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Infinitesimal Hopf algebras

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Abstract. Infinitesimal bialgebras were introduced by Joni and Rota [J-R]. An infinitesimal bialgebra is at the same time an algebra and a coalgebra, in such a way that the comultiplication is a derivation. In this paper we define infinitesimal Hopf algebras, develop their basic theory and present several examples.

It turns out that many properties of ordinary Hopf algebras possess an infinitesimal version. We introduce bicrossproducts, quasitriangular infinitesimal bialgebras, the corresponding infinitesimal Yang-Baxter equation and a notion of Drinfeld's double for infinitesimal Hopf algebras.

1. Introduction

An infinitesimal bialgebra is a triple (A, m, Δ) where (A, m) is an associative algebra, (A, Δ) is a coassociative coalgebra and for each $a, b \in A$,

$$\Delta(ab) = \sum ab_1 \otimes b_2 + \sum a_1 \otimes a_2 b \ .$$

Infinitesimal bialgebras were introduced by Joni and Rota [J-R] in order to provide an algebraic framework for the calculus of divided differences. Several new examples are introduced in section 2. In particular, it is shown that the path algebra of an arbitrary quiver admits a canonical structure of infinitesimal bialgebra.

In this paper we define the notion of antipode for infinitesimal bialgebras and develop the basic theory of *infinitesimal Hopf algebras*. Surprisingly, many of the usual properties of ordinary Hopf algebras possess an infinitesimal version. For instance, the antipode satisfies

$$S(xy) = -S(x)S(y) \quad \text{and} \quad \sum S(x_1) \otimes S(x_2) = -\sum S(x)_1 \otimes S(x)_2,$$

among other properties (section 3).

The existence of the antipode is closely related to the possibility of exponentiating a certain canonical derivation $D : A \to A$ that is carried by any ϵ-bialgebra. This and other related results are discussed in section 4.

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In section 6 we introduce the analog of "matched pairs" of groups or Hopf algebras for associative algebras, and the corresponding bicrossproduct construction. Some interesting examples are given.

Recall that a Lie bialgebra is a triple $(\mathfrak{g}, [,], \delta)$ where $(\mathfrak{g}, [,])$ is a Lie algebra, $(\mathfrak{g}, \delta)$ is a Lie coalgebra and $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ is a derivation (in the Lie sense). Therefore, infinitesimal bialgebras may also be seen as an associative analog of Lie bialgebras. This analogy is reinforced in section 5 where we introduce quasitriangular infinitesimal bialgebras and the corresponding associative Yang-Baxter equation:

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0 \text{ for } r \in A\otimes A \ .$$

Again, most properties of ordinary quasitriangular bialgebras and Hopf algebras admit an analog in the infinitesimal context. For instance the antipode satisfies

$$(S\otimes S)(r) = r = (S^{-1}\otimes S^{-1})(r) \ .$$

But perhaps the most important of these properties is the fact that there is a notion of *Drinfeld's double* for infinitesimal bialgebras, satisfying all the properties one can expect. Drinfeld's double is defined and studied in section 7. It is an important example of the bicrossproduct construction of section 6.

Recall that the underlying space of the double of a Lie bialgebra $\mathfrak{g}$ and of an ordinary Hopf algebra H is respectively

$$D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^* \text{ and } D(H) = H\otimes H^* \ .$$

The underlying space of the double of an ϵ-bialgebra A turns out to be

$$D(A) = (A\otimes A^*) \oplus A \oplus A^*$$

This is yet another manifestation of the fact that the theory of ϵ-bialgebras possesses aspects of both theories of Lie and ordinary bialgebras. Further connections between Lie and infinitesimal bialgebras, as well as a deeper study of bicrossproducts and quasitriangular infinitesimal bialgebras, will be presented in [A2].

An important motivation for studying infinitesimal Hopf algebra arises in the study of the **cd**-index of polytopes in combinatorics. Related examples will be presented in this paper but the main application (an algebraic proof of the existence of the **cd**-index of polytopes) will be presented in [A1]. One of these examples is provided by the infinitesimal Hopf algebra of all non-trivial posets. This is discussed to some extent in sections 2 and 4.

It is often assumed that all vector spaces and algebras are over a fixed field k. Sum symbols are often omitted from Sweedler's notation: we write $\Delta(a) = a_1\otimes a_2$ when Δ is a coassociative comultiplication. Composition of maps is written simply as fg. The symbol $\circ$ is reserved for the circular product on an algebra (section 3).

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2. Infinitesimal bialgebras. Basic properties and examples

DEFINITION 2.1. An infinitesimal bialgebra (abbreviated ϵ-bialgebra) is a triple (A, m, Δ) where (A, m) is an associative algebra (possibly without unit), (A, Δ) is a coassociative coalgebra (possibly without counit) and, for each $a, b \in A$,

$$\Delta(ab) = ab_1\otimes b_2 + a_1\otimes a_2 b \ . \tag{2.1}$$

Condition 2.1 can be written as follows:

$$\Delta m = (m \otimes id_A)(id_A \otimes \Delta) + (id_A \otimes m)(\Delta \otimes id_A)$$

Equivalently, $\Delta : A \to A\otimes A$ is a derivation of the algebra (A, m) with values on the A-bimodule $A\otimes A$, or $m : A\otimes A \to A$ is a coderivation [Doi] from the A-bicomodule $A\otimes A$ with values on the coalgebra (A, Δ).

Here $A\otimes A$ is viewed as A-bimodule via $a \cdot (x\otimes y) = ax\otimes y$ and $(x\otimes y) \cdot b = x\otimes yb$. Dually, $A\otimes A$ is an A-bicomodule via $A\otimes A \xrightarrow{\Delta\otimes id_A} A\otimes(A\otimes A)$ and $A\otimes A \xrightarrow{id_A\otimes\Delta} (A\otimes A)\otimes A$.

Remark 2.2. If an ϵ-bialgebra has a unit $1 \in A$ then $\Delta(1) = 0$. In fact, any derivation $D : A \to M$ annihilates 1, since $D(1) = D(1 \cdot 1) = 1 \cdot D(1) + D(1) \cdot 1 = 2D(1)$, hence $D(1) = 0$.

If an ϵ-bialgebra has both a unit $1 \in A$ and a counit $\varepsilon \in A^*$ then $A = 0$. In fact, $1 = (id\otimes\varepsilon)\Delta(1) = 0$.

Infinitesimal bialgebras were introduced by Joni and Rota (under the name *infinitesimal coalgebras*) [J-R, section XII]. Ehrenborg and Readdy have called them *newtonian coalgebras* [E-R]. The present terminology emphasizes the analogy with the notion of ordinary bialgebras, and does not favor either the algebra or coalgebra structure over the other; as we will see next, the notion is self-dual.

Since the notions of derivation and coderivation correspond to each other by duality, it follows immediately that if (A, m, Δ) is a finite dimensional ϵ-bialgebra then the dual space A^* is an ϵ-bialgebra with multiplication

$$A^*\otimes A^* \cong (A\otimes A)^* \xrightarrow{\Delta^*} A^*$$

and comultiplication

$$A^* \xrightarrow{m^*} (A\otimes A)^* \cong A^*\otimes A^* \ .$$

If (A, m, Δ) is an arbitrary ϵ-bialgebra, then so are $(A, -m, \Delta)$, $(A, m, -\Delta)$, $(A, -m, -\Delta)$ and also $(A, m^{op}, \Delta^{cop})$, where

$$m^{op} = m\tau,\ \Delta^{cop} = \tau\Delta \text{ and } \tau(a\otimes b) = b\otimes a.$$

In the context of Drinfeld's double (section 7), these basic constructions will have to be combined.

Examples 2.3.

1. Any algebra (A, m) becomes a ϵ-bialgebra by setting $\Delta = 0$. Dually, any coalgebra (A, Δ) becomes an ϵ-bialgebra with $m = 0$.
2. Let Q be an arbitrary quiver. Then the *path algebra* kQ carries a canonical ϵ-bialgebra structure. Recall that $kQ = \oplus_{n=0}^{\infty} kQ_n$ where Q_n is the set of paths γ in Q of length n:

$$\gamma : e_0 \xrightarrow{a_1} e_1 \xrightarrow{a_2} e_2 \xrightarrow{a_3} \ldots e_{n-1} \xrightarrow{a_n} e_n \ .$$

In particular, Q_0 is the set of vertices and Q_1 is the set of arrows. The multiplication is concatenation of paths whenever possible; otherwise is zero. The comultiplication is defined on a path $\gamma = a_1 a_2 \ldots a_n$ as above by

$$\Delta(\gamma) = e_0\otimes a_2 a_3 \ldots a_n + a_1\otimes a_3 \ldots a_n + \ldots + a_1 \ldots a_{n-1}\otimes e_n \ .$$

In particular, $\Delta(e) = 0$ for every vertex $e \in Q_0$ and $\Delta(a) = s(a)\otimes t(a)$ for every arrow $a \in Q_1$.

3. The polynomial algebra $k[\mathbf{x}]$ is an ϵ-bialgebra with

$$\Delta(1) = 0,\ \Delta(\mathbf{x}^n) = \mathbf{x}^{n-1} \otimes 1 + \mathbf{x}^{n-2} \otimes \mathbf{x} + \ldots + \mathbf{x} \otimes \mathbf{x}^{n-2} + 1 \otimes \mathbf{x}^{n-1} \quad \text{for } n \geq 1 \ .$$

This is the path ϵ-bialgebra corresponding to the quiver

$$\underset{1}{\overset{\mathbf{x}}{\circlearrowright}}$$

as in example 2.

Notice that the comultiplication can also be described as the map

$$\Delta : k[\mathbf{x}] \to k[\mathbf{x}, \mathbf{y}],\ \Delta(f(\mathbf{x})) = \frac{f(\mathbf{x}) - f(\mathbf{y})}{\mathbf{x} - \mathbf{y}} \ ;$$

in other words, $\Delta(f(\mathbf{x}))$ is the *Newton divided difference* of $f(\mathbf{x})$. For this reason, this structure was called the *Newtonian coalgebra* in [J-R]. Joni and Rota proposed the general notion of ϵ-bialgebra in order to axiomatize the situation of this example. For a long time this remained the only example of ϵ-bialgebra appearing in the literature. The only work in the area seems to have been that of Hirschhorn and Raphael [H-R], where the ϵ-bialgebra $k[\mathbf{x}]$ was studied in detail in connection with the calculus of divided differences.

4. It was only recently that another natural example of ϵ-bialgebras arose, again in combinatorics, but in a different context (that of the **cd**-index of polytopes).

The ϵ-bialgebra $\mathcal{P}$ of all non-trivial posets is defined as follows. As a vector space, $\mathcal{P}$ has a basis consisting of the isomorphism classes of all finite posets P with top element 1_P and bottom element 0_P, except for the one-element poset $\{\bullet\}$. Thus $0_P \neq 1_P$ always. The multiplication of two such posets P and Q is

$$P * Q = \Big(P - \{1_P\}\Big) \cup \Big(Q - \{0_Q\}\Big)$$

where $x \leq y$ iff $\begin{cases} x, y \in P \text{ and } x \leq y \text{ in } P, \\ x, y \in Q \text{ and } x \leq y \text{ in } Q, \text{ or} \\ x \in P \text{ and } y \in Q. \end{cases}$

This algebra possesses a unit element, namely the poset $B_1 = \{0 < 1\}$. Moreover, $\mathcal{P}$ is an ϵ-bialgebra with comultiplication

$$\Delta(P) = \sum_{0_P < x < 1_P} [0_P, x] \otimes [x, 1_P] \ .$$

Here if x and y are two elements of a poset P, then $[x, y]$ denotes the isomorphism class of the poset $\{z \in P \ / \ x \leq z \leq y\}$.

This ϵ-bialgebra was first considered by Ehrenborg and Hetyei [E-H], and further studied by Billera, Ehrenborg and Readdy in connection with the **cd**-index of polytopes [E-R, B-E-R]. This study is continued in example 4.7.3 and more deeply in [A1], where simple coalgebraic ideas are used to provide a proof of the existence of the **cd**-index of polytopes.

