contain both positive and negative entries and hence there will be monomials in the $q^{a_{ij}}$'s which equal 1.

**Example 20.22.** Now we briefly consider the colored linear order species of Section 14.5. Let $\mathbf{L}_{(r)}$ and $\mathbf{L}^*_{(r)}$ denote the image of $\mathbf{L}$ and $\mathbf{L}^*$ under one of the bistrong functors of Proposition 14.4. They are the same as the $Q$-Hopf monoids $\mathbf{L}_{1_{r,r}}$ and $\mathbf{L}^*_{1_{r,r}}$. Now using Proposition 20.5,
$$\mathcal{K}_Q(\mathbf{L}^*_{(r)}) = \overline{\mathcal{K}}_{1_{r,r}}(\mathbf{L}^*_{(r)} \times \mathbf{L}_Q) = \overline{\mathcal{K}}_{1_{r,r}}(\mathbb{L}_Q),$$
where $\mathbb{L}_Q$ is the colored Hopf monoid based on pairs of linear orders (Section 14.7.1). The above object is a $Q$-Hopf algebra indexed by $r$-signed permutations. It is a higher-dimensional analogue of the Hopf algebra of permutations and its deformed version which we had obtained by similar means in Example 16.32. For $Q = 1_{r,r}$, this appears in the work of Baumann and Hohlweg [35].

Now let $Q$ be a log-antisymmetric matrix. Then again using Proposition 20.5,
$$\overline{\mathcal{K}}_Q(\mathbf{L}_{(r)}) = \mathcal{K}_Q(\mathbf{E}_{(r)}).$$
This is the free algebra on $r$ generators which we saw in Example 20.21.

We have arrived at very interesting objects starting simply from the Hopf monoids $\mathbf{E}$ and $\mathbf{L}$ in species. One is compelled to wonder what Hopf algebras might arise when starting from the other Hopf monoids in species discussed in this monograph, or when working directly with Hopf monoids in colored species that need not come from Hopf monoids in species. The theory presented in this monograph provides a uniform way for constructing these Hopf algebras and for understanding their interrelationships. On the other hand, as the above examples suggest, describing each of them in explicit terms should be a difficult problem.

# Appendices

APPENDIX A

# Categorical Preliminaries

We review some basic notions from category theory, including products, adjunctions, equivalences, colimits, Kan extensions and comma categories.

## A.1. Products, coproducts and biproducts

The notions reviewed here can also be found in [250, Sections III.3–5 and VIII.2].

**A.1.1. Products.** Let $A$ and $B$ be objects of a category $\mathsf{C}$. An object $A \times B$ along with arrows

$$A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$$

is said to be a *product* of $A$ and $B$, if it satisfies the following property: given arrows

$$A \xleftarrow{f} C \xrightarrow{g} B,$$

there exists a unique arrow $(f, g) : C \to A \times B$ for which

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "{(f,g)}"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

commutes. If the product exists, it is unique up to isomorphism.

The product of an arbitrary family of objects is defined similarly. The product of an empty family is a *terminal object*: an object $J$ with a unique arrow from any other object in the category.

Suppose that every pair of objects in $\mathsf{C}$ has a product. In this situation, choosing a product $A \times B$ for each pair $(A, B)$ yields a functor

$$\times \colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}.$$

Given morphisms $f : A \to A'$ and $g : B \to B'$, we set

$$f \times g := (f\pi_A,\, g\pi_B) \colon A \times B \to A' \times B'.$$

If in addition $\mathsf{C}$ has a terminal object, then it has all finite products [250, Proposition III.5.1]. They are obtained from iterations of the functor $\times$.

**A.1.2. Coproducts.** Dually, a *coproduct* of two objects $A$ and $B$ is an object $A \amalg B$ with arrows

$$A \xrightarrow{\iota_A} A \amalg B \xleftarrow{\iota_B} B$$

such that given any arrows

$$A \xrightarrow{f} C \xleftarrow{g} B,$$

there is a unique arrow $\begin{pmatrix} f \\ g \end{pmatrix} : A \amalg B \to C$ for which

$$\begin{tikzcd} A \arrow[d, "\iota_A"'] \arrow[dr, "f"] & \\ A \amalg B \arrow[r, "\binom{f}{g}"] & C \\ B \arrow[u, "\iota_B"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

An *initial* object $I$ has a unique arrow to any other object in the category.

**A.1.3. Biproducts.** Suppose that $\mathsf{C}$ has all finite products and finite coproducts. In this case, given arrows

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "p"] & \\ C & & D \\ & B \arrow[ul, "g"] \arrow[ur, "q"'] & \end{tikzcd}$$

the following arrows $A \amalg B \to C \times D$ coincide:

$$\left( \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right) \quad \text{and} \quad \begin{pmatrix} (f, p) \\ (g, q) \end{pmatrix}.$$

We denote this arrow by the symbol

$$\begin{pmatrix} f & p \\ g & q \end{pmatrix} : A \amalg B \to C \times D.$$

Note that, contrary to standard matrix notation, $A$ and $B$ index the rows, while $C$ and $D$ index the columns.

If the unique arrow

$$I \to J$$

from the initial to the final object is invertible, so that $I$ and $J$ are isomorphic, we say that either one is a *zero object* in $\mathsf{C}$.

Suppose this is the case. Then, for any object $A$, we may consider the composite

$$A \to J \to I \to B.$$

We call it the *zero arrow* from $A$ to $B$ and denote it by $0_{A,B}$. We employ these arrows to define a map

$$\begin{pmatrix} \mathrm{id}_A & 0_{A,B} \\ 0_{B,A} & \mathrm{id}_B \end{pmatrix} : A \amalg B \to A \times B. \tag{A.1}$$

If this arrow is invertible, we say that either $A \amalg B$ or $A \times B$ is a *biproduct* of $A$ and $B$.

If all arrows $A \amalg B \to A \times B$ of this form are invertible, we say that $\mathsf{C}$ is a category with finite biproducts. We write

$$A \oplus B \text{ instead of } A \amalg B \text{ or } A \times B,$$

and

$$Z \text{ instead of } I \text{ or } J.$$

Categories with finite biproducts are sometimes also called *semi-additive*. An example is provided by the category of vector spaces, where $A \oplus B$ is direct sum.

**Remark A.1.** Suppose $\mathsf{C}$ is a category with finite products and finite coproducts, and there is a family of isomorphisms

$$A \amalg B \cong A \times B$$

natural in $A$ and $B$. A recent result of Lack [217, Theorem 5] states that then $\mathsf{C}$ has finite biproducts. In other words, the above assumption implies $I \cong J$ and the invertibility of the canonical map (A.1).

## A.2. Adjunction and equivalence

We review the notions of adjunction between functors, and equivalence between categories.

**A.2.1. Adjunctions.** Various equivalent formulations of the notion of adjunction are discussed in Mac Lane's book [250, Section IV.1, Theorem 2]. We briefly discuss two of these.

**Definition A.2.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ be a pair of functors. We say that $\mathcal{F}$ is a *left adjoint* to $\mathcal{G}$ or that $\mathcal{G}$ is a *right adjoint* to $\mathcal{F}$ if there exists a natural isomorphism

$$\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{C}}\big(-, \mathcal{G}(-)\big)$$

of functors $\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$.

In other words, the functors are adjoint if there exists a bijection

$$\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), X) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{C}}\big(A, \mathcal{G}(X)\big) \tag{A.2}$$

which is natural in $A$ and $X$.

We use the notation

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

to indicate that the functor on the top is left adjoint to the functor on the bottom.

**Proposition A.3.** *Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ be functors. Then $\mathcal{F}$ is left adjoint to $\mathcal{G}$ (or $\mathcal{G}$ is a right adjoint to $\mathcal{F}$) if and only if there exist natural transformations $\eta\colon \mathrm{id} \Rightarrow \mathcal{G}\mathcal{F}$ and $\xi\colon \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}$ such that the following diagrams commute.*

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(\eta_A)"] \arrow[dr, "\mathrm{id}"'] & \mathcal{F}\mathcal{G}\mathcal{F}(A) \arrow[d, "\xi_{\mathcal{F}(A)}"] \\ & \mathcal{F}(A) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}\mathcal{F}\mathcal{G}(X) \arrow[d, "\mathcal{G}(\xi_X)"'] & \mathcal{G}(X) \arrow[l, "\eta_{\mathcal{G}(X)}"'] \arrow[dl, "\mathrm{id}"] \\ \mathcal{G}(X) \end{tikzcd} \tag{A.3}$$

The tuple $(\mathcal{F}, \mathcal{G}, \eta, \xi)$ is called an *adjunction*. The transformations $\eta$ and $\xi$ are the *unit* and *counit* of the adjunction respectively.

Proof. Starting with (A.2), one sets $X = \mathcal{F}(A)$ and defines $\eta_A$ as the morphism corresponding to the identity. Setting $A = \mathcal{G}(X)$ one obtains $\xi_X$. Conversely, given $\eta$ and $\xi$, one defines (A.2) by sending $f\colon \mathcal{F}(A) \to X$ to the composite

$$A \xrightarrow{\eta_A} \mathcal{G}\mathcal{F}(A) \xrightarrow{\mathcal{G}(f)} \mathcal{G}(X). \tag{A.4}$$

The inverse correspondence sends $g\colon A \to \mathcal{G}(X)$ to the composite

$$\mathcal{F}(A) \xrightarrow{\mathcal{F}(g)} \mathcal{F}\mathcal{G}(X) \xrightarrow{\xi_X} X. \tag{A.5}$$

□

**Definition A.4.** A subcategory $\mathsf{C}'$ of $\mathsf{C}$ is said to be *full* if

$$\mathrm{Hom}_{\mathsf{C}'}(A,B) = \mathrm{Hom}_{\mathsf{C}}(A,B)$$

for any objects $A$, $B$ of $\mathsf{C}'$; *isomorphism-dense* if for any object $C$ of $\mathsf{C}$ there is an object $C'$ of $\mathsf{C}'$ and an isomorphism $C \cong C'$ in $\mathsf{C}$; a *skeleton* of $\mathsf{C}$ if it is full, isomorphism-dense, and no two distinct objects of $\mathsf{C}'$ are isomorphic.

The definitions imply:

**Proposition A.5.** *Let $(\mathcal{F}, \mathcal{G}, \eta, \xi)$ be an adjunction between $\mathsf{C}$ and $\mathsf{D}$. Let $\mathsf{D}'$ be a full subcategory of $\mathsf{D}$ such that the image of $\mathcal{F}$ lies in $\mathsf{D}'$. Then $(\mathcal{F}|^{\mathsf{D}'}, \mathcal{G}|_{\mathsf{D}'}, \eta', \xi')$ is an adjunction between $\mathsf{C}$ and $\mathsf{D}'$.*

Here, $\mathcal{F}|^{\mathsf{D}'}$ and $\mathcal{G}|_{\mathsf{D}'}$ are the functors such that

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] \arrow[dr, dashed, "\mathcal{F}|^{\mathsf{D}'}"'] & \mathsf{D} \\ & \mathsf{D}' \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D} \arrow[r, "\mathcal{G}"] & \mathsf{C}, \\ \mathsf{D}' \arrow[u, hook] \arrow[ur, dashed, "\mathcal{G}|_{\mathsf{D}'}"'] & \end{tikzcd}$$

where $\mathsf{D}' \hookrightarrow \mathsf{D}$ is the inclusion functor. Similarly for $\eta'$ and $\xi'$.

A similar statement holds if we have a full subcategory $\mathsf{C}'$ of $\mathsf{C}$.

### A.2.2. Equivalences.

**Definition A.6.** An *equivalence* of categories $\mathsf{C}$ and $\mathsf{D}$ is a tuple $(\mathcal{F}, \mathcal{G}, \eta, \xi)$, where

$$\mathcal{F}\colon \mathsf{C} \to \mathsf{D} \quad \text{and} \quad \mathcal{G}\colon \mathsf{D} \to \mathsf{C}$$

are functors, and

$$\eta\colon \mathrm{id} \Rightarrow \mathcal{G}\mathcal{F} \quad \text{and} \quad \xi\colon \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}$$

and natural isomorphisms.

An *adjoint equivalence* is an adjunction that is also an equivalence.

If $(\mathcal{F}, \mathcal{G}, \eta, \xi)$ is an adjoint equivalence, then so is $(\mathcal{G}, \mathcal{F}, \xi^{-1}, \eta^{-1})$. Not every equivalence is an adjoint equivalence. However, given an equivalence $(\mathcal{F}, \mathcal{G}, \eta, \xi)$, there is always an adjoint equivalence of the form $(\mathcal{F}, \mathcal{G}, \eta', \xi')$. In fact, one may always choose $\eta' = \eta$ or $\xi' = \xi$ (but not both at the same time); see the comments following the proof of [58, Proposition 3.4.3].

**Proposition A.7.** *A subcategory $\mathsf{C}'$ of $\mathsf{C}$ is full and isomorphism-dense if and only if the inclusion functor $\mathsf{C}' \to \mathsf{C}$ is an equivalence.*

Proof. This follows from [250, Theorem IV.4.1]. □

In this monograph, we encounter adjunctions, equivalences, and adjoint equivalences not only between categories but also between monoidal categories and other structured categories. Bicategories and 2-categories, discussed in Section C.1, provide a general context for these notions. Various types of adjunctions between monoidal categories are discussed in detail in Section 3.9.

## A.3. Colimits of functors

We review the concept of colimit. More information can be found in the books by Borceux [58, Chapter 2] or Mac Lane [250, Chapters III and V].

### A.3.1. Cones and colimits.

**Definition A.8.** Let $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ be a functor. Consider an object $V$ in $\mathsf{C}$ equipped with morphisms $\tau_Y : \mathcal{F}(Y) \to V$, one for each object $Y$ in $\mathsf{D}$, and such that for each morphism $f\colon Y \to Z$ in $\mathsf{D}$ the following diagram commutes.

$$\begin{tikzcd} & V & \\ \mathcal{F}(Y) \arrow[ur, "\tau_Y"] \arrow[rr, bend right, "\mathcal{F}(f)"'] & & \mathcal{F}(Z) \arrow[ul, "\tau_Z"'] \end{tikzcd}$$

Such a structure is called a *cone* from the base $\mathcal{F}$ to the vertex $V$.

**Definition A.9.** The colimit of a functor $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ is an object of $\mathsf{C}$, denoted

$$\operatorname{colim} \mathcal{F} \quad \text{or} \quad \operatorname*{colim}_X \mathcal{F}(X),$$

together with morphisms $\iota_Y : \mathcal{F}(Y) \to \operatorname{colim} \mathcal{F}$ for each object $Y$ in $\mathsf{D}$, satisfying the following properties.

- The maps $\iota_Y$ form a cone from the base $\mathcal{F}$ to the vertex $\operatorname{colim} \mathcal{F}$. In other words,
$$\iota_Z \mathcal{F}(f) = \iota_Y$$
for each morphism $f\colon Y \to Z$ in $\mathsf{D}$.
- For any cone from $\mathcal{F}$ to a vertex $V$ in $\mathsf{C}$, there is a unique morphism $\operatorname{colim} \mathcal{F} \to V$, such that for each object $Y$ in $\mathsf{D}$ the following diagram commutes, where $\tau_Y$ is the structure map of the cone to $V$.

$$\begin{tikzcd} \mathcal{F}(Y) \arrow[r, "\iota_Y"] \arrow[dr, "\tau_Y"'] & \operatorname{colim} \mathcal{F} \arrow[d, dashed] \\ & V \end{tikzcd}$$

The cone with vertex $\operatorname{colim} \mathcal{F}$ is called the *limiting cone* or *universal cone* (from $\mathcal{F}$).

Let $\mathcal{G}\colon \mathsf{E} \to \mathsf{D}$ be another functor and $\mathcal{F}\mathcal{G}\colon \mathsf{E} \to \mathsf{C}$ the composite. Suppose the colimit of $\mathcal{G}$ exists and let $\iota_Y : \mathcal{G}(Y) \to \operatorname{colim} \mathcal{G}$ be a universal cone. Then

$$\mathcal{F}\mathcal{G}(\iota_Y)\colon \mathcal{F}\mathcal{G}(Y) \to \mathcal{F}\left(\operatorname{colim} \mathcal{G}\right)$$

is a cone from $\mathcal{F}\mathcal{G}$ to $\mathcal{F}(\operatorname{colim}\mathcal{G})$. Therefore, there is a canonical map

$$\operatorname{colim}\mathcal{F}\mathcal{G} \to \mathcal{F}(\operatorname{colim}\mathcal{G}).$$

When this map is an isomorphism, or equivalently when the above cone is universal, we say that $\mathcal{F}$ *preserves* the colimit of $\mathcal{G}$, and we have

$$\operatorname{colim}\mathcal{F}\mathcal{G} \cong \mathcal{F}(\operatorname{colim}\mathcal{G}).$$

We state some basic properties of colimits.

**Proposition A.10 ([250, Theorems V.3.1 and V.5.1, Section IX.2]).**

(i) *The colimit may not exist, but if it does, then it is unique up to isomorphism.*
(ii) *A functor which admits a right adjoint, or which is part of an equivalence, preserves colimits.*
(iii) *In a functor category, colimits may be calculated pointwise.*
(iv) *Let* $\mathcal{F}\colon \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{D}$ *be a functor and suppose its colimit* $\operatorname{colim}_{X,Y}\mathcal{F}(X,Y)$ *exists. Then the iterated colimits exist and*

$$\operatorname*{colim}_X \operatorname*{colim}_Y \mathcal{F}(X,Y) \cong \operatorname*{colim}_{X,Y}\mathcal{F}(X,Y) \cong \operatorname*{colim}_Y \operatorname*{colim}_X \mathcal{F}(X,Y).$$

**Example A.11.** Colimits of functors into vector spaces commute with tensor products and direct sums. The precise meaning of these statements is explained below.

For any vector space $V$, the functor $(-)\otimes V\colon \mathsf{Vec} \to \mathsf{Vec}$ has a right adjoint given by the functor $\operatorname{Hom}_{\mathsf{Vec}}(V,-)$: For any vector spaces $U, V$ and $W$, there is a natural bijection

$$\operatorname{Hom}_{\mathsf{Vec}}(U\otimes V, W) \cong \operatorname{Hom}_{\mathsf{Vec}}\big(U, \operatorname{Hom}_{\mathsf{Vec}}(V,W)\big). \tag{A.6}$$

Therefore, by item (ii) in Proposition A.10 it preserves all colimits. Explicitly, if $\mathcal{F}\colon \mathsf{C} \to \mathsf{Vec}$ is a functor then

$$\big(\operatorname{colim}\mathcal{F}\big)\otimes V \cong \operatorname*{colim}_X \big(\mathcal{F}(X)\otimes V\big).$$

The dual situation is discussed in Example A.15.

Now suppose $\mathcal{F}_i\colon \mathsf{C}_i \to \mathsf{Vec}$ are functors, $i = 1, 2$. Then

$$\big(\operatorname{colim}\mathcal{F}_1\big)\otimes\big(\operatorname{colim}\mathcal{F}_2\big) \cong \operatorname*{colim}_{X_1,X_2}\big(\mathcal{F}_1(X_1)\otimes\mathcal{F}_2(X_2)\big).$$

This follows by combining item (iv) in Proposition A.10 with the preceding discussion.

We now turn to direct sum. The functor

$$\oplus : \mathsf{Vec}\times\mathsf{Vec} \to \mathsf{Vec} \qquad (V,W) \mapsto V\oplus W$$

has a right adjoint given by $V \mapsto (V,V)$. Therefore it preserves all colimits. This can be rephrased as follows.

Suppose $\mathcal{F}, \mathcal{G}\colon \mathsf{C} \to \mathsf{Vec}$ are functors. Then

$$\big(\operatorname{colim}\mathcal{F}\big)\oplus\big(\operatorname{colim}\mathcal{G}\big) \cong \operatorname*{colim}_X\big(\mathcal{F}(X)\oplus\mathcal{G}(X)\big).$$

This fact can also be deduced from item (iv) in Proposition A.10, since direct sums are special colimits (Section A.3.2).

**A.3.2. Special colimits.** Coproducts, coequalizers and coinvariants are special cases of the notion of colimit.

Let $\mathsf{D}$ be the category with two objects 1 and 2 and no arrows other than identities. A choice of two objects $A$ and $B$ of a category $\mathsf{C}$ uniquely determines a functor $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$, and the colimit of $\mathcal{F}$ is the coproduct $A \amalg B$ as defined in Section A.1.2. More generally, if $\mathsf{D}_1, \mathsf{D}_2$ and $\mathsf{D}$ are categories such that

$$\mathsf{D} = \mathsf{D}_1 \sqcup \mathsf{D}_2$$

(no arrows between $\mathsf{D}_1$ and $\mathsf{D}_2$), then

$$\operatorname{colim} \mathcal{F} = \operatorname{colim} \mathcal{F}|_{\mathsf{D}_1} \amalg \operatorname{colim} \mathcal{F}|_{\mathsf{D}_2}.$$

A category is *discrete* if every morphism is an identity. Let $\mathsf{D}$ be the discrete category whose objects are indexed by a set $J$. The coproduct of a family $\{X_j\}_{j\in J}$ of objects in $\mathsf{C}$ is the colimit of the functor $\mathsf{D} \to \mathsf{C}$ which sends $j$ to $X_j$. It is denoted by

$$\coprod_{j\in J} X_j.$$

We say that $\mathsf{C}$ has countable coproducts if the coproduct exists for one (and hence all) countable sets $J$.

Let $\mathsf{D}$ be the category with two objects 1 and 2, their identities, and two arrows from 1 to 2:

$$1 \rightrightarrows 2.$$

A functor $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ is uniquely determined by a diagram of the form

$$A \overset{f}{\underset{g}{\rightrightarrows}} B.$$

in $\mathsf{C}$. The colimit of $\mathcal{F}$ is the *coequalizer* of $f$ and $g$.

Let $G$ be a group. Let $\mathsf{D}$ be the category with one object, say $*$, with morphisms $\mathrm{Hom}_{\mathsf{D}}(*,*)$ indexed by the elements of $G$ and composition given by the group law. Let $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ be a functor and set $V := \mathcal{F}(*)$. Then we say that $V$ has a $G$-action and the *coinvariants* of $V$ with respect to $G$, denoted $V_G$, is the colimit of this functor.

For example, if $\mathsf{C} = \mathsf{Set}$, then $V$ is a $G$-set and $V_G$ is the set of orbits, and if $\mathsf{C} = \mathsf{Vec}$, then $V$ is a $G$-module and $V_G$ is the space of coinvariants as in Section 2.5.1.

A category $\mathsf{D}$ is *indiscrete* if there is a unique morphism between any two objects. The colimit of a functor $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ always exists: choose an object $X$ of $\mathsf{D}$, let $V := \mathcal{F}(X)$, and let $\iota_Y$ be the image under $\mathcal{F}$ of the unique arrow $Y \to X$ in $\mathsf{D}$. This defines a universal cone from $\mathcal{F}$ to $V$.

**A.3.3. Colimits from groupoids.** A *groupoid* is a category in which every arrow is invertible. Given objects $A$ and $B$ in a small groupoid $\mathsf{G}$, write

$$A \sim B \text{ if there exists an arrow } A \to B \text{ in } \mathsf{G}.$$

Then $\sim$ is an equivalence relation on the set of objects. Let $\pi_0(\mathsf{G})$ denote the set of equivalence classes. Also, let

$$\Omega_A \mathsf{G} := \mathrm{End}_{\mathsf{G}}(A)$$

be the group of automorphisms of $A$. If $A \sim B$, then the groups $\Omega_A \mathsf{G}$ and $\Omega_B \mathsf{G}$ are isomorphic.

View each group $\Omega_A \mathsf{G}$ as a category as in Section A.3.2. The category

$$\bigsqcup_{[A]\in\pi_0(\mathsf{G})} \Omega_A \mathsf{G}$$

is a skeleton of $\mathsf{G}$. It follows that the colimit of any functor $\mathcal{F}\colon \mathsf{G} \to \mathsf{C}$ can be calculated in terms of coproducts and coinvariants as follows:

$$\operatorname{colim} \mathcal{F} = \coprod_{[A]\in\pi_0(\mathsf{G})} \mathcal{F}(A)_{\Omega_A \mathsf{G}}. \tag{A.7}$$

**Example A.12.** A discrete category $\mathsf{G}$ with set of objects $J$ is a groupoid with $\pi_0(\mathsf{G}) = J$ and every group $\Omega_j \mathsf{G}$ trivial. An indiscrete category $\mathsf{G}$ is also a groupoid. In this case $\pi_0(\mathsf{G})$ reduces to one equivalence class and again every automorphism group is trivial. Formula (A.7) recovers the colimit descriptions for such categories of Section A.3.2.

**Example A.13.** A species with values in $\mathsf{C}$ is a functor of the form

$$\mathcal{F}\colon \mathsf{Set}^{\times} \to \mathsf{C},$$

where we recall that $\mathsf{Set}^{\times}$ is the category of finite sets with bijections as the morphisms. The category $\mathsf{Set}^{\times}$ is a groupoid; $\pi_0(\mathsf{Set}^{\times}) = \{\emptyset, [1], [2], \dots\}$ and $\Omega_{[n]}\mathsf{Set}^{\times} = \mathrm{S}_n$, the symmetric group on $n$ letters. Therefore, if the category $\mathsf{C}$ has coequalizers and countable coproducts, the colimit of $\mathcal{F}$ exists and is given by

$$\operatorname{colim} \mathcal{F} \cong \coprod_{n\geq 0} \mathcal{F}[n]_{\mathrm{S}_n}, \tag{A.8}$$

where $\mathcal{F}[n]$ denotes the image of the set $[n]$ under $\mathcal{F}$.

**Example A.14.** Let a monoid $G$ act on a set $X$. The *category of elements* $\mathsf{el}_G(X)$ is defined as follows. The set of objects is $X$. A morphism from $x$ to $y$ is a pair $(x, g)$ such that $g \cdot x = y$. Composition is $(y, h) \circ (x, g) = (x, hg)$ and the identity of $x$ is $(x, 1)$. This is illustrated below.

$$\begin{tikzcd}
x \arrow[r, bend left, "{(x,g)}"] \arrow[rr, bend right, "{(x,hg)}"'] & g \cdot x \arrow[r, bend left, "{(g\cdot x,h)}"] & h \cdot g \cdot x
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[loop above, "{(x,1)}"]
\end{tikzcd}$$

If $G$ is a group, then $\mathsf{el}_G(X)$ is a groupoid. Thus, the colimit of a functor $\mathcal{F}\colon \mathsf{el}_G(X) \to \mathsf{C}$ can be calculated as

$$\operatorname{colim} \mathcal{F} = \coprod_{[x]\in\pi_0(\mathsf{el}_G(X))} \mathcal{F}(x)_{\Omega_x(\mathsf{el}_G(X))}.$$

In this case, $\pi_0(\mathsf{el}_G(X))$ is the set of orbits for the action, and $\Omega_x(\mathsf{el}_G(X))$ is the stabilizer of $x$.

The category $\mathsf{el}_G(X)$ is a special comma category; see Example A.23.

**A.3.4. Limits.** The notion of *limit* of a functor is dual to that of colimit (Definition A.9). If $\mathcal{G}\colon \mathsf{E} \to \mathsf{D}$ and $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ are functors, there is a canonical map

$$\mathcal{F}(\lim \mathcal{G}) \to \lim \mathcal{F}\mathcal{G}. \tag{A.9}$$

When this map is an isomorphism, we say that $\mathcal{F}$ *preserves* the limit of $\mathcal{G}$.

**Example A.15.** Let $V$ be a vector space. In Example A.11 we noted that the functor $(-) \otimes V\colon \mathsf{Vec} \to \mathsf{Vec}$ preserves all colimits. If $V$ is finite-dimensional, then (A.6) implies

$$\mathrm{Hom}_{\mathsf{Vec}}(U \otimes V^*, W) \cong \mathrm{Hom}_{\mathsf{Vec}}(U, V \otimes W).$$

Thus, in this case the functor $V \otimes (-)\colon \mathsf{Vec} \to \mathsf{Vec}$ has a left adjoint and hence it preserves all limits.

If $V$ is infinite-dimensional, the functor $V \otimes (-)$ does not preserve all limits. In fact, let $W_n = \Bbbk$ for each $n \in \mathbb{N}$, and consider the canonical map

$$V \otimes \big(\prod_{n\in\mathbb{N}} W_n\big) \to \prod_{n\in\mathbb{N}} V \otimes W_n.$$

The canonical projections yield maps

$$p_n\colon \prod_{n\in\mathbb{N}} (V \otimes W_n) \twoheadrightarrow V \otimes W_n \cong V.$$

If $x$ is a fixed vector in $V \otimes \big(\prod_{n\in\mathbb{N}} W_n\big)$, then the subspace of $V$ generated by $\{p_n(x)\}_{n\in\mathbb{N}}$ is finite-dimensional. On the other hand, if $\{v_n\}_{n\in\mathbb{N}}$ is a sequence of linearly independent vectors in $V$, and $y := (v_n \otimes 1)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} V \otimes W_n$, then the subspace of $V$ generated by $\{p_n(y)\}_{n\in\mathbb{N}}$ is infinite-dimensional. Thus, the canonical map is not surjective and $V \otimes (-)$ does not preserve the product of the $W$'s.

**A.3.5. Completeness.** The category $\mathsf{D}$ is said to be *(co)complete* if the (co)limit of every functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ exists, for every small category $\mathsf{C}$.

Suppose $\mathsf{D}$ is (co)complete and $\mathsf{C}$ is small. It follows from item (iii) in Proposition A.10 that the category of functors $\mathsf{C} \to \mathsf{D}$ is also (co)complete. In particular, the category of species with values in $\mathsf{D}$ is (co)complete.

**Example A.16.** The category $\mathsf{Vec}$ of vector spaces and linear maps is complete and cocomplete. Hence so is the category $\mathsf{Sp}$ of vector species. The same is true of the categories of sets and of set species.

## A.4. Kan extensions

Kan extensions are discussed in [58, Section 3.7] and [250, Chapter X]. We briefly review this notion and apply it to the calculation of certain colimits.

**Definition A.17.** Let $\mathcal{F}\colon \mathsf{A} \to \mathsf{C}$ and $\mathcal{P}\colon \mathsf{A} \to \mathsf{B}$ be two functors defined on the same category. A *left Kan extension of* $\mathcal{F}$ *along* $\mathcal{P}$ consists of a functor $\mathcal{L}\colon \mathsf{B} \to \mathsf{C}$ and a natural transformation $\alpha\colon \mathcal{F} \Rightarrow \mathcal{L}\mathcal{P}$ that are initial among such pairs.

