

THE STEINBERG TORUS OF A WEYL GROUP AS A MODULE OVER THE COXETER COMPLEX

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ABSTRACT. Associated to each irreducible crystallographic root system Φ , there is a certain cell complex structure on the torus obtained as the quotient of the ambient space by the coroot lattice of Φ . This is the Steinberg torus. A main goal of this paper is to exhibit a module structure on (the set of faces of) this complex over the (set of faces of the) Coxeter complex of Φ . The latter is a monoid under the Tits product of faces. The module structure is obtained from geometric considerations involving affine hyperplane arrangements. As a consequence, a module structure is obtained on the space spanned by affine descent classes of a Weyl group, over the space spanned by ordinary descent classes. The latter constitute a subalgebra of the group algebra, the classical descent algebra of Solomon. We provide combinatorial models for the module of faces when Φ is of type A or C .

CONTENTS

Introduction	1
1. Hyperplane arrangements. Faces: mundane and celestial	2
2. Root systems, Coxeter complexes and Steinberg tori	7
3. Solomon's descent ring and the module of affine descent classes	16
4. Combinatorial models	24
References	36

INTRODUCTION

Let Φ be a root system and W the associated Coxeter group. The *descent set* of an element $w \in W$ keeps track of the simple roots whose image under w is negative. When Φ is irreducible and crystallographic, a notion of *affine descent* may be defined. This is due to Cellini [11, Section 2]. The affine descent set enlarges the ordinary descent set by recording the behavior of w on the (unique) highest root of Φ .

Lumping group elements according to ordinary descent sets leads to *Solomon's descent algebra* (or ring) [24], denoted in this paper by $\text{Sol}(\Phi)$. It is a subring of the group ring $\mathbb{Z}W$. One goal of this paper is to arrive at a related algebraic structure on the additive subgroup of $\mathbb{Z}W$ spanned by affine descent classes and denoted $\overline{\text{Sol}}(\Phi)$. We show in Theorem 11 that the multiplication of $\mathbb{Z}W$ turns $\overline{\text{Sol}}(\Phi)$ into a left module over $\text{Sol}(\Phi)$.

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This module structure is part of a family of such structures introduced by Moszkowski [20]; see Remark 10. Our main interest lies however in more general geometric considerations. We follow the approach of Tits (in his appendix [26] to Solomon's paper [24]), as developed by Bidigare [6] and Brown [9, Section 4.8]. These works relate the algebraic structure of $\text{Sol}(\Phi)$ to the geometric structure of the Coxeter complex $\Sigma(\Phi)$. This relation is based on the following key points:

- (i) the set $\Sigma(\Phi)$ is a monoid under the Tits product;
- (ii) the group W acts on $\Sigma(\Phi)$ and the W -orbits are in bijection with descent sets;
- (iii) $\text{Sol}(\Phi)$ is (anti-isomorphic to) the subring of W -invariants in the monoid ring $\mathbb{Z}\Sigma(\Phi)$.

Work of Dilks, Petersen, and Stembridge [12] uncovered a relation analogous to (ii) between affine descents and the structure of the *Steinberg torus* $\overline{\Sigma}(\Phi)$. This cell complex was introduced by Steinberg in [25]. It is obtained as the quotient of the affine Coxeter complex $\widetilde{\Sigma}(\Phi)$ by the coroot lattice $\mathbb{Z}\Phi^\vee$. We provide analogs of the remaining points. We show that $\text{Sol}(\Phi)$ acts on $\widetilde{\Sigma}(\Phi)$, and that this action passes through the quotient by $\mathbb{Z}\Phi^\vee$ to an action on the Steinberg torus $\overline{\Sigma}(\Phi)$. Finally, we show that the elements of the module $\overline{\text{Sol}}(\Phi)$ arise as the W -invariants in the module $\mathbb{Z}\overline{\Sigma}(\Phi)$.

The action on affine faces can be defined in the general context of real affine hyperplane arrangements. Such an arrangement splits the ambient space into a set of affine faces. The hyperplane *at infinity* is similarly decomposed into a set of *celestial* faces. The latter set is a semigroup under the Tits product and the former is a right module over it. In the case of the affine arrangement of Φ , celestial faces constitute the Coxeter complex $\Sigma(\Phi)$ and affine faces constitute $\widetilde{\Sigma}(\Phi)$.

The contents of the paper are as follows. Celestial faces and other geometric aspects are discussed in Section 1. Section 2 provides background on both finite and affine Coxeter complexes and arrives at the module structure on the set of faces of the Steinberg torus. Section 3 relates the geometric actions to the module structures over the descent algebra. Section 4 reviews the known combinatorial models for the Coxeter complex when Φ is of type A or C , and introduces analogous models for the Steinberg torus of such root systems. They involve certain cyclic partitions that we call *spin necklaces* in type A , and similar structures in type C . The partial order among faces as well as the product of faces are described in these terms.

We thank Ken Brown for illuminating discussions that got this project underway. The idea of employing celestial faces is due to him.