5. The free algebra $A = k\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \rangle$ is an ϵ-bialgebra with

$$\Delta(\mathbf{x}_n) = \sum_{i=0}^{n-1} \mathbf{x}_i \otimes \mathbf{x}_{n-1-i} = 1 \otimes \mathbf{x}_{n-1} + \mathbf{x}_1 \otimes \mathbf{x}_{n-2} + \ldots + \mathbf{x}_{n-1} \otimes 1 \ ,$$

where we set $\mathbf{x}_0 = 1$.

6. The algebra of dual numbers $k[\varepsilon]/(\varepsilon^2)$ is an ϵ-bialgebra with
$$\Delta(1) = 0,\ \Delta(\varepsilon) = \varepsilon \otimes \varepsilon\ .$$
7. The algebra of matrices $A = M_2(k)$ admits many ϵ-bialgebra structures. One such is
$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}\ .$$
Other structures on $M_2(k)$ will be discussed later (examples 5.4).

3. Antipodes and infinitesimal Hopf algebras

Recall that if an ϵ-bialgebra A possesses both a unit and a counit then $A = 0$ (remark 2.2). This simple observation shows that one cannot hope to define a notion of antipode for ϵ-bialgebras as one does for ordinary bialgebras H, since for this one must refer to both the unit and counit of H. Recall that the antipode of an ordinary bialgebra H is defined as the inverse of id_H in the space $\mathsf{Hom}_k(H, H)$, which is an algebra under the convolution product, with unit $u_H \varepsilon_H$ (where $u_H : k \to H$ is the unit map $u_H(1) = 1$).

If A is an ϵ-bialgebra, then the space $\mathsf{Hom}_k(A, A)$ is still an algebra under convolution, but it does not have a unit element in general. However, one may formally adjoin a unit to this algebra and then consider invertible elements. It turns out that this simple algebraic device will provide the right notion of antipode for ϵ-bialgebras, as will become clear from the examples to be discussed in this work. We recall this concept next.

Let R be any k-algebra, not necessarily unital. The *circular product* on R is
$$a \circ b = ab + a + b$$
It is easy to check directly that this turns R into an associative unital monoid, with unit $0 \in R$. This can also be seen as follows: if we adjoin a unit to R to form $R^+ = R \oplus k$, with associative multiplication
$$(a, \lambda) \cdot (b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$$
and unit element $(0, 1)$, then the subset $\{(a, 1) \in R^+ \ /\ a \in R\}$ is closed under the multiplication of R^+ and contains its unit. This monoid is isomorphic to R equipped with the circular product.

Now let A be an ϵ-bialgebra. The space $\mathsf{Hom}_k(A, A)$ is an algebra under convolution
$$f * g = m(f \otimes g)\Delta$$
(recall that concatenation denotes composition of maps). The circular product on this (in general, nonunital) algebra will be called the *circular convolution* and denoted by the symbol $\circledast$. Explicitly,
$$f \circledast g = f * g + f + g \quad \text{or} \quad (f \circledast g)(a) = f(a_1)g(a_2) + f(a) + g(a)\ .$$

Definition 3.1. An infinitesimal bialgebra A is called an infinitesimal Hopf algebra if the identity map $id \in \mathsf{Hom}_k(A, A)$ is invertible with respect to circular convolution. In this case, the inverse $S \in \mathsf{Hom}_k(A, A)$ of id is called the antipode of A. It is characterized by the equations

$$\text{(A)} \qquad S(a_1)a_2 + S(a) + a = 0 = a_1 S(a_2) + a + S(a)\ \forall\ a \in A\ .$$

EXAMPLES 3.2.

1. The algebra of polynomials $k[\mathbf{x}]$ is an ϵ-Hopf algebra. The antipode is
$$S(\mathbf{x}^n) = -(\mathbf{x}-1)^n, \text{ that is } S(p(\mathbf{x})) = -p(\mathbf{x}-1) \ .$$
In fact, since $\Delta(\mathbf{x}^n) = \sum_{i+j=n-1} \mathbf{x}^i \otimes \mathbf{x}^j$, equations (A) become
$$\sum_{i+j=n-1} -\mathbf{x}^i(\mathbf{x}-1)^j + \mathbf{x}^n - (\mathbf{x}-1)^n = 0$$
which follows from the basic identity
$$a^n - b^n = (a-b) \sum_{i+j=n-1} a^i b^j \ .$$
Notice that S is bijective with $S^{-1}(p(\mathbf{x})) = -p(\mathbf{x}+1)$. More generally, for any $m \in \mathbb{Z}$,
$$S^m(p(\mathbf{x})) = (-1)^m p(\mathbf{x}-m) \ .$$
In particular, S has infinite order.
2. More generally, for any quiver Q the path algebra kQ is an ϵ-Hopf algebra with antipode
$$S(e) = -e \ \forall \ e \in Q_0 \text{ and } S(a) = \begin{cases} e-a \text{ if } s(a) = t(a) = e, \\ -a \text{ if } s(a) \neq t(a). \end{cases}$$
These assertions follow from a general result on the existence of antipodes (corollary 4.3, example 4.7.2). The antipode is uniquely determined by the formulas above according to proposition 3.7.
3. The algebra $\mathcal{P}$ of non-trivial posets is an ϵ-Hopf algebra. An explicit formula for the antipode is:
$$S(P) = \sum_{n=1}^{\infty} (-1)^n \sum_{0_P < x_1 < \ldots < x_{n-1} < 1_P} [0_P, x_1][x_1, x_2] \ldots [x_{n-1}, 1_P] \ .$$
This will discussed in detail in example 4.7.3.
4. The ϵ-bialgebra $A = k\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \rangle$ of example 2.3.5 is an ϵ-Hopf algebra with antipode
$$S(\mathbf{x}_n) = \sum_{k=1}^{n+1} (-1)^k \sum_{(n_1, \ldots, n_k) \in \mathcal{C}^+(n+1,k)} \mathbf{x}_{n_1-1}\mathbf{x}_{n_2-1} \ldots \mathbf{x}_{n_k-1}$$
where $\mathcal{C}^+(n+1,k) = \{(n_1, \ldots, n_k) \ / \ n_i \in \mathbb{Z}^+, \ n_1 + \ldots + n_k = n+1\}$ is the set of *strict compositions of* $n+1$ *into* k *parts*. See example 4.7.4.
5. The algebra of dual numbers (example 2.3.6) is an ϵ-Hopf algebra. The antipode is simply $S = -id$. The same is true for the ϵ-bialgebra $M_2(k)$ of example 2.3.7.
6. Not every ϵ-bialgebra possesses an antipode. Consider the following comultiplication on the polynomial algebra $k[\mathbf{x}]$:

$$\Delta(1) = 0, \ \Delta(\mathbf{x}^n) = \mathbf{x}^n \otimes \mathbf{x} + \mathbf{x}^{n-1} \otimes \mathbf{x}^2 + \ldots + \mathbf{x}^2 \otimes \mathbf{x}^{n-1} + \mathbf{x} \otimes \mathbf{x}^n \quad \text{for } n > 0 \ .$$

It is easy to see that this endows $k[\mathbf{x}]$ with the structure of an ϵ-bialgebra (different from that of example 3, but closely related to its *graded dual*). In particular $\Delta(\mathbf{x}) = \mathbf{x}\otimes\mathbf{x}$. If there were an antipode S, then we would have

$$S(\mathbf{x})\mathbf{x} + S(\mathbf{x}) + \mathbf{x} = 0 \Rightarrow S(\mathbf{x}) = \frac{-\mathbf{x}}{1+\mathbf{x}} \notin k[\mathbf{x}]$$

which is a contradiction.

REMARK 3.3. In all previous examples, $S(1) = -1$. More generally, for any ϵ-Hopf algebra A and $u \in \mathsf{Ker}\Delta$, $S(u) = -u$. In fact, equation (A) gives

$$0 = u_1 S(u_2) + u + S(u) \Rightarrow S(u) = -u\ .$$

The antipode of an ϵ-Hopf algebra satisfies many properties analogous to those of the antipode of an ordinary Hopf algebra, which we will present next.

We need some basic general results first.

LEMMA 3.4. *Let A, B be algebras and C, D coalgebras.*

(a) *If $\phi : C \to D$ is a morphism of coalgebras then $\phi^* : \mathsf{Hom}_k(D, A) \to \mathsf{Hom}_k(C, A)$, $\phi^*(f) = f\phi$, is a morphism of (circular) convolution monoids.*
(b) *If $\phi : A \to B$ is a morphism of algebras then $\phi_* : \mathsf{Hom}_k(C, A) \to \mathsf{Hom}_k(C, B)$, $\phi_*(f) = \phi f$, is a morphism of (circular) convolution monoids.*

PROOF. Any morphism of algebras preserves the corresponding circular products, so it is enough to check that ordinary convolution is preserved in either case. This is well-known. □

The next lemma is meaningful for nonunital algebras (or noncounital coalgebras) only, since a unital multiplication is always surjective (and a counital comultiplication injective) .

LEMMA 3.5. (a) *Let C and D be coalgebras, $u \in \mathsf{Ker}\Delta_C$ and $v \in \mathsf{Ker}\Delta_D$. Then $C\otimes D$ is a coalgebra with*

$$\Delta(c\otimes d) = (c_1\otimes v)\otimes(c_2\otimes d) + (c\otimes d_1)\otimes(u\otimes d_2)\ .$$

(b) *Let A and B be algebras, $\gamma \in (\mathsf{Coker} m_A)^*$ and $\delta \in (\mathsf{Coker} m_B)^*$. Then $A\otimes B$ is an algebra with*

$$(a\otimes b)\cdot(a'\otimes b') = \delta(b)aa'\otimes b' + \gamma(a')a\otimes bb'\ .$$

PROOF. To prove (a) we calculate

$$(id\otimes\Delta)\Delta(c\otimes d) = (c_1\otimes v)\otimes\Delta(c_2\otimes d) + (c\otimes d_1)\otimes\Delta(u\otimes d_2)$$
$$= (c_1\otimes v)\otimes(c_2\otimes v)\otimes(c_3\otimes d) + (c_1\otimes v)\otimes(c_2\otimes d_1)\otimes(u\otimes d_2) + (c\otimes d_1)\otimes(u\otimes d_2)\otimes(u\otimes d_3)\ ,$$

and

$$(\Delta\otimes id)\Delta(c\otimes d) = \Delta(c_1\otimes v)\otimes(c_2\otimes d) + \Delta(c\otimes d_1)\otimes(u\otimes d_2)$$
$$= (c_1\otimes v)\otimes(c_2\otimes v)\otimes(c_3\otimes d) + (c_1\otimes v)\otimes(c_2\otimes d_1)\otimes(u\otimes d_2) + (c\otimes d_1)\otimes(u\otimes d_2)\otimes(u\otimes d_3)\ .$$

thus $(id\otimes\Delta)\Delta = (\Delta\otimes id)\Delta$ as needed. Case (b) is dual. □

Recall that if an ϵ-bialgebra has a unit 1 then $\Delta(1) = 0$. Dually, if it has a counit ϵ then $\epsilon(\mathsf{Im} m) = 0$, so we can view $\epsilon \in (\mathsf{Coker} m)^*$. Thus, lemma 3.5 may be applied as follows.

LEMMA 3.6. *Let (A, m, Δ) be an ϵ-bialgebra.*

(a) *Suppose that A has a unit 1. View $A \otimes A$ as a coalgebra as in lemma 3.5 (a) with $u = v = 1$. Then $m : A \otimes A \to A$ is a morphism of coalgebras.*
(b) *Suppose that A has a counit ϵ. View $A \otimes A$ as an algebra as in lemma 3.5 (b) with $\gamma = \delta = \epsilon$. Then $\Delta : A \to A \otimes A$ is a morphism of algebras.*

Proof. To prove (a) we need to show that
$$\begin{array}{ccc} A\otimes A & \xrightarrow{m} & A \\ \Delta\downarrow & & \downarrow\Delta \\ (A\otimes A)\otimes(A\otimes A) & \xrightarrow[m\otimes m]{} & A\otimes A \end{array}$$
commutes.