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\mathcal{P}"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\xRightarrow{\alpha}"] & \mathsf{B} \arrow[dl, "\mathcal{L}"] \\ \mathsf{C} & \end{tikzcd}$$

In other words, given another functor $\mathcal{L}'\colon \mathsf{B} \to \mathsf{C}$ and natural transformation $\alpha'\colon \mathcal{F} \Rightarrow \mathcal{L}'\mathcal{P}$, there exists a unique natural transformation $\gamma\colon \mathcal{L} \Rightarrow \mathcal{L}'$ such that

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\alpha_A"] \arrow[dr, "\alpha'_A"'] & \mathcal{L}\mathcal{P}(A) \arrow[d, "\gamma_{\mathcal{P}(A)}"] \\ & \mathcal{L}'\mathcal{P}(A) \end{tikzcd}$$

commutes, for every object $A$ of $\mathsf{A}$.

When a left Kan extension exists, it is unique up to isomorphism. Sometimes we refer to the functor $\mathcal{L}$ as the left Kan extension, letting the transformation $\alpha$ implicit.

**Proposition A.18.** *Let $\mathcal{L}$ be a left Kan extension of $\mathcal{F}$ along $\mathcal{P}$. If one of* $\operatorname{colim}\mathcal{F}$ *or* $\operatorname{colim}\mathcal{L}$ *exists, so does the other, and*

$$\operatorname{colim}\mathcal{F} \cong \operatorname{colim}\mathcal{L}.$$

Proof. This follows from [250, Exercise X.4.3 and Theorem X.7.1]. □

**Example A.19.** Let a group $G$ act on a set $X$ and consider the category $\mathsf{el}_G(X)$ as in Example A.14. Let $\{*\}$ be a singleton. The unique map $X \to \{*\}$ induces a functor

$$\mathcal{P}\colon \mathsf{el}_G(X) \to \mathsf{el}_G(\{*\}).$$

The latter category has only one object and the morphisms are the elements of $G$.

Let $\mathsf{C}$ be a cocomplete category and suppose a functor $\mathcal{F}\colon \mathsf{el}_G(X) \to \mathsf{C}$ is given. Define a functor $\mathcal{L}\colon \mathsf{el}_G(\{*\}) \to \mathsf{C}$ as follows. On the unique object,

$$\mathcal{L}(*) := \coprod_{x\in X} \mathcal{F}(x).$$

On a morphism $g \in G$, $\mathcal{L}(g)\colon \mathcal{L}(*) \to \mathcal{L}(*)$ is defined through the universal property of coproducts from the maps

$$\mathcal{F}(x) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(g\cdot x) \to \coprod_{y\in X} \mathcal{F}(y) = \mathcal{L}(*).$$

According to Example A.14, we have

$$\operatorname{colim}\mathcal{F} = \coprod_{[x]\in\pi_0(\mathsf{el}_G(X))} \mathcal{F}(x)_{\Omega_x(\mathsf{el}_G(X))} \quad \text{and} \quad \operatorname{colim}\mathcal{L} = \Big(\coprod_{x\in X}\mathcal{F}(x)\Big)_G.$$

There is a natural transformation $\alpha\colon \mathcal{F} \Rightarrow \mathcal{L}\mathcal{P}$ with $\alpha_x$ equal to the canonical map

$$\mathcal{F}(x) \to \coprod_{y\in X}\mathcal{F}(y) = \mathcal{L}(*) = \mathcal{L}\mathcal{P}(x).$$

It is easy to see that $\mathcal{L}$ and $\alpha$ constitute a left Kan extension of $\mathcal{F}$ along $\mathcal{P}$. It follows from Proposition A.18 that

$$\coprod_{[x]\in\pi_0(\mathsf{el}_G(X))} \mathcal{F}(x)_{\Omega_x(\mathsf{el}_G(X))} \cong \Big(\coprod_{x\in X}\mathcal{F}(x)\Big)_G.$$

**Example A.20.** Let $\mathsf{D}$ be a category and $X$ an object of $\mathsf{D}$. The *slice category over* $X$, denoted $\mathsf{D} \downarrow X$, is defined as follows. The objects are the morphisms

$$h : Y \to X$$

of $\mathsf{D}$ with target $X$. A morphism from $h : Y \to X$ to $h' : Y' \to X$ is a morphism $f : Y \to Y'$ in $\mathsf{D}$ such that

$$\begin{tikzcd} Y \arrow[d, "f"'] \arrow[dr, "h"] & \\ Y' \arrow[r, "h'"'] & X \end{tikzcd}$$

commutes. Composition and identities are inherited from $\mathsf{D}$.

The category $\mathsf{D} \downarrow X$ is a special comma category; see Example A.22.

There is a functor

$$\mathcal{P} \colon \mathsf{D} \downarrow X \to \mathsf{D}$$

which sends $h \colon Y \to X$ to $Y$, and is the identity on morphisms.

Let $\mathsf{D}^{\times}$ be the category with the same objects as $\mathsf{D}$ and with morphisms the isomorphisms of $\mathsf{D}$. It is a groupoid. Consider also the groupoid $(\mathsf{D} \downarrow X)^{\times}$ and the induced functor

$$\mathcal{P}^{\times} \colon (\mathsf{D} \downarrow X)^{\times} \to \mathsf{D}^{\times}.$$

Let $\mathsf{C}$ be a cocomplete category and suppose a functor $\mathcal{F} \colon (\mathsf{D} \downarrow X)^{\times} \to \mathsf{C}$ is given. Define a functor $\mathcal{L} \colon \mathsf{D}^{\times} \to \mathsf{C}$ as follows. On an object $Y$,

$$\mathcal{L}(Y) := \coprod_{k \colon Y \to X} \mathcal{F}(k).$$

On a morphism $\sigma \colon Y \to Y'$, $\mathcal{L}(\sigma) \colon \mathcal{L}(Y) \to \mathcal{L}(Y')$ is defined through the universal property of coproducts from the maps

$$\mathcal{F}(h) \xrightarrow{\mathcal{F}(\sigma)} \mathcal{F}(h \circ \sigma^{-1}) \to \coprod_{k \colon Y' \to X} \mathcal{F}(k) = \mathcal{L}(Y').$$

There is a natural transformation $\alpha \colon \mathcal{F} \Rightarrow \mathcal{L}\mathcal{P}^{\times}$ which for $h : Y \to X$ has $\alpha_h$ equal to the canonical map

$$\mathcal{F}(h) \to \coprod_{k \colon Y \to X} \mathcal{F}(k) = \mathcal{L}(Y) = \mathcal{L}\mathcal{P}^{\times}(h).$$

It is easy to see that $\mathcal{L}$ and $\alpha$ constitute a left Kan extension of $\mathcal{F}$ along $\mathcal{P}^{\times}$. It follows from Proposition A.18 that

$$\operatorname*{colim}_{h} \mathcal{F}(h) \cong \operatorname*{colim}_{Y} \coprod_{k \colon Y \to X} \mathcal{F}(k). \tag{A.10}$$

## A.5. Comma categories

Several categories considered in the preceding sections are particular comma categories. We briefly review this notion, following [250, Section II.6].

**Definition A.21.** Let $\mathcal{F} \colon \mathsf{A} \to \mathsf{C}$ and $\mathcal{G} \colon \mathsf{B} \to \mathsf{C}$ be two functors to the same category. The *comma category* $\mathcal{F} \downarrow \mathcal{G}$ is defined as follows. The objects are triples $(A, \gamma, B)$ where $A$ is an object of $\mathsf{A}$, $B$ is an object of $\mathsf{B}$, and

$$\gamma \colon \mathcal{F}(A) \to \mathcal{G}(B)$$

is a morphism in $\mathsf{C}$. A morphism $(A, \gamma, B) \to (A', \gamma', B')$ is a pair $(\alpha, \beta)$ where $\alpha\colon A \to A'$ is a morphism in $\mathsf{A}$ and $\beta\colon B \to B'$ is a morphism in $\mathsf{B}$ such that

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\gamma"] \arrow[d, "\mathcal{F}(\alpha)"'] & \mathcal{G}(B) \arrow[d, "\mathcal{G}(\beta)"] \\ \mathcal{F}(A') \arrow[r, "\gamma'"'] & \mathcal{G}(B') \end{tikzcd}$$

commutes. Composition and identities are coordinatewise.

**Example A.22.** If $\mathsf{A} = \mathsf{B} = \mathsf{C}$ and both $\mathcal{F}$ and $\mathcal{G}$ are the identity functor, then the comma category is the category of arrows of $\mathsf{C}$ of Section 3.11.1:

$$\mathrm{id}_{\mathsf{C}} \downarrow \mathrm{id}_{\mathsf{C}} = \mathsf{D}^{(2)}.$$

Let $\mathsf{I}$ denote the one-arrow category. Given an object $X$ of $\mathsf{C}$, let $\mathcal{F}_X\colon \mathsf{I} \to \mathsf{C}$ be the functor that sends the unique object of $\mathsf{I}$ to $X$. The comma category $\mathrm{id}_{\mathsf{C}} \downarrow \mathcal{F}_X$ is the slice category over $X$ of Example A.20:

$$\mathrm{id}_{\mathsf{C}} \downarrow \mathcal{F}_X = \mathsf{C} \downarrow X.$$

The slice category of objects *under* $X$ is

$$X \downarrow \mathsf{C} := \mathcal{F}_X \downarrow \mathrm{id}_{\mathsf{C}}.$$

Given objects $C$ and $A$ of $\mathsf{C}$, the comma category

$$\mathcal{F}_C \downarrow \mathcal{F}_A$$

is the discrete category corresponding to the set $\mathrm{Hom}(C, A)$ (also see Section 3.4.5).

**Example A.23.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{Set}$ be a functor. Let $\{*\}$ be a singleton and $\mathcal{F}_{\{*\}}\colon \mathsf{I} \to \mathsf{Set}$ the corresponding functor. The category of elements of $\mathcal{F}$ is defined as

$$\mathsf{el}(\mathcal{F}) := \mathcal{F}_{\{*\}} \downarrow \mathcal{F}.$$

If $G$ is a monoid and $\mathsf{C}$ is the corresponding one-object category (as in Section A.3.2), then $\mathcal{F}$ amounts to a $G$-set $X$, and

$$\mathsf{el}(\mathcal{F}) = \mathsf{el}_G(X),$$

the category of elements considered in Example A.14.

The category of elements $\mathsf{el}(\mathcal{F})$ is also called the *Grothendieck construction.* It can be carried out in a more general setting than the above; see [58, Definition 1.6.4] or [251, Section VII.6, p. 386].

APPENDIX B

# Operads

There are various approaches to the notion of operad. One consists in viewing operads in relation to species as algebras sit in relation to vector spaces. We adopt this point of view in this appendix.

We do not go into the theory of operads in any depth, but we address a number of basic points whose discussion is not always explicit in the literature. We discuss a number of examples of positive (co)operads in concrete combinatorial terms and a construction of positive cooperads of Schmitt (Sections B.1, B.2 and B.3). We provide a careful discussion of the substitution product for general (not necessarily positive) species. In contrast with the case of positive species, there are two different notions of substitution for general species. One notion of substitution gives rise to the general notion of operad and the other to the general notion of cooperad. We discuss these ideas in Section B.4. We also study the interaction between the former substitution product and the Hadamard product and discuss the notion of Hopf operad (Section B.6).

The main use for operads in this monograph occurs in Chapter 4, where we explain that to each operad corresponds a type of monoid (in a monoidal category) and a type of monoidal functor (between monoidal categories). Types of monoids may also be understood in terms of modules over operads, as discussed in Section B.5. The link between monoids and modules is made via the internal Hom for the former substitution product, which we describe explicitly.

The language of monoidal and 2-monoidal categories (Part I) is employed throughout this appendix.

## B.1. Positive operads

Positive operads are positive species with extra structure: that of a monoid with respect to the substitution product (8.8). We make this structure more explicit in Section B.1.2, after reviewing the substitution product in Section B.1.1. We then discuss a number of examples in Section B.1.4.

**B.1.1. Substitution for positive species.** Let $\mathbf{m}$ be a positive species (Section 8.9.2) and $\mathbf{p}$ an arbitrary species. The substitution of $\mathbf{m}$ in $\mathbf{p}$ is the species $\mathbf{p} \circ \mathbf{m}$ with components

$$(\mathbf{p} \circ \mathbf{m})[I] := \bigoplus_{X \vdash I} \mathbf{p}[X] \otimes \left( \bigotimes_{S \in X} \mathbf{m}[S] \right), \tag{B.1}$$

where the direct sum is over all partitions $X$ of $I$, and $I$ is a finite set. We considered this operation briefly in (8.8).

We now discuss associativity. In order to do this, fix a finite set $I$ and consider partitions $X$ and $Y$ of $I$, with $Y$ refining $X$. As in Section 10.1.4, we write $X \leq Y$.

There is a map $f : Y \to X$ which sends $T \in Y$ to the unique block $S \in X$ such that $T \subseteq S$. The fiber of $f$ over $S \in X$ is the set whose elements are the blocks of $Y$ which refine the block $S$ of $X$. We write

$$\mathbf{p}[X : Y] := \bigotimes_{S \in X} \mathbf{p}[f^{-1}(S)].$$

Let $\hat{0}$ denote the unique partition of $I$ into one block and $\hat{1}$ the unique partition into singletons. Note that

$$\mathbf{p}[\hat{0} : X] = \mathbf{p}[X] \quad \text{and} \quad \mathbf{p}[X : \hat{1}] = \mathbf{p}(X),$$

the latter as defined in Notation 11.1. Thus,

$$(\mathbf{p} \circ \mathbf{m})[I] = \bigoplus_{\hat{0} \leq X \leq \hat{1}} \mathbf{p}[\hat{0} : X] \otimes \mathbf{m}[X : \hat{1}].$$

Let $\mathbf{n}$ be a second positive species. Consider the species $\mathbf{p} \circ \mathbf{m} \circ \mathbf{n}$ defined by

$$(\mathbf{p} \circ \mathbf{m} \circ \mathbf{n})[I] := \bigoplus_{\hat{0} \leq X \leq Y \leq \hat{1}} \mathbf{p}[\hat{0} : X] \otimes \mathbf{m}[X : Y] \otimes \mathbf{n}[Y : \hat{1}].$$

Associativity follows from the fact that

$$(\mathbf{p} \circ \mathbf{m}) \circ \mathbf{n}[I] \cong (\mathbf{p} \circ \mathbf{m} \circ \mathbf{n})[I] \cong \mathbf{p} \circ (\mathbf{m} \circ \mathbf{n})[I].$$

The former holds because partitions of $I$ which are coarser than $Y$ may be identified with partitions of $Y$. The latter holds because partitions of $I$ which refine $X$ may be identified with a family of partitions, one partition of $S$ for each block $S$ of $X$.

There is only one partition of the empty set, the partition with no blocks. In this case, (B.1) is to be understood as follows:

$$(\mathbf{p} \circ \mathbf{m})[\emptyset] := \mathbf{p}[\emptyset].$$

Therefore, if $\mathbf{p}$ is positive, then so is $\mathbf{p} \circ \mathbf{m}$. It follows that the category $\mathsf{Sp}_+$ of positive species is a monoidal category under substitution. The unit object is the positive species $\mathbf{X}$ characteristic of singletons (8.3).

The monoidal category $(\mathsf{Sp}_+, \circ, \mathbf{X})$ is not braided: we may consider monoids and comonoids, but not bimonoids or (co)commutative (co)monoids.

**B.1.2. Definition of positive operad.** A *positive operad* is a monoid in $(\mathsf{Sp}_+, \circ, \mathbf{X})$; a morphism of positive operads is a morphism of monoids.

Thus, a positive operad is a positive species $\mathbf{p}$ together with morphisms of species

$$\gamma \colon \mathbf{p} \circ \mathbf{p} \to \mathbf{p} \qquad \text{and} \qquad \eta \colon \mathbf{X} \to \mathbf{p}$$

which are associative and unital. We refer to $\gamma$ as the *operadic composition* and to $\eta$ as the *operadic unit*.

A morphism $(\mathbf{p}_1, \gamma_1, \eta_1) \to (\mathbf{p}_2, \gamma_2, \eta_2)$ of positive operads is a map $\mathbf{p}_1 \to \mathbf{p}_2$ of species which commutes with $\gamma_1$ and $\gamma_2$, and $\eta_1$ and $\eta_2$ respectively.

A partition $X \vdash I$ determines a surjection $I \twoheadrightarrow X$ such that

$$i \mapsto S \iff i \in S \in X.$$

Conversely, a surjection $f \colon I \twoheadrightarrow J$ between finite sets determines a partition $X$ of $I$ given by

$$X := \{f^{-1}(j) \mid j \in J\}.$$

Using this correspondence one may reformulate (B.1) in terms of surjections between finite sets. This approach is taken in [260, Definition 1.63].

In the same vein, a positive operad is a positive species $\mathbf{p}$ with the following structure. For each surjection $f\colon I \twoheadrightarrow X$ between nonempty finite sets, there is a linear map

$$\gamma_f\colon \mathbf{p}[X] \otimes \bigotimes_{x\in X} \mathbf{p}[f^{-1}(x)] \to \mathbf{p}[I], \tag{B.2}$$

and for each singleton $\{*\}$, there is a linear map

$$\eta_*\colon \Bbbk \to \mathbf{p}[\{*\}]. \tag{B.3}$$

These structure maps are subject to the conditions that follow.

**Naturality.** Given a commutative diagram of finite sets

$$\begin{tikzcd} I \arrow[r, "\sigma"] \arrow[d, "f"'] & J \arrow[d, "g"] \\ X \arrow[r, "\tau"'] & Y \end{tikzcd}$$

where $\sigma$ and $\tau$ are bijections and $f$ and $g$ are surjections, let

$$\sigma_x\colon f^{-1}(x) \to g^{-1}\big(\tau(x)\big)$$

denote the restriction of $\sigma$. The following diagram must commute.

$$\begin{tikzcd} \mathbf{p}[X] \otimes \bigotimes_{x\in X} \mathbf{p}[f^{-1}(x)] \arrow[r, "\gamma_f"] \arrow[d, "\mathbf{p}[\tau]\otimes \bigotimes_{x\in X}\mathbf{p}[\sigma_x]"'] & \mathbf{p}[I] \arrow[d, "\mathbf{p}[\sigma]"] \\ \mathbf{p}[Y] \otimes \bigotimes_{y\in Y} \mathbf{p}[g^{-1}(y)] \arrow[r, "\gamma_g"'] & \mathbf{p}[J]. \end{tikzcd}$$

In addition, for any singletons $\{x\}$ and $\{y\}$, the following diagram must commute

$$\begin{tikzcd} \mathbf{p}[\{x\}] \arrow[rr, "\mathbf{p}[\sigma_{x,y}]"] & & \mathbf{p}[\{y\}] \\ & \Bbbk \arrow[ul, "\eta_x"] \arrow[ur, "\eta_y"'] & \end{tikzcd}$$

where $\sigma_{x,y}$ is the unique map $\{x\} \to \{y\}$.

**Associativity.** Given a commutative diagram of finite sets and surjections

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "f"'] & & Y \arrow[dl, "h"] \\ & X, & \end{tikzcd}$$

let

$$g_x\colon f^{-1}(x) \to h^{-1}(x)$$

denote the restriction of $g$. The following diagram must commute.

$$\begin{tikzcd}
\Big(\mathbf{p}[X]\otimes\bigotimes\limits_{x\in X}\mathbf{p}[h^{-1}(x)]\Big)\otimes\bigotimes\limits_{y\in Y}\mathbf{p}[g^{-1}(y)] \arrow[d, "\gamma_h\otimes\mathrm{id}"'] & = & \mathbf{p}[X]\otimes\bigotimes\limits_{x\in X}\Big(\mathbf{p}[h^{-1}(x)]\otimes\bigotimes\limits_{y\in h^{-1}(x)}\mathbf{p}[g^{-1}(y)]\Big) \arrow[d, "\mathrm{id}\otimes\bigotimes\limits_{x\in X}\gamma_{g_x}"] \\
\mathbf{p}[Y]\otimes\bigotimes\limits_{y\in Y}\mathbf{p}[g^{-1}(y)] \arrow[dr, "\gamma_g"'] & & \mathbf{p}[X]\otimes\bigotimes\limits_{x\in X}\mathbf{p}[f^{-1}(x)] \arrow[dl, "\gamma_f"] \\
& \mathbf{p}[Z] &
\end{tikzcd}$$

**Unitality.** For each nonempty finite sets $X$ and $I$, the following diagrams must commute.

$$\begin{tikzcd}
\mathbf{p}[X]\otimes\bigotimes\limits_{x\in X}\Bbbk \arrow[r, "\mathrm{id}\otimes\bigotimes\limits_{x\in X}\eta_x"] \arrow[dr, equal] & \mathbf{p}[X]\otimes\bigotimes\limits_{x\in X}\mathbf{p}[\{x\}] \arrow[d, "\gamma_{\mathrm{id}_X}"] \\
& \mathbf{p}[X]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk\otimes\mathbf{p}[I] \arrow[r, "\eta_*\otimes\mathrm{id}"] \arrow[dr, equal] & \mathbf{p}[\{*\}]\otimes\mathbf{p}[I] \arrow[d, "\gamma_{I\to\{*\}}"] \\
& \mathbf{p}[I].
\end{tikzcd}$$

A morphism $(\mathbf{p}_1,\gamma_1,\eta_1)\to(\mathbf{p}_2,\gamma_2,\eta_2)$ of positive operads is a map $\mathbf{p}_1\to\mathbf{p}_2$ of species such that the following diagrams commute, for each surjection $f\colon I\to X$ between finite sets, and for each singleton $\{*\}$.

$$\begin{tikzcd}
\mathbf{p}_1[X]\otimes\bigotimes\limits_{x\in X}\mathbf{p}_1[f^{-1}(x)] \arrow[r, "(\gamma_1)_f"] \arrow[d] & \mathbf{p}_1[I] \arrow[d] \\
\mathbf{p}_2[X]\otimes\bigotimes\limits_{x\in X}\mathbf{p}_2[f^{-1}(x)] \arrow[r, "(\gamma_2)_f"'] & \mathbf{p}_2[I]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{p}_1[\{*\}] \arrow[rr] & & \mathbf{p}_2[\{*\}] \\
& \Bbbk \arrow[ul, "(\eta_1)_*"] \arrow[ur, "(\eta_2)_*"'] &
\end{tikzcd}
\tag{B.4}$$

This completes the explicit definition of a positive operad. A similar description of the operad axioms is given in [260, Theorem 1.60].

**B.1.3. Set positive operads and linearized operads.** The substitution operation can also be defined for positive set species, replacing direct sums and tensor products of vector spaces by disjoint unions and Cartesian product of sets; see Section 8.7.1.

A *set positive operad* is a monoid in the category of positive set species with the substitution product.

The linearization functor (Section 8.7) turns a set operad Q into an operad $\mathbf{q} := \Bbbk\mathrm{Q}$. A *linearized operad* is an operad of this form. In this situation, for any finite set $I$, $\mathbf{q}[I]$ comes equipped with a canonical linear basis.

### B.1.4. Examples of positive operads.

**Example B.1.** Consider the positive exponential species $\mathbf{E}_+$: for each nonempty finite set $I$,

$$\mathbf{E}_+[I] = \Bbbk\{*_I\}.$$

It carries a structure of a positive operad as follows. The operadic composition $\mathbf{E}_+ \circ \mathbf{E}_+ \to \mathbf{E}_+$ has components

$$\mathbf{E}_+[X] \otimes \bigotimes_{S \in X} \mathbf{E}_+[S] \to \mathbf{E}_+[I], \qquad *_X \otimes \bigotimes_{S \in X} *_S \mapsto *_I$$

which simply identify the two one-dimensional spaces on the left by means of their distinguished basis elements. The components of the unit $\mathbf{X} \to \mathbf{E}_+$ are

$$\mathbf{X}[I] \to \mathbf{E}_+[I], \qquad \begin{cases} 1 \mapsto *_I & \text{if } I \text{ is a singleton,} \\ 0 & \text{otherwise.} \end{cases}$$

This turns $\mathbf{E}_+$ into a positive operad; it is the *positive commutative operad*, denoted by $\mathbf{Com}_+$.

**Example B.2.** Consider the positive species $\mathbf{L}_+$ of linear orders. It is the same as $\mathbf{L}$ (Example 8.3) except that it is zero on the empty set. It is a positive operad as follows.

Let $X$ be a partition of a finite set $I$. Suppose we are given a linear order $l_X$ on $X$ and linear orders $l_S$ on each $S \in X$. The *ordinal sum* of these is the linear order $l_I$ on the set $I$ for which $i < j$ if

$$i \in S,\ j \in T, \text{ and either } S = T \text{ and } i < j \text{ in } l_S, \text{ or } S < T \text{ in } l_X.$$

The operadic composition has components

$$\mathbf{L}_+[X] \otimes \bigotimes_{S \in X} \mathbf{L}_+[S] \to \mathbf{L}_+[I], \qquad l_X \otimes \bigotimes_{S \in X} l_S \mapsto l_I$$

The unit map is defined as for the commutative operad (Example B.1).

This is the *positive associative operad*, denoted $\mathbf{As}_+$.

**Example B.3.** We define an operad structure on the species $\mathbf{L}_+$ which is different from the associative operad.

Given a partition $X$ of $I$ and linear orders $l_X$ on $X$ and $l_S$ on $S$ for each $S \in X$, the operadic composition yields the sum of all the linear orders $l$ on $I$ such that:

- The restriction of $l$ to $S$ is $l_S$ for every $S \in X$,
- The order induced by $l$ on the subset $X' := \{\min S : S \in X\}$ of $I$ coincides with $l_X$ under the bijection $X \to X'$, $S \mapsto \min S$.

The unit map is defined as for the commutative or associative operads.

This turns $\mathbf{L}_+$ into an operad. It is the *Zinbiel operad*, denoted **Zinb**.

**Example B.4.** Let $\mathbf{e}$ be the species of elements (Section 8.13.7), that is, $\mathbf{e}[I]$ is the vector space with linear basis $I$; in particular $\mathbf{e}[\emptyset] = 0$.

The species of elements is a positive operad as follows. The operadic composition has components

$$\mathbf{e}[X] \otimes \bigotimes_{S \in X} \mathbf{e}[S] \to \mathbf{e}[I], \qquad T \otimes \bigotimes_{S \in X} a_S \mapsto a_T,$$

where $T$ is a block of the partition $X$ and $a_S \in S$ for each $S \in X$. The unit map is defined as for the commutative operad.

This is the *permutative operad*, denoted **Perm**. It was introduced by Chapoton [80].

The commutative, associative, and permutative operads are linearized. The Zinbiel operad is not, since in this case the operadic composition involves a sum.

The commutative operad is the terminal object in the category of positive linearized operads. For instance, there are morphisms of operads

$$\mathbf{As}_+ \to \mathbf{Com}_+ \qquad \text{and} \qquad \mathbf{Perm} \to \mathbf{Com}_+$$

which map all basis elements of $\mathbf{As}_+[I]$ or $\mathbf{Perm}[I]$ to $*_I$ (the basis element of $\mathbf{Com}_+[I]$).

There is also a morphism of operads $\mathbf{Com}_+ \to \mathbf{Zinb}$ which sends $*_I$ to the sum of all linear orders on $I$.

**Example B.5.** We now proceed to define a related example which is that of the **Lie** operad [260, Definition 1.28]. Assume that $\Bbbk$ is a field of characteristic different from 2. A *bracket sequence* on $I$ is a way to parenthesize the elements of $I$, each one appearing exactly once. For example, $\big[[x\ y][w\ z]\big]$ is a bracket sequence on the set $\{x, y, z, w\}$. The space $\mathbf{Lie}[I]$ is the span of all bracket sequences of $I$ subject to the relations generated by antisymmetry and the Jacobi identity. For example, $\mathbf{Lie}[\{x, y\}]$ is the span of $[x\ y]$ and $[y\ x]$ subject to the relation

$$[x\ y] = -[y\ x];$$

hence it is one-dimensional. Also $\mathbf{Lie}[\{*\}] = \Bbbk$.

The operadic composition is as follows. Suppose we are given a bracket sequence $a_X$ on $X$ and bracket sequences $a_S$ on each $S \in X$. Then this yields a bracket sequence on $I$ by replacing each element $S \in X$ in the bracket sequence $a_X$ by the bracket sequence $a_S$. The unit map is defined to be the obvious isomorphism.

There is a morphism of operads

$$\mathbf{Lie} \to \mathbf{As}_+.$$

It is obtained by replacing each bracket symbol $[x\ y]$ by $xy - yx$ and concatenating. For example,

$$\big[[x\ y]\ z\big] \mapsto xyz - yxz - zxy + zyx.$$

where the notation, say $yxz$, denotes the linear order $y|x|z$ on the set $\{x, y, z\}$.

**B.1.5. Algebras as operads.** Let $\mathbf{p}$ be a species concentrated in degree one; in other words, $\mathbf{p}[I] = 0$ if $I$ is not a singleton. Suppose that an operad structure on $\mathbf{p}$ is given. In this situation, the operadic compositions $\gamma_f$ (B.2) must be 0, except when $f$ is a bijection between singletons. The axioms in Section B.1.2 then simply express the fact that $\mathbf{p}[*]$ is an associative algebra with product $\gamma_{\mathrm{id}_*}$ and unit $\eta_*$, plus the fact that this algebra is independent of the chosen singleton. In summary, given a species $\mathbf{p}$ concentrated in degree one, we have that

$$\mathbf{p} \text{ is a positive operad} \iff \mathbf{p}[1] \text{ is a unital associative algebra.}$$

In this manner, the category of operads concentrated in degree one is equivalent to the category of unital associative algebras.

## B.2. Positive cooperads

Reversing the arrows in the definition of positive operads results in that of positive cooperads. We discuss this notion and provide several examples.

**B.2.1. Definition.** A *positive cooperad* is a comonoid in $(\mathsf{Sp}_+, \circ, \mathbf{X})$. A morphism of positive cooperads is a morphism of comonoids.

Explicitly, a positive cooperad consists of a positive species $\mathbf{p}$ with linear maps

$$\mathbf{p}[I] \to \mathbf{p}[X] \otimes \bigotimes_{S \in X} \mathbf{p}[S], \qquad \text{and} \qquad \mathbf{p}[\{*\}] \to \Bbbk$$

one for each surjection $I \twoheadrightarrow X$ and each singleton $\{*\}$ respectively, subject to axioms which are dual to those in Section B.1.2.

**B.2.2. Duality between operads and cooperads.** The dual $\mathbf{p}^*$ of a species $\mathbf{p}$ is defined in Section 8.6.1.

Let $\mathbf{p}$ and $\mathbf{m}$ be positive species and $I$ a finite set. There is a canonical inclusion

$$\begin{array}{c} (\mathbf{p}^* \circ \mathbf{m}^*)[I] = \bigoplus_{X \vdash I} \mathbf{p}[X]^* \otimes \Big( \bigotimes_{S \in X} \mathbf{m}[S]^* \Big) \\ \Big\downarrow \\ \Big( \bigoplus_{X \vdash I} \mathbf{p}[X] \otimes \Big( \bigotimes_{S \in X} \mathbf{m}[S] \Big) \Big)^* = (\mathbf{p} \circ \mathbf{m})^*[I] \end{array}$$

and an identification $\mathbf{X}^* = \mathbf{X}$. These turn duality into a lax monoidal functor

$$(\mathsf{Sp}_+^{\mathrm{op}}, \circ) \to (\mathsf{Sp}_+, \circ).$$

In particular, the dual of a positive cooperad (a monoid in $(\mathsf{Sp}_+^{\mathrm{op}}, \circ)$) is a positive operad.

There are finitely many partitions of a finite set. Therefore, if $\mathbf{p}$ and $\mathbf{m}$ are finite-dimensional (Definition 8.2), then so is $\mathbf{p} \circ \mathbf{m}$. Restricted to the category of finite-dimensional species, duality is a strong monoidal functor

$$(\mathsf{Sp}_+^{\mathrm{op}}, \circ) \to (\mathsf{Sp}_+, \circ),$$

and an involution. In particular, the dual of a finite-dimensional positive operad is a positive cooperad.

**B.2.3. Examples of positive cooperads.**

**Example B.6.** The positive exponential species $\mathbf{E}_+$ carries a structure of a cooperad as follows. The coproduct $\mathbf{E}_+ \to \mathbf{E}_+ \circ \mathbf{E}_+$ has components

$$\mathbf{E}_+[I] \to \mathbf{E}_+[X] \otimes \bigotimes_{S \in X} \mathbf{E}_+[S], \qquad *_I \mapsto *_X \otimes \bigotimes_{S \in X} *_S,$$

which simply identify the two one-dimensional spaces on the left by means of their distinguished basis elements. The components of the counit $\mathbf{E}_+ \to \mathbf{X}$ are

$$\mathbf{E}_+[I] \to \mathbf{X}[I], \qquad *_I \mapsto \begin{cases} 1 & \text{if } I \text{ is a singleton,} \\ 0 & \text{otherwise.} \end{cases}$$

This is the *positive commutative cooperad* $\mathbf{Com}_+^*$. It is dual to the positive operad $\mathbf{Com}_+$ of Example B.1.

**Example B.7.** Consider the positive species $\mathbf{L}_+$ of linear orders.

Given a linear order $l$ on $I$ and $S \subseteq I$, we say that $S$ is an *interval* (with respect to $l$) if $x \le y \le z$ and $x, z \in S$ imply $y \in S$. We say that a partition $X \vdash I$ is *into intervals* if every block of $X$ is an interval (with respect to $l$). In this case, the order $l$ induces a linear order on $X$ in which $S < S'$ if one (and hence any) element of $S$ precedes one (and hence any) element of $S'$ with respect to $l$. We denote this order by $l/_X$.

Define a coproduct $\mathbf{L}_+ \to \mathbf{L}_+ \circ \mathbf{L}_+$ by letting its components

$$\mathbf{L}_+[I] \to \mathbf{L}_+[X] \otimes \bigotimes_{S \in X} \mathbf{L}_+[S], \qquad l \mapsto \begin{cases} l/_X \otimes \bigotimes_{S \in X} l|_S & \text{if } X \text{ is into intervals,} \\ 0 & \text{otherwise,} \end{cases}$$

where $l|_S$ is the linear order $l$ restricted to the subset $S$, and $l/_X$ is as above. The counit $\mathbf{L}_+ \to \mathbf{X}$ is defined as for the commutative cooperad above.

This is the *positive associative cooperad* $\mathbf{As}_+^*$. It is dual to the positive operad $\mathbf{As}_+$ of Example B.2.

**Example B.8.** Consider again the positive species $\mathbf{L}_+$. It carries another cooperad structure as follows. The coproduct has components

$$\mathbf{L}_+[I] \to \mathbf{L}_+[X] \otimes \bigotimes_{S \in X} \mathbf{L}_+[S], \qquad l \mapsto l/_X \otimes \bigotimes_{S \in X} l|_S,$$

where $l|_S$ is the linear order $l$ restricted to the subset $S$, and $l/_X$ is defined as follows. The order induced by $l$ on the subset $X' := \{\min S : S \in X\}$ of $I$ coincides with $l/_X$ under the bijection $X \to X'$, $S \mapsto \min S$. The counit $\mathbf{L}_+ \to \mathbf{X}$ is defined as for the associative cooperad above.

This is the *Zinbiel positive cooperad* $\mathbf{Zinb}^*$. It is dual to the positive operad $\mathbf{Zinb}$ of Example B.3.

**Example B.9.** We now discuss a cooperad of graphs. For each finite set $I$, let $\mathbf{G}_c[I]$ be the vector space with basis the set of connected simple graphs with vertex set $I$. We ignore the empty set and view $\mathbf{G}_c$ as a positive species. It is a subspecies of the species $\mathbf{G}$ of simple graphs (Section 13.2.1).

Let $g$ be such a graph with vertex set $I$. Given a nonempty subset $S \subseteq I$, let $g|_S$ be the graph with vertex set $S$ whose edges are all edges of $g$ which join elements of $S$. The graph $g|_S$ is simple but not necessarily connected. Given a partition $X \vdash I$, let $g/_X$ be the graph with vertex set $X$ such that there is an edge between $S, T \in X$ if there is at least one edge in $g$ between an element of $S$ and an element of $T$. In other words, $g/_X$ is obtained from $g$ by identifying all vertices in each block of $X$ and removing all loops and multiple edges that may arise as a result. The graph $g/_X$ is always simple and connected. The *lattice of contractions* of $g$ is the set

$$L(g) := \{X \vdash I \mid \text{the graph } g|_S \text{ is connected for each } S \in X\}. \tag{B.5}$$

The positive species $\mathbf{G}_c$ carries a structure of cooperad as follows. The coproduct has components

$$\mathbf{G}_c[I] \to \mathbf{G}_c[X] \otimes \bigotimes_{S \in X} \mathbf{G}_c[S], \qquad g \mapsto \begin{cases} g/_X \otimes \bigotimes_{S \in X} g|_S & \text{if } X \in L(g), \\ 0 & \text{otherwise.} \end{cases}$$

The counit is as for the positive commutative cooperad.

Note that the only those partitions of the vertex set that are contractions of the graph $g$ are involved in the coproduct.

This cooperad is implicit in work of Schmitt [323, Section 14].

**Example B.10.** Let $\mathbf{a}$ be the species of rooted trees, as in Example 19.32. Thus, $\mathbf{a}[I]$ is the vector space with basis the set of rooted trees (connected acyclic graphs with a distinguished vertex) with vertex set $I$. The species $\mathbf{a}$ is positive. We proceed to endow it with a structure of cooperad.

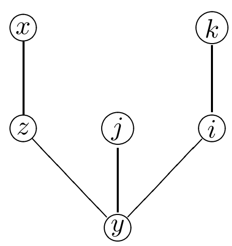
Let $t$ be such a rooted tree with vertex set $I$. Given a nonempty subset $S \subseteq I$, let $t|_S$ be the graph with vertex set $S$ whose edges are all edges of $t$ which join elements of $S$. The graph $t|_S$ is acyclic but not necessarily connected. Let $L(t)$ have the same meaning as in (B.5). In other words, $L(t)$ consists of those partitions $X \vdash I$ such that $t|_S$ is connected (and hence a tree) for each $S \in X$. In this situation, we declare that the root of $t|_S$ is its vertex that is closest to the root of $t$. Given a partition $X \in L(t)$, let $t/_X$ be the graph obtained by contracting each edge of $t$ which joins two elements in one same block of $X$. In other words, $t/_X$ is obtained from $t$ by identifying all vertices in each block of $X$. The vertices of $t/_X$ are thus labeled by the blocks of $X$. The graph $t/_X$ is acylic and connected, so it is a tree. We turn it into a rooted tree by declaring that its root is the block of $X$ to which the root of $t$ belongs.

The cooperad structure is as follows. The coproduct has components

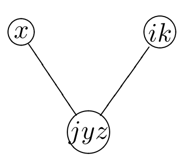
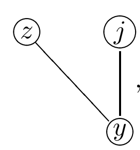
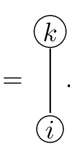
$$\mathbf{a}[I] \to \mathbf{a}[X] \otimes \bigotimes_{S \in X} \mathbf{a}[S], \qquad t \mapsto \begin{cases} t/_X \otimes \bigotimes_{S\in X} t|_S & \text{if } X \in L(t), \\ 0 & \text{otherwise.} \end{cases}$$

The counit is as for the positive commutative cooperad.

For example, if

$t =$  and $X = \{x, jyz, ik\}$,

then

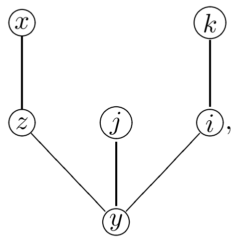
$t/_X =$  , $t|_x = x$, $t|_{jyz} =$  , $t|_{ik} =$  .

This is the *pre-Lie positive cooperad*. The dual positive operad was explicitly described by Chapoton and Livernet [81].

**Example B.11.** We consider now a different positive cooperad structure on the species $\mathbf{a}$ of rooted trees. Given a rooted tree $t$ with vertex set $I$, let

$$R(t) := \{X \in L(t) \mid \text{the roots of the trees } t|_S, \text{ for } S \in X, \text{ are adjacent in } t\}.$$

Thus, $R(t)$ consists of those partitions of the vertex set of $t$ such that each block is a subtree and in addition the roots of the blocks form a subtree. For example, if

$t =$  , $\qquad X = \{x, jyz, ik\}, \qquad$ and $\qquad Y = \{xyz, j, ik\},$

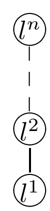
then $X \in L(t)$ but $X \notin R(t)$ (because $x$ is not adjacent to $y$ in $t$), while $Y \in R(t)$.

The map

$$\mathbf{a}[I] \to \mathbf{a}[X] \otimes \bigotimes_{S \in X} \mathbf{a}[S], \qquad t \mapsto \begin{cases} t/_X \otimes \bigotimes_{S \in X} t|_S & \text{if } X \in R(t), \\ 0 & \text{otherwise.} \end{cases}$$

defines a new coproduct on $\mathbf{a}$. The counit is as for the positive commutative cooperad. This defines a new positive cooperad structure on $\mathbf{a}$. The dual is the *nonassociative permutative positive operad* introduced by Livernet in [232, Section 2.2].

**B.2.4. Some morphisms of positive cooperads.** Consider the positive cooperads of Examples B.7, B.9 and B.10. A linear order $l = l^1|l^2|\cdots|l^n$ gives rise to a rooted tree in which $l^1$ is the root and $l^i$ is the only child of $l^{i-1}$ for $i \geq 2$:

$l^1|l^2|\cdots|l^n \longmapsto$ 

In turn, a rooted tree may be viewed as a simple graph by forgetting the root. This defines maps

$$\mathbf{L}_+ \xrightarrow{\lambda_1} \mathbf{a} \xrightarrow{\lambda_2} \mathbf{G}_c$$

(the former is the same as the map in Section 13.3.4).

Now consider a partition $X \vdash I$. Clearly $X$ is into intervals with respect to $l$ if and only if $X \in L\big(\lambda_1(l)\big)$, and $X \in L(t)$ if and only if $X \in L\big(\lambda_2(t)\big)$. It follows that $\lambda_1$ and $\lambda_2$ are morphisms of positive cooperads. Note that $\lambda_2$ is not injective, since all trees differing only in the choice of root map to the same graph.

Consider the positive cooperads of Examples B.6 and B.8. There is a morphism

$$\mathbf{L}_+ \twoheadrightarrow \mathbf{E}_+$$

which sends any linear order $l$ on $I$ to $*_I$.

## B.3. Hereditary species. Schmitt's construction of positive cooperads

The first of Schmitt's constructions, discussed in Section 8.7.8, produces comonoids in $(\mathsf{Sp}, \cdot)$. We turn to another construction of Schmitt from [322, Section 4] which produces comonoids in $(\mathsf{Sp}_+, \circ)$ (positive cooperads). Schmitt originally formulated his construction as one of graded bialgebras, but the ingredients for a cooperad are present in his work. The construction takes for input a heriditary species (see definition below). This notion is the same as that of a $\Omega$-set in the sense of Pirashvili.

**B.3.1. Heriditary species.** Consider the category of nonempty finite sets with *partially defined surjections* as morphisms. A partially defined surjection from $I$ to $J$ is a pair $(S, f)$ where $S$ is a subset of $I$ and $f\colon S \twoheadrightarrow J$ is a surjection. The composite of two such morphisms

$$I \xrightarrow{(S,f)} J \xrightarrow{(T,g)} K$$

is $(f^{-1}(T), g \circ f|_{f^{-1}(T)})$.

A *hereditary species* [322] is a functor from the category of nonempty finite sets and partially defined surjections to the category of sets.

**Remark B.12.** Given a finite set $I$, let $I^+ := I \sqcup \{*_I\}$, as in Section 8.11.1. The element $*_I$ is the *base point* of $I^+$. Partially defined surjections $I \to J$ are in one-to-one correspondence with base-point preserving surjections $I_+ \to J_+$: to $(S, f)\colon I \to J$ one associates $f_+\colon I_+ \to J_+$ given by

$$f_+(i) := \begin{cases} f(i) & \text{if } i \in S, \\ *_J & \text{otherwise.} \end{cases}$$

The category of nonempty finite sets and partially defined surjections identifies in this manner with the category $\Omega$ considered by Pirashvili in [296, Section 2]. A hereditary species in the sense of Schmitt is thus a $\Omega$-set in the sense of Pirashvili.

Pirashvili's $\Omega$ is a subcategory of the opposite of Segal's category $\Gamma$. The latter is defined in [327, Definition 1.1]. A $\Gamma$-set is a contravariant functor from $\Gamma$ to the category of sets. Therefore, every $\Gamma$-set gives rise to a hereditary species.

We thank Clemens Berger for help with this remark.

*Warning.* Pirashvili renames Segal's category, so that Segal's $\Gamma$ is Pirashvili's $\Gamma^{\mathrm{op}}$.

**B.3.2. Schmitt's construction of positive cooperads.** Let $\mathrm{Q}$ be a hereditary species. Given a nonempty finite set $I$ and a nonempty subset $S$ of $I$, let

$$\rho_{I,S}\colon \mathrm{Q}[I] \to \mathrm{Q}[S]$$

denote the image under the functor $\mathrm{Q}$ of the partially defined surjection $(S, \mathrm{id}_S)\colon I \to S$. In addition, given a partition $X$ of $I$, let

$$\rho_{I,X}\colon \mathrm{Q}[I] \to \mathrm{Q}[X]$$

denote the image under the functor $\mathrm{Q}$ of the canonical projection to the quotient $I \twoheadrightarrow X$ (which is a partially defined surjection).

Consider the linearization $\mathbf{q} = \Bbbk\mathrm{Q}$, which is in particular a positive species. Given a nonempty finite set $I$ and a partition $X \vdash I$, define a map

$$\Delta\colon \mathbf{q}[I] \to \mathbf{q}[X] \otimes \bigotimes_{S\in X} \mathbf{q}[S] \qquad \text{by} \qquad x \mapsto \rho_{I,X}(x) \otimes \bigotimes_{S\in X} \rho_{I,S}(x)$$

for all $x \in \mathrm{Q}[I]$. Also, define a map $\epsilon\colon \mathbf{q}[I] \to \mathbf{X}[I]$ by sending all elements of $\mathrm{Q}[I]$ to 1 if $I$ is a singleton and to 0 otherwise. This defines morphisms

$$\Delta\colon \mathbf{q} \to \mathbf{q} \circ \mathbf{q} \qquad \text{and} \qquad \epsilon\colon \mathbf{q} \to \mathbf{X}.$$

**Proposition B.13 (Schmitt).** *Let* $\mathrm{Q}$ *be a hereditary species. The triple* $(\mathbf{q}, \Delta, \epsilon)$*, as defined above, is a positive cooperad.*

**B.3.3. Examples.** The positive exponential species $\mathbf{E}_+$ can be turned into a hereditary species: send a partially defined surjection $I \to J$ to the unique map that sends $*_I$ to $*_J$. The cooperad yielded by Proposition B.13 in this case is the positive commutative cooperad of Example B.6.

The positive linear order species $\mathbf{L}_+$ can also be turned into a hereditary species: for a partially defined surjection $f\colon I \to J$ the map $\mathbf{L}_+[I] \to \mathbf{L}_+[J]$ sends $l$ to $l_J$, where $l_J$ is defined as follows. Let $X := \{f^{-1}(j) : j \in J\}$ be the set of fibers of $f$. The order induced by $l$ on the subset $X' := \{\min S : S \in X\}$ of $I$ coincides with $l_J$ under the bijection $J \to X'$, $j \mapsto \min f^{-1}(j)$. The cooperad yielded by Proposition B.13 in this case is the Zinbiel cooperad of Example B.8.

Consider the cooperad of graphs of Example B.9. Since certain components of the coproduct of a graph $g$ are 0 (those corresponding to partitions of the vertex set which are not contractions of $g$), this cooperad structure does not arise from Schmitt's construction.

For a similar reason, the cooperads of Examples B.7, B.10 and B.11 do not arise from this construction either.

## B.4. General operads and cooperads

In Section 8.1.2, we discussed various monoidal structures on species. Among these, the substitution operation was defined only in the case when the second argument was a positive species. This special case gives rise to the notion of positive (co)operad studied in Sections B.1 and B.2.

The goal of this section is to discuss the substitution operation in full generality. In contrast with the case of positive species, there are two different notions of substitution for general species. They are discussed in Sections B.4.2 and B.4.4. The first one is given as a colimit; see (B.10) and (B.11) below. This operation gives rise to the general notion of operad (Section B.4.3). The second one is given as a limit and gives rise to the general notion of cooperad (Section B.4.5). Duality between the two notions is discussed in Sections B.4.6 and B.4.7. The connection between the general notion of (co)operad and that of positive (co)operad is discussed in Section B.4.8.

A careful discussion of associativity for the colimit version of substitution operation is given by Kelly [197]. Our presentation differs from his and is based on an explicit description of the iterated substitution products. This approach becomes essential when discussing the limit version of substitution, since this operation is not strongly associative. We employ the notion of lax monoidal category to overcome this difficulty (Definition D.3).

**B.4.1. Divided set powers and related functors.** We recall the notion of divided set power of a species due to Joyal [181, Section 2.2].

Let $\mathbf{m}$ be a species and $X$ a finite set. The *divided* $X$*-power* of $\mathbf{m}$ is the species $\mathbf{m}^{\cdot X}$ defined by

$$\mathbf{m}^{\cdot X}[I] := \bigoplus_{f\colon I \to X} \bigotimes_{x \in X} \mathbf{m}[f^{-1}(x)]. \tag{B.6}$$

The sum is over all functions from the finite set $I$ to the finite set $X$. Given a bijection $\sigma\colon I \to J$, the map

$$\mathbf{m}^{\cdot X}[\sigma]\colon \mathbf{m}^{\cdot X}[I] \to \mathbf{m}^{\cdot X}[J]$$

has components

$$\bigotimes_{x\in X} \mathbf{m}[f^{-1}(x)] \xrightarrow{\otimes\, \mathbf{m}\left[\sigma|_{f^{-1}(x)}\right]} \bigotimes_{x\in X} \mathbf{m}[g^{-1}(x)]$$

where $f\colon I \to X$ is given and $g := f\sigma^{-1}\colon J \to X$. In this manner, $\mathbf{m}^{\cdot X}$ is a species.

Recall the Cauchy product of species (8.6). A function $f\colon I \to [k]$ is equivalent to an ordered decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ via $S_x = f^{-1}(x)$, and therefore

$$\mathbf{m}^{\cdot k}[I] = \bigoplus_{I=S_1\sqcup\cdots\sqcup S_k} \mathbf{m}[S_1] \otimes \cdots \otimes \mathbf{m}[S_k].$$

Thus the species $\mathbf{m}^{\cdot k}$ is isomorphic to the Cauchy product of $\mathbf{m}$ with itself $k$ times, as suggested by the notation. More generally,

$$\mathbf{m}^{\cdot X} \cong \mathop{\cdot}_{x\in X} \mathbf{m}. \tag{B.7}$$

In other words, the divided $X$-power of $\mathbf{m}$ is the same as the unordered tensor product of $\mathbf{m}$ over $X$ (Section 1.4).

A bijection $\tau\colon X \to Y$ induces an isomorphism of species

$$\mathbf{m}^{\cdot \tau}\colon \mathbf{m}^{\cdot X} \to \mathbf{m}^{\cdot Y}$$

with components

$$\bigotimes_{x\in X} \mathbf{m}[f^{-1}(x)] \xrightarrow{\cong} \bigotimes_{y\in Y} \mathbf{m}[g^{-1}(y)]$$

where $f\colon I \to X$ is given and $g := \tau f\colon I \to Y$. The above map is the isomorphism between unordered tensor products induced by $\tau$. In addition, a morphism of species $\mathbf{m} \to \mathbf{n}$ induces a morphism of species $\mathbf{m}^{\cdot X} \to \mathbf{n}^{\cdot X}$. These constructions combine to give a functor

$$\mathsf{Sp} \times \mathsf{Set}^{\times} \to \mathsf{Sp}, \qquad (\mathbf{m}, X) \mapsto \mathbf{m}^{\cdot X}.$$

In particular, for each fixed species $\mathbf{m}$, there is a functor

$$\mathsf{Set}^{\times} \to \mathsf{Sp}, \qquad X \mapsto \mathbf{m}^{\cdot X}.$$

Let $\mathbf{p}$ be another species. We then have a species $\mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}$ whose value on a finite set $I$ is $\mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I]$. Moreover, in view of the above, we obtain a functor

$$\mathsf{Set}^{\times} \to \mathsf{Sp}, \qquad X \mapsto \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X} \tag{B.8}$$

for fixed species $\mathbf{p}$ and $\mathbf{m}$.

**B.4.2. Substitution of species: the colimit version.** Let $\mathbf{p}$ and $\mathbf{m}$ be arbitrary species. The *substitution* of $\mathbf{m}$ in $\mathbf{p}$ is the species $\mathbf{p} \circ \mathbf{m}$ defined by

$$\mathbf{p} \circ \mathbf{m} := \operatorname*{colim}_{X} \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}. \tag{B.9}$$

This is the colimit of the functor (B.8). Since the category $\mathsf{Sp}$ is cocomplete (Example A.16), the colimit exists. Moreover, since $\mathsf{Sp}$ is a functor category, the colimit can be calculated pointwise (item (iii) in Proposition A.10):

$$(\mathbf{p} \circ \mathbf{m})[I] = \operatorname*{colim}_{X} \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I]. \tag{B.10}$$

Note that even when $\mathbf{p}$ and $\mathbf{m}$ are finite-dimensional species, the colimit $\mathbf{p} \circ \mathbf{m}$ may not be so.

Using formula (A.8), equation (B.9) can be written in a more explicit form as

$$\mathbf{p} \circ \mathbf{m} = \bigoplus_{k \geq 0} (\mathbf{p}[k] \otimes \mathbf{m}^{\cdot k})_{\mathrm{S}_k}. \tag{B.11}$$

Since $\mathbf{p}$ is a species, the symmetric group $\mathrm{S}_k$ acts on $\mathbf{p}[k]$ for all $k$. Since $(\mathsf{Sp}, \cdot)$ is a symmetric monoidal category, $\mathrm{S}_k$ acts on $\mathbf{m}^{\cdot k}$ by permuting the tensor factors. The coinvariants in the above formula are taken with respect to the diagonal action of $\mathrm{S}_k$.

Formula (B.11) can be found in [137, Section 1.3.5].

We turn to associativity of substitution. Given finite sets $X$ and $Y$, let us write

$$\mathbf{p}[X : Y] := \bigoplus_{f : Y \to X} \bigotimes_{x \in X} \mathbf{p}[f^{-1}(x)]. \tag{B.12}$$

The direct sum is over all functions $f$ from $Y$ to $X$. With this notation, formula (B.10) may be rewritten as

$$(\mathbf{p} \circ \mathbf{m})[I] \cong \operatorname*{colim}_X \mathbf{p}[\{*\} : X] \otimes \mathbf{m}[X : I].$$

Here $\{*\}$ denotes a fixed singleton, so that there is a unique map $X \to \{*\}$.

Let $\mathbf{n}$ be a third species. Consider the species $\mathbf{p} \circ \mathbf{m} \circ \mathbf{n}$ defined by

$$(\mathbf{p} \circ \mathbf{m} \circ \mathbf{n})[I] := \operatorname*{colim}_{X,Y} \mathbf{p}[\{*\} : X] \otimes \mathbf{m}[X : Y] \otimes \mathbf{n}[Y : I].$$

Here both $X$ and $Y$ vary in the category $\mathsf{Set}^{\times}$, independently of each other.

**Lemma B.14.** *We have*

$$(\mathbf{p} \circ \mathbf{m}) \circ \mathbf{n}[I] \cong (\mathbf{p} \circ \mathbf{m} \circ \mathbf{n})[I] \cong \mathbf{p} \circ (\mathbf{m} \circ \mathbf{n})[I].$$

Proof. We first calculate with the left hand side.

$$\begin{aligned}
(\mathbf{p} \circ \mathbf{m}) \circ \mathbf{n}[I] &\cong \operatorname*{colim}_Y (\mathbf{p} \circ \mathbf{m})[\{*\} : Y] \otimes \mathbf{n}[Y : I] \\
&\cong \operatorname*{colim}_Y (\mathbf{p} \circ \mathbf{m})[Y] \otimes \mathbf{n}[Y : I] \\
&\cong \operatorname*{colim}_Y \operatorname*{colim}_X \mathbf{p}[\{*\} : X] \otimes \mathbf{m}[X : Y] \otimes \mathbf{n}[Y : I] \\
&\cong \operatorname*{colim}_{X,Y} \mathbf{p}[\{*\} : X] \otimes \mathbf{m}[X : Y] \otimes \mathbf{n}[Y : I] = (\mathbf{p} \circ \mathbf{m} \circ \mathbf{n})[I]
\end{aligned}$$

We have used that tensor products preserve colimits (Example A.11) and that double colimits can be calculated iteratively (Proposition A.10, item (iv)).

We now work with the right hand side.

$$
\begin{aligned}
&\mathbf{p}\circ(\mathbf{m}\circ\mathbf{n})[I]\\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[\{*\}:X]\otimes(\mathbf{m}\circ\mathbf{n})[X:I]\\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[X]\otimes\Big(\bigoplus_{f:I\to X}\bigotimes_{x\in X}(\mathbf{m}\circ\mathbf{n})[f^{-1}(x)]\Big)\\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[X]\otimes\Big(\bigoplus_{f:I\to X}\bigotimes_{x\in X}\operatorname*{colim}_{Y_x}\mathbf{m}[\{*\}:Y_x]\otimes\mathbf{n}[Y_x:f^{-1}(x)]\Big)\\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[X]\otimes\Big(\bigoplus_{f:I\to X}\bigotimes_{x\in X}\operatorname*{colim}_{Y_x}\mathbf{m}[Y_x]\otimes\big(\bigoplus_{g_x:f^{-1}(x)\to Y_x}\bigotimes_{y\in Y_x}\mathbf{n}[g_x^{-1}(y)]\big)\Big)\\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[X]\otimes\Big(\operatorname*{colim}_{(Y_x)_{x\in X}}\bigoplus_{f:I\to X}\bigotimes_{x\in X}\mathbf{m}[Y_x]\otimes\big(\bigoplus_{g_x:f^{-1}(x)\to Y_x}\bigotimes_{y\in Y_x}\mathbf{n}[g_x^{-1}(y)]\big)\Big)
\end{aligned}
$$

Above, we commuted colimits first with tensor products and then with direct sums as in Example A.11. This results in the inner-most colimit in the previous formula, which is calculated over the product category $(\mathsf{Set}^{\times})^X$. We proceed by commuting the tensor product over elements $x$ with the direct sums over functions $g_x$.

$$
\begin{aligned}
&\mathbf{p}\circ(\mathbf{m}\circ\mathbf{n})[I]\\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[X]\otimes\Big(\operatorname*{colim}_{(Y_x)_{x\in X}}\bigoplus_{f:I\to X}\bigoplus_{(g_x:f^{-1}(x)\to Y_x)_{x\in X}}\bigotimes_{x\in X}\big(\mathbf{m}[Y_x]\otimes\bigotimes_{y\in Y_x}\mathbf{n}[g_x^{-1}(y)]\big)\Big)\\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[X]\otimes\Big(\operatorname*{colim}_{(Y_x)_{x\in X}}\bigoplus_{f:I\to X}\bigoplus_{(g_x:f^{-1}(x)\to Y_x)_{x\in X}}\big(\bigotimes_{x\in X}\mathbf{m}[Y_x]\big)\otimes\big(\bigotimes_{\substack{x\in X\\ y\in Y_x}}\mathbf{n}[g_x^{-1}(y)]\big)\Big)
\end{aligned}
$$

Now, the category $\mathsf{Set}^X$ is equivalent to the slice category $\mathsf{Set}\downarrow X$. The equivalence sends the object $(Y_x)_{x\in X}$ to $h: Y\to X$ given by

$$
Y := \coprod_{x\in X} Y_x \quad\text{and}\quad h|_{Y_x}\equiv x,
$$

and the arrow $(g_x)_{x\in X}:\big(f^{-1}(x)\big)_{x\in X}\to(Y_x)_{x\in X}$ to $g: I\to Y$ given by $g|_{f^{-1}(x)} = g_x$. Note that

$$
\begin{tikzcd}
I \arrow[rr, "f"] \arrow[dr, "g"'] & & X \\
& Y \arrow[ur, "h"'] &
\end{tikzcd}
$$

commutes, so that $f$ is determined by $h$ and $g$. It follows that $(\mathsf{Set}^{\times})^X$ is equivalent to the groupoid $(\mathsf{Set}\downarrow X)^{\times}$.