We calculate
$$\Delta m(x \otimes y) = \Delta(xy) \overset{(2.1)}{=} x_1 \otimes x_2 y + x y_1 \otimes y_2,$$
and
$$(m \otimes m)\Delta(x \otimes y) = (m \otimes m)\Big((x_1 \otimes 1) \otimes (x_2 \otimes y) + (x \otimes y_1) \otimes (1 \otimes y_2)\Big) = x_1 \otimes x_2 y + x y_1 \otimes y_2\ ,$$
as needed. Case (b) is dual. □

The previous result does not say that A is an ordinary bialgebra, since the coalgebra or algebra structures on $A \otimes A$ are not the usual tensor product structures.

The antipode of an ordinary Hopf algebra reverses multiplications and comultiplications. The analogous result for ϵ-Hopf algebras is as follows.

Proposition 3.7. *Let A be an ϵ-Hopf algebra with antipode S. Then*
(a) $S(xy) = -S(x)S(y)$,
(b) $S(x_1) \otimes S(x_2) = -S(x)_1 \otimes S(x)_2$.

Proof. We present the proof of (a), (b) being dual.

Suppose first that A has a unit 1. View $A \otimes A$ as a coalgebra as in lemma 3.6 (a). Then $m : A \otimes A \to A$ is a morphism of coalgebras, so by lemma 3.4 (a), $m^* : \mathsf{Hom}_k(A, A) \to \mathsf{Hom}_k(A \otimes A, A)$ preserves circular convolutions. Hence $m = m^*(id)$ is invertible with inverse $m^*(S)$ (with respect to circular convolution).

On the other hand, let $\nu \in \mathsf{Hom}_k(A \otimes A, A)$ be $\nu(x \otimes y) = -S(x)S(y)$. We need to show that $\nu = m^*(S)$ (since $m^*(S)(x \otimes y) = S(xy)$). Since $m^*(S)$ is the inverse of m, it suffices to check that $m \circledast \nu = 0$. We calculate
$$\begin{aligned}(m \circledast \nu)(x \otimes y) &= m(x_1 \otimes 1)\nu(x_2 \otimes y) + m(x \otimes y_1)\nu(1 \otimes y_2) + m(x \otimes y) + \nu(x \otimes y)\\ &= -x_1 S(x_2)S(y) - x y_1 S(1) S(y_2) + xy - S(x)S(y)\\ &= (-x_1 S(x_2) - S(x))S(y) + x y_1 S(y_2) + xy\\ &\overset{(A)}{=} xS(y) + x y_1 S(y_2) + xy = x(S(y) + y_1 S(y_2) + y) \overset{(A)}{=} 0.\end{aligned}$$
as needed. (We used that $S(1) = -1$, which we know from remark 3.3.)

This completes the proof when A has a unit. The general case can de reduced to this one as follows: adjoin a unit to A to form the unital algebra $A^+ = A \oplus k$ as in the paragraph preceding definition 3.1. It is easy to check that A^+ is an ϵ-Hopf algebra, with comultiplication $\Delta(a, \lambda) = (a_1, 0) \otimes (a_2, 0)$ and antipode $S(a, \lambda) = (S(a), -\lambda)$. Since the result holds for A^+, it also does for its ϵ-Hopf subalgebra A. □

A morphism of ϵ-bialgebras is a linear map $\phi : A \to B$ that is both a morphism of algebras and coalgebras:
$$m_B(\phi \otimes \phi) = \phi m_A \text{ and } (\phi \otimes \phi)\Delta_A = \Delta_B \phi\ .$$

For instance, proposition 3.7 says precisely that $S : (A, m, \Delta) \to (A, -m, -\Delta)$ is a morphism of ϵ-bialgebras.

A morphism of ϵ-Hopf algebras is a morphism of ϵ-bialgebras that furthermore preserves the antipodes: $\phi S_A = S_B \phi$. As for ordinary Hopf algebras, this turns out to be automatic.

Proposition 3.8. *Let A and B be ϵ-Hopf algebras and $\phi : A \to B$ a morphism of ϵ-bialgebras. Then $\phi S_A = S_B \phi$, i.e. ϕ is a morphism of ϵ-Hopf algebras.*

Proof. By lemma 3.4, there are morphisms of monoids

$$\phi^* : \mathsf{Hom}_k(B, B) \to \mathsf{Hom}_k(A, B),\ \ \phi^*(f) = f\phi$$

and

$$\phi_* : \mathsf{Hom}_k(A, A) \to \mathsf{Hom}_k(A, B),\ \ \phi_*(f) = \phi f\ .$$

Since $\phi^*(id_B) = f = \phi_*(id_A)$ and inverses are preserved, we must have $\phi^*(S_B) = \phi_*(S_A)$, i.e. $S_B\phi = \phi S_A$. □

Example 3.9. Let $A = k\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots \rangle$ and $B = k[\mathbf{x}]$ be the algebras of examples 2.3.5 and 2.3.3. Recall from examples 3.2 that

$$S(\mathbf{x}^n) = -(\mathbf{x} - 1)^n \text{ and } S(\mathbf{x}_n) = \sum_{k=1}^{n+1} (-1)^k \sum_{(n_1,\dots,n_k)\in \mathcal{C}^+(n+1,k)} \mathbf{x}_{n_1-1}\mathbf{x}_{n_2-1}\dots\mathbf{x}_{n_k-1}\ .$$

The map $\phi : A \to B$, $\phi(\mathbf{x}_n) = \mathbf{x}^n$, is clearly a morphism of ϵ-bialgebras. Since it must preserve the antipodes, we deduce that

$$\begin{aligned} -(\mathbf{x} - 1)^n &= \sum_{k=1}^{n+1} (-1)^k \sum_{(n_1,\dots,n_k)\in \mathcal{C}^+(n+1,k)} \mathbf{x}^{n_1-1}\mathbf{x}^{n_2-1}\dots\mathbf{x}^{n_k-1} \\ &= \sum_{k=1}^{n+1} (-1)^k \#\mathcal{C}(n+1,k)\mathbf{x}^{n+1-k} = -\sum_{k=0}^{n} (-1)^k \#\mathcal{C}(n+1,k+1)\mathbf{x}^{n-k} \end{aligned}$$

from which we obtain the basic fact that the number of strict compositions of $n+1$ into $k+1$ parts is

$$\#\mathcal{C}(n+1, k+1) = \binom{n}{k}\ .$$

A finite dimensional subbialgebra of an ordinary Hopf algebra is necessarily a Hopf subalgebra. This is a consequence of the following basic fact: if R is a finite dimensional unital subalgebra of a unital algebra S and $x \in R$ is invertible in S, then x is already invertible in R. To deduce the corresponding property of ϵ-Hopf algebras, first note that if R is a finite dimensional subalgebra of an arbitrary (nonunital) algebra S and $x \in R$ is circular invertible in S, then x is already circular invertible in R. This follows from the previous fact applied to R^+, S^+ and the element $(x, 1)$.

Proposition 3.10. *If B is a finite dimensional ϵ-subbialgebra of an ϵ-Hopf algebra A, then B is an ϵ-Hopf subalgebra.*

Proof. Let $i : B \to A$ be the inclusion. $\mathsf{Hom}_k(B, B)$ is a finite dimensional subalgebra of $\mathsf{Hom}_k(B, A)$ via i_*. Considering $i^* : \mathsf{Hom}_k(A, A) \to \mathsf{Hom}_k(B, A)$ we see that i is circular invertible in $\mathsf{Hom}_k(B, A)$. By the preceding remark, id_B is invertible in $\mathsf{Hom}_k(B, B)$. $\square$

We turn to the study of antipodes in relation to the basic constructions of section 2.

Proposition 3.11. *Let (A, m, Δ) be an ϵ-Hopf algebra with antipode S.*

1. *S is the antipode for $(A, -m, -\Delta)$.*
2. *If S is bijective, then S^{-1} is the antipode for $(A, -m, \Delta)$ and $(A, m, -\Delta)$.*
3. *Conversely, if $(A, -m, \Delta)$ or $(A, m, -\Delta)$ admit an antipode $\bar{S}$, then $\bar{S}$ is the inverse of S with respect to composition.*

Proof. 1. Equations (A) coincide for (A, m, Δ) and $(A, -m, -\Delta)$, so this assertion is clear.

2. We first show that for any $a, b \in A$,

$$S^{-1}(a)S^{-1}(b) = -S^{-1}(ab) \tag{$*$}$$

Since S is bijective, we can write $a = S(x)$ and $b = S(y)$. By proposition 3.7, $S(xy) = -S(x)S(y)$. Hence $xy = -S^{-1}\Big(S(x)S(y)\Big)$, which rewrites as $S^{-1}(a)S^{-1}(b) = -S^{-1}(ab)$, as needed.

Now from (A) we deduce

$$\begin{aligned} 0 &= a_1S(a_2) + a + S(a) \\ \Rightarrow 0 &= S^{-1}\Big(a_1S(a_2)\Big) + S^{-1}(a) + S^{-1}S(a) \\ \overset{(*)}{\Rightarrow} 0 &= -S^{-1}(a_1)a_2 + S^{-1}(a) + a \ . \end{aligned}$$

Similarly, from the other half of (A) one deduces $0 = -a_1S^{-1}(a_2) + a + S^{-1}(a)$. These say that S^{-1} is the antipode for both $(A, -m, \Delta)$ and $(A, m, -\Delta)$.

3. Suppose $(A, -m, \Delta)$ admits an antipode $\bar{S}$.

By proposition 3.7 (a), $S : (A, -m) \to (A, m)$ is a morphism of algebras. Hence, by lemma 3.4 (a),

$$S_* : \mathsf{Hom}_k((A, \Delta), (A, -m)) \to \mathsf{Hom}_k((A, \Delta), (A, m)), \quad f \mapsto Sf \ ,$$

is a morphism of circular convolution monoids. Now, $S_*(id) = S$ and $S_*(\bar{S}) = S\bar{S}$. We deduce that $S\bar{S}$ is the inverse of S with respect to circular convolution. Hence $S\bar{S} = id$.

One deduces similarly that $\bar{S}S = id$, by using the morphism S^*.

$\square$

Proposition 3.12. *Let (A, m, Δ) be an ϵ-Hopf algebra with antipode S. Then so is $(A, m^{op}, \Delta^{cop})$, with the same antipode S.*

Proof. The convolution product on $\mathsf{Hom}_k(A, A)$ is opposite to the convolution product on $\mathsf{Hom}_k(A^{cop}, A^{op})$:

$$m(f \otimes g)\Delta = m\tau(g \otimes f)\tau\Delta = m^{op}(g \otimes f)\Delta^{cop} \ .$$

Hence the same is true for the circular products. In particular, the inverse of id is the same in both monoids. $\square$

PROPOSITION 3.13. *If A is a finite dimensional ϵ-Hopf algebra with antipode S, then so is A^*, with antipode S^*.*

PROOF. For any coalgebra C and finite dimensional algebra B, the map

$$\mathsf{Hom}_k(C,B) \to \mathsf{Hom}_k(B^*,C^*),\ f \mapsto f^*$$

is a morphism of (circular) convolution monoids:

$$(f*g)^* = (m_B(f \otimes g)\Delta_C)^* = \Delta_C^*(f\otimes g)^* m_B^* = m_{C^*}(f^*\otimes g^*)\Delta_{B^*} = f^* * g^* \ .$$

When $C = B = A$, this morphism sends id_A to id_{A^*} and S_A to S_A^*. Hence S_A^* is the inverse of id_{A^*} with respect to circular convolution, i.e. the antipode of A^*. □

Recall that if H is an ordinary Hopf algebra, B is any algebra and $f : H \to B$ is a morphism of algebras, then f is convolution-invertible in $\mathsf{Hom}_k(H,B)$ and the inverse is fS_H. We close this section with the analogous property for ϵ-Hopf algebras.

PROPOSITION 3.14. *Let A be an ϵ-Hopf algebra, B an algebra and C a coalgebra.*

(a) *If $g \in \mathsf{Coalg}_k(C,A)$ then g is invertible in $\mathsf{Hom}_k(C,A)$ with respect to circular convolution, its inverse is $S_A g$. Moreover, $-S_A g \in \mathsf{Coalg}_k(C,A)$.*
(b) *If $f \in \mathsf{Alg}_k(A,B)$ then f is invertible in $\mathsf{Hom}_k(A,B)$ with respect to circular convolution, its inverse is fS_A. Moreover, $-fS_A \in \mathsf{Alg}_k(A,B)$.*

PROOF. By lemma 3.4 (b), there is a morphism of monoids

$$f_* : \mathsf{Hom}_k(A,A) \to \mathsf{Hom}_k(A,B),\ h \mapsto fh \ .$$

Hence $fS_A = f_*(S_A)$ is the inverse of $f = f_*(id_A)$ in $\mathsf{Hom}_k(A,B)$. Also, since $-S_A \in \mathsf{Alg}_k(A,A)$ (by proposition 3.7 (a)), $-fS_A \in \mathsf{Alg}_k(A,B)$. This proves (b). Part (a) is dual. □

In particular, choosing $C = k$ with its usual coalgebra structure ($\Delta(1) = 1\otimes 1$) we obtain that if $x \in A$ is group-like then it is circular invertible in A, with circular inverse $S(x)$, and $-S(x)$ is group-like.

4. The canonical derivation and the existence of the antipode

In this section we derive a result that shows that many ϵ-bialgebras do possess an antipode. This applies to most examples considered in this paper; in particular to the path ϵ-Hopf algebra kQ of an arbitrary quiver Q and the ϵ-Hopf algebra $\mathcal{P}$ of non-trivial posets.

We start with a basic result on circular inverses.

LEMMA 4.1. *Let R be a ring. If $a \in R$ is nilpotent then it is circular invertible, with inverse $\sum_{n=1}^{\infty}(-a)^n$.*

PROOF. In the unital ring $R^+ = R \oplus \mathbb{Z}$, $1 + a$ is invertible with inverse $\sum_{n=0}^{\infty}(-1)^n a^n$. The result follows by considering the injective morphism of monoids $R \to R^+$, $a \mapsto 1 + a$. □

The result of lemma 4.1 can be extended to the more general situation of topological rings, where the series $\sum_{n=1}^{\infty}(-a)^n$ may converge under weaker assumptions. In particular the result may be applied to convolution rings of the form $\mathsf{Hom}_k(C,A)$ and linear maps $a : C \to A$ that are locally nilpotent with respect to convolution (i.e. for each $c \in C$ there is some $n \in \mathbb{N}$ such that $a^{*n}(c) := a(c_1)a(c_2)\dots a(c_{n+1}) = 0$).

Corollary 4.2. *Let A be an ϵ-bialgebra. If $id : A \to A$ is locally nilpotent with respect to convolution then A is an ϵ-Hopf algebra with antipode*

$$S = \sum_{n=1}^{\infty} (-1)^n id^{*n} \ .$$

Moreover, S is bijective and $S^{-1} = -\sum_{n=1}^{\infty} id^{*n}$.

Proof. As explained in the preceding paragraph, the map $S = \sum_{n=1}^{\infty}(-1)^n id^{*n}$ is the circular inverse of id. To prove the remaining assertion, consider the ϵ-bialgebra $\bar{A} := (A, -m, \Delta)$. We have

$$id_{\bar{A}}^{*n} = m_{\bar{A}}^{(n-1)} id_{\bar{A}}^{\otimes n} \Delta_{\bar{A}}^{(n-1)} = (-1)^{n-1} m_A^{(n-1)} id_A^{\otimes n} \Delta_A^{(n-1)} = (-1)^{n-1} id_A^{*n} \ .$$

Therefore $id_{\bar{A}}$ is locally nilpotent and, by the result just proved, $\bar{A}$ is an ϵ-Hopf algebra with antipode

$$S_{\bar{A}} = \sum_{n=1}^{\infty} (-1)^n id_{\bar{A}}^{*n} = -\sum_{n=1}^{\infty} id_A^{*n} \ .$$

By proposition 3.11.3, S is bijective and $S^{-1} = S_{\bar{A}}$. □

Corollary 4.3. *Let (A, m, Δ) be an ϵ-bialgebra for which there is a sequence A_n of subspaces with the following properties:*

1. $A = \cup_{n=0}^{\infty} A_n$,
2. $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$, *and*
3. $\Delta(A_n) \subseteq \cup_{i+j<n} A_i \otimes A_j$.

Then A is an ϵ-Hopf algebra with bijective antipode.

Proof. Notice that $id * id = m\Delta$ and, by induction,

$$id^{*(n+1)} = id^{*n} * id = m^{(n-1)}\Delta^{(n-1)} * id = m\Big(m^{(n-1)}\Delta^{(n-1)} \otimes id\Big)\Delta = m^{(n)}\Delta^{(n)} \ .$$

Now, under the present hypothesis, Δ is locally nilpotent, in the sense that if $a \in A_n$ then $\Delta^{(n+1)}(a) = 0$. Therefore id is locally nilpotent with respect to convolution and corollary 4.2 applies to give the result. □

The expression for the antipode in corollary 4.2 admits another formulation in terms of exponentials. In order to explain it, we first show that every ϵ-bialgebra A carries a canonical *biderivation* $D : A \to A$, i.e. a map that is both a derivation and a coderivation.

Proposition 4.4. *Let (A, m, Δ) be an ϵ-bialgebra. Then the map*

$$D = m\Delta : A \to A$$

is both a derivation and a coderivation. Moreover, for every $n \geq 0$,

$$D^n = n! m^{(n)} \Delta^{(n)}$$

where $m^{(n)}$ and $\Delta^{(n)}$ are the iterated multiplications and comultiplications and D^n is the iterated selfcomposition of D.

Proof. We calculate $D(ab) \overset{(2.1)}{=} m(ab_1 \otimes b_2 + a_1 \otimes a_2 b = ab_1 b_2 + a_1 a_2 b = aD(b) + D(a)b$.

Alternatively, we may notice that $\Delta : A \to A \otimes A$ is a derivation and $m : A \otimes A \to A$ is a morphism of A-bimodules (this is equivalent to associativity). Composing a derivation with a morphism of bimodules yields a derivation. Thus, the map $D = m\Delta$ is a derivation. This argument can be dualized to obtain that D is a coderivation.

To prove the remaining assertion first note that for any derivation D of an algebra A and elements $a, b, \ldots, z \in A$,

$$(*) \qquad D(ab\cdots z) = D(a)b\cdots z + aD(b)\cdots z + \ldots + ab\cdots D(z)\ .$$

Write $\Delta^{(n)}(a) = \sum_{(a)} a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}$, using Sweedler's notation. We will show that

$$D^n(a) = n! \sum_{(a)} a_1 a_2 \ldots a_{n+1}$$

by induction on n. For $n = 0$ or 1 there is nothing to prove. If $n \geq 2$ then by induction hypothesis

$$D^n(a) = DD^{n-1}(a) = (n-1)! D(\sum_{(a)} a_1 a_2 \ldots a_n)$$

$$\stackrel{(*)}{=} (n-1)! \sum_{(a)} \Big(D(a_1)a_2 \ldots a_n + a_1 D(a_2) \ldots a_n + \ldots + a_1 \ldots a_{n-1} D(a_n) \Big)$$

$$= (n-1)! \Big(\sum_{(a)} D(a_1)a_2 \ldots a_n + \sum_{(a)} a_1 D(a_2) \ldots a_n + \ldots + \sum_{(a)} a_1 \ldots a_{n-1} D(a_n) \Big)\ .$$

Now, by coassociativity and associativity, each of the n sums above is equal to

$$\sum_{(a)} a_1 a_2 \ldots a_n a_{n+1}\ .$$

Hence

$$D^n(a) = n! \sum_{(a)} a_1 a_2 \ldots a_{n+1}$$

as needed. □

Let A be an algebra and $T : A \to A$ a linear map. If A is a finite dimensional real or complex algebra, or if T is locally nilpotent (i.e. for each $a \in A$ there is some $n \in \mathbb{N}$ such that $T^n(a) = 0$) and the characteristic of the base field is zero, then the series

$$e^T := \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

converges in the algebra $\mathsf{End}(A)$, i.e. there is a well-defined linear map $e^T : A \to A$ such that $e^T(a) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n(a)$.

PROPOSITION 4.5. *Let (A, m, Δ) be an ϵ-bialgebra over the field k and $D = m\Delta$. Suppose that either*

(a) *$k = \mathbb{R}$ or $\mathbb{C}$ and A is finite dimensional, or*
(b) *D is locally nilpotent and $\mathsf{char}(k) = 0$.*

Then A is an ϵ-Hopf algebra with bijective antipode $S = -e^{-D}$.

Proof. By proposition 4.4, $D^n = n!m^{(n)}\Delta^{(n)} = n!id^{*(n+1)}$. Hence

$$-e^{-D} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}D^n = -\sum_{n=0}^{\infty}(-1)^n id^{*(n+1)} = \sum_{n=1}^{\infty}(-1)^n id^{*n} = S$$

by corollary 4.2. □

Remark 4.6. If $D : A \to A$ is an arbitrary derivation of an algebra A then $e^D : A \to A$ is an automorphism of algebras (when defined) [Jac, section I.2]. Dually, if $D : C \to C$ is a coderivation of a coalgebra C then e^D is an automorphism of coalgebras.

Let (A, m, Δ) be an ϵ-bialgebra and $D = m\Delta$ the canonical biderivation. The exponentials e^D and $e^{-D} : A \to A$ are then automorphisms of ϵ-bialgebras (assuming the hypothesis of proposition 4.5). This confirms the result of proposition 3.7 that $-S$ is an automorphism of ϵ-bialgebras, in the particular case when S is given as an exponential.

Examples 4.7.

1. Consider the ϵ-bialgebra $k[\mathbf{x}]$, where $\Delta(\mathbf{x}) = 1\otimes 1$. We have $D(\mathbf{x}) = 1$, therefore $D = \frac{\mathbf{d}}{\mathbf{dx}}$ is the usual derivative. Hence

$$\begin{aligned} e^D(\mathbf{x}^n) &= (id + D + \frac{D^2}{2!} + \ldots + \frac{D^n}{n!})(\mathbf{x}^n) \\ &= \mathbf{x}^n + n\mathbf{x}^{n-1} + \frac{n(n-1)}{2}\mathbf{x}^{n-2} + \ldots + \frac{n!}{n!}\mathbf{x}^0 \\ &= \sum_{i=0}^{n}\binom{n}{i}\mathbf{x}^i = (\mathbf{x}+1)^n \ . \end{aligned}$$

It follows that e^D is the shift operator

$$e^D(p(\mathbf{x})) = p(\mathbf{x}+1) \ .$$

A similar calculation shows that the antipode $S = -e^{-D}$ is

$$S(p(\mathbf{x})) = -p(\mathbf{x}-1) \ .$$

This is an alternative derivation of the expression for S found in example 3.2.1. Notice that to obtain this result one may avoid any assumptions on k and the use of exponentials, by rephrasing the above argument in terms of corollary 4.3 only. In fact, in this example, and in many others examples of interest, even stronger assumptions than those of corollary 4.3 hold, as follows. Suppose that the ϵ-bialgebra A admits a decomposition $A = \oplus_{n=0}^{\infty} A_n$ such that

$$(*) \qquad \Delta(A_n) \subseteq \sum_{i+j=n-1} A_i \otimes A_j \ .$$

Then the sequence of subspaces $A'_n = \oplus_{i=0}^{n} A_i$ satisfies the hypothesis of the corollary and hence A is an ϵ-Hopf algebra with antipode $S = \sum_{n=1}^{\infty}(-id)^{*n}$.

Notice that, when $(*)$ holds, one could redefine the degree in order to obtain a degree-preserving comultiplication. However, in most examples, there is a natural notion of degree for which both $(*)$ and the additional condition $m(A_i\otimes A_j) \subseteq A_{i+j}$ hold. It is for this reason that we do not shift the degree.