In view of the above, the preceding calculation can be continued as follows.

$$
\mathbf{p}\circ(\mathbf{m}\circ\mathbf{n})[I]\cong \operatorname*{colim}_X \mathbf{p}[X]\otimes\Big(\operatorname*{colim}_{h:Y\to X}\bigoplus_{g:I\to Y}\bigotimes_{x\in X}\mathbf{m}[h^{-1}(x)]\otimes\bigotimes_{y\in Y}\mathbf{n}[g^{-1}(y)]\Big)
$$

The inner-most colimit is over the groupoid $(\mathsf{Set} \downarrow X)^{\times}$. We use (A.10) to calculate it, and proceed as follows.

$$\begin{aligned}
&\mathbf{p} \circ (\mathbf{m} \circ \mathbf{n})[I] \\
&\quad\cong \operatorname*{colim}_X \mathbf{p}[X] \otimes \Big( \operatorname*{colim}_Y \bigoplus_{h: Y \to X} \bigoplus_{g: I \to Y} \bigotimes_{x \in X} \mathbf{m}[h^{-1}(x)] \otimes \bigotimes_{y \in Y} \mathbf{n}[g^{-1}(y)] \Big) \\
&\quad\cong \operatorname*{colim}_X \operatorname*{colim}_Y \mathbf{p}[X] \otimes \big( \bigoplus_{h: Y \to X} \bigotimes_{x \in X} \mathbf{m}[h^{-1}(x)] \big) \otimes \big( \bigoplus_{g: I \to Y} \bigotimes_{y \in Y} \mathbf{n}[g^{-1}(y)] \big) \\
&\quad\cong \operatorname*{colim}_{X,Y} \mathbf{p}[\{*\} : X] \otimes \mathbf{m}[X : Y] \otimes \mathbf{n}[Y : I] = (\mathbf{p} \circ \mathbf{m} \circ \mathbf{n})[I]
\end{aligned}$$

This completes the proof. □

It follows that $(\mathsf{Sp}, \circ, \mathbf{X})$ is a monoidal category, where the unit object $\mathbf{X}$ is the species characteristic of singletons (8.3).

**B.4.3. Definition of operad: the general case.** An *operad* is a monoid in $(\mathsf{Sp}, \circ, \mathbf{X})$; a morphism of operads is a morphism of monoids.

An operad is thus a species $\mathbf{p}$ together with morphisms of species

$$\gamma \colon \mathbf{p} \circ \mathbf{p} \to \mathbf{p} \qquad \text{and} \qquad \eta \colon \mathbf{X} \to \mathbf{p}$$

which are associative and unital. As for positive operads, this can be made very explicit. The structure amounts to a linear map

$$\gamma_f \colon \mathbf{p}[X] \otimes \bigotimes_{x \in X} \mathbf{p}[f^{-1}(x)] \to \mathbf{p}[I], \tag{B.13}$$

for each (arbitrary) map $f \colon I \to X$ between finite sets $I$ and $X$, and a linear map

$$\eta_* \colon \Bbbk \to \mathbf{p}[\{*\}]. \tag{B.14}$$

for each singleton $\{*\}$. These maps are subject to the same list of axioms as in Section B.1.2, where now arbitrary maps are used instead of surjections.

**Example B.15.** The exponential species $\mathbf{E}$ carries an operad structure. The definition is as in Example B.1, with $\gamma_f$ and $\eta_*$ being the obvious isomorphisms. This is the *commutative operad*, denoted $\mathbf{Com}$.

The species $\mathbf{L}$ of linear orders is an operad. The operadic composition is given by ordinal sum, as in Example B.2. This is the *associative operad*, denoted $\mathbf{As}$.

The term operad and its first formal definition is due to May [261]; however this concept had appeared implicitly earlier or about the same time in the works of many others such as Boardman and Vogt [56], Lazard [222, 223], Lambek [219] and Stasheff [345]. Kelly [197] noted that operads may be viewed as monoids in $(\mathsf{Sp}, \circ, \mathbf{X})$; this is also done by Smirnov [330] and Joyal [181]. More recent references that treat this point are [137, Part 1], [260, Definition 1.67] and [226, Section A.2]. For generalizations of the notion of operad, see [34] and [226, Part II].

**B.4.4. Substitution of species: the limit version.** It is possible to define another operation on arbitrary species as follows:

$$\mathbf{p} \circ' \mathbf{m} := \lim_X \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}. \tag{B.15}$$

This is the limit of the functor (B.8). Since the category $\mathsf{Sp}$ is complete (Example A.16), the limit exists and it can be calculated pointwise:

$$(\mathbf{p} \circ' \mathbf{m})[I] = \lim_{X} \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I]. \tag{B.16}$$

Even when $\mathbf{p}$ and $\mathbf{m}$ are finite-dimensional species, the limit $\mathbf{p} \circ' \mathbf{m}$ may not be so.

Using (the dual of) formula (A.8), equation (B.15) can be written in a more explicit form as

$$\mathbf{p} \circ' \mathbf{m} = \prod_{k \geq 0} (\mathbf{p}[k] \otimes \mathbf{m}^{\cdot k})^{\mathrm{S}_k}. \tag{B.17}$$

Since $\mathbf{p}$ is a species, the symmetric group $\mathrm{S}_k$ acts on $\mathbf{p}[k]$ for all $k$. Since $(\mathsf{Sp}, \cdot)$ is a symmetric monoidal category, $\mathrm{S}_k$ acts on $\mathbf{m}^{\cdot k}$ by permuting the tensor factors. The invariants in the above formula are taken with respect to the diagonal action of $\mathrm{S}_k$.

Let us consider associativity for the operation $\circ'$. Since limits commute with finite direct sums, we have

$$(\mathbf{p} \circ' \mathbf{m})[I] \cong \lim_{X} \mathbf{p}[\{*\} : X] \otimes \mathbf{m}[X : I],$$

with $\mathbf{p}[X : Y]$ as in (B.12). Given a third species $\mathbf{n}$, consider the species $\mathbf{p} \circ \mathbf{m} \circ \mathbf{n}$ defined by

$$(\mathbf{p} \circ' \mathbf{m} \circ' \mathbf{n})[I] := \lim_{X,Y} \mathbf{p}[\{*\} : X] \otimes \mathbf{m}[X : Y] \otimes \mathbf{n}[Y : I].$$

The canonical maps (A.9) yield maps

$$(\mathbf{p} \circ' \mathbf{m}) \circ' \mathbf{n}[I] \to (\mathbf{p} \circ' \mathbf{m} \circ' \mathbf{n})[I] \leftarrow \mathbf{p} \circ' (\mathbf{m} \circ' \mathbf{n})[I].$$

At this point, the situation regarding the operation $\circ'$ is no longer as for the operation $\circ$: these maps are not isomorphisms in general, and the operation $\circ'$ is not strongly associative. The proof of Lemma B.14, which gave the result for $\circ$, breaks down in view of the fact that tensor products do not preserve limits (Example A.15).

However, the above definitions of $\mathbf{p} \circ' \mathbf{m}$ and $\mathbf{p} \circ' \mathbf{m} \circ' \mathbf{n}$ can be extended to any number of factors in the obvious manner. These satisfy the conditions in Definition D.3 and thus turn $(\mathsf{Sp}, \circ', \mathbf{X})$ into a lax monoidal category. The unit object is the species $\mathbf{X}$ characteristic of singletons (8.3).

In the context of lax monoidal categories it is still possible to consider the notion of (co)monoid and also of (co)lax monoidal functors. (We do not provide the details of this.)

**B.4.5. Definition of cooperad: the general case.** A *cooperad* is a comonoid in the lax monoidal category $(\mathsf{Sp}, \circ', \mathbf{X})$.

A cooperad structure on a species $\mathbf{p}$ amounts to a linear map

$$\mathbf{p}[I] \to \mathbf{p}[X] \otimes \bigotimes_{x \in X} \mathbf{p}[f^{-1}(x)] \tag{B.18}$$

for each (arbitrary) map $f : I \to X$ between finite sets $I$ and $X$, and a linear map

$$\mathbf{p}[\{*\}] \to \Bbbk \tag{B.19}$$

for each singleton $\{*\}$. The axioms are obtained from those in Section B.1.2 by reversing the arrows.

**Example B.16.** The exponential species $\mathbf{E}$ carries a cooperad structure, for which the maps (B.18) and (B.19) are the obvious isomorphisms. This is the commutative cooperad, denoted $\mathbf{Com}^*$.

The species $\mathbf{L}$ is also a cooperad, with structure maps defined in the same manner as in Example B.7. This the associative cooperad, denoted $\mathbf{As}^*$.

These cooperads are dual to the operads $\mathbf{Com}$ and $\mathbf{As}$, in the sense of Sections B.4.6 and B.4.7 below.

**Remark B.17.** Our definition of cooperad differs from the one by Fresse [137, Section 1.2.17]. His cooperads are comonoids in $(\mathsf{Sp}, \circ, \mathbf{X})$. The norm map (Section 2.5) defines a transformation

$$(\mathbf{p}[k] \otimes \mathbf{m}^{\cdot k})_{\mathrm{S}_k} \to (\mathbf{p}[k] \otimes \mathbf{m}^{\cdot k})^{\mathrm{S}_k}$$

This gives rise to a colax monoidal functor

$$(\mathsf{Sp}, \circ, \mathbf{X}) \to (\mathsf{Sp}, \circ', \mathbf{X})$$

which is the identity on objects. Hence, every cooperad in Fresse's sense is a cooperad in ours. The converse is not true. The commutative and associative cooperads are not cooperads in Fresse's sense.

**B.4.6. The dual of a cooperad.** Let $\mathbf{p}$ and $\mathbf{m}$ be arbitrary species (not necessarily positive or finite-dimensional).

**Lemma B.18.** *For each finite set $I$ and each $k \geq 0$, the $\Bbbk\mathrm{S}_k$-module*

$$\mathbf{m}^{\cdot k}[I]$$

*is free, where $\mathrm{S}_k$ acts on $\mathbf{m}^{\cdot k}$ by permuting the tensor factors.*

Proof. We employ Notation 11.1. We have

$$\mathbf{m}^{\cdot k}[I] = \bigoplus_{\substack{F \vDash I \\ \deg(F)=k}} \mathbf{m}(F) = \bigoplus_{\substack{X \vdash I \\ \deg(X)=k}} \bigoplus_{\substack{F \vDash I \\ \mathrm{supp}(F)=X}} \mathbf{m}(F).$$

For each fixed $X$, the space

$$\bigoplus_{\substack{F \vDash I \\ \mathrm{supp}(F)=X}} \mathbf{m}(F)$$

is stable under the action of $\mathrm{S}_k$ and isomorphic to

$$\mathbf{m}(X) \otimes \Bbbk\{F \vDash I \mid \mathrm{supp}(F) = X\}$$

as $\Bbbk\mathrm{S}_k$-modules, where $\mathrm{S}_k$ acts trivially on $\mathbf{m}(X)$ and permutes the basis of the space $\Bbbk\{F \vDash I \mid \mathrm{supp}(F) = X\}$. Permuting the parts of a set composition always yields a different set composition. Hence, the latter action has no fixed points, and the $\Bbbk\mathrm{S}_k$-modules above are free. □

It is worth pointing out that the analogue of the latter fact for vector spaces $V$ is not true: the action of $\mathrm{S}_k$ on $V^{\otimes k}$ is not free.

Using now Lemma 2.17 we deduce that the $\Bbbk\mathrm{S}_k$-module

$$\mathbf{p}[k] \otimes \mathbf{m}^{\cdot k}[I],$$

where $\mathrm{S}_k$ acts diagonally, also is free. Then, by Lemma 2.21, item (b), there is a canonical isomorphism

$$\Big(\big(\mathbf{p}[k] \otimes \mathbf{m}^{\cdot k}[I]\big)^*\Big)_{\mathrm{S}_k} \cong \Big(\big(\mathbf{p}[k] \otimes \mathbf{m}^{\cdot k}[I]\big)^{\mathrm{S}_k}\Big)^*.$$

We make use of this isomorphism to form the composite below. We also use (B.11), (B.17) and canonical inclusions.

$$\begin{array}{ccc}
(\mathbf{p}^* \circ \mathbf{m}^*)[I] & & \\
\| & & \\
\bigoplus_{k\geq 0}\Big(\mathbf{p}[k]^* \otimes \mathbf{m}^{\cdot k}[I]^*\Big)_{\mathrm{S}_k} & \hookrightarrow \bigoplus_{k\geq 0}\Big(\big(\mathbf{p}[k]\otimes \mathbf{m}^{\cdot k}[I]\big)^*\Big)_{\mathrm{S}_k} & \\
& \| \wr & \\
& \bigoplus_{k\geq 0}\Big(\big(\mathbf{p}[k]\otimes \mathbf{m}^{\cdot k}[I]\big)^{\mathrm{S}_k}\Big)^* & \hookrightarrow \Big(\prod_{k\geq 0}\big(\mathbf{p}[k]\otimes \mathbf{m}^{\cdot k}[I]\big)^{\mathrm{S}_k}\Big)^* \\
& & \| \\
& & (\mathbf{p}\circ' \mathbf{m})[I]^*
\end{array}$$

The above maps define a natural transformation:

$$\mathbf{p}^* \circ \mathbf{m}^* \to (\mathbf{p} \circ' \mathbf{m})^*.$$

We endow the duality functor with this transformation and with the canonical identification $\mathbf{X} \cong \mathbf{X}^*$.

**Proposition B.19.** *With the above structure, the duality functor*

$$(\mathsf{Sp}, \circ')^{\mathrm{op}} \to (\mathsf{Sp}, \circ)$$

*is lax monoidal.*

As a consequence:

**Corollary B.20.** *Let* $\mathbf{p}$ *be an arbitrary cooperad. Then* $\mathbf{p}^*$ *is an operad.*

**B.4.7. The dual of an operad.** Assume now that both species $\mathbf{p}$ and $\mathbf{m}$ are finite-dimensional (Definition 8.2). In this situation, we may construct the following composite.

$$\begin{array}{ccc}
(\mathbf{p} \circ \mathbf{m})[I]^* & & \\
\| & & \\
\Big(\bigoplus_{k\geq 0}(\mathbf{p}[k]\otimes \mathbf{m}^{\cdot k}[I])_{\mathrm{S}_k}\Big)^* & \cong \prod_{k\geq 0}\Big(\big(\mathbf{p}[k]\otimes \mathbf{m}^{\cdot k}[I]\big)_{\mathrm{S}_k}\Big)^* & \\
& \| \wr & \\
& \prod_{k\geq 0}\Big(\big(\mathbf{p}[k]\otimes \mathbf{m}^{\cdot k}[I]\big)^*\Big)^{\mathrm{S}_k} & \cong \prod_{k\geq 0}\big(\mathbf{p}[k]^*\otimes \mathbf{m}^{\cdot k}[I]^*\big)^{\mathrm{S}_k} \\
& & \| \\
& & (\mathbf{p}^* \circ' \mathbf{m}^*)[I]
\end{array}$$

The first isomorphism is canonical. The vertical isomorphism uses Lemma 2.21, item (a). The third isomorphism in the chain uses finite-dimensionality.

These maps define an invertible transformation which is natural in the finite-dimensional species $\mathbf{p}$ and $\mathbf{m}$:

$$(\mathbf{p} \circ \mathbf{m})^* \to \mathbf{p}^* \circ' \mathbf{m}^*.$$

**Proposition B.21.** *When restricted to finite-dimensional species, and with the above structure, the duality functor*

$$(\mathsf{Sp}, \circ)^{\mathrm{op}} \to (\mathsf{Sp}, \circ')$$

*is (co)strong monoidal.*

As a consequence:

**Corollary B.22.** *Let* $\mathbf{p}$ *be a finite-dimensional operad. Then* $\mathbf{p}^*$ *is a cooperad.*

**B.4.8. Positive (co)operads versus general (co)operads.** In the preceding sections, we have defined two operations $\circ$ and $\circ'$ on the category of species. We now check that when restricted to positive species, they both coincide with the substitution product defined in Section B.1.1 (and which we also denote by $\circ$).

Let $\mathbf{p}$ be an arbitrary species and $\mathbf{m}$ a positive one. We analyze the divided powers of $\mathbf{m}$. Since $\mathbf{m}[\emptyset] = 0$, the only functions $f\colon I \to X$ that contribute to the sum (B.6) are the surjective ones. For such $f$,

$$\widetilde{X} := \{f^{-1}(x) \mid x \in X\}$$

is a partition of $I$ (no empty blocks). The map $f$ induces a bijection $\widetilde{X} \cong X$ and

$$\mathbf{p}[\widetilde{X}] \otimes \mathbf{m}^{\cdot \widetilde{X}}[I] \cong \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I].$$

Composing with the canonical maps we obtain

$$\prod_{\widetilde{X} \vdash I} \mathbf{p}[\widetilde{X}] \otimes \Big(\bigotimes_{S \in \widetilde{X}} \mathbf{m}[S]\Big) \to \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I] \to \bigoplus_{\widetilde{X} \vdash I} \mathbf{p}[\widetilde{X}] \otimes \Big(\bigotimes_{S \in \widetilde{X}} \mathbf{m}[S]\Big)$$

where the product and the sum are over all partitions $\widetilde{X}$ of $I$. If $\tau\colon X \to Y$ is a bijection and $g = \tau f$, then $\{f^{-1}(x) \mid x \in X\}$ and $\{g^{-1}(y) \mid y \in Y\}$ are the same partition of $I$. It follows that

$$\lim_{X} \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I] = \prod_{X \vdash I} \mathbf{p}[X] \otimes \Big(\bigotimes_{S \in X} \mathbf{m}[S]\Big).$$

and

$$\operatorname*{colim}_{X} \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I] = \bigoplus_{X \vdash I} \mathbf{p}[X] \otimes \Big(\bigotimes_{S \in X} \mathbf{m}[S]\Big).$$

Since there are only finitely-many partitions of $I$, the above sum and product coincide. Thus, all three definitions (B.1), (B.11) and (B.15) coincide in this case. More formally:

**Proposition B.23.** *The inclusion functor* $inc\colon \mathsf{Sp}_+ \to \mathsf{Sp}$ *is strong monoidal in two ways: As a functor*

$$(\mathsf{Sp}_+, \circ, \mathbf{X}) \to (\mathsf{Sp}, \circ, \mathbf{X})$$

*and as a functor*

$$(\mathsf{Sp}_+, \circ, \mathbf{X}) \to (\mathsf{Sp}, \circ', \mathbf{X}).$$

Recall from Proposition 3.94 that the left adjoint of a strong functor carries a canonical colax structure while the right adjoint carries a canonical lax structure. Further recall from (8.57) that the functor $(-)_+\colon \mathsf{Sp} \to \mathsf{Sp}_+$ of (8.56) is a two-sided adjoint of $inc\colon \mathsf{Sp}_+ \to \mathsf{Sp}$. It then follows that

$$(-)_+\colon (\mathsf{Sp}, \circ, \mathbf{X}) \to (\mathsf{Sp}_+, \circ, \mathbf{X}) \quad \text{and} \quad (-)_+\colon (\mathsf{Sp}, \circ', \mathbf{X}) \to (\mathsf{Sp}_+, \circ, \mathbf{X})$$

carry both a lax and a colax monoidal structure. Since (co)lax functors preserve (co)monoids (Proposition 3.29), we deduce:

**Corollary B.24.** *If* $\mathbf{p}$ *is a positive (co)operad, then it is also a (co)operad. If* $\mathbf{p}$ *is a (co)operad, then* $\mathbf{p}_+$ *is a positive (co)operad.*

As an illustration, the commutative and associative operads **Com** and **As** give rise to the positive commutative and associative operads $\mathbf{Com}_+$ and $\mathbf{As}_+$.

## B.5. Modules over operads and monoids in species

In Chapter 4 we defined a type of monoid for each operad and discussed many explicit examples. Now consider such types of monoid in the monoidal category $(\mathsf{Sp}, \cdot)$ of species under Cauchy product (8.6). It turns out in this case that the same notion arises in a different manner: if $\mathbf{p}$ is an operad, then a $\mathbf{p}$-monoid structure on a species is the same as a $\mathbf{p}$-module structure on that species.

In this section, we first review modules, then discuss the internal Hom for the substitution product, and finally use it to derive the equivalence between monoids and modules.

**B.5.1. Modules over operads.** Let $\mathbf{p}$ be an operad. Since operads are monoids in a certain monoidal category, there is a corresponding notion of module (Definition 1.12). Thus, a (left) $\mathbf{p}$-module is a species $\mathbf{m}$ with a map $\chi\colon \mathbf{p} \circ \mathbf{m} \to \mathbf{m}$ which is associative and unital.

A morphism $\mathbf{m} \to \mathbf{n}$ of $\mathbf{p}$-modules is a map $\mathbf{m} \to \mathbf{n}$ of species which commutes with the module structure maps.

More explicitly, a $\mathbf{p}$-module is a species $\mathbf{m}$ with the following structure. For each map $f\colon I \to X$ between finite sets, there is a linear map

$$\chi_f\colon \mathbf{p}[X] \otimes \bigotimes_{x \in X} \mathbf{m}[f^{-1}(x)] \to \mathbf{m}[I]. \tag{B.20}$$

These maps are subject to conditions similar to those for an operad given in Section B.1.2. (Replace the last instance of $\mathbf{p}$ by $\mathbf{m}$ in each entry of all diagrams, and omit the first unitality diagram.)

**Proposition B.25.** *For an operad* $\mathbf{p}$, *the free* $\mathbf{p}$*-module over a species* $\mathbf{m}$ *is* $\mathbf{p} \circ \mathbf{m}$.

The $\mathbf{p}$-module structure on $\mathbf{p} \circ \mathbf{m}$ is

$$\mathbf{p} \circ \mathbf{p} \circ \mathbf{m} \xrightarrow{\gamma \circ \mathrm{id}} \mathbf{p} \circ \mathbf{m}$$

where $\gamma$ is the operadic composition of $\mathbf{p}$. The inclusion of generators is

$$\mathbf{m} = \mathbf{X} \circ \mathbf{m} \xrightarrow{\eta \circ \mathrm{id}} \mathbf{p} \circ \mathbf{m},$$

where $\eta$ is the operadic unit of $\mathbf{p}$.

The above result is stated in [137, Section 2.1.10]; it holds in the general context of monoidal categories.

**B.5.2. Internal Hom for the substitution of species.** The discussion of this section complements that on substitution of species in Section B.4.2.

Given species $\mathbf{m}$ and $\mathbf{n}$, define a species $\mathcal{H}^\circ(\mathbf{m}, \mathbf{n})$ by

$$\mathcal{H}^\circ(\mathbf{m}, \mathbf{n})[X] := \mathrm{Hom}_{\mathsf{Sp}}(\mathbf{m}^{\cdot X}, \mathbf{n}),$$

for any finite set $X$. This is the set of all maps of species from the divided power $\mathbf{m}^{\cdot X}$ to $\mathbf{n}$.

A bijection $\sigma\colon X \to Y$ induces an isomorphism $\mathbf{m}^{\cdot X} \to \mathbf{m}^{\cdot Y}$ and hence a bijection

$$\mathcal{H}^\circ(\mathbf{m}, \mathbf{n})[X] \to \mathcal{H}^\circ(\mathbf{m}, \mathbf{n})[Y].$$

In this manner, $\mathcal{H}^\circ(\mathbf{m}, \mathbf{n})$ is a species.

**Proposition B.26.** *For any species* $\mathbf{p}, \mathbf{m}$, *and* $\mathbf{n}$, *there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Sp}}(\mathbf{p} \circ \mathbf{m}, \mathbf{n}) \cong \mathrm{Hom}_{\mathsf{Sp}}\big(\mathbf{p}, \mathcal{H}^\circ(\mathbf{m}, \mathbf{n})\big).$$

PROOF. Start from a map $\mathbf{p} \circ \mathbf{m} \to \mathbf{n}$. For each finite set $I$, we are given a map

$$\operatorname*{colim}_X \mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I] \to \mathbf{n}[I],$$

and this family of maps is natural in $I$. By universality of the colimit (Definition A.9), this is equivalent to a family of maps

$$\mathbf{p}[X] \otimes \mathbf{m}^{\cdot X}[I] \to \mathbf{n}[I]$$

which are natural in $I$ and form a cone with respect to $X$. By the familiar adjunction between $\otimes$ and $\mathrm{Hom}_{\mathsf{Vec}}$ (A.6), this is equivalent to a family of maps

$$\mathbf{p}[X] \to \mathrm{Hom}_{\mathsf{Vec}}(\mathbf{m}^{\cdot X}[I], \mathbf{n}[I])$$

subject to the naturality and cone conditions. Naturality in $I$ implies that this is equivalent to a map

$$\mathbf{p}[X] \to \mathrm{Hom}_{\mathsf{Sp}}(\mathbf{m}^{\cdot X}, \mathbf{n}),$$

and the cone property translates into the naturality in $X$ of this map. In conclusion, the given data is equivalent to a map

$$\mathbf{p} \to \mathcal{H}^{\circ}(\mathbf{m}, \mathbf{n}). \qquad \square$$

Proposition B.26 says that $\mathcal{H}^{\circ}$ is the internal Hom for the monoidal category $(\mathsf{Sp}, \circ)$, in the sense of Section 1.3. It follows from Proposition 1.28 that the species

$$\mathcal{E}^{\circ}(\mathbf{m}) := \mathcal{H}^{\circ}(\mathbf{m}, \mathbf{m}) \tag{B.21}$$

is an operad and that, if $\mathbf{p}$ is an operad, then a $\mathbf{p}$-module structure on a species $\mathbf{m}$ is equivalent to a morphism of operads

$$\mathbf{p} \to \mathcal{E}^{\circ}(\mathbf{m}).$$

**B.5.3. Equivalence between monoids and modules.** In Section 4.2.1 we defined the endomorphism operad of an object in an arbitrary symmetric linear monoidal category. Choosing the latter to be the category $(\mathsf{Sp}, \cdot)$ of species under Cauchy product, we obtain the endomorphism operad $\mathbf{End}_{\mathbf{m}}$ of a species $\mathbf{m}$. From (4.6) we have that

$$\mathbf{End}_{\mathbf{m}}[I] := \mathrm{Hom}_{\mathsf{Sp}}(\mathbf{m}^{\cdot I}, \mathbf{m}).$$

On the other hand, we have the operad $\mathcal{E}^{\circ}(\mathbf{m})$ of (B.21).

**Proposition B.27.** *Let $\mathbf{m}$ be a species. The operads $\mathcal{E}^{\circ}(\mathbf{m})$ and $\mathbf{End}_{\mathbf{m}}$ are the same.*

PROOF. Equation (B.7) shows that

$$\mathcal{E}^{\circ}(\mathbf{m})[X] = \mathrm{Hom}_{\mathsf{Sp}}(\mathbf{m}^{\cdot X}, \mathbf{m}) = \mathbf{End}_{\mathbf{m}}[X].$$

Thus $\mathcal{E}^{\circ}(\mathbf{m})$ and $\mathbf{End}_{\mathbf{m}}$ agree as species. The verification that their operad structures agree as well is straightforward, and we omit it. $\square$

It follows that the operad structure of $\mathcal{E}^{\circ}(\mathbf{m})$ can be described purely in terms of the Cauchy product on species.

Let $\mathbf{p}$ be an operad. In Section 4.2.2 we discussed the notion of $\mathbf{p}$-monoids in an arbitrary symmetric linear monoidal category. Choosing the latter to be the category $(\mathsf{Sp}, \cdot)$ of species under Cauchy product, we obtain the notion of $\mathbf{p}$-monoids in species.

It turns out that this coincides with the notion of $\mathbf{p}$-modules of Section B.5.1.

**Corollary B.28.** *Let* $\mathbf{p}$ *be an operad and* $\mathbf{m}$ *a species. A* $\mathbf{p}$*-module structure on a species* $\mathbf{m}$ *is equivalent to a* $\mathbf{p}$*-monoid structure on* $\mathbf{m}$*.*

PROOF. As discussed in Section B.5.2, a $\mathbf{p}$-module structure on $\mathbf{m}$ is equivalent to a morphism of operads $\mathbf{p} \to \mathcal{E}^{\circ}(\mathbf{m})$. By Proposition B.27, this is the same as a morphism of operads $\mathbf{p} \to \mathbf{End}_{\mathbf{m}}$, which according to Definition 4.16 is a $\mathbf{p}$-monoid structure on $\mathbf{m}$. $\square$

It follows from Corollary B.28 that the category of $\mathbf{p}$-modules in $(\mathsf{Sp}, \cdot)$ is equivalent to the category of $\mathbf{p}$-monoids in $(\mathsf{Sp}, \cdot)$. In particular, Proposition B.25 also describes the free $\mathbf{p}$-monoid over a species $\mathbf{m}$.

**Example B.29.** Consider the associative operad $\mathbf{As}$ (Example B.15). Let $\mathbf{m}$ be an $\mathbf{As}$-module, or equivalently, an $\mathbf{As}$-monoid. We found in Example 4.18 that such a structure is equivalent to a monoid structure on the object $\mathbf{m}$ in the monoidal category $(\mathsf{Sp}, \cdot)$. Proposition B.25 says that the free monoid in $(\mathsf{Sp}, \cdot)$ over a species $\mathbf{m}$ is given by $\mathbf{L} \circ \mathbf{m}$. This is in agreement with the construction of Section 11.2.1.

**Example B.30.** Consider the Zinbiel operad $\mathbf{Zinb}$ (Example B.3). The underlying species is $\mathbf{L}_+$, the positive part of the species of linear orders. Since any operad is a module over itself, $\mathbf{L}_+$ is a module over the operad $\mathbf{Zinb}$. Hence, by Corollary B.28, $\mathbf{L}_+$ is a $\mathbf{Zinb}$-monoid in $(\mathsf{Sp}_+, \cdot)$, or according to Table 4.1, a Zinbiel monoid, as in Definition 4.2.