2. Hypothesis $(*)$ are also satisfied in the case of the path ϵ-bialgebra $A = kQ$ of a quiver Q, taking $A_n = kQ_n$. The formula of corollary 4.3 immediately yields the expression for the antipode S given in example 3.2.2.
3. Let us illustrate a few of the previous results for the ϵ-bialgebra $\mathcal{P}$ of posets. First of all, since $\Delta(P) = \sum_{0_P<x<1_P}[0_P, x]\otimes[x, 1_P]$, we have

$$id^{*n}(P) = m^{(n-1)}\Delta^{(n-1)}(P) = \sum_{0_P<x_1<\ldots<x_{n-1}<1_P}[0_P, x_1][x_1, x_2]\ldots[x_{n-1}, 1_P] \ ;$$

therefore, by corollary 4.2,

$$S(P) = \sum_{n=1}^{\infty}(-1)^n \sum_{0_P<x_1<\ldots<x_{n-1}<1_P}[0_P, x_1][x_1, x_2]\ldots[x_{n-1}, 1_P] \ .$$

This is the formula announced in example 3.2.3.

Now consider the linear functionals $\zeta : \mathcal{P} \to k$ and $\mu : \mathcal{P} \to k$ defined by

$$\zeta(P) = 1 \ \forall \text{ poset } P \in \mathcal{P} \text{ and } \mu = \zeta S \ .$$

Since ζ is a morphism of algebras, proposition 3.14 implies that it is circular invertible with inverse μ, in other words,

$$0 = \mu \circledast \zeta = \mu * \zeta + \mu + \zeta \ .$$

Evaluating at a poset $P \in \mathcal{P}$ we find

$$0 = \sum_{0_P<x<1_P}\mu[0_P, x]\zeta[x, 1_P] + \mu(P) + \zeta(P)$$
$$\Rightarrow \mu[0_P, 1_P] = -1 - \sum_{0_P<x<1_P}\mu[0_P, x] \ .$$

This shows that μ is the usual *Möbius function* of posets (since this is its defining recursion).

Applying ζ to both sides of the explicit formula for S above we find

$$\mu(P) = \sum_{n=1}^{\infty}(-1)^n\#\{0_P < x_1 < \ldots < x_{n-1} < 1_P\} \ ;$$

the well-known formula of P.Hall giving the Möbius function in terms of numbers of chains.

Finally, proposition 3.14 also says that $-\mu$ is a morphism of algebras, in other words that

$$\mu(PQ) = -\mu(P)\mu(Q) \ ,$$

another well-known property of the product of posets under consideration.

Further consideration of the ϵ-bialgebra structure of $\mathcal{P}$ enables one to obtain a simple algebraic proof of the existence of the **cd**-index of polytopes. This important application is explained in detail in [A1].
4. For the ϵ-bialgebra $k\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\rangle$ of example 2.3.5, we have

$$\Delta(\mathbf{x}_n) = \sum_{i=0}^{n-1}\mathbf{x}_i\otimes\mathbf{x}_{n-1-i} = \sum_{(n_1,n_2)\in\mathcal{C}^+(n+1,2)}\mathbf{x}_{n_1-1}\otimes\mathbf{x}_{n_2-1} \ ,$$

where $\mathcal{C}^+(n+1,k) = \{(n_1,\dots,n_k)\ /\ n_i \in \mathbb{Z}^+,\ n_1+\dots+n_k = n+1\}$. Hypothesis $(*)$ are therefore satisfied if we set $\deg(\mathbf{x}_n) = n$ and $\deg(uv) = \deg(u)\deg(v)$.

It follows by induction that

$$id^{*k}(\mathbf{x}_n) = m^{(k-1)}\Delta^{(k-1)}(\mathbf{x}_n) = \sum_{(n_1,\dots,n_k)\in\mathcal{C}^+(n+1,k)} \mathbf{x}_{n_1-1}\mathbf{x}_{n_2-1}\dots\mathbf{x}_{n_k-1}$$

and hence, by corollary 4.2,

$$S(\mathbf{x}_n) = \sum_{k=1}^{n+1}(-1)^k \sum_{(n_1,\dots,n_k)\in\mathcal{C}^+(n+1,k)} \mathbf{x}_{n_1-1}\mathbf{x}_{n_2-1}\dots\mathbf{x}_{n_k-1}$$

as announced in example 3.2.4.

5. The associative Yang-Baxter equation

Let A be an associative algebra and M an A-bimodule. There is a derivation $\Delta_r : A \to M$ associated to each element $r \in M$ as follows:

$$\Delta_r(a) = a\cdot r - r\cdot a \ \ \forall\ a \in A\ .$$

Such derivations are called *principal*. In this section we discuss when a principal derivation $\Delta : A \to A{\otimes}A$ satisfies the coassociativity condition

$$(\Delta{\otimes}id_A)\circ\Delta = (id_A{\otimes}\Delta)\circ\Delta\ ,$$

and therefore endows A with a ϵ-bialgebra structure. Given an element $r = \sum_i u_i{\otimes}v_i \in A{\otimes}A$, let

$$r_{12} = \sum_i u_i{\otimes}v_i{\otimes}1,\ r_{13} = \sum_i u_i{\otimes}1{\otimes}v_i \ \text{ and }\ r_{23} = \sum_i 1{\otimes}u_i{\otimes}v_i \in A^+{\otimes}A^+{\otimes}A^+\ ,$$

where $A^+ = A\oplus k$ is the result of adjoining a unit element to A.

Recall that an element $w \in W$ of an A-bimodule W is called *A-invariant* if

$$a\cdot w = w\cdot a\ \forall\ a\in A\ .$$

Below, we view $A{\otimes}A{\otimes}A$ as an A-bimodule via

$$a\cdot(x{\otimes}y{\otimes}z) = ax{\otimes}y{\otimes}z \ \text{ and }\ (x{\otimes}y{\otimes}z)\cdot b = x{\otimes}y{\otimes}zb\ .$$

Proposition 5.1. *A principal derivation $\Delta_r : A \to A{\otimes}A$ is coassociative if and only if the element*

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} \in A{\otimes}A{\otimes}A$$

is A-invariant.

Proof. Keeping the above notation, we compute

$$\begin{aligned}
&(\Delta_r{\otimes}id_A)\circ\Delta_r(a) = (\Delta_r{\otimes}id_A)(a\cdot r - r\cdot a)\\
&= (\Delta_r{\otimes}id_A)\Big(\sum_i au_i{\otimes}v_i - u_i{\otimes}v_ia\Big) = \sum_i (au_i\cdot r - r\cdot au_i){\otimes}v_i - (u_i\cdot r - r\cdot u_i){\otimes}v_ia\\
&\qquad = \sum_{i,j} au_iu_j{\otimes}v_j{\otimes}v_i - u_j{\otimes}v_jau_i{\otimes}v_i - u_iu_j{\otimes}v_j{\otimes}v_ia + u_j{\otimes}v_ju_i{\otimes}v_ia\\
&\qquad\qquad = a\cdot r_{13}r_{12} - r_{12}(1{\otimes}a{\otimes}1)r_{23} - r_{13}r_{12}\cdot a + r_{12}r_{23}\cdot a\ .
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&(id_A\otimes\Delta_r)\circ\Delta_r(a) = (id_A\otimes\Delta_r)(a\cdot r - r\cdot a)\\
&= (id_A\otimes\Delta_r)\Big(\sum_i au_i\otimes v_i - u_i\otimes v_i a\Big) = \sum_i au_i\otimes(v_i\cdot m - m\cdot v_i) - u_i\otimes(v_i a\cdot m - m\cdot v_i a)\\
&\qquad = \sum_{i,j} au_i\otimes v_i u_j\otimes v_j - au_i\otimes u_j\otimes v_j v_i - u_i\otimes v_i a u_j\otimes v_j + u_i\otimes u_j\otimes v_j v_i a\\
&\qquad\qquad = a\cdot r_{12}r_{23} - a\cdot r_{23}r_{13}\cdot - r_{12}(1\otimes a\otimes 1)r_{23} + r_{23}r_{13}\cdot a\ .
\end{aligned}
$$

Comparing the two expressions above we see that Δ_r is coassociative if and only if

$$a\cdot r_{13}r_{12} - r_{13}r_{12}\cdot a + r_{12}r_{23}\cdot a - a\cdot r_{12}r_{23} + a\cdot r_{23}r_{13} - r_{23}r_{13}\cdot a = 0\ ,$$

i.e. if and only if $r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13}$ is A-invariant. □

Remark 5.2. The situation above parallels one encountered in the theory of quantum groups [Dri], as we now recall. If $\mathfrak{g}$ is a Lie algebra and $r \in \mathfrak{g}\otimes\mathfrak{g}$ is an element, then the principal derivation $\delta_r : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ is coassociative if and only if the element

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$$

is $\mathfrak{g}$-invariant. Here "principal derivations" and "invariants" are taken in the sense of Lie theory, and each r_{ij} above lives in $U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes U(\mathfrak{g})$.

The *classical Yang-Baxter equation* is the equality

$$\text{(CYB)}\qquad [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\ .$$

Solutions to this equation give rise to Lie bialgebras and quantum groups [Dri].

By analogy with the above situation, we are led to consider solutions $r \in A\otimes A$ to the equation

$$\text{(AYB)}\qquad r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0\ ,$$

which we call the *associative Yang-Baxter equation*. For each solution r, the principal derivation $\Delta_r : A \to A\otimes A$ endows A with the structure of a ϵ-bialgebra, according to proposition 5.1.

Definition 5.3. A quasitriangular ϵ-bialgebra is a pair (A, r) where A is an associative algebra and $r \in A\otimes A$ is a solution to (AYB).

As explained above, in this case the triple (A, m, Δ_r) is indeed an ϵ-bialgebra. We present some examples next.

Examples 5.4.

1. Let A be any unital algebra possessing an element $b \in A$ such that $b^2 = 0$. Then $r = 1\otimes b$ satisfies (AYB). The corresponding ϵ-bialgebra structure is

$$\Delta_r(a) = a\otimes b - 1\otimes ba\ \forall\ a \in A\ .$$

The ϵ-bialgebra of dual numbers of example 2.3.6 is a particular case.

2. The polynomial algebra $k[\mathbf{x}]$ is not quasitriangular. However, let us regard the element

$$r := \frac{1}{\mathbf{x}-\mathbf{y}} \in k(\mathbf{x}, \mathbf{y})$$

as belonging to a certain completed tensor product $k[\mathbf{x}]\hat{\otimes}k[\mathbf{y}]$. Then (AYB) holds for r:

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = \frac{1}{\mathbf{x}-\mathbf{z}} \cdot \frac{1}{\mathbf{x}-\mathbf{y}} - \frac{1}{\mathbf{x}-\mathbf{y}} \cdot \frac{1}{\mathbf{y}-\mathbf{z}} + \frac{1}{\mathbf{y}-\mathbf{z}} \cdot \frac{1}{\mathbf{x}-\mathbf{z}}$$
$$= \frac{\mathbf{y}-\mathbf{z}-(\mathbf{x}-\mathbf{z})+\mathbf{x}-\mathbf{y}}{(\mathbf{x}-\mathbf{z})(\mathbf{x}-\mathbf{y})(\mathbf{y}-\mathbf{z})} = 0 \ .$$

The corresponding comultiplication is indeed the Newton divided difference:

$$\Delta_r(f) = f \cdot r - r \cdot f = \frac{f(\mathbf{x})}{\mathbf{x}-\mathbf{y}} - \frac{f(\mathbf{y})}{\mathbf{x}-\mathbf{y}} \ .$$

In this sense, $k[\mathbf{x}]$ is "essentially" quasitriangular.

3. More generally, let A be a Frobenius k-algebra and $t \in A\otimes A$ a Casimir element. Then the k-algebra $A[\mathbf{x}]$ is "essentially" quasitriangular with

$$r = \frac{t}{\mathbf{x}-\mathbf{y}} \ .$$

This is analogous to Drinfeld's solution of (CYB) for the loop Lie algebra $\mathfrak{g}[\mathbf{x}]$ of a semisimple finite dimensional Lie algebra $\mathfrak{g}$ in terms of the Casimir tensor $t \in \mathfrak{g}\otimes\mathfrak{g}$ [Dri, example 3.3].

4. Suppose $a \in A$ is an element such that $a^2 = 0$, and A is an arbitrary algebra. Then $r = a\otimes a$ is a solution to (AYB). For instance, if $A = M_2(k)$ we may take $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The corresponding ϵ-bialgebra structure on $M_2(k)$ is the one described on example 2.3.7.

5. The solutions to (AYB) for $M_2(\mathbb{C})$ can be explicitly described. Notice that if $r = \sum_i u_i\otimes v_i$ is a solution, then so are its transpose $\sum_i v_i^t\otimes u_i^t$ and any conjugate $\sum_i xu_ix^{-1}\otimes xv_ix^{-1}$. It can be shown that the following is the complete list of solutions to (AYB) for $M_2(\mathbb{C})$, up to conjugates and transposes:
   (a) $r_0 = 0$;
   (b) $r_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;
   (c) $r_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;
   (d) $r_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and
   (e) $r_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

   The comultiplication corresponding to the last solution is

$$\Delta_{r_3}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-d & b \\ c & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ .$$

This turns out to be an ϵ-Hopf algebra with antipode

$$S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a-c & a-b+c-d \\ -c & c-d \end{bmatrix}$$

as can be easily checked.

6. For the algebra $A = k^X$ of all functions $X \to k$, where X is a finite set and k a field, the only solution to (AYB) is $r = 0$. In fact, the same conclusion holds for any algebra A such that $A\otimes A$ has no nilpotents (other than zero), since it follows immediately from (AYB) that $r^2 = 0$ necessarily.

Next we present some properties of quasitriangular ϵ-bialgebras. The first one shows that this notion is also analogous to that of ordinary quasitriangular bialgebras. The remaining properties are analogs of well-known properties of ordinary quasitriangular bialgebras or Hopf algebras.

PROPOSITION 5.5. *Let (A,r) be a quasitriangular ϵ-bialgebra and $\Delta = \Delta_r$. Then*

$$\Delta(a) = a\cdot r - r\cdot a \ \forall\ a \in A\ ; \tag{5.1}$$

$$(\Delta\otimes id)(r) = -r_{23}r_{13} \ \text{and} \tag{5.2}$$

$$(id\otimes\Delta)(r) = r_{13}r_{12}\ . \tag{5.3}$$

Conversely, if an ϵ-bialgebra (A,m,Δ) satisfies (5.1), (5.2) *and* (5.3) *for some $r \in A\otimes A$, then (A,r) is a quasitriangular ϵ-bialgebra and $\Delta = \Delta_r$.*

PROOF. Property (5.1) is just a restatement of $\Delta = \Delta_r$. Assuming this, we compute

$$(\Delta\otimes id)(r) = \sum_i \Delta(u_i)\otimes v_i = \sum_{i,j} u_iu_j\otimes v_j\otimes v_i - u_j\otimes v_ju_i\otimes v_i$$
$$= r_{13}r_{12} - r_{12}r_{23} \overset{(AYB)}{=} - r_{23}r_{13}$$

and

$$(id\otimes\Delta)(r) = \sum_i u_i\otimes\Delta(v_i) = \sum_{i,j} u_i\otimes v_iu_j\otimes v_j - u_i\otimes u_j\otimes v_jv_i$$
$$= r_{12}r_{23} - r_{23}r_{13} \overset{(AYB)}{=} r_{13}r_{12}$$

which proves (5.2) and (5.3). The same calculation shows that the converse holds. □

Given a finite dimensional ϵ-bialgebra $A = (A,m,\Delta)$, we are interested in the ϵ-bialgebras

$$A' := (A^*, \Delta^{*^{op}}, -m^{*^{cop}}) \ \text{ and } \ {'\!A} := (A^*, -\Delta^{*^{op}}, m^{*^{cop}})$$

(recall the basic constructions of section 2).

PROPOSITION 5.6. *Let (A,r) be a finite dimensional quasitriangular ϵ-bialgebra, $r = \sum_i u_i\otimes v_i$. Then the maps*

$$\lambda_r : {'\!A} \to A\ ,\quad f \mapsto \sum_i f(u_i)v_i \ \text{and}$$
$$\rho_r : A' \to A\ ,\quad f \mapsto \sum_i u_if(v_i)$$

are morphisms of ϵ-bialgebras.

PROOF. First consider $\lambda_r : {'\!A} \to A$. The multiplication in ${'\!A}$ is $fg = -(g\otimes f)\Delta_A$. Therefore

$$\lambda_r(fg) = \sum_i (fg)(u_i)v_i = -\sum_i (g\otimes f)\Delta_A(u_i)v_i = -(g\otimes f\otimes id)(\Delta_A\otimes id)(r)$$
$$\overset{(5.2)}{=} (g\otimes f\otimes id)(r_{23}r_{13}) = \sum_{i,j} g(u_j)f(u_i)v_iv_j = \lambda_r(f)\lambda_r(g)\ .$$

Thus λ_r is a morphism of algebras.

The comultiplication in ${}'\!A$ is

$$\Delta_{'A}(f) = f_1 \otimes f_2 \iff f(ab) = f_1(b) f_2(a) \ \forall \ a, b \in A \ .$$

Therefore

$$(\lambda_r \otimes \lambda_r)\Delta_{'A}(f) = \lambda_r(f_1)\lambda_r(f_2) = \sum_{i,j} f_1(u_i) f_2(u_j) v_i \otimes v_j = (f \otimes id \otimes id)(r_{13} r_{12})$$

$$\overset{(5.3)}{=} (f \otimes id \otimes id)(id \otimes \Delta_A)(r) = \sum_i f(u_i)\Delta_A(v_i) = \Delta_A\Big(\sum_i f(u_i) v_i\Big) = \Delta_A \lambda_r(f) \ .$$

Thus λ_r is also a morphism of coalgebras.

We will reduce the assertion regarding $\rho_r : A' \to A$ to the one just proved, by using that $\rho_r = \lambda_{\tau(r)}$, where $\tau(a \otimes b) = b \otimes a$.

First we claim that $\tau(r)$ is a solution to (AYB) for the algebra (A, m^{op}). To see this, let $\sigma(x \otimes y \otimes z) = z \otimes y \otimes x$. Then

$$\tau(r)_{13} \cdot^{op} \tau(r)_{12} = \sum_{i,j} v_i \cdot^{op} v_j \otimes u_j \otimes u_i = \sum_{i,j} v_j v_i \otimes u_j \otimes u_i = \sigma(u_i \otimes u_j \otimes v_j v_i) = \sigma(r_{23} r_{13}) \ .$$

Similarly, $\tau(r)_{12} \cdot^{op} \tau(r)_{23} = \sigma(r_{12} r_{23})$ and $\tau(r)_{23} \cdot^{op} \tau(r)_{13} = \sigma(r_{13} r_{12})$. Thus

$$\begin{aligned} &\tau(r)_{13} \cdot^{op} \tau(r)_{12} - \tau(r)_{12} \cdot^{op} \tau(r)_{23} + \tau(r)_{23} \cdot^{op} \tau(r)_{13} \\ &\qquad = \sigma(r_{13} r_{12} - r_{12} r_{23} + r_{23} r_{13}) = 0 \end{aligned}$$

as claimed.

Now, the comultiplication on (A, m^{op}) corresponding to $\tau(r)$ is

$$\begin{aligned} \Delta_{\tau(r)}(a) &= a \cdot^{op} \tau(r) - \tau(r) \cdot^{op} a = \sum_i v_i a \otimes u_i - \sum_i v_i \otimes a u_i \\ &= -\tau(a \cdot r - r \cdot a) = -\tau \Delta_r(a) = -\Delta_r^{cop}(a) \ . \end{aligned}$$

Thus, the quasitriangular ϵ-bialgebra corresponding to $\tau(r)$ is $B := (A, m^{op}, -\Delta_r^{cop})$. It follows from the part of the statement already proved that $\lambda_{\tau(r)} : {}'B \to B$ is a morphism of ϵ-bialgebras. A trivial inspection of the various basic constructions reveals that this is the same thing as saying that $\rho_r : A' \to A$ is a morphism of ϵ-bialgebras. $\square$

Corollary 5.7. *Let (A, r) be a quasitriangular ϵ-Hopf algebra with bijective antipode S. Then*

(a) $(id \otimes S)(r) = (S^{-1} \otimes id)(r)$*;*
(b) $(S \otimes id)(r) = (id \otimes S^{-1})(r)$ *and*
(c) $(S \otimes S)(r) = r = (S^{-1} \otimes S^{-1})(r)$*.*

Proof. We know from propositions 5.6 and 3.8 that $\lambda_r : {}'\!A \to A$ is a morphism of ϵ-Hopf algebras. By propositions 3.11, 3.12 and 3.13, the antipode of ${}'\!A$ is $(S^{-1})^*$. Therefore, $\forall \ f \in A^*$,

$$\begin{aligned} &\lambda_r (S^{-1})^*(f) = S\lambda_r(f) \Rightarrow \lambda_r(fS^{-1}) = S\Big(\sum_i f(u_i) v_i\Big) \\ &\qquad \Rightarrow \sum_i (fS^{-1})(u_i) v_i = \sum_i f(u_i) S(v_i) \ . \end{aligned}$$

Hence

$$\sum_i S^{-1}(u_i)\otimes v_i = \sum_i u_i\otimes S(v_i) \Rightarrow (S^{-1}\otimes id)(r) = (id\otimes S)(r) \ .$$

This proves (a).

For the same reasons, $\rho_r : A' \to A$ is a morphism of ϵ-Hopf algebras. Therefore, $\forall\ f \in A^*$,

$$\rho_r(S^{-1})^*(f) = S\rho_r(f) \Rightarrow \rho_r(fS^{-1}) = S\Big(\sum_i u_i f(v_i)\Big)$$
$$\Rightarrow \sum_i u_i(fS^{-1})(v_i) = \sum_i S(u_i)f(v_i) \ .$$

Hence

$$\sum_i u_i\otimes S^{-1}(v_i) = \sum_i S(u_i)\otimes v_i \Rightarrow (id\otimes S^{-1})(r) = (S\otimes id)(r) \ ,$$

proving (b). Part (c) follows from (a) by applying $S\otimes id$ and $id\otimes S^{-1}$ to both sides. □

Example 5.8. The results of corollary 5.7 can be verified directly for the "essentially" quasitriangular ϵ-Hopf algebra $k[\mathbf{x}]$. We have

$$\begin{aligned} S\otimes id : k[\mathbf{x},\mathbf{y}] \to k[\mathbf{x},\mathbf{y}] \ &, \ f(x,y) \mapsto -f(x-1,y) \ ; \\ id\otimes S : k[\mathbf{x},\mathbf{y}] \to k[\mathbf{x},\mathbf{y}] \ &, \ f(x,y) \mapsto -f(x,y-1) \ ; \\ S^{-1}\otimes id : k[\mathbf{x},\mathbf{y}] \to k[\mathbf{x},\mathbf{y}] \ &, \ f(x,y) \mapsto -f(x+1,y) \ ; \\ id\otimes S^{-1} : k[\mathbf{x},\mathbf{y}] \to k[\mathbf{x},\mathbf{y}] \ &, \ f(x,y) \mapsto -f(x,y+1) \ . \end{aligned}$$

Therefore

$$\begin{aligned} (id\otimes S)(r) &= \frac{-1}{\mathbf{x}-(\mathbf{y}-1)} = \frac{-1}{\mathbf{x}+1-\mathbf{y}} = (S^{-1}\otimes id)(r) \ ; \\ (S\otimes id)(r) &= \frac{-1}{\mathbf{x}-1-\mathbf{y}} = \frac{-1}{\mathbf{x}-(\mathbf{y}+1)} = (id\otimes S^{-1})(r) \ ; \\ (S\otimes S)(r) &= \frac{1}{\mathbf{x}-1-(\mathbf{y}-1)} = \frac{1}{\mathbf{x}-\mathbf{y}} = r \ ; \\ (S^{-1}\otimes S^{-1})(r) &= \frac{1}{\mathbf{x}+1-(\mathbf{y}+1)} = \frac{1}{\mathbf{x}-\mathbf{y}} = r \ . \end{aligned}$$

Remark 5.9. There are many other interesting properties of quasitriangular ϵ-Hopf algebras. In particular one can show that if the antipode S is bijective then $-S$ is necessarily given as circular conjugation by the canonical element

$$u := -\sum_i S(u_i)v_i = -\sum_i u_i S^{-1}(v_i) \ .$$

This is the analog of Drinfeld's element for ordinary quasitriangular Hopf algebras. These properties, as well as the connection between ϵ-bialgebras and Lie bialgebras are discussed in detail in [A2].