The Zinbiel monoid structure of $\mathbf{L}_+$ can be explicitly described. The structure map $\mu\colon \mathbf{L}_+ \cdot \mathbf{L}_+ \to \mathbf{L}_+$ is as follows. Choose a finite set $I$ and a decomposition $I = S \sqcup T$ into nonempty disjoint subsets. The corresponding component of $\mu$ is

$$\mu_{S,T}\colon \mathbf{L}_+[S] \otimes \mathbf{L}_+[T] \to \mathbf{L}_+[I], \qquad l_1 \otimes l_2 \mapsto \sum_{l} l,$$

where the sum is over all linear orders $l \in \mathbf{L}[I]$ such that

$$l_{|_S} = l_1, \quad l_{|_T} = l_2, \quad \text{and} \quad \min(l_1) < \min(l_2) \text{ with respect to } l.$$

The latter condition can be equivalently formulated by saying that the minimum element of $I$ according to the order $l$ belongs to $S$.

The commutativity of the diagram in Definition 4.2 boils down to the following fact. Given a decomposition $I = S \sqcup T \sqcup U$ into nonempty disjoint subsets, the set of linear orders $l$ on $I$ with prescribed restrictions to each of the three subsets and such that $\min(l) \in S$ can be decomposed into two classes, according to whether $\min(l_{|_T}) < \min(l_{|_U})$ or $\min(l_{|_T}) > \min(l_{|_U})$ (always with respect to the order $l$). Thus, $\mathbf{L}_+$ is a Zinbiel monoid.

Proposition 4.5 implies that $\mu + \mu\beta$ defines a nonunital commutative monoid structure on $\mathbf{L}_+$. Note that $\mu\beta$ is given by a similar sum to that above, but involving those linear orders $l$ such that $\min(l_1) > \min(l_2)$. It follows that $\mu + \mu\beta$ is given by the sum of all shuffles of $l_1$ and $l_2$.

This is not the monoid structure of $\mathbf{L}_+$ of Example 8.16, but rather the commutative monoid structure of $\mathbf{L}_+^*$ of Example 8.24, after the canonical identification of species $\mathbf{L} \cong \mathbf{L}^*$, $l \mapsto l^*$ for all linear orders $l$.

## B.6. Hopf operads

There is an interchange law between the substitution and Hadamard products on species which leads to the notion of Hopf operad. Hopf operads first appeared in work of Getzler and Jones [145] (also see [260, Definition 3.135]). Some recent papers dealing with this notion are [233, 234, 241].

**B.6.1. Substitution and Hadamard products on positive species.** Consider the operations $\circ$ (substitution) and $\times$ (Hadamard product) defined in Section 8.1.2. They restrict to the category $\mathsf{Sp}_+$ of positive species. The category $(\mathsf{Sp}_+, \circ, \mathbf{X})$ is monoidal (Section B.1.1) and the category $(\mathsf{Sp}_+, \times, \mathbf{E}_+)$ is braided monoidal (the unit object is the positive exponential species of Example B.1). We now show that these structures combine into that of a 2-monoidal category (Section 6.1).

Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ be positive species. Consider the species $(\mathbf{p} \times \mathbf{q}) \circ (\mathbf{r} \times \mathbf{s})$ and $(\mathbf{p} \circ \mathbf{r}) \times (\mathbf{q} \circ \mathbf{s})$. Their $I$-components, for $I$ a nonempty finite set, are

$$\bigoplus_{X \vdash I} (\mathbf{p}[X] \otimes \mathbf{q}[X]) \otimes \Big( \bigotimes_{S \in X} \mathbf{r}[S] \otimes \mathbf{s}[S] ) \Big)$$

and

$$\bigoplus_{X,Y \vdash I} \mathbf{p}[X] \otimes \Big( \bigotimes_{S \in X} \mathbf{r}[S] \Big) \otimes \mathbf{q}[Y] \otimes \Big( \bigotimes_{T \in Y} \mathbf{s}[T] \Big).$$

Each summand of the former appears also in the latter (for $Y = X$, $T = S$). Rearranging the middle factors we thus obtain a map

$$\zeta \colon (\mathbf{p} \times \mathbf{q}) \circ (\mathbf{r} \times \mathbf{s}) \to (\mathbf{p} \circ \mathbf{r}) \times (\mathbf{q} \circ \mathbf{s}). \tag{B.22}$$

Consider also the maps

$$\Delta_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}, \qquad \mu_{\mathbf{E}_+} \colon \mathbf{E}_+ \circ \mathbf{E}_+ \to \mathbf{E}_+, \qquad \iota_{\mathbf{E}_+} = \epsilon_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{E}_+. \tag{B.23}$$

The first map is defined to be the obvious isomorphism. The second and third maps are the operadic composition and unit of the positive commutative operad, as in Example B.1.

**Proposition B.31.** *With the structure maps* (B.22) *and* (B.23), $(\mathsf{Sp}_+, \circ, \mathbf{X}, \times, \mathbf{E}_+)$ *is a* 2*-monoidal category. Moreover, it is* $\times$*-braided.*

The proof is a straightforward verification of the axioms. By applying the contragredient construction, one sees that $(\mathsf{Sp}_+, \times, \circ)$ is also a 2-monoidal category.

**B.6.2. Hadamard product of positive operads.** From Propositions B.31 and 6.35, we obtain that

$$(\mathsf{Mon}(\mathsf{Sp}_+, \circ), \times) \qquad \text{and} \qquad (\mathsf{Comon}(\mathsf{Sp}_+, \times), \circ)$$

are both monoidal categories. We expand further on these statements.

First, recall that $\mathsf{Comon}(\mathsf{Sp}, \times)$ is the category of species with values in $\mathsf{Coalg}$, the category of coalgebras (Section 8.2.2). Hence the second statement above says that the substitution product can be extended to species with values in $\mathsf{Coalg}$.

Next, recall that $\mathsf{Mon}(\mathsf{Sp}_+, \circ)$ is the category of positive operads. Hence the first statement above says that the category of positive operads has a monoidal structure given by the Hadamard product. In particular, if $\mathbf{p}$ and $\mathbf{q}$ are positive operads, then so is $\mathbf{p} \times \mathbf{q}$. The structure maps of $\mathbf{p} \times \mathbf{q}$ are obtained by tensoring the structure maps of $\mathbf{q}$ and $\mathbf{p}$. Explicitly, let

$$\mathbf{p}[X] \otimes \bigotimes_{x\in X} \mathbf{p}[f^{-1}(x)] \to \mathbf{p}[I], \qquad \Bbbk \to \mathbf{p}[\{*\}]$$

$$a \otimes \bigotimes_{x\in X} a_x \mapsto c \qquad 1 \mapsto i$$

and

$$\mathbf{q}[X] \otimes \bigotimes_{x\in X} \mathbf{q}[f^{-1}(x)], \to \mathbf{q}[I] \qquad \Bbbk \to \mathbf{q}[\{*\}]$$

$$b \otimes \bigotimes_{x\in X} b_x \mapsto d \qquad 1 \mapsto j$$

be the generic notation for the structure maps of $\mathbf{p}$ and $\mathbf{q}$ respectively, as in (4.7). Then

$$(\mathbf{p}\times\mathbf{q})[X] \otimes \bigotimes_{x\in X} (\mathbf{p}\times\mathbf{q})[f^{-1}(x)] \to (\mathbf{p}\times\mathbf{q})[I], \qquad \Bbbk \cong \Bbbk\otimes\Bbbk \to (\mathbf{p}\times\mathbf{q})[\{*\}]$$

$$(a\otimes b) \otimes \bigotimes_{x\in X} (a_x \otimes b_x) \mapsto c\otimes d \qquad 1\otimes 1 \mapsto i \otimes j$$

yield the structure maps of $\mathbf{p}\times\mathbf{q}$.

We note that

$$\mathbf{q}\times\mathbf{p} \cong \mathbf{p}\times\mathbf{q} \qquad \text{and} \qquad \mathbf{E}\times\mathbf{p} \cong \mathbf{p}$$

as operads.

**B.6.3. Pointing of operads.** The pointing operation is defined in Section 8.13.7. We saw in Example B.4 that the species of elements $\mathbf{e}$ carries a positive operad structure, denoted **Perm**. Since $\mathbf{e}$ is self-dual, this species also carries a positive cooperad structure, denoted $\mathbf{Perm}^*$. Since $\mathbf{p}^\bullet = \mathbf{p}\times\mathbf{e}$, we may use this (co)operad structure to turn pointing into a lax and colax monoidal functor

$$(-)^\bullet \colon (\mathsf{Sp}_+, \circ) \to (\mathsf{Sp}_+, \circ).$$

In particular, if $\mathbf{p}$ is a positive (co)operad, then so is $\mathbf{p}^\bullet$ and $(\mathbf{p}^\bullet)^* \cong (\mathbf{p}^*)^\bullet$ as positive (co)operads.

We have $\mathbf{Com}_+^\bullet = \mathbf{Perm}$. The operad $\mathbf{As}_+^\bullet$ is the *diassociative* operad. Algebras over this operad are the dialgebras of Section 10.10.3.

**B.6.4. Positive Hopf operads.** The 2-monoidal structure of Proposition B.31 allows us to consider bimonoids in $(\mathsf{Sp}_+, \circ, \mathbf{X}, \times, \mathbf{E}_+)$. (Bimonoids in 2-monoidal categories are defined in Section 6.5.1.)

**Definition B.32.** A *positive Hopf operad* is a bimonoid in the 2-monoidal category $(\mathsf{Sp}_+, \circ, \mathbf{X}, \times, \mathbf{E}_+)$. A bimonoid in the 2-monoidal category $(\mathsf{Sp}_+, \times, \mathbf{E}_+, \circ, \mathbf{X})$ is a *positive Hopf cooperad*.

Explicitly, a positive Hopf operad is a positive species $\mathbf{p}$ with maps

$$\mu\colon \mathbf{p}\circ\mathbf{p} \to \mathbf{p}, \qquad \iota\colon \mathbf{X} \to \mathbf{p}, \qquad \Delta\colon \mathbf{p} \to \mathbf{p}\times\mathbf{p}, \qquad \epsilon\colon \mathbf{p} \to \mathbf{E}$$

satisfying axioms (6.3)–(6.7).

In light of Proposition 6.36, there are two other interpretations one may give for a Hopf operad. Namely, it is a comonoid in the category of operads, or it is an operad with values in $\mathsf{Coalg}$.

**Example B.33.** Let P be a positive set operad and $\mathbf{p} = \Bbbk\mathrm{P}$ its linearization (Section B.1.3). The diagonal map $\mathrm{P} \to \mathrm{P} \times \mathrm{P}$ is a morphism of set operads. There is also a (unique) morphism of set operads $\mathrm{P} \to \mathrm{E}$, which sends all elements of $\mathrm{P}[I]$ to $*_I$. Linearizing, one obtains morphisms $\mathbf{p} \to \mathbf{p} \times \mathbf{p}$ and $\mathbf{p} \to \mathbf{E}$ which turn $\mathbf{p}$ into a positive Hopf operad.

**Example B.34.** Recall from Section B.1.5 that an algebra can be viewed as an operad concentrated in degree 1. Similarly, a bialgebra can be viewed as a positive Hopf operad concentrated in degree 1.

This may be seen as follows. The functor

$$\mathbf{X}_{(-)} : (\mathsf{Vec}, \otimes, \otimes) \to (\mathsf{Sp}_+, \circ, \times) \qquad V \mapsto \mathbf{X}_V$$

(with $\mathbf{X}_V$ as in (8.4)) is bilax. In fact, it is strong-costrong. In other words, there are isomorphisms

$$\mathbf{X}_V \circ \mathbf{X}_W \xrightarrow{\cong} \mathbf{X}_{V \otimes W} \qquad \text{and} \qquad \mathbf{X}_{V \otimes W} \xrightarrow{\cong} \mathbf{X}_V \times \mathbf{X}_W.$$

Further, the costrong structure is braided, that is, the braidings commute with the second isomorphism. Since a bilax functor preserves bimonoids, $\mathbf{X}_{(-)}$ takes bialgebras to positive Hopf operads.

**B.6.5. Substitution and Hadamard products on general species. Hopf operads.** The considerations of Sections B.6.1 and B.6.4 can be carried out in the context of general (not necessarily positive) species with minor adjustments.

Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ be (general) species. From (B.6), note that there is a diagonal embedding

$$(\mathbf{r} \times \mathbf{s})^{\cdot X} \to \mathbf{r}^{\cdot X} \times \mathbf{s}^{\cdot X}.$$

This induces a map

$$\mathbf{p}[X] \otimes \mathbf{q}[X] \otimes (\mathbf{r} \times \mathbf{s})^{\cdot X} \to (\mathbf{p}[X] \otimes \mathbf{r}^{\cdot X}) \otimes (\mathbf{q}[X] \otimes \mathbf{s}^{\cdot X}).$$

Passing to colimits (B.9), one obtains a map

$$\zeta \colon (\mathbf{p} \times \mathbf{q}) \circ (\mathbf{r} \times \mathbf{s}) \to (\mathbf{p} \circ \mathbf{r}) \times (\mathbf{q} \circ \mathbf{s}). \tag{B.24}$$

In addition, there are maps

$$\Delta_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}, \qquad \mu_{\mathbf{E}} \colon \mathbf{E} \circ \mathbf{E} \to \mathbf{E}, \qquad \iota_{\mathbf{E}} = \epsilon_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{E}. \tag{B.25}$$

The first map is as in (B.23); the remaining two are the operadic composition and unit of the commutative operad (Example B.15).

**Proposition B.35.** *With the structure maps* (B.24) *and* (B.25), $(\mathsf{Sp}, \circ, \mathbf{X}, \times, \mathbf{E})$ *is a* 2*-monoidal category. Moreover, it is* $\times$*-braided.*

Extending the considerations of Section B.4.8, we have the following result.

**Proposition B.36.** *The inclusion functor*

$$inc \colon (\mathsf{Sp}_+, \circ, \times) \to (\mathsf{Sp}, \circ, \times)$$

*is strong-costrong.*

A *Hopf operad* is a bimonoid in the 2-monoidal category $(\mathsf{Sp}, \circ, \mathbf{X}, \times, \mathbf{E})$. In view of Proposition B.36, every positive Hopf operad is in particular a Hopf operad.

It is also possible to construct a 2-monoidal category involving the operations $\circ'$ and $\times$. To this end, one starts from the projection

$$\mathbf{r}^{\cdot X} \times \mathbf{s}^{\cdot X} \to (\mathbf{r} \times \mathbf{s})^{\cdot X}$$

which sends the non-diagonal terms to 0. Passing to limits one obtains an interchange law

$$\zeta\colon (\mathbf{p} \circ' \mathbf{r}) \times (\mathbf{q} \circ' \mathbf{s}) \to (\mathbf{p} \times \mathbf{q}) \circ' (\mathbf{r} \times \mathbf{s})$$

which turns $(\mathsf{Sp}, \times, \mathbf{E}, \circ', \mathbf{X})$ into a 2-monoidal category.

A *Hopf cooperad* is a bimonoid in $(\mathsf{Sp}, \times, \mathbf{E}, \circ', \mathbf{X})$. Every positive Hopf cooperad is in particular a Hopf cooperad.

## B.7. Nonsymmetric operads

Nonsymmetric operads relate to graded vector spaces the same way as operads relate to species.

This seems to be the choice of terminology for most authors, including [260]. Some authors emphasize the distinction between nonsymmetric operads and operads by referring to the latter as *symmetric* operads. Other authors [226] use operad for our nonsymmetric operads, and symmetric operad for our operads.

**B.7.1. Substitution for graded vector spaces.** The substitution operation $\circ$ for graded vector spaces is discussed in Section 2.1.1. Recall that

$$(V \circ W)_n := \bigoplus_{k \geq 0} V_k \otimes \Bigg( \bigoplus_{i_1 + \cdots + i_k = n} W_{i_1} \otimes \cdots \otimes W_{i_k} \Bigg).$$

It defines a monoidal structure on the category $\mathsf{gVec}$ for which the unit object is $X$ (2.7).

A *nonsymmetric operad* is a monoid in the monoidal category $(\mathsf{gVec}, \circ)$. The terminology is that in [260, Definition 1.14].

Explicitly, a nonsymmetric operad consists of a graded vector space $V$ with a linear map

$$V_k \otimes V_{i_1} \otimes \cdots \otimes V_{i_k} \to V_{i_1 + \cdots + i_k}$$

for any nonnegative integers $k$, $i_1$, …, $i_k$, and a linear map

$$\Bbbk \to V_1$$

satisfying associativity and unitality axioms (Definition 1.9).

There is a different operation on graded vector spaces defined as follows:

$$(V \circ' W)_n := \prod_{k \geq 0} V_k \otimes \Bigg( \bigoplus_{i_1 + \cdots + i_k = n} W_{i_1} \otimes \cdots \otimes W_{i_k} \Bigg)$$

It defines a lax monoidal structure on $\mathsf{gVec}$ for which the unit object is still $X$.

A *nonsymmetric cooperad* is a comonoid in the lax monoidal category $(\mathsf{gVec}, \circ')$. There is an explicit description in terms of maps as the one above for operads, obtained by reversing the arrows.

On positively graded vector spaces, the operations $\circ$ and $\circ'$ coincide.

**B.7.2. The full Fock functor $\mathcal{K}$ and substitution.** We relate the substitution product on species to that on graded vector spaces via the full Fock functor $\mathcal{K}$ (Definition 15.1). We limit our attention to the operation $\circ$; similar considerations apply to the operation $\circ'$.

Let $n$ be a nonnegative integer and $(i_1, \ldots, i_k)$ a weak composition of $n$ (Section 10.1.1). Let $f\colon [n] \to [k]$ be the map that sends the initial segment of size $i_1$ to 1, the next segment of size $i_2$ to 2, and so on. In other words, $f$ is order-preserving and $|f^{-1}(j)| = i_j$ for all $j \in [k]$.

Let $\mathbf{q}$ be a species. The order-preserving bijections

$$[i_1] \to [i_1], \qquad [i_2] \to [i_1+1, i_1+i_2], \dots,$$

with $[i_j]$ mapping to the $j$-th fiber of $f$, induce a map

$$\mathbf{q}[i_1] \otimes \mathbf{q}[i_2] \otimes \cdots \otimes \mathbf{q}[i_k] \xrightarrow{\cong} \mathbf{q}[i_1] \otimes \mathbf{q}[i_1+1, i_1+i_2] \otimes \cdots \otimes \mathbf{q}[i_1+\cdots+i_{k-1}+1, n].$$

According to (B.6), we have a canonical inclusion

$$\mathbf{q}[i_1] \otimes \mathbf{q}[i_1+1, i_1+i_2] \otimes \cdots \otimes \mathbf{q}[i_1+\cdots+i_{k-1}+1, n] = \bigotimes_{j\in[k]} \mathbf{q}[f^{-1}(j)] \hookrightarrow \mathbf{q}^{\cdot k}[n].$$

Let $\mathbf{p}$ be another species. The above gives rise to a map

$$\mathbf{p}[k] \otimes \mathbf{q}[i_1] \otimes \cdots \otimes \mathbf{q}[i_k] \to \mathbf{p}[k] \otimes \mathbf{q}^{\cdot k}[n].$$

From (B.11) we also have a canonical map

$$\mathbf{p}[k] \otimes \mathbf{q}^{\cdot k}[n] \to (\mathbf{p} \circ \mathbf{q})[n].$$

Composing, we obtain a map

$$\mathbf{p}[k] \otimes \mathbf{q}[i_1] \otimes \cdots \otimes \mathbf{q}[i_k] \to (\mathbf{p} \circ \mathbf{q})[n].$$

The vector space in the left-hand side is a direct summand of the degree $n$ component of $\mathcal{K}(\mathbf{p}) \circ \mathcal{K}(\mathbf{q})$, while the other space is the degree $n$ component of $\mathcal{K}(\mathbf{p} \circ \mathbf{q})$.

We define a map

$$\varphi_{\mathbf{p},\mathbf{q}} \colon \mathcal{K}(\mathbf{p}) \circ \mathcal{K}(\mathbf{q}) \to \mathcal{K}(\mathbf{p} \circ \mathbf{q})$$

by letting its components be the above maps. We also let

$$\varphi_0 \colon X \to \mathcal{K}(\mathbf{X})$$

be the identity.

**Proposition B.37.** *The functor $(\mathcal{K}, \varphi, \varphi_0) \colon (\mathsf{Sp}, \circ, \mathbf{X}) \to (\mathsf{gVec}, \circ, X)$ is lax monoidal.*

Proposition 3.29 now implies the familiar fact that every operad is a nonsymmetric operad if one forgets the symmetric group actions:

**Corollary B.38.** *If $\mathbf{p}$ is an operad, then $\mathcal{K}(\mathbf{p})$ is a nonsymmetric operad.*

In Example 6.23 we discussed the 2-monoidal category $(\mathsf{gVec}, \circ, \times)$, which is analogous to the 2-monoidal category $(\mathsf{Sp}, \circ, \times)$ of Section B.6.5. The two constructions can be related via the following result, which extends that of Proposition B.37.

**Proposition B.39.** *The functor $\mathcal{K} \colon (\mathsf{Sp}, \circ, \times) \to (\mathsf{gVec}, \circ, \times)$ is bilax. In fact, it is lax-costrong.*

APPENDIX C

# Pseudomonoids and the Looping Principle

The set of endomorphisms of an object in a category is an ordinary monoid under composition. This is the first instance of a very general principle in category theory, the *looping principle*. We discuss instances of this principle in Section C.4.

We are mainly interested in a 2-dimensional version of the principle which relates pseudomonoids (in a monoidal 2-category) to bicategories (enriched in the same monoidal 2-category). We arrive at this in Section C.4.4.

A monoidal 2-category is a 2-category (with objects, arrows, and 2-cells) with a compatible monoidal structure. These notions are discussed in Section C.1. Pseudomonoids are discussed in Section C.2 and enrichment in Section C.3.

This appendix provides context to Section 6.11 on 2-monoidal categories. In Proposition 6.73 we showed that 2-monoidal categories can be viewed as pseudomonoids (in two different monoidal 2-categories). It follows that 2-monoidal categories arise as loops in certain enriched bicategories. In Section C.5 we illustrate this point with some concrete examples. In particular, in Section C.5.1 we describe the enriched bicategory of bipartite graphs, where loops give rise to the 2-monoidal category of graphs of Example 6.17.

## C.1. 2-categories and bicategories

In a category there are objects and arrows between objects. In a bicategory there are also cells between arrows. A 2-category is a similar but simpler structure. Bicategories go back to Bénabou [38] and 2-categories to Ehresmann [114, 115]. We review these notions in Sections C.1.1 and C.1.2. More details may be found in [216], [250, Chapter XII] or [58, Chapter 7]; other references are given below.

Of the 2-dimensional analogues of the notion of monoidal category, we are mainly interested in monoidal 2-categories. We sketch the definition in Section C.1.3. Monoidal 2-categories are a special class of the more general monoidal bicategories.

Monoidal 2-categories provide the context for pseudomonoids, as monoidal categories do for monoids. Pseudomonoids are the object of Section C.2.

**C.1.1. 2-categories.** Let $\mathsf{I}$ be the one-arrow category (Example 1.3).

**Definition C.1.** A 2-*category* $\mathsf{C}$ consists of the following data:

- A class of *objects* $A$, $B$, $C$,...
- For each pair of objects $A$, $B$, a category

$$\mathrm{Hom}_{\mathsf{C}}(A, B).$$

- For each object $A$, a functor

$$\mathsf{I} \xrightarrow{\iota_A} \mathrm{Hom}_{\mathsf{C}}(A, A).$$

- For each triple of objects $A, B, C$, a functor
$$\mathrm{Hom}_{\mathsf{C}}(A,B) \times \mathrm{Hom}_{\mathsf{C}}(B,C) \xrightarrow{\mu_{A,B,C}} \mathrm{Hom}_{\mathsf{C}}(A,C).$$

The following diagrams must commute, for all objects $A, B, C, D$.

$$\begin{tikzcd}
\mathsf{I} \times \mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[r, "\iota_A \times \mathrm{id}"] & \mathrm{Hom}_{\mathsf{C}}(A,A) \times \mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[d, "\mu_{A,A,B}"] \\
& \mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[ul, "\cong"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(A,B) \times \mathrm{Hom}_{\mathsf{C}}(B,B) \arrow[d, "\mu_{A,B,B}"'] & \mathrm{Hom}_{\mathsf{C}}(A,B) \times \mathsf{I} \arrow[l, "\mathrm{id} \times \iota_B"'] \\
\mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(A,B) \times \mathrm{Hom}_{\mathsf{C}}(B,C) \times \mathrm{Hom}_{\mathsf{C}}(C,D) \arrow[r, "\mathrm{id} \times \mu_{B,C,D}"] \arrow[d, "\mu_{A,B,C} \times \mathrm{id}"'] & \mathrm{Hom}_{\mathsf{C}}(A,B) \times \mathrm{Hom}_{\mathsf{C}}(B,D) \arrow[d, "\mu_{A,B,D}"] \\
\mathrm{Hom}_{\mathsf{C}}(A,C) \times \mathrm{Hom}_{\mathsf{C}}(C,D) \arrow[r, "\mu_{A,C,D}"'] & \mathrm{Hom}_{\mathsf{C}}(A,D)
\end{tikzcd}$$

Let $\mathsf{C}$ be a 2-category. The objects of $\mathsf{C}$ are also called 0-*cells*. For each pair of objects $A$, $B$ of $\mathsf{C}$, the objects of the category $\mathrm{Hom}_{\mathsf{C}}(A,B)$ are *arrows* or 1-*cells* of $\mathsf{C}$ and the arrows of $\mathrm{Hom}_{\mathsf{C}}(A,B)$ are 2-*cells* of $\mathsf{C}$. The diagram

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & B
\end{tikzcd}$$

shows two objects $A$ and $B$, two arrows $f$ and $g$, and one 2-cell $\alpha$ in a 2-category $\mathsf{C}$.

The functors $\mu_{A,B,C}$ allow us to *horizontally compose* 1 and 2-cells in $\mathsf{C}$. The 1 and 2-cells in the images of the functors $\iota_A$ are *identities*. Composition in the category $\mathrm{Hom}_{\mathsf{C}}(A,B)$ allows us to *vertically compose* 2-cells in $\mathsf{C}$. Functoriality of $\mu$ implies that the possible iterations of horizontal and vertical composition in the following diagram yield the same result.

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f_1"] \arrow[r, "f_2" description] \arrow[r, bend right=50, "f_3"'] \arrow[r, phantom, shift left=3, "\Downarrow \alpha_1"] \arrow[r, phantom, shift right=3, "\Downarrow \alpha_2"] & B \arrow[r, bend left=50, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=50, "g_3"'] \arrow[r, phantom, shift left=3, "\Downarrow \beta_1"] \arrow[r, phantom, shift right=3, "\Downarrow \beta_2"] & C
\end{tikzcd}$$

For more details, see [58, Section 7.1], [151, Chapter I.2], [199, Section 1.2] or [250, Section XII.3].

**Example C.2.** There is a 2-category whose 0-cells, 1-cells and 2-cells are categories, functors and natural transformations respectively. We denote this 2-category by

$\mathsf{Cat}$. We are also interested in 2-categories of a similar nature whose 0-cells are monoidal categories rather than categories, see Tables 6.3 and 6.4.

One can define adjunctions, equivalences, and adjoint equivalences in any 2-category $\mathsf{C}$: in Definition A.2, one replaces the functors $\mathcal{F}$ and $\mathcal{G}$ by 1-cells in $\mathsf{C}$ and the natural transformations $\eta$ and $\xi$ by 2-cells. For more information, see [151, Chapter I.6]. Setting $\mathsf{C} := \mathsf{Cat}$ recovers Definition A.2.

Using the other 2-categories mentioned in Example C.2, one obtains notions of adjunctions, equivalences, and adjoint equivalences for monoidal categories, some of which are discussed in Section 3.9.2.

In the world of categories one can define functors and natural transformations. Similarly, in the world of 2-categories one can define functors, transformations and modifications. The definitions can be found in [58, Section 7.2], [199], [250, Chapter XII], or [227], along with a discussion of other commonly used terms for these notions. One can in fact consider various kinds of functors: lax, colax, strong, strict. For example, for a lax functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ between 2-categories there must be given 2-cells in $\mathsf{D}$ of the form

$$\mathcal{F}(f)\mathcal{F}(g) \Rightarrow \mathcal{F}(fg) \qquad \text{and} \qquad \mathrm{id}_{\mathcal{F}(A)} \Rightarrow \mathcal{F}(\mathrm{id}_A),$$

for any composable 1-cells $f$ and $g$ in $\mathsf{C}$ and object $A$ in $\mathsf{C}$. These 2-cells are subject to certain axioms. When they are invertible, the functor is strong; when they are identities, the functor is strict. Strict functors are also called 2-functors.

Similarly, transformations can be lax, strong, or strict (also called 2-natural). A lax transformation $\sigma\colon \mathcal{F} \Rightarrow \mathcal{G}$ between functors $\mathcal{F}, \mathcal{G}\colon \mathsf{C} \to \mathsf{D}$ consists of 1-cells $\sigma_A$ and 2-cells $\sigma_f$ as in the diagram below.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\sigma_A"'] \arrow[dr, phantom, "\sigma_f \nearrow"] & \mathcal{F}(B) \arrow[d, "\sigma_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B)
\end{tikzcd}$$

These are subject to certain axioms. The transformation is strong if the 2-cells $\sigma_f$ are invertible and strict if they are identities. The transformation is invertible if both $\sigma_A$ and $\sigma_f$ are invertible.

**C.1.2. Bicategories.** The notion of *bicategory* generalizes that of a 2-category. Bicategories were first considered by Bénabou [38]. We do not provide a complete definition, but refer to [38], [58, Section 7.7], [151, Chapter I.3], [250, Section XII.6], [226, Section 1.5] and [227].

As a 2-category, a bicategory consists of 0, 1, and 2-cells. The feature which distinguishes a bicategory from a 2-category is that horizontal composition is not required to be strictly associative or unital. Instead, it satisfies a weak form of these axioms, in which invertible 2-cells intervene. These 2-cells are in turn subject to further compatiblity conditions, similar to (1.1)–(1.2) in the definition of monoidal category.

Bicategories are also called *weak 2-categories*.

As for 2-categories, functors between bicategories may be lax, strong, or strict. Strong functors are usually called *homomorphisms*. Leinster [227] also uses *morphism* for lax functor.

**C.1.3. Monoidal 2-categories and bicategories (sketch).** A monoidal bicategory is a *tricategory* with a single object. A complete definition of tricategories is given by Gordon, Power and Street [148, Chapter 2]. A monoidal 2-category is a monoidal bicategory whose underlying bicategory is a 2-category [148, Section 2.6].

An early reference to monoidal 2-categories is [371]. Closely related notions are considered by Kapranov and Voevodsky [188, Section 4], including that of a *semistrict monoidal* 2*-category* [188, Section 4.3]. The latter notion is the same as that of a *Gray monoid* [93]. We mention in passing that [189, Section 4] discusses braided semistrict monoidal 2-categories and [264, Appendix A] discusses braided monoidal bicategories; we will not make use of these notions.

We now describe some of the main ingredients in the definition of monoidal 2-categories, following [148, Chapter 2]. Let $\mathsf{I}$ be the one-arrow category.

A *monoidal* 2*-category* is a 2-category $\mathsf{C}$ with strong functors

$$\bullet \colon \mathsf{C} \times \mathsf{C} \to \mathsf{C} \quad \text{and} \quad \mathsf{I} \to \mathsf{C},$$

and constraints $(\mathrm{a}, \mathrm{l}, \mathrm{r}, \mathrm{A}, \mathrm{U})$ as specified below. Let $I$ be the object of $\mathsf{C}$ which is the image of the unique object of $\mathsf{I}$ under the latter functor. The constraints consist of invertible transformations $(\mathrm{a}, \mathrm{l}, \mathrm{r})$ as below

$$\mathrm{a}_{A,B,C} \colon (A \bullet B) \bullet C \to A \bullet (B \bullet C), \quad \mathrm{l}_A \colon A \to I \bullet A \quad \text{and} \quad \mathrm{r}_A \colon A \to A \bullet I,$$

and invertible modifications $(\mathrm{A}, \mathrm{U})$ as below

(C.1)
$$\begin{tikzcd}
 & & (A \bullet B) \bullet (C \bullet D) \arrow[drr, "\mathrm{a}_{A,B,C \bullet D}"] & & \\
\big((A \bullet B) \bullet C\big) \bullet D \arrow[urr, "\mathrm{a}_{A \bullet B,C,D}"] \arrow[dr, "\mathrm{a}_{A,B,C} \bullet \mathrm{id}_D"'] & & \Downarrow \mathrm{A}_{A,B,C,D} & & A \bullet \big(B \bullet (C \bullet D)\big) \\
 & \big(A \bullet (B \bullet C)\big) \bullet D \arrow[rr, "\mathrm{a}_{A,B \bullet C,D}"'] & & A \bullet \big((B \bullet C) \bullet D\big) \arrow[ur, "\mathrm{id}_A \bullet \mathrm{a}_{B,C,D}"'] &
\end{tikzcd}$$

(C.2)
$$\begin{tikzcd}
(A \bullet I) \bullet B \arrow[rr, "\mathrm{a}_{A,I,B}"] & \Swarrow \mathrm{U}_{A,B} & A \bullet (I \bullet B) \\
 & A \bullet B \arrow[ul, "\mathrm{r}_A \bullet \mathrm{id}_B"] \arrow[ur, "\mathrm{id}_A \bullet \mathrm{l}_B"'] &
\end{tikzcd}$$

subject to certain axioms which we do not provide. As for monoidal categories, we refer to $\bullet$ as the monoidal operation and to $I$ as the unit object of $\mathsf{C}$.

We mention that two more modifications are given as part of the definition in [148, Chapter 2]; however, they are uniquely determined by the rest of the structure [148, Remark 2.3].

## C.2. Pseudomonoids

The notion of pseudomonoid (in a monoidal 2-category) is a 2-dimensional analogue of the notion of monoid (in a monoidal category). The relevant definitions are discussed in Section C.2.1 and C.2.2. In Section C.2.4 we discuss the examples of monoidal 2-categories and pseudomonoids of interest to this monograph.

Pseudomonoids appear at various places in the literature. In the context of Gray monoids, they are defined by Day and Street [93, Section 3]; in the context of monoidal 2-categories, by McCrudden [264, Section 2]. We follow these authors below. Pseudomonoids are called *monoidales* in [82] and *monoidal objects* in [245]. They generalize the *tensor objects* of Joyal and Street [184, Definition 5.1] and are a special case of the lax monoids of Day and Street (Section D.2).

**C.2.1. Definition.** Let $(\mathsf{C}, \bullet, I)$ be a monoidal 2-category with structure transformations $(\mathrm{a}, \mathrm{l}, \mathrm{r})$ and modifications $(\mathrm{A}, \mathrm{U})$ as in Section C.1.3.

**Definition C.3.** A *pseudomonoid* in $(\mathsf{C}, \bullet, I)$ is an object $A$ in $\mathsf{C}$, along with 1-cells $\mu\colon A \bullet A \to A$ and $\iota\colon I \to A$, and invertible 2-cells $\alpha$, $\lambda$, and $\rho$ as below.

(C.3)
$$\begin{tikzcd}
(A \bullet A) \bullet A \arrow[rr, "\mu\bullet\mathrm{id}"] \arrow[dd, "\mathrm{a}"'] & & A \bullet A \arrow[dr, "\mu"] & \\
& \Downarrow \alpha & & A \\
A \bullet (A \bullet A) \arrow[rr, "\mathrm{id}\bullet\mu"'] & & A \bullet A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

(C.4)
$$\begin{tikzcd}
I \bullet A \arrow[r, "\iota\bullet\mathrm{id}"] \arrow[dr, "\mathrm{l}^{-1}"'] \arrow[r, phantom, shift right=6, "\lambda \Nearrow"] & A \bullet A \arrow[d, "\mu"] & A \bullet I \arrow[l, "\mathrm{id}\bullet\iota"'] \arrow[dl, "\mathrm{r}^{-1}"] \arrow[l, phantom, shift left=6, "\Swarrow \rho"] \\
& A &
\end{tikzcd}$$

The following diagrams of 2-cells are required to commute (the operation $\bullet$ has been omitted).

$$
\begin{tikzcd}
 & & & A & & & \\
 & AA \arrow[urr, "\mu"] & & AA \arrow[u, "\mu"] & & AA \arrow[ull, "\mu"'] & \\
(AA)A \arrow[ur, "\mu\,\mathrm{id}"] \arrow[rr, "\mathrm{a}"] & & A(AA) \arrow[ur, "\mathrm{id}\,\mu"] & & (AA)A \arrow[ul, "\mu\,\mathrm{id}"'] \arrow[rr, "\mathrm{a}"] & & (AA)A \arrow[ul, "\mathrm{id}\,\mu"'] \\
 & & (AA)A \arrow[uul, "\mu\,\mathrm{id}"'] \arrow[rr, "\mathrm{a}"] & & A(AA) \arrow[uur, "\mathrm{id}\,\mu"] & & \\
 & & & (AA)(AA) \arrow[uul, "\mu\,\mathrm{id}"] \arrow[uur, "\mathrm{id}\,\mu"'] \arrow[drrr, "\mathrm{a}"] & & & \\
\big((AA)A\big)A \arrow[uuu, "(\mu\,\mathrm{id})\,\mathrm{id}"] \arrow[urrr, "\mathrm{a}"] \arrow[drr, "\mathrm{a}"'] & & & & & & A\big(A(AA)\big) \arrow[uuu, "\mathrm{id}(\mathrm{id}\,\mu)"'] \\
 & & \big(A(AA)\big)A \arrow[uuu, "(\mathrm{id}\,\mu)\,\mathrm{id}"] \arrow[rr, "\mathrm{a}"'] & & A\big((AA)A\big) \arrow[uuu, "\mathrm{id}(\mu\,\mathrm{id})"] \arrow[urr, "\mathrm{a}"'] & &
\end{tikzcd}
\tag{C.5}
$$

$$
\begin{tikzcd}
(AA)A \arrow[rrr, "\mathrm{a}"] \arrow[dr, "\mu\,\mathrm{id}"'] & & & A(AA) \arrow[dr, "\mathrm{id}\,\mu"] & \\
 & AA \arrow[dr, "\mu"] & & & AA \arrow[dll, "\mu"'] \\
 & & A & & \\
(AI)A \arrow[uuu, "(\mathrm{id}\,\iota)\,\mathrm{id}"] \arrow[rrr, "\mathrm{a}"] \arrow[dr, "\mathrm{r}^{-1}\,\mathrm{id}"'] & & & A(IA) \arrow[uuu, "\mathrm{id}(\mathrm{id}\,\iota)"'] \arrow[dr, "\mathrm{id}\,\mathrm{l}^{-1}"'] & \\
 & AA \arrow[uuu, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & & & AA \arrow[uuu, "\mathrm{id}"'] \arrow[dll, "\mathrm{id}"] \\
 & & AA \arrow[uuu, "\mu"] & &
\end{tikzcd}
\tag{C.6}
$$

In these diagrams, each face is either one of the 2-cells in (C.3)–(C.4), or one of the structure modifications in (C.1)–(C.2), or the identity.

**C.2.2. Morphisms of pseudomonoids.** Let $(\mathsf{C}, \bullet, I)$ be a monoidal 2-category as in Section C.1.3, and let $A$ and $B$ be pseudomonoids in $(\mathsf{C}, \bullet, I)$.

**Definition C.4.** A *lax morphism* from $A$ to $B$ is a 1-cell $f\colon A \to B$ in $\mathsf{C}$, along with 2-cells $\varphi$ and $\varphi_0$ as below

$$\begin{tikzcd}
A \bullet A \arrow[r, "\mu"] \arrow[d, "f \bullet f"'] \arrow[dr, phantom, "\varphi \nearrow"] & A \arrow[d, "f"] \\
B \bullet B \arrow[r, "\mu"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dr, "f"] \arrow[d, phantom, "\varphi_0 \Uparrow"] & \\
I \arrow[ur, "\iota"] \arrow[rr, "\iota"'] & {} & B
\end{tikzcd} \tag{C.7}$$

and such that the following diagrams of 2-cells commute.

$$\begin{tikzcd}[column sep=small]
(AA)A \arrow[rrrr, "\mathrm{a}"] \arrow[dr, "\mu\,\mathrm{id}"] \arrow[ddd, "(ff)f"'] & & & & A(AA) \arrow[dr, "\mathrm{id}\,\mu"] \arrow[ddd, "f(ff)"] & \\
 & AA \arrow[drr, "\mu"] \arrow[ddd, "ff"'] & & & & AA \arrow[dll, "\mu"'] \arrow[ddd, "ff"] \\
 & & & A \arrow[ddd, "f"] & & \\
(BB)B \arrow[rrrr, "\mathrm{a}"] \arrow[dr, "\mu\,\mathrm{id}"'] & & & & B(BB) \arrow[dr, "\mathrm{id}\,\mu"'] & \\
 & BB \arrow[drr, "\mu"'] & & & & BB \arrow[dll, "\mu"] \\
 & & & B & &
\end{tikzcd} \tag{C.8}$$

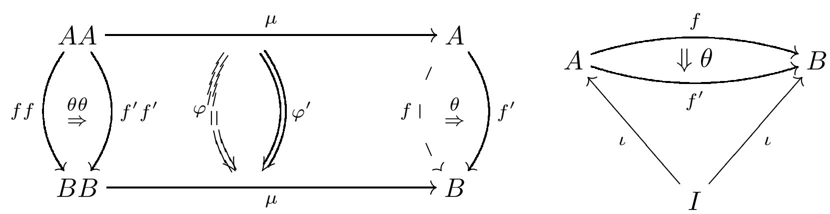
$$\begin{tikzcd}
 & A \arrow[rr, "f"] & & B \\
 & AA \arrow[u, "\mu"'] \arrow[rr, "ff"] & & BB \arrow[u, "\mu"'] \\
IA \arrow[uur, "\mathrm{l}^{-1}"] \arrow[ur, "\mathrm{id}\,\iota"'] \arrow[rr, "\mathrm{id}\,f"'] & & IB \arrow[uur, "\mathrm{l}^{-1}"] \arrow[ur, "\iota\,\mathrm{id}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[rr, "f"] & & B \\
 & AA \arrow[u, "\mu"'] \arrow[rr, "ff"] & & BB \arrow[u, "\mu"'] \\
AI \arrow[uur, "\mathrm{r}^{-1}"] \arrow[ur, "\mathrm{id}\,\iota"'] \arrow[rr, "f\,\mathrm{id}"'] & & BI \arrow[uur, "\mathrm{r}^{-1}"] \arrow[ur, "\mathrm{id}\,\iota"'] &
\end{tikzcd} \tag{C.9}$$

In these diagrams, each face is either one of the 2-cells in (C.3)–(C.4), or one of the 2-cells in (C.7), possibly tensored with an identity 2-cell.

A *colax morphism* from $A$ to $B$ is a 1-cell $A \to B$ in $\mathsf{C}$, along with 2-cells as in (C.7) but with reverse directions such that the above diagrams (with the 2-cells drawn appropriately) commute.

**Definition C.5.** A morphism between (co)lax morphisms $f, f' \colon A \to B$ is a 2-cell $\theta \colon f \Rightarrow f'$ in $\mathsf{C}$ such that the following diagrams of 2-cells commute.

(C.10)




The cylinder has two side faces labeled $\varphi$ and $\varphi'$ and two bases labeled $\theta$ and $\theta \bullet \theta$, as shown. The back face of the cone is $\varphi_0$, the front face is $\varphi'_0$ and the base is $\theta$.

A *strong morphism* of pseudomonoids is a (co)lax morphism for which the 2-cells in (C.7) are invertible. Morphisms between strong morphisms are defined as in Definition C.5.

**C.2.3. The lax, colax, and strong constructions.** Combining the preceding definitions we obtain the 2-categories $\mathsf{l}(\mathsf{C})$ and $\mathsf{c}(\mathsf{C})$, whose objects are pseudomonoids in $\mathsf{C}$, 1-cells are lax and colax morphisms respectively, and 2-cells are morphisms between them. Similarly, the 2-category $\mathsf{s}(\mathsf{C})$ has pseudomonoids in $\mathsf{C}$ for objects and strong morphisms for 1-cells.

We refer to the passages from a monoidal 2-category $\mathsf{C}$ to the 2-categories $\mathsf{l}(\mathsf{C})$, $\mathsf{c}(\mathsf{C})$, and $\mathsf{s}(\mathsf{C})$ as the *lax*, *colax*, and *strong constructions* respectively.

One may wonder when the 2-categories $\mathsf{l}(\mathsf{C})$, $\mathsf{c}(\mathsf{C})$, and $\mathsf{s}(\mathsf{C})$ are themselves monoidal; in which case these constructions could be iterated. We refrain from addressing this question in full generality, which would require a discussion of braidings on monoidal 2-categories, but we note that for $\mathsf{C} = \mathsf{Cat}$, the monoidal structure (Cartesian product) is indeed inherited. The results of the iteration of the constructions are discussed below.

**C.2.4. Examples of monoidal 2-categories and their pseudomonoids.** We discuss a few instances of the notions discussed in the preceding sections.

Let $\mathsf{C}$ be a monoidal category. We may view it as a monoidal 2-category in which all 2-cells are identities. In this case, a pseudomonoid (Definition C.3) is a monoid in the sense of Definition 1.9 and lax and colax morphisms (Definition C.4) coincide with ordinary morphisms of monoids. Thus,

$$\mathsf{l}(\mathsf{C}) = \mathsf{c}(\mathsf{C}) = \mathsf{Mon}(\mathsf{C}).$$

Let us regard the 2-category $\mathsf{Cat}$ (Example C.2) as a monoidal 2-category under Cartesian product. A pseudomonoid in $\mathsf{Cat}$ is a monoidal category in the sense of Definition 1.1. Moreover, applying the lax construction (Section C.2.3) to $\mathsf{Cat}$ we obtain the 2-category $\mathsf{l}(\mathsf{Cat})$ whose 0-cells are monoidal categories, 1-cells are lax functors and 2-cells are morphisms between lax functors. Similarly, $\mathsf{c}(\mathsf{Cat})$ is the 2-category whose 1-cells are colax functors between monoidal categories. These assertions are verified in Proposition 6.72. Equivalently, in terms of the notation introduced in Section 3.3.3, $\mathsf{l}(\mathsf{Cat}) = \mathsf{lCat}$ and $\mathsf{c}(\mathsf{Cat}) = \mathsf{cCat}$. (Elsewhere in the text, we prefer the latter notation to the former.)

The 2-category $\mathsf{l}(\mathsf{Cat})$ is again a monoidal 2-category under Cartesian product. A pseudomonoid in $\mathsf{l}(\mathsf{Cat})$ is a 2-monoidal category; moreover, $\mathsf{l}(\mathsf{l}(\mathsf{Cat}))$ is the 2-category whose 0-cells are 2-monoidal categories, 1-cells are double lax functors, and

2-cells are morphisms between double lax functors. These assertions are verified in Propositions 6.73 and 6.75.

Since the 2-categories $\mathsf{l}(\mathsf{Cat})$ and $\mathsf{c}(\mathsf{Cat})$ are again monoidal (under Cartesian product), the lax and colax constructions can in this context be iterated and combined. This leads to higher monoidal categories and higher monoidal functors, as discussed in more depth in Sections 6.11 and 7.9.

Applying the strong construction to $\mathsf{Cat}$ we obtain the 2-category $\mathsf{s}(\mathsf{Cat})$ consisting of strong monoidal categories, strong monoidal functors, and morphisms of such. It is monoidal under Cartesian product. A pseudomonoid in $\mathsf{s}(\mathsf{Cat})$ is a braided monoidal category. Iterating this construction leads to symmetric monoidal categories, at which point further iterations yield nothing new. This is reviewed in more detail in Section 7.9.2.

Let $\mathsf{C}$ be a 2-category. We proceed to define a new 2-category which we denote $\mathsf{C}^{(l)}$. It is analogous to the category of arrows $\mathsf{D}^{(2)}$ which is associated to an arbitrary category $\mathsf{D}$ and which we discussed in Section 3.11.1. An object of $\mathsf{C}^{(l)}$ is a triple $(A, f, B)$ where $A$ and $B$ are objects of $\mathsf{C}$ and $f\colon A \to B$ is a 1-cell in $\mathsf{C}$. A 1-cell from $(A, f, B)$ to $(C, g, D)$ is a triple $(h, \theta, k)$ where $h$ and $k$ are 1-cells and $\theta$ is a 2-cell in $\mathsf{C}$ as below.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow \theta"] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
$$

Finally, given 1-cells $(h, \theta, k)$ and $(h', \theta', k')$ both from $(A, f, B)$ to $(C, g, D)$, a 2-cell from $(h, \theta, k)$ to $(h', \theta', k')$ is a pair $(\eta, \kappa)$ of 2-cells in $\mathsf{C}$ making the following cylinder commute.

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, bend right=50, "h"'] \arrow[d, bend left=50, "h'"] \arrow[d, phantom, "\overset{\eta}{\Rightarrow}"] & \arrow[d, phantom, "\theta \Downarrow \quad \Downarrow \theta'"] & B \arrow[d, bend right=50, dashed, "k"'] \arrow[d, bend left=50, "k'"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
C \arrow[rr, "g"'] & & D
\end{tikzcd}
$$

This means that the 2-cells from $(A, kf, D)$ to $(A, gh', D)$ obtained respectively by pasting $\theta$ to $\eta$ and $\kappa$ to $\theta'$ must coincide.

If the 2-category $\mathsf{C}$ is monoidal, then so is $\mathsf{C}^{(l)}$. On objects, the tensor product of $\mathsf{C}^{(l)}$ is

$$(A, f, B) \bullet (A', f', B') := (A \bullet A', f \bullet f', B \bullet B').$$

It is easy to see that, in analogy to Proposition 3.110, a pseudomonoid in $\mathsf{C}^{(l)}$ consists of a pair of pseudomonoids $A$ and $B$ in $\mathsf{C}$ and a lax morphism of pseudomonoids $f\colon A \to B$ (Definition C.4).

If in the above construction we change the definition of 1-cells and 2-cells by requiring that $\theta$, $\eta$ and $\kappa$ map in the opposite direction, we obtain a monoidal 2-category $\mathsf{C}^{(c)}$ in which a pseudomonoid consists of a pair of pseudomonoids $A$ and $B$ in $\mathsf{C}$ and a colax morphism of pseudomonoids $f\colon A \to B$.

TABLE C.1. Pseudomonoids in various monoidal 2-categories.

| Monoidal 2-category | Pseudomonoid |
|---|---|
| $\mathsf{Cat}$ | monoidal category |
| $\mathsf{l}(\mathsf{Cat})$ | 2-monoidal category |
| $\mathsf{c}(\mathsf{Cat})$ | 2-monoidal category |
| $\mathbf{s}(\mathsf{Cat})$ | braided monoidal category |
| $\mathbf{s}(\mathbf{s}(\mathsf{Cat}))$ | symmetric monoidal category |
| $\mathsf{Cat}^{(l)}$ | lax monoidal functor |
| $\mathsf{Cat}^{(c)}$ | colax monoidal functor |

In particular, pseudomonoids in $\mathsf{Cat}^{(l)}$ and $\mathsf{Cat}^{(c)}$ are lax and colax monoidal functors, respectively (Definitions 3.1 and 3.2).

Table C.1 summarizes some of the examples discussed in this section.

## C.3. Enrichment

Enriched categories originated in the work of Eilenberg and Kelly [117]. A detailed study can be found in [196]. A more general theory of category enrichment is given in [225] and [226, Sections 1.3 and 6.8].

We review the definition of enriched categories (by a monoidal category) in Section C.3.1. A 2-dimensional analogue of this notion involving enrichment by a monoidal 2-category is outlined in Section C.3.2.

**C.3.1. Enriched categories.** Let $(\mathsf{V}, \bullet, I)$ be a monoidal category. We employ the notation of Definition 1.1.

**Definition C.6.** A category *enriched by* $(\mathsf{V}, \bullet, I)$, or more simply a $\mathsf{V}$-*category*, denoted $\mathsf{C}$, consists of the following data:

- A class of *objects* $A$, $B$, $C$,...
- For each pair of objects $A$, $B$, an object

$$\mathrm{Hom}_{\mathsf{C}}(A, B)$$

  in the category $\mathsf{V}$.
- For each object $A$, an arrow

$$I \xrightarrow{\iota_A} \mathrm{Hom}_{\mathsf{C}}(A, A)$$

  in $\mathsf{V}$.
- For each triple of objects $A, B, C$, an arrow

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \bullet \mathrm{Hom}_{\mathsf{C}}(B, C) \xrightarrow{\mu_{A,B,C}} \mathrm{Hom}_{\mathsf{C}}(A, C)$$

  in $\mathsf{V}$.

The following diagrams must commute, for all objects $A, B, C, D$.

(C.11)
$$\begin{tikzcd}
I \bullet \mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[r, "\iota_A \bullet \mathrm{id}"] & \mathrm{Hom}_{\mathsf{C}}(A,A) \bullet \mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[d, "\mu_{A,A,B}"] \\
 & \mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[ul, "\cong", "\lambda"']
\end{tikzcd}$$

(C.12)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(A,B) \bullet \mathrm{Hom}_{\mathsf{C}}(B,B) \arrow[d, "\mu_{A,B,B}"'] & \mathrm{Hom}_{\mathsf{C}}(A,B) \bullet I \arrow[l, "\mathrm{id} \bullet \iota_B"'] \\
\mathrm{Hom}_{\mathsf{C}}(A,B) \arrow[ur, "\cong", "\rho"'] &
\end{tikzcd}$$

(C.13)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(A,B) \bullet \mathrm{Hom}_{\mathsf{C}}(B,C) \bullet \mathrm{Hom}_{\mathsf{C}}(C,D) \arrow[r, "\mathrm{id} \bullet \mu_{B,C,D}"] \arrow[d, "\mu_{A,B,C} \bullet \mathrm{id}"'] & \mathrm{Hom}_{\mathsf{C}}(A,B) \bullet \mathrm{Hom}_{\mathsf{C}}(B,D) \arrow[d, "\mu_{A,B,D}"] \\
\mathrm{Hom}_{\mathsf{C}}(A,C) \bullet \mathrm{Hom}_{\mathsf{C}}(C,D) \arrow[r, "\mu_{A,C,D}"'] & \mathrm{Hom}_{\mathsf{C}}(A,D)
\end{tikzcd}$$

**Example C.7.** Consider the monoidal category $\mathsf{Set}$ of sets under Cartesian product (Example 1.3). A category enriched by $\mathsf{Set}$ is just an ordinary category.

A category enriched by $(\mathsf{Vec}, \otimes, \Bbbk)$ is precisely a $\Bbbk$-linear category (Definition 1.6).

**Example C.8.** Let $\mathsf{Cat}$ be the category whose objects are categories and whose morphisms are functors. It is a monoidal category under Cartesian product $\times$; the unit object is the one-arrow category $\mathsf{I}$ (Example 1.3). Comparing Definitions C.6 and C.1 we see that a 2-category is precisely a category enriched by $(\mathsf{Cat}, \times, \mathsf{I})$.

**Remark C.9.** In general, a $\mathsf{V}$-category is not necessarily an ordinary category. On the other hand, if there is given a lax monoidal functor $\mathcal{F}\colon \mathsf{V} \to \mathsf{Set}$, then any $\mathsf{V}$-category $\mathsf{C}$ gives rise to an ordinary category $\mathcal{F}(\mathsf{C})$ with the same objects and with arrows
$$\mathrm{Hom}_{\mathcal{F}(\mathsf{C})}(A,B) := \mathcal{F}\big(\mathrm{Hom}_{\mathsf{C}}(A,B)\big).$$
More generally, if $\mathcal{F}\colon \mathsf{V} \to \mathsf{V}'$ is a lax monoidal functor and $\mathsf{C}$ is a $\mathsf{V}$-category, then $\mathcal{F}(\mathsf{C})$ is a $\mathsf{V}'$-category. This is a straightforward generalization of Proposition 3.29.

Now suppose that $C$ is a comonoid in $\mathsf{V}$. Then the functor
$$\mathrm{Hom}_{\mathsf{V}}(C,-)\colon \mathsf{V} \to \mathsf{Set}$$
is lax monoidal (in view of Example 3.17 and Proposition 3.25). In particular, we may choose $C = I$, and use the functor $\mathrm{Hom}_{\mathsf{V}}(I,-)$ to turn any $\mathsf{V}$-category into an ordinary category.

**Example C.10.** The forgetful functor $\mathcal{F}\colon \mathsf{Vec} \to \mathsf{Set}$ is lax monoidal via the canonical maps
$$V \times W \to V \otimes W, \quad (v,w) \mapsto v \otimes w.$$
If $\mathsf{C}$ is a $\Bbbk$-linear category, then $\mathcal{F}(\mathsf{C})$ is the underlying ordinary category.

**Example C.11.** Let $\mathsf{sCat}$ denote the full subcategory of $\mathsf{Cat}$ consisting of *small* categories (categories for which the class of objects is a set). It is a monoidal subcategory under Cartesian product. An $\mathsf{sCat}$-category is a 2-category in which for any two objects $A$ and $B$, the 1-cells from $A$ to $B$ form a set.

Let $\mathcal{F}\colon \mathsf{sCat} \to \mathsf{Set}$ be the functor which sends a small category to its set of objects. It is a strong monoidal functor with respect to Cartesian product. If $\mathsf{C}$ is a 2-category as in the preceding paragraph, then $\mathcal{F}(\mathsf{C})$ is the underlying category (obtained from $\mathsf{C}$ by forgetting the 2-cells).

**C.3.2. Enriched bicategories.** Let $(\mathsf{V}, \bullet, I)$ be a monoidal 2-category, as in Section C.1.3. We may consider a notion of enrichment by such categories $\mathsf{V}$, as outlined below. We do not know of a reference for a complete definition.

A *bicategory enriched by* $(\mathsf{V}, \bullet, I)$, or more simply a $\mathsf{V}$-*bicategory*, denoted $\mathsf{C}$, consists of a class of *objects* $A$, $B$, $C$,..., an object

$$\mathrm{Hom}_{\mathsf{C}}(A, B)$$

in the category $\mathsf{V}$ for each pair of objects $A$ and $B$ in $\mathsf{C}$, an arrow

$$I \xrightarrow{\iota_A} \mathrm{Hom}_{\mathsf{C}}(A, A)$$

in $\mathsf{V}$ for each object $A$ in $\mathsf{C}$, an arrow

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \bullet \mathrm{Hom}_{\mathsf{C}}(B, C) \xrightarrow{\mu_{A,B,C}} \mathrm{Hom}_{\mathsf{C}}(A, C)$$

in $\mathsf{V}$ for each triple of objects $A, B, C$ in $\mathsf{C}$. Instead of the commutativity of diagrams (C.11)–(C.13), the existence of invertible 2-cells in $\mathsf{V}$ filling in those diagrams is required. These 2-cells are in turn subject to further compatiblity conditions (similar to those alluded to in Section C.1.2).

**Example C.12.** Let $\mathsf{V} := \mathsf{Cat}$ viewed as a monoidal 2-category under Cartesian product (combining the structures in Examples C.2 and C.8). Then a $\mathsf{V}$-bicategory is the same as an ordinary bicategory.

We are mainly interested in bicategories enriched by $\mathsf{lCat}$ or $\mathsf{cCat}$; see Section C.5 for concrete examples. As for enriched categories, an enriched bicategory need not be an ordinary bicategory.

We mention that one may envision a more general notion of enrichment by monoidal bicategories (rather than monoidal 2-categories). We do not need this notion for our purposes.

## C.4. The looping principle

The *looping principle* can be loosely stated as follows. Given a certain notion of "higher category" involving cells of various dimensions, such as the notions of bicategory or tricategory, define a corresponding notion of "higher monoid" as a higher category with only one 0-cell. Now suppose an arbitrary higher category $\mathsf{C}$ is given and a 0-cell $A$ in $\mathsf{C}$ is chosen. Consider "loops" based at $A$, denoted $\Omega_A\mathsf{C}$, by keeping only the higher cells incident to $A$ and no other 0-cell. The principle then claims simply that the higher category structure on $\mathsf{C}$ induces a higher monoid structure on $\Omega_A\mathsf{C}$:

$$\text{higher category } \mathsf{C} \overset{\text{loops}}{\longmapsto} \text{higher monoid } \Omega_A\mathsf{C}.$$

This idea has been considered by Kapranov and Voevodsky [189, Section 2.10]; see also Baez and Dolan [28, Section V].