We close this section with an application of proposition 5.1, which complements the result of example 5.4.6.

Proposition 5.10. *Let X be a finite set and k^X the algebra of functions on X. Then the only ϵ-bialgebra structure on k^X is the trivial one: $\Delta = 0$.*

Proof. Let $\Delta : k^X \to k^X \otimes k^X$ be a coassociative derivation. Since k^X is a separable algebra, $H^1(k^X, M) = 0$ for any k^X-bimodule M, i.e. any derivation is principal [Wei, theorem 9.2.11]. Thus there is some element

$$r = \sum_{x,y} r(x,y) e_x \otimes e_y \in k^X \otimes k^X \text{ such that } \Delta = \Delta_r \ ,$$

where $\{e_x\}$ denotes the canonical basis of orthogonal idempotents of k^X. By proposition 5.1,

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} =$$
$$\sum_{x,y,z} r(x,z)r(x,y)e_x \otimes e_y \otimes e_z - \sum_{x,y,z} r(x,y)r(y,z)e_x \otimes e_y \otimes e_z + \sum_{x,y,z} r(x,z)r(y,z)e_x \otimes e_y \otimes e_z$$

is an k^X-invariant element of $k^X \otimes k^X \otimes k^X$. Applying the morphism of k^X-bimodules $id \otimes m : k^X \otimes k^X \otimes k^X \to k^X \otimes k^X$ we deduce that

$$\sum_{x,y} r(x,y)^2 e_x \otimes e_y$$

is an k^X-invariant element of $k^X \otimes k^X$. Hence, acting with e_z from both sides on this element we must have that

$$\sum_y r(z,y)^2 e_z \otimes e_y = \sum_x r(x,z)^2 e_x \otimes e_z \ .$$

It follows that $r(x,y) = 0 \ \forall \ x \neq y$. Thus

$$r = \sum_x r(x,x) e_x \otimes e_x \ .$$

But such an element is clearly k^X-invariant, so $\Delta_r = 0$. □

6. Bicrossproducts of associative algebras

In this section we present a notion analogous to that of "matched pairs" of groups or Hopf algebras [Kas, definitions IX.1.1, IX.2.2], and the corresponding bicrossproduct construction, for associative nonunital algebras. Drinfeld's double for ϵ-bialgebras will be obtained as a particular case in section 7. Matched pairs of groups are also called "double groups" and we choose this terminology for the analogous notion for associative algebras.

Definition 6.1. A double algebra is a pair (A, B) of associative nonunital algebras together with

(6.1) a left B-module structure on A: $B \times A \to A$, $(b,a) \mapsto b \to a$

(6.2) a right A-module structure on B: $B \times A \to B$, $(b,a) \mapsto b \leftarrow a$

such that

$$b \to aa' = (b \to a)a' + (b \leftarrow a) \to a' \tag{6.3}$$

$$bb' \leftarrow a = b(b' \leftarrow a) + b \leftarrow (b' \to a) \ . \tag{6.4}$$

Proposition 6.2. *Given a double algebra* (A, B)*, there is a unique associative multiplication on the space*

$$A{\bigcirc}B := (A{\otimes}B) \oplus A \oplus B$$

such that

(a) *A and B are subalgebras of $A{\bigcirc}B$,*
(b) *$a \cdot b = a{\otimes}b$ and*
(c) *$b \cdot a = b \to a + b \leftarrow a$.*

Proof. We define a multiplication on $A{\bigcirc}B$ by formulas (a), (b) and (c) together with:

(d) $a \cdot (a'{\otimes}b) = aa'{\otimes}b, \quad (a{\otimes}b) \cdot b' = a{\otimes}bb'$;
(e) $(a{\otimes}b) \cdot a' = a(b \to a') + a{\otimes}(b \leftarrow a'), \quad b \cdot (a{\otimes}b') = (b \to a){\otimes}b' + (b \leftarrow a)b'$;
(f) $(a{\otimes}b) \cdot (a'{\otimes}b') = a(b \to a'){\otimes}b' + a{\otimes}(b \leftarrow a')b'$.

Associativity plus (a), (b) and (c) clearly force us to define the multiplication in this way; so uniqueness is guaranteed, once we prove that this multiplication is actually associative.

It is enough to check the associativity axiom on the generators $a \in A$ and $b \in B$ of $A{\bigcirc}B$. There are four cases to consider:

$$\begin{array}{lll}
\text{(i)} & a(a'a'') = (aa')a'', & b(b'b'') = (bb')b''; \\
\text{(ii)} & a(a'b) = (aa')b, & a(bb') = (ab)b'; \\
\text{(iii)} & a(ba') = (ab)a', & (ab') = (ba)b'; \\
\text{(iv)} & b(aa') = (ba)a', & b(b'a) = (bb')a.
\end{array}$$

(i) holds by (a), i.e. by definition. Similarly, (ii) and (iii) hold by definitions (a)-(e). (iv) is the only case that requires verification. For the first half of (iv), we have

$$\begin{aligned}
(ba)a' &\stackrel{(c)}{=} (b \to a + b \leftarrow a)a' = (b \to a)a' + (b \leftarrow a)a' \\
&\stackrel{(c)}{=} (b \to a)a' + (b \leftarrow a) \to a' + (b \leftarrow a) \leftarrow a' \\
&\stackrel{(6.3),\,(6.2)}{=} b \to aa' + b \leftarrow aa' \\
&\stackrel{(c)}{=} b(aa') \ .
\end{aligned}$$

Similarly the other half of (iv) follows from (c), (6.4) and (6.1). □

Remark 6.3. The proof shows that the following converse of proposition 6.2 holds: given a pair of algebras (A, B) and linear maps $(b, a) \mapsto b \to a$ and $(b, a) \mapsto b \leftarrow a$, if there is an associative multiplication on $A{\bigcirc}B$ satisfying (a), (b) and (c) then axioms (6.1)-(6.4) hold, i.e. (A, B) is a double algebra.

Examples 6.4.

For the purposes of this paper, the most important example of a double algebra is provided by an ϵ-bialgebra and its dual, since this gives rise to Drinfeld's double (see section 7). Other interesting examples are discussed below.

1. Let A be any associative algebra and $B := \mathsf{End}(A)$, an algebra under composition. Define

$$T \to a := T(a) \text{ and } T \leftarrow a := TL_a - L_{T(a)}$$

where $L_a \in \mathsf{End}(A)$ is $L_a(b) = ab$. The reader can easily check that then $(A, \mathsf{End}(A))$ is a double algebra. The resulting algebra structure on the subalgebra $A \otimes \mathsf{End}(A)$ of $A \bigcirc \mathsf{End}(A)$ is

$$(a \otimes T) \cdot (b \otimes S) = aT(b) \otimes S + a \otimes T L_b S - a \otimes L_{T(b)} S \ .$$

2. Let A and B be any associative algebras and define

$$b \to a := 0 \text{ and } b \leftarrow a := 0 \ .$$

Then (A, B) is trivially a double algebra. The resulting algebra structure on $A \bigcirc B$ can be described as follows. Consider first the direct sum of algebras $R := A \oplus B$:

$$(a + b) \cdot (a' + b') = aa' + bb'$$

and view the space $M := A \otimes B$ as an $A \oplus B$-bimodule via

$$(a + b) \cdot a' \otimes b' = aa' \otimes b' \text{ and } a' \otimes b' \cdot (a + b) = a' \otimes b'b \ .$$

The map

$$f : R \times R \to M, \ \ f(a + b, a' + b') = a \otimes b'$$

is then a Hochschild 2-cocycle and the corresponding Hochschild extension is precisely the algebra $A \bigcirc B$ of proposition 6.2

$$0 \to M \to A \bigcirc B \to R \to 0 \ .$$

We close this section with the universal property of the bicrossproduct construction. This says that $A \bigcirc B$ is the free product of algebras $A * B$ modulo the relation $b \cdot a = b \to a + b \leftarrow a$.

Proposition 6.5. *Let (A, B) be a double algebra, C another algebra and $f : A \to C$ and $g : B \to C$ morphisms of algebras such that $\forall\ a \in A,\ b \in B$,*

$$g(b) f(a) = f(b \to a) + g(b \leftarrow a) \ . \tag{6.5}$$

Then there exists a unique morphism of algebras $h : A \bigcirc B \to C$ such that $h_{|_A} = f$ and $h_{|_B} = g$.

Proof. Since $a \otimes b = a \cdot b$, h must be defined by

$$h(a) = f(a),\ \ h(b) = g(b) \text{ and } h(a \otimes b) = f(a) g(b) \ .$$

Thus, uniqueness is clear. To show that h is indeed a morphism of algebras one has to check the multiplicativity property

$$h(\alpha) h(\beta) = h(\alpha \beta) \tag{$*$}$$

in each of the following nine cases:

Case	i	ii	iii	iv	v	vi	vii	viii	ix
α in	A	B	A	A	$A \otimes B$	B	$A \otimes B$	B	$A \otimes B$
β in	A	B	B	$A \otimes B$	B	A	A	$A \otimes B$	$A \otimes B$

In cases i and ii, $(*)$ holds by hypothesis. In case iii it holds by definition of h:

$$h(a \cdot b) = h(a \otimes b) = f(a) g(b) = h(a) h(b) \ .$$

Cases iv and v follow formally from i, ii and iii. For instance, case iv is

$$h(a \cdot (a' \otimes b)) = h(aa' \otimes b) = h(aa'b) \stackrel{iii}{=} h(aa') h(b) \stackrel{i}{=} h(a) h(a') h(b) \stackrel{iii}{=} h(a) h(a' \otimes b) \ .$$

The crucial case is vi:

$$h(ba) = h(b \to a) + h(b \leftarrow a) = f(b \to a) + g(b \leftarrow a) = g(b)f(a) = h(b)h(a)$$

by hypothesis.

Cases vii, viii and ix again reduce to the previous cases, because of the fact that $B \cdot A \subseteq A + B$. For instance, case viii is

$$\begin{aligned} h(b \cdot a \otimes b') &= h(bab') = h((b \to a)b') + h((b \leftarrow a)b') \\ &\overset{ii,iii}{=} h(b \to a)h(b') + h(b \leftarrow a)h(b') \\ &= h(b \to a + b \leftarrow a)h(b') = h(ba)h(b') \overset{vi}{=} h(b)h(a)h(b') \overset{iii}{=} h(b)h(ab') \ . \end{aligned}$$

□

Remark 6.6. Double algebras are studied in more detail in [A2]. In particular it is discussed under which conditions $A \bigcirc B$ is an ϵ-bialgebra.

7. Drinfeld's double for infinitesimal Hopf algebras

For ordinary Hopf algebras, the double $D(H)$ contains H and $H^{*^{op}}$ as subalgebras. The relevant version of the dual for (finite dimensional) ϵ-bialgebras turns out to be

$$A' := (A^*, \Delta^{*^{op}}, -m^{*^{cop}}) \ ,$$

as already considered on section 5. In terms of the given ϵ-bialgebra structure on (A, m, Δ), the structure on A' is:

$$(f \cdot g)(a) = g(a_1)f(a_2) \ \forall \ a \in A, \ f, g \in A' \text{ and} \tag{7.1}$$

$$\Delta(f) = f_1 \otimes f_2 \iff f(ab) = -f_2(a)f_1(b) \ \forall \ f \in A', \ a, b \in A \ . \tag{7.2}$$

Below we always refer to this structure when dealing with multiplications or comultiplications of elements of A'.

Proposition 7.1. *Consider the maps $A' \times A \to A$ and $A' \times A \to A'$:*

$$f \to a = f(a_1)a_2 \text{ and } f \leftarrow a = -f_2(a)f_1 \tag{7.3}$$

or equivalently

$$g(f \to a) = (gf)(a) \text{ and } (f \leftarrow a)(b) = f(ab) \ ; \tag{7.4}$$

in terms of the multiplication and comultiplication of A'. Then (A, A') is a double algebra.