In this section we make this principle precise in a number of special cases. We are mainly interested in two instances in which 2-monoidal categories arise as higher monoids.

**C.4.1. Loops in topological spaces.** We start with a topological version of the looping principle, for motivation purposes only.

Let $X$ be a topological space and $x$ a point in $X$. A *loop based at* $x$ is a continuous map

$$\gamma\colon [0,1] \to X \quad \text{such that } \gamma(0) = \gamma(1) = x,$$

where $[0,1]$ denotes the unit interval. Let

$$\Omega_x X$$

denote the set of loops based at $x$. The set $\Omega_x X$ is given the compact-open topology.

In order to concatenate two loops based at $x$, one must make a choice of reparametrization. This leads to an operation

$$\Omega_x X \times \Omega_x X \to \Omega_x X$$

which is associative and unital *up to homotopy*. In this manner, $\Omega_x X$ becomes an *$H$-space*. For more information, see [160, Sections 4.3 and 4.J].

**C.4.2. Loops in ordinary categories.** Let $\mathsf{C}$ be a category and $A$ an object of $\mathsf{C}$. The endomorphism set

$$\Omega_A \mathsf{C} := \mathrm{End}_{\mathsf{C}}(A)$$

is an ordinary monoid under composition of arrows in $\mathsf{C}$. Elements of $\Omega_A \mathsf{C}$ are sometimes called *loops based at* $A$.

The passage from the category $\mathsf{C}$ to the monoid $\Omega_A \mathsf{C}$ is a first instance of the looping principle.

**C.4.3. Loops in enriched categories.** Let $\mathsf{V}$ be a monoidal category and $\mathsf{C}$ a $\mathsf{V}$-category, as in Section C.3.1. As before, we set

$$\Omega_A \mathsf{C} := \mathrm{End}_{\mathsf{C}}(A).$$

This is an object of $\mathsf{V}$ and it follows from Definition C.6 that $(\Omega_A \mathsf{C}, \mu_{A,A}, \iota_A)$ is a monoid in $\mathsf{V}$. Indeed, axioms (C.11)–(C.13) specialize to the axioms in Definition 1.9.

If $\mathsf{V} = \mathsf{Set}$, this recovers the looping principle for ordinary categories mentioned in Section C.4.2. If $\mathsf{V} = \mathsf{Cat}$ viewed as a monoidal category as in Example C.8, then $\mathsf{C}$ is a 2-category and $\Omega_A \mathsf{C}$ is a strict monoidal category.

**C.4.4. Loops in enriched bicategories.** Let $\mathsf{V}$ be a monoidal 2-category and $\mathsf{C}$ a $\mathsf{V}$-bicategory, as in Section C.3.2. Let

$$\Omega_A \mathsf{C} := \mathrm{End}_{\mathsf{C}}(A).$$

This is an object of $\mathsf{V}$. The looping principle states in this context that $\Omega_A \mathsf{C}$ is in a natural way a pseudomonoid in $\mathsf{V}$.

If $\mathsf{V} = \mathsf{Cat}$ viewed as a monoidal 2-category as in Example C.12, then $\mathsf{C}$ is a bicategory and $\Omega_A \mathsf{C}$ is a monoidal category (not necessarily strict).

If $\mathsf{V} = \mathsf{lCat}$ or $\mathsf{cCat}$, then $\Omega_A \mathsf{C}$ is a 2-monoidal category (since according to Section C.2.4, a pseudomonoid in any of these two monoidal 2-categories is precisely a 2-monoidal category). We discuss concrete examples in Section C.5.

## C.5. Bipartite graphs, spans, and bimodules

We discuss some examples of $\mathsf{V}$-enriched bicategories (Section C.3.2) where the monoidal 2-category $\mathsf{V}$ is either $\mathsf{lCat}$ or $\mathsf{cCat}$ (consisting of monoidal categories and lax or colax monoidal functors, respectively). The looping principle then yields pseudomonoids in $\mathsf{V}$, which are 2-monoidal categories (Table C.1). In particular, we discuss a bicategory of bipartite graphs where looping gives rise to the 2-monoidal category of directed graphs of Example 6.17.

The consideration of such enriched structures and their connection to 2-monoidal categories was suggested to us by Steve Chase.

**C.5.1. The enriched bicategory of bipartite graphs.** Let $X$ and $Y$ be two sets. A *bipartite graph* with vertex set $(Y, X)$ is a triple $(A, s, t)$ where

$$\begin{tikzcd} & A \arrow[dl, "t"'] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}$$

are two maps. The elements of $A$ may be visualized as arrows directed from $Y$ to $X$. There are as many arrows with source $y$ and target $x$ as the cardinality of the set $s^{-1}(y) \cap t^{-1}(x)$.

A morphism $(A, s, t) \to (B, s, t)$ of bipartite graphs is a map $f : A \to B$ such that both triangles below commute.

$$\begin{tikzcd} & A \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, "f"] & \\ X & & Y \\ & B \arrow[ul, "t"] \arrow[ur, "s"'] & \end{tikzcd}$$

This defines the category of bipartite graphs with vertex set $(Y, X)$.

Given two bipartite graphs $(A, s, t)$ and $(B, s, t)$ with vertex set $(Y, X)$, define a new one $(A \star B, s, t)$ by

$$\begin{gathered} A \star B := \{(a, b) \in A \times B : s(a) = s(b) \quad \text{and} \quad t(a) = t(b)\}, \\ s(a, b) := s(a) = s(b) \qquad \text{and} \qquad t(a, b) := t(a) = t(b). \end{gathered}$$

This operation turns the category of bipartite graphs with vertex set $(Y, X)$ into a monoidal category. The unit object is the bipartite graph $(Y \times X, p_1, p_2)$ with $p_1(y, x) = y$ and $p_2(y, x) = x$.

Recall that $\mathsf{cCat}$ denotes the monoidal 2-category whose 0-cells are monoidal categories, 1-cells are colax monoidal functors, and 2-cells are morphisms of colax monoidal functors. The monoidal structure is Cartesian product of categories.

We proceed to define a $\mathsf{cCat}$-bicategory (Section C.3.2) of all bipartite graphs, denoted $\mathsf{C}$. The objects are sets $X, Y, Z, \ldots$. For each pair of objects $X$, $Y$, we let $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ be the monoidal category of bipartite graphs with vertex set $(Y, X)$ defined above. The functor

$$\iota_X : \mathsf{I} \to \mathrm{Hom}_{\mathsf{C}}(X, X)$$

sends the unique object of $\mathsf{I}$ to the discrete graph $(X, \mathrm{id}, \mathrm{id})$, and the unique arrow to the identity. The functor

$$\mu_{X,Y,Z}\colon \operatorname{Hom}_{\mathsf{C}}(X,Y) \times \operatorname{Hom}_{\mathsf{C}}(Y,Z) \to \operatorname{Hom}_{\mathsf{C}}(X,Z)$$

sends a pair of bipartite graphs $(A,s,t)$ and $(B,s,t)$ to $(A \diamond B, s, t)$ where

$$A \diamond B := \{(a,b) \in A \times B : s(a) = t(b)\},$$
$$s(a,b) := s(b) \qquad \text{and} \qquad t(a,b) := t(a).$$

Schematically,

$$\begin{tikzcd}
 & & A \diamond B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & A \arrow[dl, "t"'] \arrow[dr, "s"] & & B \arrow[dl, "t"'] \arrow[dr, "s"] & \\
X & & Y & & Z
\end{tikzcd}$$

We need to turn $\iota_X$ and $\mu_{X,Y,Z}$ into colax monoidal functors. The colax structure of $\iota_X$ boils down to the map

$$X \to X \times X, \quad x \mapsto (x,x),$$

which is a morphism of bipartite graphs $(X, \mathrm{id}, \mathrm{id}) \to (X \times X, p_1, p_2)$. The colax structure of $\mu_{X,Y,Z}$ boils down to

$$(A_1 \star A_2) \diamond (B_1 \star B_2) \to (A_1 \diamond B_1) \star (A_2 \diamond B_2) \qquad (a_1,a_2,b_1,b_2) \mapsto (a_1,b_1,a_2,b_2),$$

where $A_1$ and $A_2$ are bipartite graphs with vertex set $(Y,X)$ and $B_1$ and $B_2$ are bipartite graphs with vertex set $(Z,Y)$.

To complete the definition of the enriched bicategory $\mathsf{C}$, we need to specify the 2-cells controlling the associativity and unitality of $\iota_X$ and $\mu_{X,Y,Z}$. These are inherited from corresponding properties of the Cartesian product of sets.

In this manner, $\mathsf{C}$ is a bicategory enriched by the monoidal 2-category $\mathsf{cCat}$.

Note that an endomorphism of an object $X$ in $\mathsf{C}$ is a directed graph with vertex set $X$. According to the looping principle (Section C.4.4), $\Omega_X\mathsf{C} = \operatorname{End}_{\mathsf{C}}(X)$ is a 2-monoidal category for any set $X$. This is the 2-monoidal category of directed graphs of Example 6.17.

**C.5.2. The enriched bicategory of spans.** Let $\mathsf{D}$ be a category with finite pullbacks (finite products and equalizers). There is a straightforward generalization of the construction of Section C.5.1 in which sets are replaced by the objects of $\mathsf{D}$. Briefly, there is a $\mathsf{cCat}$-bicategory whose objects are the objects of $\mathsf{D}$, the objects of the monoidal category $\operatorname{Hom}_{\mathsf{C}}(X,Y)$ are *spans* in $\mathsf{D}$, that is triples $(A,s,t)$ where

$$\begin{tikzcd}
 & A \arrow[dl, "t"'] \arrow[dr, "s"] & \\
X & & Y
\end{tikzcd}$$

is a diagram in $\mathsf{D}$; the tensor product of two such objects $(A,s,t)$ and $(B,s,t)$ is the equalizer of the maps

$$A \xrightarrow{(s,t)} X \times Y \quad \text{and} \quad B \xrightarrow{(s,t)} X \times Y,$$

where $\times$ denotes the product in $\mathsf{D}$; horizontal composition of $(A, s, t)$ from $X$ to $Y$ and $(B, s, t)$ from $Y$ to $Z$ is the pull-back of the diagram below.

$$\begin{tikzcd} A \arrow[dr, "s"'] & & B \arrow[dl, "t"] \\ & Y & \end{tikzcd}$$

This bicategory of spans has often been considered in the literature; see for instance [151, Section I.3.4] and [250, Section XII.7]. We do not know if the finer $\mathsf{cCat}$-bicategory structure has been pointed out.

**C.5.3. The enriched bicategory of bimodules.** Recall that $\mathsf{lCat}$ denotes the monoidal 2-category whose 0-cells are monoidal categories, 1-cells are lax monoidal functors, and 2-cells are morphisms of lax monoidal functors. The monoidal structure is Cartesian product of categories.

We now explain how the 2-monoidal category of $K$-bimodules of Example 6.18 arises from the looping principle. For this, we need to construct an $\mathsf{lCat}$-bicategory $\mathsf{C}$ of bimodules over commutative algebras. This is done as follows.

We fix a ground field $\Bbbk$. The objects are commutative $\Bbbk$-algebras $F, G, K, \ldots$. The monoidal category $\mathrm{Hom}_{\mathsf{C}}(F, G)$ consists of $F$-$G$-bimodules and their morphisms, the tensor product being

$$M \diamond N := M \otimes_{F \otimes G} N.$$

Here, we make use of commutativity to view $M$ as a right $F \otimes G$-module and $N$ as a left $F \otimes G$-module. Horizontal composition of an $F$-$G$-bimodule $M$ and a $G$-$K$-bimodule $N$ is

$$M \star N := M \otimes_G N.$$

The lax structure of horizontal composition $\star$ with respect to the monoidal structure $\diamond$ is defined as in (6.21).

It is clear that looping the $\mathsf{lCat}$-bicategory $\mathsf{C}$ at the object $K$ yields the 2-monoidal category of $K$-bimodules of Example 6.18.

APPENDIX D

# Monoids and the Simplicial Category

Mac Lane's simplicial category $\mathsf{M}_\Delta$ is universal for monoids: it is a monoidal category, it contains a distinguished monoid object, and given a monoid in another monoidal category $\mathsf{C}$, there is a strong monoidal functor $\mathsf{M}_\Delta \to \mathsf{C}$ sending one monoid to the other; moreover, this functor is unique up to equivalence. In Sections D.1 and D.2, we review the definition of $\mathsf{M}_\Delta$ and the above property. Relaxing the conditions on the functor leads to generalizations of the notion of monoid. Two such notions, lax monoids and homotopy monoids, are outlined in the latter section, following Day and Street, and Leinster, respectively.

In Section D.3 we discuss lax monoidal categories (a special case of lax monoids). Lax monoidal categories play only a minor role in this monograph; they appear briefly in Section B.4.4 and tangentially in Remark 6.45. Nevertheless, the notion is discussed here in some detail.

The set of maps from a comonoid to a monoid forms an ordinary monoid under convolution. In Section D.4, we study the analogous structure on the natural transformations from a colax monoidal functor to a lax monoidal functor; we find it to be that of an augmented simplicial set with a lax monoidal structure (as a functor on Mac Lane's simplicial category). This is an example of a homotopy monoid.

## D.1. Mac Lane's simplicial category

The *simplicial category of Mac Lane* [250, Section VII.5], also called the *algebraist's simplicial category*, has for objects the nonnegative integers $0, 1, 2, \ldots$. The morphisms from $n$ to $m$ are the order-preserving functions

$$g\colon [n] \to [m].$$

(Mac Lane uses $\{0, 1, \ldots, n-1\}$ instead of $[n] = \{1, 2, \ldots, n\}$, but this makes no difference.) Morphisms are composed as ordinary functions. We let $\mathsf{M}_\Delta$ denote this category.

**Remark D.1.** Recall the simplicial category $\Delta$ of Section 5.1.1. This is sometimes called the *topologist's simplicial category*. It differs from the algebraist's simplicial category $\mathsf{M}_\Delta$ defined above. There is a functor

$$\Delta \to \mathsf{M}_\Delta, \qquad n \mapsto n+1$$

that embeds $\Delta$ as a full subcategory of $\mathsf{M}_\Delta$. However, this connection is not relevant to our purposes.

*Warning.* Mac Lane uses $\Delta$ for $\mathsf{M}_\Delta$ [250, Section VII], while Leinster uses $\mathbb{D}$ [226, Example 1.2.2]. We reserve $\Delta$ for the topologist's simplicial category.

Given nonnegative integers $m$ and $n$, let us write

$$m \bullet n := m + n.$$

The *concatenation* of two functions $f\colon m \to m'$ and $g\colon n \to n'$ is

$$f \bullet g : m \bullet n \to m' \bullet n', \qquad (f \bullet g)(i) := \begin{cases} f(i) & \text{if } 1 \leq i \leq m, \\ g(i-m) + m' & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

Concatenation $\bullet$ and composition $\circ$ are related as follows: for each double pair of composable order-preserving functions

$$[n_1] \xrightarrow{g_1} [m_1] \xrightarrow{f_1} [l_1] \quad \text{and} \quad [n_2] \xrightarrow{g_2} [m_2] \xrightarrow{f_2} [l_2],$$

we have

$$(f_1 \bullet f_2) \circ (g_1 \bullet g_2) = (f_1 \circ g_1) \bullet (f_2 \circ g_2). \tag{D.1}$$

Thus, addition of numbers and concatenation of functions define a functor

$$\mathsf{M}_{\Delta} \times \mathsf{M}_{\Delta} \xrightarrow{\bullet} \mathsf{M}_{\Delta}.$$

This turns $\mathsf{M}_{\Delta}$ into a strict monoidal category with tensor product $\bullet$ and unit object the number 0.

A weak composition of $n$ is a sequence $\alpha = (j_1, \ldots, j_m)$ of nonnegative integers such that

$$j_1 + \cdots + j_m = n.$$

In this case, we say that $\alpha$ has $m$ parts and write $\alpha \vDash n$.

Given an order-preserving function $g\colon [n] \to [m]$, the sequence of fiber cardinalities

$$\big(g^{-1}(1), \ldots, g^{-1}(m)\big) \tag{D.2}$$

is a weak composition of $n$ with $m$ parts. The weak composition determines the map, since each fiber must precede the next. Therefore, we may identify the set of morphisms in $\mathsf{M}_{\Delta}$ from $n$ to $m$ with

$$\{\alpha \vDash n \mid \alpha \text{ has } m \text{ parts}\}.$$

In this notation, tensor product of morphisms is simply concatenation of weak compositions,

$$(m \xrightarrow{(i_1,\ldots,i_{m'})} m') \bullet (n \xrightarrow{(j_1,\ldots,j'_n)} n') = m + n \xrightarrow{(i_1,\ldots,i_{m'},j_1,\ldots,j_{n'})} m' + n'$$

and composition of morphisms is as follows:

$$\begin{tikzcd} & m \arrow[dr, "{(i_1,\ldots,i_l)}"] & \\ n \arrow[ur, "{(j_1,\ldots,j_m)}"] \arrow[rr, dashed, "{(k_1,\ldots,k_l)}"'] & & l \end{tikzcd}$$

where given $(i_1, \ldots, i_l) \vDash m$ and $(j_1, \ldots, j_m) \vDash n$, we define

$$\begin{aligned} k_1 &:= j_1 + \cdots + j_{i_1}, \\ k_2 &:= j_{i_1+1} + \cdots + j_{i_2}, \\ &\vdots \\ k_l &:= j_{i_1+\cdots+i_{l-1}+1} + \cdots + j_{i_1+\cdots+i_{l-1}+i_l}. \end{aligned} \tag{D.3}$$

Note that

$$k_1 + \cdots + k_l = j_1 + \cdots + j_m = n$$

so $(k_1, \ldots, k_l)$ is a morphism from $n$ to $l$. For example, composing $(2, 1)$ with $(1, 2, 1)$ yields the morphism $(3, 1)\colon 4 \to 2$.

For each object $n \geq 1$, the identity of $n$ is the composition $(1, 1, \ldots, 1)$ with $n$ parts equal to 1. The identity of the object 0 is the unique weak composition with no parts. The object 0 is initial: for any object $n$ there is a unique map $0 \to n$, namely,

$$0 \xrightarrow{(0,\ldots,0)} n.$$

A generating set of morphisms is shown below [250, Proposition VII.5.2].

$$\begin{tikzcd}
0 \arrow[r, "(0)"] & 1 \arrow[r, shift left, "(1,0)"] \arrow[r, shift right, "(0,1)"'] & 2 \arrow[r, shift left=2, "(1,1,0)"] \arrow[r, "(1,0,1)" description] \arrow[r, shift right=2, "(0,1,1)"'] & 3 \arrow[r, dotted] & 4
\end{tikzcd}$$

$$\begin{tikzcd}
0 & 1 & 2 \arrow[l, "(2)"] & 3 \arrow[l, shift right, "(1,2)"'] \arrow[l, shift left, "(2,1)"] & 4 \arrow[l, shift right=2, "(1,1,2)"'] \arrow[l, "(1,2,1)" description] \arrow[l, shift left=2, "(2,1,1)"] & \arrow[l, dotted]
\end{tikzcd}$$

The morphisms in the first row are called *face maps* and those in the second row are called *degeneracies*.

An *augmented simplicial set* is a contravariant functor

$$\mathsf{M}_\Delta \to \mathsf{Set}.$$

This terminology extends that of a simplicial set which we recall is a contravariant functor $\Delta \to \mathsf{Set}$.

## D.2. Monoids, lax monoids, and homotopy monoids

In this section, we review the universal property of Mac Lane's simplicial category and two related generalizations.

**D.2.1. Monoids.** The object 0 is initial in the category $\mathsf{M}_\Delta$, and the object 1 is terminal. Therefore, the maps

$$2 \xrightarrow{(2)} 1 \quad \text{and} \quad 0 \xrightarrow{(0)} 1$$

define a monoid structure on 1.

Let $(A, \mu, \iota)$ be a monoid in a monoidal category $(\mathsf{C}, \bullet, I)$.

We define a functor $\mathcal{F}\colon \mathsf{M}_\Delta \to \mathsf{C}$ as follows. On objects,

$$\mathcal{F}(0) := I \;\text{ and for } n \geq 1,\; \mathcal{F}(n) := A^{\bullet(n)},$$

where

$$A^{\bullet(n)} := \underbrace{A \bullet \cdots \bullet A}_{n}$$

is the unbracketed tensor product as in Section 1.4. Given an order-preserving map $g : [n] \to [m]$, let $(n_1, \ldots, n_m)$ be the sequence of fiber cardinalities (D.2). We define

$$\mathcal{F}(g)\colon \mathcal{F}(n) \to \mathcal{F}(m)$$

as the following composite:

$$A^{\bullet(n)} \cong A^{\bullet(n_1)} \bullet \cdots \bullet A^{\bullet(n_m)} \xrightarrow{\mu^{(n_1)}\bullet\cdots\bullet\mu^{(n_m)}} A^{\bullet(m)}.$$

The first map is the canonical isomorphism between unbracketed tensor products; the second map is built out of the iterated products of $A$ (which are well-defined by associativity and unitality). There are canonical isomorphisms

$$\mathcal{F}(m \bullet n) \cong \mathcal{F}(m) \bullet \mathcal{F}(n)$$

and the functor $\mathcal{F}$ is strong monoidal.

Conversely, given a strong monoidal functor $\mathcal{F}\colon \mathsf{M}_{\Delta} \to \mathsf{C}$, the object $A := \mathcal{F}(1)$ is a monoid in $\mathsf{C}$ (as in Section 3.4.3).

These constructions define an equivalence between the category of monoids in $\mathsf{C}$ and the category of strong monoidal functors from Mac Lane's simplicial category to $\mathsf{C}$.

**Proposition D.2** ([250, Propositon VII.5.1])**.** *Monoids in* $\mathsf{C}$ *and strong monoidal functors* $\mathsf{M}_{\Delta} \to \mathsf{C}$ *are equivalent notions.*

This result is the starting point for various generalizations of the notion of monoid, as we briefly review in the rest of the section.

**D.2.2. Lax monoids.** Let $\mathsf{C}$ be a monoidal 2-category (Section C.1.3). Day and Street [94, Section 2] define a *lax monoid* in $\mathsf{C}$ as a *strict monoidal lax functor* $\mathcal{F}\colon \mathsf{M}_{\Delta} \to \mathsf{C}$. Here "lax" refers to the fact that $\mathcal{F}$ need not preserve compositions and identities on the nose. Instead, there are 2-cells in $\mathsf{C}$

$$\mathcal{F}(f) \circ \mathcal{F}(g) \Rightarrow \mathcal{F}(f \circ g) \quad \text{and} \quad \mathrm{id}_{\mathcal{F}(n)} \Rightarrow \mathcal{F}_{\mathrm{id}_n}$$

for each pair of composable order-preserving functions $[n] \xrightarrow{g} [m] \xrightarrow{f} [l]$ and for each nonnegative integer $n$, respectively. These are subject to various axioms. We make this explicit in the case when $\mathsf{C} = \mathsf{Cat}$ in Section D.3.1.

One may also consider *strong monoids*: these are strict monoidal *strong* functors $\mathcal{F}\colon \mathsf{M}_{\Delta} \to \mathsf{C}$, that is, lax monoids for which the 2-cells above are invertible. It may be shown [94, Section 2, Example 2] that strong monoids and pseudomonoids (Section C.2) are equivalent notions. This is a 2-dimensional analogue of Proposition D.2.

**D.2.3. Homotopy monoids.** In a different direction, Leinster has proposed a definition of homotopy monoids in an arbitrary monoidal category $\mathsf{C}$ [229, Section 2.2]. A *homotopy monoid* in $\mathsf{C}$ is a colax monoidal functor $\mathcal{F}\colon \mathsf{M}_{\Delta} \to \mathsf{C}$. The colax structure maps

$$\mathcal{F}(m \bullet n) \to \mathcal{F}(m) \bullet \mathcal{F}(n) \quad \text{and} \quad \mathcal{F}(0) \to I$$

must come from a specified class of arrows in $\mathsf{C}$. Different choices for this class lead to different classes of homotopy monoids. Two extreme cases are on the one hand the class of all isomorphisms in $\mathsf{C}$, and on the other, the class of all morphisms in $\mathsf{C}$ [229, Section 2.1, Examples a,b]. In the former case, homotopy monoids and monoids are equivalent notions [229, Theorem 1.6.1]. We give an example of the latter kind of homotopy monoid in Section D.4. Leinster's ideas are summarized in [228]. In [229], he in fact develops the homotopy version of the **p**-monoids of Section 4.2.2, where **p** is an operad.

## D.3. Lax monoidal categories

There is a more general notion than that of monoidal category, in which the invertibility of the associativity and unit constraints is not required. To compensate for this, one assumes the existence of $n$-ary tensor products for $n \geq 0$ and transformations between the various bracketed tensor products of a string of objects. We discuss this notion next.

**D.3.1. Lax monoidal categories as lax monoids.** View $\mathsf{Cat}$ as a monoidal 2-category under Cartesian product. Day and Street [94, Section 2] define a lax monoidal category as a lax monoid in $\mathsf{Cat}$, as in Section D.2.2. Below we provide an explicit translation of this definition. We make use of the notion of concatenation of order-preserving functions, as discussed in Section D.1.

Given a category $\mathsf{C}$ and $n \in \mathbb{N}$, let

$$\mathsf{C}^n := \begin{cases} \underbrace{\mathsf{C} \times \cdots \times \mathsf{C}}_{n} & \text{(Cartesian product), if } n > 0, \\ \mathsf{I} & \text{the one-arrow category, if } n = 0. \end{cases}$$

**Definition D.3.** A category $\mathsf{C}$ is *lax monoidal* if there is given a family of functors

$$\mathcal{M}_g \colon \mathsf{C}^n \to \mathsf{C}^m,$$

one for each order-preserving function $g : [n] \to [m]$; a family of natural transformations $\alpha_{f,g} : \mathcal{M}_f \circ \mathcal{M}_g \Rightarrow \mathcal{M}_{f \circ g}$,

$$\begin{tikzcd} & \mathsf{C}^m \arrow[dr, bend left, "\mathcal{M}_f"] \arrow[d, phantom, "\Downarrow \alpha_{f,g}"] & \\ \mathsf{C}^n \arrow[ur, bend left, "\mathcal{M}_g"] \arrow[rr, bend right, "\mathcal{M}_{f \circ g}"'] & {} & \mathsf{C}^l \end{tikzcd}$$

one for each pair of composable order-preserving functions $[n] \xrightarrow{g} [m] \xrightarrow{f} [l]$; and a sequence of natural transformations $\iota_n : \mathrm{id}_{\mathsf{C}^n} \Rightarrow \mathcal{M}_{\mathrm{id}_n}$,

$$\begin{tikzcd} \mathsf{C}^n \arrow[r, bend left, "\mathrm{id}_{\mathsf{C}^n}"] \arrow[r, bend right, "\mathcal{M}_{\mathrm{id}_n}"'] \arrow[r, phantom, "\Downarrow \iota_n"] & \mathsf{C}^n \end{tikzcd}$$

one for each $n \in \mathbb{N}$. These are subject to the following conditions.

**Associativity.** For each triple of composable order-preserving functions

$$[n] \xrightarrow{g} [m] \xrightarrow{f} [l] \xrightarrow{e} [k]$$

the following diagram commutes.

$$\begin{tikzcd} \mathcal{M}_e \circ \mathcal{M}_f \circ \mathcal{M}_g \arrow[r, "\alpha_{e,f} \circ \mathrm{id}"] \arrow[d, "\mathrm{id} \circ \alpha_{f,g}"'] & \mathcal{M}_{e \circ f} \circ \mathcal{M}_g \arrow[d, "\alpha_{e \circ f, g}"] \\ \mathcal{M}_e \circ \mathcal{M}_{f \circ g} \arrow[r, "\alpha_{e, f \circ g}"'] & \mathcal{M}_{e \circ f \circ g} \end{tikzcd}$$

**Unitality.** For each order-preserving function $[n] \xrightarrow{g} [m]$, the following diagrams commute.

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{id}_m} \circ \mathcal{M}_g \arrow[r, "\alpha_{\mathrm{id}_m, g}"] & \mathcal{M}_g \\
\mathrm{id}_{\mathsf{C}^m} \circ \mathcal{M}_g \arrow[u, "\eta_m \circ \mathrm{id}"] \arrow[ur, equals] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_g \circ \mathcal{M}_{\mathrm{id}_n} \arrow[r, "\alpha_{g, \mathrm{id}_n}"] & \mathcal{M}_g \\
\mathcal{M}_g \circ \mathrm{id}_{\mathsf{C}^n} \arrow[u, "\mathrm{id} \circ \eta_n"] \arrow[ur, equals] &
\end{tikzcd}$$

**Multiplicativity.** For each pair of functions

$$[n_1] \xrightarrow{g_1} [m_1] \quad \text{and} \quad [n_2] \xrightarrow{g_2} [m_2],$$

the functors $\mathcal{M}_{g_1 \bullet g_2}$ and $\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$ are equal. In other words, the following diagram commutes.

$$\begin{tikzcd}
\mathsf{C}^{n_1} \times \mathsf{C}^{n_2} \arrow[r, "\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}"] \arrow[d, equals] & \mathsf{C}^{m_1} \times \mathsf{C}^{m_2} \arrow[d, equals] \\
\mathsf{C}^{n_1 + n_2} \arrow[r, "\mathcal{M}_{g_1 \bullet g_2}"'] & \mathsf{C}^{m_1 + m_2}
\end{tikzcd}$$

For each double pair of composable order-preserving functions

$$[n_1] \xrightarrow{g_1} [m_1] \xrightarrow{f_1} [l_1] \quad \text{and} \quad [n_2] \xrightarrow{g_2} [m_2] \xrightarrow{f_2} [l_2],$$

the transformations $\alpha_{f_1 \bullet f_2, g_1 \bullet g_2}$ and $\alpha_{f_1, g_1} \times \alpha_{f_2, g_2}$ are equal. In other words, the following diagram commutes. (We make use of (D.1).)

$$\begin{tikzcd}
\mathcal{M}_{f_1 \bullet f_2} \circ \mathcal{M}_{g_1 \bullet g_2} \arrow[r, "\alpha_{f_1 \bullet f_2, g_1 \bullet g_2}"] \arrow[d, equals] & \mathcal{M}_{(f_1 \bullet f_2) \circ (g_1 \bullet g_2)} \arrow[d, equals] \\
(\mathcal{M}_{f_1} \times \mathcal{M}_{f_2}) \circ (\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}) \arrow[d, equals] & \mathcal{M}_{(f_1 \circ g_1) \bullet (f_2 \circ g_2)} \arrow[d, equals] \\
(\mathcal{M}_{f_1} \circ \mathcal{M}_{g_1}) \times (\mathcal{M}_{f_2} \circ \mathcal{M}_{g_2}) \arrow[r, "\alpha_{f_1, g_1} \times \alpha_{f_2, g_2}"'] & \mathcal{M}_{f_1 \circ g_1} \times \mathcal{M}_{f_2 \circ g_2}
\end{tikzcd}$$

For each pair of nonnegative integers $n_1$ and $n_2$, the transformations $\eta_{n_1 + n_2}$ and $\eta_{n_1} \times \eta_{n_2}$ are equal. In other words, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{id}_{\mathsf{C}^{n_1 + n_2}} \arrow[r, "\eta_{n_1 + n_2}"] \arrow[d, equals] & \mathcal{M}_{\mathrm{id}_{n_1 \bullet n_2}} \arrow[d, equals] \\
\mathrm{id}_{\mathsf{C}^{n_1}} \times \mathrm{id}_{\mathsf{C}^{n_2}} \arrow[r, "\eta_{n_1} \times \eta_{n_2}"'] & \mathcal{M}_{\mathrm{id}_{n_1}} \times \mathcal{M}_{\mathrm{id}_{n_2}}
\end{tikzcd}$$

This completes the definition of lax monoidal category.