Proof. We have to check the conditions in definition 6.1. First, it is clear that $f \leftarrow a$ defines a right A-module structure on A' and that $f \to a$ defines a left A'-module structure on A (i.e. the right A^*-module structure corresponding to the left A-module structure $a \mapsto a_1 \otimes a_2$). It remains to verify axioms (6.3) and (6.4). We have

$$\begin{aligned} f \to ab &\overset{(2.1),(7.3)}{=} f(ab_1)b_2 + f(a_1)a_2 b \\ &\overset{(7.4)}{=} (f \leftarrow a)(b_1)b_2 + f(a_1)a_2 b \overset{(7.3)}{=} (f \leftarrow a) \to b + (f \to a)b \end{aligned}$$

which proves (6.3). Similarly,

$$\begin{aligned}
(fg \leftarrow a)(b) &\overset{(7.4)}{=} (fg)(ab) \\
&\overset{(7.1),(2.1)}{=} g(ab_1)f(b_2) + g(a_1)f(a_2b) \overset{(7.4)}{=} (g \leftarrow a)(b_1)f(b_2) + f\Big(g(a_1)a_2b\Big) \\
&\overset{(7.1),(7.3)}{=} \Big(f(g \leftarrow a)\Big)(b) + f\Big((g \rightarrow a)b\Big) \overset{(7.4)}{=} \Big(f(g \leftarrow a) + f \leftarrow (g \rightarrow a)\Big)(b)
\end{aligned}$$

which proves (6.4). □

LEMMA 7.2. *Let A be a finite dimensional ϵ-bialgebra, $\{e_i\}$ be a linear basis of A and $\{f_i\}$ the dual basis of A'. Then $\forall\ a \in A$ and $f \in A'$,*

$$\sum_i f_i(a)e_i = a \ \text{ and } \ \sum_i f(e_i)f_i = f \tag{7.5}$$

$$\sum_i ae_i \otimes f_i = \sum_i e_i \otimes (f_i \leftarrow a) \tag{7.6}$$

$$\sum_i (f \rightarrow e_i) \otimes f_i = \sum_i e_i \otimes f_i f\ . \tag{7.7}$$

PROOF. Equations 7.5 are immediate from the definition of dual bases. To prove (7.6) we evaluate on $b \in A$:

$$\sum_i f_i(b)ae_i \overset{(7.5)}{=} ab \overset{(7.5)}{=} \sum_i f_i(ab)e_i \overset{(7.4)}{=} \sum_i (f_i \leftarrow a)(b)e_i\ ,$$

as needed. Similarly, to prove (7.7) we evaluate on $a \in A$:

$$\begin{aligned}
\sum_i (f_i f)(a)e_i &\overset{(7.1)}{=} \sum_i f(a_1)f_i(a_2)e_i \overset{(7.5)}{=} f(a_1)a_2 \overset{(7.3)}{=} f \rightarrow a \\
&\overset{(7.5)}{=} \Big(\sum_i f(e_i)f_i\Big) \rightarrow a = \sum_i f_i(a)(f \rightarrow e_i)\ ,
\end{aligned}$$

as needed. □

THEOREM 7.3. *Let A be a finite dimensional ϵ-bialgebra, consider the vector space*

$$D(A) := (A \otimes A') \oplus A \oplus A'$$

and denote the element $a \otimes f \in A \otimes A' \subseteq D(A)$ by $a \bowtie f$. Let $\{e_i\}$ be a linear basis of A and $\{f_i\}$ the dual basis of A'.

1. *$D(A)$ is an associative algebra with multiplication determined by*
   (a) *A and A' are subalgebras,*
   (b) *$a \cdot f = a \bowtie f$ and*
   (c) *$f \cdot a = f \rightarrow a + f \leftarrow a$.*
2. *Let*

$$r = -\sum_i e_i \otimes f_i \in A \otimes A' \subseteq D(A) \otimes D(A)\ .$$

   Then $(D(A), r)$ is a quasitriangular ϵ-bialgebra.
3. *The corresponding coassociative comultiplication on $D(A)$ is determined by*
   (d) *A and A' are subcoalgebras;*

(e) $\Delta(a \bowtie f) = (a \bowtie f_1) \otimes f_2 + a_1 \otimes (a_2 \bowtie f)$.

Proof. Part 1 follows immediately from propositions 7.1 and 6.2. To prove 2 we must show that (AYB) holds. We compute

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = \sum_{i,j} e_j e_i \otimes f_i \otimes f_j - \sum_{i,j} e_i \otimes f_i e_j \otimes f_j + \sum_{i,j} e_i \otimes e_j \otimes f_j f_i$$

$$\overset{(c)}{=} \sum_{i,j} e_j e_i \otimes f_i \otimes f_j - \sum_{i,j} e_i \otimes (f_i \leftarrow e_j) \otimes f_j - \sum_{i,j} e_i \otimes (f_i \rightarrow e_j) \otimes f_j + \sum_{i,j} e_i \otimes e_j \otimes f_j f_i$$

$$= \sum_{i,j} \Big(e_j e_i \otimes f_i - e_i \otimes (f_i \leftarrow e_j) \Big) \otimes f_j - \sum_{i,j} e_i \otimes \Big((f_i \rightarrow e_j) \otimes f_j - e_j \otimes f_j f_i \Big) = 0$$

by (7.6) (applied to $a = e_j$) and (7.7) (applied to $f = f_i$).

It only remains to check that $\Delta = \Delta_r$ verifies (d) and (e). Since Δ is a derivation, (e) follows from (b) and (d). Now, for any $a \in A$ we have

$$a \cdot r - r \cdot a = -\sum_i a e_i \otimes f_i + \sum_i e_i \otimes f_i a$$

$$\overset{(c)}{=} -\sum_i a e_i \otimes f_i + \sum_i e_i \otimes (f_i \leftarrow a) + \sum_i e_i \otimes (f_i \rightarrow a)$$

$$\overset{(7.6)}{=} \sum_i e_i \otimes (f_i \rightarrow a) \overset{(7.3)}{=} \sum_i e_i \otimes f_i(a_1) a_2 \overset{(7.5)}{=} a_1 \otimes a_2 = \Delta_A(a)\ .$$

This proves that A is a subcoalgebra of $D(A)$. Similarly, for any $f \in A'$,

$$f \cdot r - r \cdot f = -\sum_i f e_i \otimes f_i + \sum_i e_i \otimes f_i f$$

$$\overset{(c)}{=} -\sum_i (f \leftarrow e_i) \otimes f_i - \sum_i (f \rightarrow e_i) \otimes f_i + \sum_i e_i \otimes f_i f$$

$$\overset{(7.7)}{=} -\sum_i (f \leftarrow e_i) \otimes f_i \overset{(7.3)}{=} \sum_i f_2(e_i) f_1 \otimes f_i \overset{(7.5)}{=} f_1 \otimes f_2 = \Delta_{A'}(f)\ .$$

This proves that A' is a subcoalgebra of $D(A)$. Thus (d) holds and the proof is complete. □

Proposition 7.4. *Let A be a finite dimensional ϵ-Hopf algebra with bijective antipode S_A. Then $D(A)$ is an ϵ-Hopf algebra with antipode S determined by*

$$S(a) = S_A(a),\ S(f) = fS_A^{-1} \text{ and } S(a \bowtie f) = -S_A(a) \bowtie fS_A^{-1}\ \forall\ a \in A,\ f \in A'.$$

Proof. As already noted in the proof of corollary 5.7, A' is an ϵ-Hopf algebra with antipode $S_{A'} = (S_A^{-1})^*$. Thus, we only need to verify the antipode axioms (A) for S on an element of the form $\alpha := a \bowtie f$. Referring to the conditions in theorem

7.3 we have

$$
\begin{aligned}
S(\alpha_1)\alpha_2 + S(\alpha) + \alpha &\overset{(e)}{=} S(a \bowtie f_1)f_2 + S_A(a_1)(a_2 \bowtie f) + S(a \bowtie f) + a \bowtie f \\
&= -\Big(S_A(a) \bowtie S_{A'}(f_1)\Big)f_2 + S_A(a_1)(a_2 \bowtie f) - S_A(a) \bowtie S_{A'}(f) + a \bowtie f \\
&\overset{(a),(b)}{=} - S_A(a) \bowtie S_{A'}(f_1)f_2 + S_A(a_1)a_2 \bowtie f - S_A(a) \bowtie S_{A'}(f) + a \bowtie f \\
&= S_A(a) \bowtie f - S_A(a) \bowtie f = 0
\end{aligned}
$$

by the antipode axioms for S_A and $S_{A'}$. Similarly one checks that

$$\alpha_1 S(\alpha_2) + \alpha + S(\alpha) = 0 \ .$$

□

Our last result shows that every quasitriangular ϵ-bialgebra is a quotient of its double. This is a familiar property of ordinary quasitriangular bialgebras.

Proposition 7.5. *Let A be a finite dimensional ϵ-bialgebra. Then A is quasitriangular if and only if the canonical inclusion $A \hookrightarrow D(A)$ splits as a morphism of ϵ-bialgebras.*

Proof. Suppose that (A, r) is quasitriangular for some $r = \sum u_i \otimes v_i \in A \otimes A$; let $\Delta = \Delta_r$. Define $\pi_r : D(A) \to A$ by

$$(*) \qquad \pi_r(a) = a, \ \pi_r(f) = -\sum_i f(u_i)v_i \text{ and } \pi_r(a \bowtie f) = -\sum_i f(u_i)av_i \ .$$

To show that π_r is a morphism of algebras, according to the universal property of double algebras, we only need to check that

$$\pi_r(f)\pi_r(a) = \pi_r(f \rightarrow a) + \pi_r(f \leftarrow a) \ \forall \ a \in A, \ f \in A' \ .$$

We have that

$$
\begin{aligned}
a_1 \otimes a_2 = \Delta(a) = a \cdot r - r \cdot a &= \sum_i a u_i \otimes v_i - \sum_i u_i \otimes v_i a \\
\Rightarrow f(a_1)a_2 &= \sum_i f(a u_i)v_i - \sum_i f(u_i)v_i a \\
\overset{(7.3),(7.4)}{\Rightarrow} f \rightarrow a &= \sum_i (f \leftarrow a)(u_i)v_i - \sum_i f(u_i)v_i a \\
\overset{(*)}{\Rightarrow} \pi_r(f \rightarrow a) &= -\pi_r(f \leftarrow a) + \pi_r(f)\pi_r(a)
\end{aligned}
$$

as needed.

Finally, to show that π_r is a morphism of coalgebras, it suffices to check that $\pi_{r|_A}$ and $\pi_{r|_{A'}}$ are morphisms of coalgebras, since A and A' generate $D(A)$ as an algebra (this is a general property of ϵ-bialgebras). Obviously $\pi_{r|_A}$ is a morphism of coalgebras. If $f \in A'$, then

$$
\begin{aligned}
&(\pi_r \otimes \pi_r)\Delta(f) \overset{(*)}{=} \sum_{i,j} f_1(u_i)v_i \otimes f_2(u_j)v_j \overset{(7.2)}{=} -\sum_{i,j} f(u_j u_i)v_i \otimes v_j \\
&= -(f \otimes id \otimes id)(r_{13}r_{12}) \overset{(5.3)}{=} -(f \otimes id \otimes id)(id \otimes \Delta)(r) = -\sum_i f(u_i)\Delta(v_i) \overset{(*)}{=} \Delta\pi_r(f)
\end{aligned}
$$

as needed.

For the converse it follows immediately from theorem 7.3 that if $\pi : D(A) \to A$ is a section of $A \hookrightarrow D(A)$, then letting

$$r := \pi \otimes \pi(-\sum_i e_i \otimes f_i) \in A \otimes A$$

one obtains a quasitriangular ϵ-bialgebra (A, r). $\square$

REMARK 7.6. The morphism $\lambda_r : A' \to A$ of proposition 5.6 is the composition of the map $'A \to A'$, $a \to -a$, (an isomorphism of ϵ-bialgebras), the canonical inclusion $A' \hookrightarrow D(A)$ and the morphism $\pi_r : D(A) \to A$ in the proof of proposition 7.5.

Modules over the double admit a simple description in terms of modules and comodules over the original ϵ-bialgebra. This will be detailed in [A2].

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