**Definition D.4.** A lax monoidal category is *normal* if the transformations $\eta_n$ are identities. A *strong monoidal category* is a lax monoidal category for which the transformations $\alpha_{f,g}$ and $\eta_n$ are invertible.

In Leinster's book, strong monoidal categories are called *unbiased* monoidal categories [226, Definition 3.1.1].

**Remark D.5.** The multiplicativity conditions in Definition D.3 imply that a certain part of the structure determines the rest, as we now explain. For each $n \in \mathbb{N}$, let

$$p_n : [n] \to [1]$$

be the unique such map. Write:

$$\mathcal{M}_n := \mathcal{M}_{p_n}.$$

Given an order-preserving map $g : [n] \to [m]$, let $(n_1, \dots, n_m)$ be the sequence of fiber cardinalities (D.2). We have

$$g = p_{n_1} \bullet \cdots \bullet p_{n_m}.$$

Therefore,

$$\mathcal{M}_g = \mathcal{M}_{n_1} \times \cdots \times \mathcal{M}_{n_m}.$$

Write also

$$\alpha_g := \alpha_{p_m, g}.$$

Since $p_m \circ g = p_n$, we have $\alpha_g \colon \mathcal{M}_m \circ \mathcal{M}_g \Rightarrow \mathcal{M}_n$.

$$\begin{tikzcd}
& \mathsf{C} \times \cdots \times \mathsf{C} = \mathsf{C}^m \arrow[dr, "\mathcal{M}_m"] \arrow[d, phantom, "\Downarrow \alpha_g"] & \\
\mathsf{C}^{n_1} \times \cdots \times \mathsf{C}^{n_1} = \mathsf{C}^n \arrow[ur, bend left, "\mathcal{M}_{n_1} \times \cdots \times \mathcal{M}_{n_m}"] \arrow[rr, bend right, "\mathcal{M}_n"'] & & \mathsf{C}^l
\end{tikzcd}$$

Given an order-preserving map $f : [m] \to [l]$, let $(m_1, \dots, m_l)$ be the sequence of fiber cardinalities (D.2). Then there are nonnegative integers $(n_1, \dots, n_l)$ and order-preserving functions $g_i : [n_i] \to [m_i]$ such that the following diagram commutes.

$$\begin{tikzcd}
n \arrow[r, "g"] \arrow[d, equal] & m \arrow[d, equal] \\
n_1 \bullet \cdots \bullet n_l \arrow[r, "g_1 \bullet \cdots \bullet g_l"'] & m_1 \bullet \cdots \bullet m_l
\end{tikzcd}$$

Therefore,

$$\alpha_{f,g} = \alpha_{g_1} \times \cdots \times \alpha_{g_l}.$$

Finally, write $\eta := \eta_1 \colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{M}_1$. Then

$$\eta_n = \overbrace{\eta \times \cdots \times \eta}^{n}.$$

This allows for a reformulation of the notion of lax monoidal category in terms of less structure; namely, the functors $\mathcal{M}_n$, the transformations $\alpha_g$ and the map $\eta$. The corresponding list of axioms is given explicitly in [94, Section 2, Example 3] or [226, Section 3.1].

**D.3.2. Monoidal categories as strong monoidal categories.** Let $\mathsf{C}$ be a monoidal category with tensor product $\bullet$, associativity constraint $\alpha$, and unit object $I$, as in Definition 1.1. We proceed to turn it into a strong monoidal category, and for this we take advantage of Remark D.5.

First of all, we define the functors $\mathcal{M}_n\colon \mathsf{C}^n \to \mathsf{C}$ by

$$\mathcal{M}_n(A_1, \ldots, A_n) := A_1 \bullet \cdots \bullet A_n.$$

This is the unbracketed tensor product of Section 1.4.

Second, with the notation as in Remark D.5, we have

$$(\mathcal{M}_m \circ \mathcal{M}_g)(A_1, \ldots, A_n) =$$
$$(A_1 \bullet \cdots \bullet A_{n_1}) \bullet (A_{n_1+1} \bullet \cdots \bullet A_{n_1+n_2}) \bullet \cdots \bullet (A_{n_1+\cdots+n_{m-1}+1} \bullet \cdots \bullet A_{n_1+\cdots+n_m}).$$

We let the transformation $\alpha_g\colon \mathcal{M}_m \circ \mathcal{M}_g \Rightarrow \mathcal{M}_n$ consist of the canonical isomorphisms between this particular bracketed tensor product and the unbracketed product $A_1 \bullet \cdots \bullet A_n$. This is constructed from the associativity constraint $\alpha$ as in Section 1.4. Notice this makes use of coherence for monoidal categories.

Finally, note that in this situation $\mathcal{M}_1 = \mathrm{id}_{\mathsf{C}}$. We let $\iota$ be the identity transformation of this functor.

With this structure, $\mathsf{C}$ is a normal strong monoidal category.

Conversely, from a strong monoidal category (not necessarily normal), one may derive a monoidal category in the sense of Definition 1.1.

When the two constructions are iterated one returns to a category that is equivalent to the original one. One thus has the following result.

**Proposition D.6** ([226, Corollary 3.2.5])**.** *Strong monoidal categories (as in Definition D.4) and monoidal categories (as in Definition 1.1) are equivalent notions.*

## D.4. The convolution homotopy monoid

We present an analogue of the convolution monoid $\mathrm{Hom}(C, A)$ associated to a comonoid $C$ and a monoid $A$ (Definition 1.13). Given a colax monoidal functor $\mathcal{F}$ and a lax monoidal functor $\mathcal{G}$, we construct an augmented simplicial set $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ (a contravariant functor on Mac Lane's simplicial category) in terms of natural transformations. This is the analogue of the set $\mathrm{Hom}(C, A)$. The role of the convolution product is played by convolution of natural transformations, which turns $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ into a lax monoidal functor (Theorem D.9). As an application, we explain how the convolution identities of Section 3.7.5 can be deduced from this theorem. It turns out that $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ is an example of a homotopy monoid; we explain this in Remark D.10.

**D.4.1. The simplicial set of natural transformations.** Let $(\mathsf{D}, \bullet)$ be an arbitrary category and $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ a functor. Let $(\mathsf{I}, \bullet)$ be the one-arrow category and let $*$ denote its unique object.

We let $\mathcal{F}_n\colon \mathsf{C}^n \to \mathsf{D}$ be the functor given by

$$\mathcal{F}_n(A_1, \ldots, A_n) := \mathcal{F}(A_1 \bullet \cdots \bullet A_n). \tag{D.4}$$

In particular, $\mathcal{F}_1 = \mathcal{F}$, $\mathcal{F}_2$ is the functor introduced in (3.1), and $\mathcal{F}_0$ is the functor introduced in (3.2), that is,

$$\mathcal{F}_0\colon \mathsf{I} \to \mathsf{D} \quad \text{is} \quad \mathcal{F}_0(*) = \mathcal{F}(I),$$

the image of the unit object of $\mathsf{C}$.

Suppose that $\mathcal{G}\colon \mathsf{C} \to \mathsf{D}$ is another functor and $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation. We let $\theta^{(n)}\colon \mathcal{F}_n \Rightarrow \mathcal{G}_n$ be the natural transformation

$$\theta^{(n)}_{A_1,\dots,A_n}\colon \mathcal{F}(A_1 \bullet \cdots \bullet A_n) \xrightarrow{\theta_{A_1\bullet\cdots\bullet A_n}} \mathcal{G}(A_1 \bullet \cdots \bullet A_n). \tag{D.5}$$

This generalizes the construction in (3.36). Observe that $\theta^{(1)} = \theta$.

Using these ideas, we proceed to define an augmented simplicial set of natural transformations. It is convenient to think of morphisms in $\mathsf{M}_{\Delta}$ in terms of weak compositions, as explained in Section D.1.

**Definition D.7.** Define a functor

$$\mathsf{N}_{\mathcal{F},\mathcal{G}}\colon \mathsf{M}_{\Delta}^{\mathrm{op}} \to \mathsf{Set}$$

as follows. On objects, we let $\mathsf{N}_{\mathcal{F},\mathcal{G}}(n)$ be the set of natural transformations from $\mathcal{F}_n$ to $\mathcal{G}_n$. Given a morphism

$$\alpha = (j_1, \dots, j_m)\colon n \to m,$$

we define a map

$$\mathsf{N}_{\mathcal{F},\mathcal{G}}(m) \to \mathsf{N}_{\mathcal{F},\mathcal{G}}(n), \qquad \theta \mapsto \theta^{\alpha},$$

by

$$\theta^{\alpha}_{A_1,\dots,A_n} := \theta_{A_1^{\alpha},\dots,A_m^{\alpha}},$$

where

$$\begin{aligned} A_1^{\alpha} &:= A_1 \bullet \cdots \bullet A_{j_1}, \\ A_2^{\alpha} &:= A_{j_1+1} \bullet \cdots \bullet A_{j_1+j_2}, \\ &\ \vdots \\ A_m^{\alpha} &:= A_{j_1+\cdots+j_{m-1}+1} \bullet \cdots \bullet A_{j_1+\cdots+j_m}. \end{aligned}$$

Note that

$$A_1^{\alpha} \bullet \cdots \bullet A_m^{\alpha} \xrightarrow{\cong} A_1 \bullet \cdots \bullet A_n,$$

so $\theta^{\alpha}$ is a natural transformation $\mathcal{F}_n \Rightarrow \mathcal{G}_n$.

For weak compositions with a single part, namely $\alpha = (n)$, note that $\theta^{\alpha}$ agrees with the construction in (D.5). Empty tensor products are taken to be equal to the unit object $I$ of $\mathsf{C}$. These arise when a part of $\alpha$ is 0. For instance,

$$\theta^{(0,\dots,0)}\colon \mathcal{F}_0(*) = \mathcal{F}(I) \xrightarrow{\theta_{I,\dots,I}} \mathcal{G}(I) = \mathcal{G}_0(*).$$

**Proposition D.8.** *Let $\mathcal{F}$ and $\mathcal{G}$ be functors from a monoidal category $\mathsf{C}$ to an arbitrary category $\mathsf{D}$. The above defines a functor*

$$\mathsf{N}_{\mathcal{F},\mathcal{G}}\colon \mathsf{M}_{\Delta}^{\mathrm{op}} \to \mathsf{Set}.$$

Thus, $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ is an augmented simplicial set.

Proof. If indices $i$, $j$, and $k$ are related as in (D.3), then grouping $n$ variables into intervals of lengths $j_r$ and then grouping the resulting $m$ variables into intervals of lengths $i_s$, yields the same result as directly grouping the $n$ variables into intervals of lengths $k_t$. Thus, $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ preserves compositions. Further,

$$\theta^{(1,1,\dots,1)} = \theta,$$

thus $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ preserves identities. $\square$

**D.4.2. The lax monoidal functor of natural transformations.** Now suppose that $(\mathcal{F}, \psi)$ is a colax monoidal functor and $(\mathcal{G}, \varphi)$ is a lax monoidal functor, both from a monoidal category $(\mathsf{C}, \bullet)$ to a monoidal category $(\mathsf{D}, \bullet)$. We claim that in this case $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ can be turned into a lax monoidal functor

$$(\mathsf{M}_{\Delta}^{\mathrm{op}}, \bullet) \to (\mathsf{Set}, \times),$$

where $(\mathsf{Set}, \times)$ is the monoidal category of sets under Cartesian product (Example 1.3).

Given natural transformations

$$\sigma\colon \mathcal{F}_m \Rightarrow \mathcal{G}_m \quad \text{and} \quad \tau\colon \mathcal{F}_n \Rightarrow \mathcal{G}_n,$$

define their *convolution*

$$\sigma * \tau\colon \mathcal{F}_{m+n} \Rightarrow \mathcal{G}_{m+n}$$

by

$$\begin{tikzcd}
\mathcal{F}(A_1\bullet\cdots\bullet A_m\bullet B_1\bullet\cdots\bullet B_n) \arrow[r, dashed, "(\sigma*\tau)_{A_1,\dots,A_m,B_1,\dots,B_n}"] \arrow[d, "\psi_{A_1\bullet\cdots\bullet A_m, B_1\bullet\cdots\bullet B_n}"] & \mathcal{G}(A_1\bullet\cdots\bullet A_m\bullet B_1\bullet\cdots\bullet B_n) \\
\mathcal{F}(A_1\bullet\cdots\bullet A_m)\bullet\mathcal{F}(B_1\bullet\cdots\bullet B_n) \arrow[r, "\sigma_{A_1,\dots,A_m}\bullet\tau_{B_1,\dots,B_n}"'] & \mathcal{G}(A_1\bullet\cdots\bullet A_m)\bullet\mathcal{G}(B_1\bullet\cdots\bullet B_n). \arrow[u, "\varphi_{A_1\bullet\cdots\bullet A_m, B_1\bullet\cdots\bullet B_n}"']
\end{tikzcd}$$

The structure map

$$\mathsf{N}_{\mathcal{F},\mathcal{G}}(m) \times \mathsf{N}_{\mathcal{F},\mathcal{G}}(n) \to \mathsf{N}_{\mathcal{F},\mathcal{G}}(m+n) \tag{D.6}$$

is defined by $(\sigma, \tau) \mapsto \sigma * \tau$. The structure map

$$\{\emptyset\} \to \mathsf{N}_{\mathcal{F},\mathcal{G}}(0)$$

sends $\emptyset$ to the map

$$\mathcal{F}_0(*) = \mathcal{F}(I) \xrightarrow{\psi_0} I \xrightarrow{\varphi_0} \mathcal{G}(I) = \mathcal{G}_0(*).$$

**Theorem D.9.** *Let $(\mathcal{F}, \psi)$ be a colax monoidal functor and $(\mathcal{G}, \varphi)$ a lax monoidal functor, both from a monoidal category $\mathsf{C}$ to a monoidal category $\mathsf{D}$. The above turns*

$$\mathsf{N}_{\mathcal{F},\mathcal{G}}\colon (\mathsf{M}_{\Delta}^{\mathrm{op}}, \bullet) \to (\mathsf{Set}, \times).$$

*into a lax monoidal functor.*

Proof. The associativity and unitality of (D.6) follow readily from those of $\varphi$ and $\psi$ (axioms (3.5) and (3.6)). Let us take a closer look at the naturality of (D.6). Take natural transformations $\sigma\colon \mathcal{F}_{m'} \to \mathcal{G}_{m'}$ and $\tau\colon \mathcal{F}_{n'} \to \mathcal{G}_{n'}$, and weak compositions $\alpha \vDash m$ and $\beta \vDash n$ with $m'$ and $n'$ parts respectively. We have to show that

$$(\sigma * \tau)^{\alpha\bullet\beta} = \sigma^{\alpha} * \tau^{\beta}.$$

This follows from the commutativity of the diagram below (in which the tensor product symbol is omitted).

$$\begin{tikzcd}
\mathcal{F}(A_1^\alpha \cdots A_{m'}^\alpha B_1^\beta \cdots B_{n'}^\beta) \arrow[r, "\cong"] \arrow[d, "\psi_{A_1^\alpha \cdots A_{m'}^\alpha, B_1^\beta \cdots B_{n'}^\beta}"'] & \mathcal{F}(A_1 \cdots A_m B_1 \cdots B_n) \arrow[d, "\psi_{A_1 \cdots A_m, B_1 \cdots B_n}"] \\
\mathcal{F}(A_1^\alpha \cdots A_{m'}^\alpha)\mathcal{F}(B_1^\beta \cdots B_{n'}^\beta) \arrow[r, "\cong"] \arrow[d, "\sigma_{A_1^\alpha, \dots, A_{m'}^\alpha} \tau_{B_1^\beta, \dots, B_{n'}^\beta}"'] & \mathcal{F}(A_1 \cdots A_m)\mathcal{F}(B_1 \cdots B_n) \arrow[d, "\sigma^\alpha_{A_1, \dots, A_m} \tau^\beta_{B_1, \dots, B_n}"] \\
\mathcal{G}(A_1^\alpha \cdots A_{m'}^\alpha)\mathcal{G}(B_1^\beta \cdots B_{n'}^\beta) \arrow[r, "\cong"] \arrow[d, "\varphi_{A_1^\alpha \cdots A_{m'}^\alpha, B_1^\beta \cdots B_{n'}^\beta}"'] & \mathcal{G}(A_1 \cdots A_m)\mathcal{G}(B_1 \cdots B_n) \arrow[d, "\varphi_{A_1 \cdots A_m, B_1 \cdots B_n}"] \\
\mathcal{G}(A_1^\alpha \cdots A_{m'}^\alpha B_1^\beta \cdots B_{n'}^\beta) \arrow[r, "\cong"] & \mathcal{G}(A_1 \cdots A_m B_1 \cdots B_n)
\end{tikzcd} \qquad \square$$

We refer to $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ as the convolution lax monoidal functor.

**Remark D.10.** A lax monoidal functor $(\mathsf{M}_{\Delta}^{\mathrm{op}}, \bullet) \to (\mathsf{C}, \bullet)$ is equivalent to a colax monoidal functor $(\mathsf{M}_{\Delta}, \bullet) \to (\mathsf{C}^{\mathrm{op}}, \bullet)$. This is precisely a homotopy monoid (Section D.2.3). Thus, we may restate Theorem D.9 by saying that convolution of natural transformations from a colax functor to a lax functor gives rise to a homotopy monoid

$$\mathsf{N}_{\mathcal{F},\mathcal{G}} \colon (\mathsf{M}_{\Delta}, \bullet) \to (\mathsf{Set}^{\mathrm{op}}, \times).$$

Recall that in Leinster's setting, the colax structure maps must come from a specified class of *equivalences*. To satisfy this requirement in our case, we may choose the class of equivalences to consist of all maps in $\mathsf{Set}^{\mathrm{op}}$ [229, Section 2.1, Example b].

We mention in passing and without further argument that if the functors $\mathcal{F}$ and $\mathcal{G}$ are braided and the underlying monoidal categories are symmetric, then $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ is a *homotopy commutative monoid* in the sense of Leinster.

The following result generalizes Proposition 3.61.

**Proposition D.11.** *Let $(\mathcal{F}, \psi)$ and $(\mathcal{F}', \psi')$ be colax functors and $(\mathcal{G}, \gamma)$ and $(\mathcal{G}', \gamma')$ be lax functors, all from $\mathsf{C}$ to $\mathsf{D}$. Let*

$$\theta \colon (\mathcal{F}', \psi') \Rightarrow (\mathcal{F}, \psi) \quad \textit{and} \quad \kappa \colon (\mathcal{G}, \gamma) \Rightarrow (\mathcal{G}', \gamma')$$

*be a morphism of colax functors and a morphism of lax functors, respectively. Then the maps*

$$\mathsf{N}_{\mathcal{F},\mathcal{G}} \Rightarrow \mathsf{N}_{\mathcal{F}',\mathcal{G}}, \qquad \sigma \mapsto \sigma\theta^{(n)}$$

*and*

$$\mathsf{N}_{\mathcal{F},\mathcal{G}} \Rightarrow \mathsf{N}_{\mathcal{F},\mathcal{G}'}, \qquad \sigma \mapsto \kappa^{(n)}\sigma,$$

*where $\sigma \in \mathsf{N}_{\mathcal{F},\mathcal{G}}(n)$, are morphisms of lax monoidal functors.*

The proof is straightforward.

**D.4.3. Application to Hopf lax functors.** We now explain how the results in Section 3.7.5 can be given succinct proofs based on Theorem D.9. Let $\mathrm{id}_m$ denote the identity natural transformation of the functor $\mathcal{F}_m$. First note that the definition of a Hopf lax functor and its associated convolution units (Definitions 3.54 and 3.57) can be reformulated as below.

**Proposition D.12.** *A Hopf lax monoidal functor $(\mathcal{F}, \varphi, \psi, \Upsilon)$ consists of a bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$ from $\mathsf{C}$ to $\mathsf{D}$ and a natural transformation $\Upsilon \colon \mathcal{F} \Rightarrow \mathcal{F}$ such that*

(D.7) $$\mathrm{id}_1 * \Upsilon * \mathrm{id}_1 = \mathrm{id}_3, \quad \Upsilon * \mathrm{id}_1 * \Upsilon = \Upsilon^{(3)}, \quad \mathrm{id}_1^{(0)} * \Upsilon^{(0)} = \Upsilon^{(0)} * \mathrm{id}_1^{(0)} = \varphi_0 \psi_0.$$

*Similarly the convolution units associated to $\mathcal{F}$ are given by*

(D.8) $$\upsilon = \mathrm{id}_1 * \Upsilon^{(0)} \qquad \text{and} \qquad \upsilon' = \Upsilon * \mathrm{id}_1^{(0)}.$$

Now Proposition 3.63 can be proved as follows. First observe that for any weak composition $\alpha$ of $n$ with $m$ parts we have

(D.9) $$(\mathrm{id}_m)^{\alpha} = \mathrm{id}_n.$$

Therefore,

$$\upsilon * \mathrm{id}_1 = \mathrm{id}_1 * \Upsilon^{(0)} * \mathrm{id}_1 = \mathrm{id}_1^{(1)} * \Upsilon^{(0)} * \mathrm{id}_1^{(1)} = (\mathrm{id}_1 * \Upsilon * \mathrm{id}_1)^{(1,0,1)} = \mathrm{id}_3^{(1,0,1)} = \mathrm{id}_2.$$

The first equality holds by associativity of (D.6) and (D.8), the third by naturality of (D.6), the fourth by (D.7), and the second and fifth by (D.9).

Regarding Proposition 3.64, we have

$$\Upsilon * \upsilon = \Upsilon^{(1)} * \mathrm{id}_1 * \Upsilon^{(0)} = (\Upsilon * \mathrm{id}_1 * \Upsilon)^{(1,1,0)} = (\Upsilon^{(3)})^{(1,1,0)} = \Upsilon^{(2)}.$$

Propositions 3.65 and 3.68 can be given similar proofs.

**D.4.4. The convolution monoid as a special case.** We close this section by explaining the sense in which the convolution lax monoidal functor $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ generalizes the convolution monoid $\mathrm{Hom}(C, A)$ (Definition 1.13).

First note that any set $X$ may be seen as a (trivial) simplicial set

$$\mathsf{T}_X \colon \mathsf{M}_{\Delta}^{\mathrm{op}} \to \mathsf{Set}$$

by defining

$$\mathsf{T}_X(n) = X \qquad \text{and} \qquad \mathsf{T}_X(\alpha) = \mathrm{id}_X$$

for any object $n$ and any morphism $\alpha$ of $\mathsf{M}_{\Delta}$. This embeds $\mathsf{Set}$ as a full subcategory of the category of augmented simplicial sets. Observe that $X$ is a monoid if and only if $\mathsf{T}_X$ is a lax monoidal functor.

Given a comonoid $C$ and a monoid $A$ in a monoidal category $\mathsf{C}$, we may consider the monoidal functors

$$\mathcal{F}_C \colon \mathsf{I} \to \mathsf{C} \qquad \text{and} \qquad \mathcal{F}_A \colon \mathsf{I} \to \mathsf{C},$$

as in Section 3.4.1. The former is colax and the latter is lax. Hence, we can consider the augmented simplicial set

$$\mathsf{N}_{\mathcal{F}_C, \mathcal{G}_A} \colon \mathsf{M}_{\Delta}^{\mathrm{op}} \to \mathsf{Set}$$

which is a lax monoidal functor (Theorem D.9). We have that

$$\mathsf{N}_{\mathcal{F}_C, \mathcal{G}_A} = \mathsf{T}_{\mathrm{Hom}(C,A)}$$

as augmented simplicial sets and as lax monoidal functors.

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# Notation Index

### Number systems



### Sets and maps



### Numbers and counting

## Symmetric group



## Matrices

## Categories and functors



**Examples of categories.**

**Monoidal functors.**



**Monoids.**



**Hopf monoids and Hopf lax functors.**

**2-monoidal categories and higher monoidal categories**



**Examples of 2-monoidal categories and higher monoidal categories.**

### Contragredient construction



### 2-categories



**Monoidal 2-categories.**



### Vector spaces

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**Multigraded vector spaces.**



**Categories related to graded vector spaces.**

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**Examples of multigraded Hopf algebras.**



**Bases elements.**

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**Species with up-down operators.**



**Categories related to species.**

**Examples of species and Hopf monoids in species.**

**Bases elements.**



**Morphisms of Hopf monoids.**

## Colored species

**Simplicial sets and homology**

**Braid arrangement**

| | |
|---|---|
| $F$, $G$, $H$ | set compositions or faces 306 |
| $(F, C)$, $(G, D)$ | linear set compositions or directed faces 330 |
| $X$, $Y$ | set partitions or flats 306 |
| $L$, $M$ | linear set partitions or directed flats 308 |
| $F \cdot G$ | concatenation of the set compositions $F$ and $G$ 309 |
| $F\|_S$ | restriction of the set composition $F$ to the subset $S$ 309 |
| $l_1 \cdot l_2$ | ordinal sum, or concatenation, of the linear orders $l_1$ and $l_2$ 250 |
| $l\|_S$ | restriction of the linear order $l$ to the subset $S$ 250 |
| $X\|_S$ | restriction of the set partition $X$ to the subset $S$ 310 |
| $X \sqcup Y$ | union of the set partitions $X$ and $Y$ 310 |
| $L\|_S$ | restriction of the linear set partition $L$ to the subset $S$ 310 |
| $L \sqcup M$ | union of the linear set partitions $L$ and $M$ 310 |
| $(C_1, D_1) \leq (C_2, D_2)$ | partial order on pairs of linear orders 327 |
| $(F, C) \leq (G, D)$ | partial order on linear set compositions 308 |
| $X \leq Y$ | partial order on set partitions ($Y$ refines $X$) 308 |
| $L \leq M$, $L \leq' M$ | partial orders on linear set partitions 308 |
| $\mathrm{supp}(F)$ | support of the set composition $F$ 308 |
| $\mathrm{supp}(F, C)$ | support of the linear set composition $(F, C)$ 309 |
| $\mathrm{base}(F, C)$ | base of the linear set composition $(F, C)$ 309 |
| $\mathrm{base}(L)$ | base of the linear set partition $L$ 309 |
| $\mathrm{type}(F)$ | type of the set composition $F$ 308 |
| $\mathrm{H}_{ij}$ | hyperplane $x_i = x_j$ in the braid arrangement 311 |
| $\Sigma[I]$ | poset of faces of the braid arrangement in $\mathbb{R}^I$ 312 |
| $\mathrm{L}[I]$ | set of chambers of the braid arrangement in $\mathbb{R}^I$ 312 |
| $\Pi[I]$ | lattice of flats in the braid arrangement in $\mathbb{R}^I$ 313 |
| $\overrightarrow{\Sigma}[I]$ | set of directed faces in the braid arrangement in $\mathbb{R}^I$ 330 |
| $\overrightarrow{\Pi}[I]$ | set of directed flats in the braid arrangement in $\mathbb{R}^I$ 330 |
| $C_{(n)}$ | canonical linear order on the set $[n]$ 312 |
| $\overline{F}$ | face opposite to the face $F$ 312 |
| $FG$ | product of the faces $F$ and $G$ 317 |
| $E \vee F$ | join of the faces $E$ and $F$ (if it exists) 318 |
| $K \cdot X$ | action of the face $K$ on the flat $X$ 319 |
| $F \sim G$ | equivalence relation on faces 319 |
| $(G, D) \sim (F, C)$ | equivalence relation on directed faces 330 |
| $K \cdot (G, D)$ | left action of the face $K$ on the directed face $(G, D)$ 333 |
| $(F, C) \cdot G$ | right action of the face $G$ on the directed face $(F, C)$ 336 |
| $X \cdot M$ | left action of the flat $X$ on the directed flat $M$ 336 |
| $L \cdot X$ | right action of the flat $X$ on the directed flat $L$ 336 |
| $\Psi(F, C)$ | top-dimensional cone of the directed face $(F, C)$ 331 |
| $\Psi(L)$ | top-dimensional cone of the directed flat $L$ 332 |
| $\mathrm{Star}(K)$ | star of the set composition $K$ in the poset of faces 338 |
| $\mathrm{Star}(X)$ | star of the partition $X$ in the lattice of flats 340 |
| $\mathrm{L}_F$ | set of chambers containing $F$ 322 |
| $C - D - E$ | minimum gallery 320 |
| $p_F$ | Tits projection map 320 |
| $b_{S\|T}$ | break map for the vertex $S\|T$ 338 |
| $j_{S\|T}$ | join map for the vertex $S\|T$ 338 |
| $b_K$ | break map for the face $K$ 338 |
| $j_K$ | join map for the face $K$ 338 |
| $\mathrm{Des}(C, D)$ | set of descents of the pair of chambers $(C, D)$ 325 |

**Simplicial complexes**



**Relations, posets, graphs, trees**

**Closure operators**



**Schubert statistic and cocycle**



**Fock functors**

**The image functor.**



**Structure transformations.**



**Morphisms between Fock functors.**

**Fock spaces.**



**Related functors.**



## Operads

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# Subject Index

**C**