1. HYPERPLANE ARRANGEMENTS. FACES: MUNDANE AND CELESTIAL

We give some general background on hyperplane arrangements, following [9, Appendix A]. We center around the notion of face of an arrangement and the Tits product of faces. We discuss the notion of celestial face and focus on the construction of a right module structure on the set of mundane faces over the semigroup of celestial faces.

The notion of celestial face may not have appeared explicitly in the literature before, but closely related ideas appear in [1, Section 11.8], in the context of buildings.

1.1. Hyperplane arrangements and the Tits product. Let V be a finite-dimensional real vector space. A *hyperplane* H is an affine subspace of V of codimension 1. A *hyperplane arrangement* \mathcal{H} in V is a collection of hyperplanes in V (not necessarily finite).

For each $H \in \mathcal{H}$ we choose a nonzero affine functional f_H so that $H = \{\lambda \in V \mid f_H(\lambda) = 0\}$. The positive and negative *halfspaces* of H are defined by

$$H^+ := \{\lambda \in V \mid f_H(\lambda) > 0\} \quad \text{and} \quad H^- := \{\lambda \in V \mid f_H(\lambda) < 0\}.$$

Also, let $H^0 := H$.

The hyperplane arrangement \mathcal{H} partitions V into a collection of nonempty disjoint convex polyhedral sets called *faces*, given by intersections of hyperplanes and their halfspaces. A face F is uniquely determined by its *sign vector*:

$$\sigma(F) = (\sigma_H(F))_{H \in \mathcal{H}},$$

where $\sigma_H(F) \in \{+, -, 0\}$ is the sign of f_H on F (the sign of $f_H(\lambda)$ on any point $\lambda \in F$). Indeed,

$$F = \bigcap_{H \in \mathcal{H}} H^{\sigma_H(F)}.$$

A sign sequence $(\sigma_H)_{H \in \mathcal{H}}$ is the sign vector of a face if and only if $\bigcap_{H \in \mathcal{H}} H^{\sigma_H} \neq \emptyset$.

We denote by $\Sigma(\mathcal{H})$ the collection of faces of \mathcal{H} . There is a partial order on faces, given by $F \leq G \Leftrightarrow \overline{F} \subseteq \overline{G}$; that is, if the closure of F is contained in the closure of G . In terms of sign vectors, this can be stated as: $F \leq G$ if and only if for each $H \in \mathcal{H}$ either $\sigma_H(F) = 0$ or $\sigma_H(F) = \sigma_H(G)$. Maximal faces C are called *chambers*, and they are characterized by the fact that $\sigma_H(C) \neq 0$ for all $H \in \mathcal{H}$. Let $\mathcal{C}(\mathcal{H})$ denote the set of chambers.

Let F and G be faces of a hyperplane arrangement \mathcal{H} , and choose any two points $\lambda \in F$ and $\lambda' \in G$. We say the ordered pair (F, G) is *multiplicable*, if there is $\epsilon > 0$ such that the line segment $\{(1-t)\lambda + t\lambda' \mid 0 < t < \epsilon\}$ crosses only finitely many hyperplanes in \mathcal{H} . If (F, G) is a multiplicable pair, their *Tits product* FG is the first face of \mathcal{H} entered upon traveling a small positive distance along the line $(1-t)\lambda + t\lambda'$. (That is, there is some $\epsilon > 0$ such that for all $0 < t < \epsilon$, we have $(1-t)\lambda + t\lambda' \in FG$.) Both notions are independent of the choice of points λ and λ' . An example is shown in Figure 1.

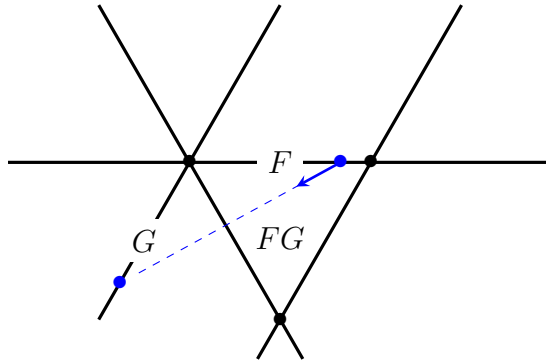


FIGURE 1. The product of faces in a line arrangement.

The product of two multiplicable faces admits a simple description in terms of sign vectors. Indeed, for small positive ϵ ,

$$f_H((1-\epsilon)\lambda + \epsilon\lambda') = (1-\epsilon)f_H(\lambda) + \epsilon f_H(\lambda')$$

has the same sign as $f_H(\lambda)$ unless $f_H(\lambda) = 0$, in which case it adopts the sign of $f_H(\lambda)$. In other words, we have

$$(1) \quad \sigma_H(FG) = \begin{cases} \sigma_H(F) & \text{if } \sigma_H(F) \neq 0, \\ \sigma_H(G) & \text{if } \sigma_H(F) = 0. \end{cases}$$

It follows that when defined, the Tits product is associative. Moreover, if $C \in \mathcal{C}(\mathcal{H})$ is a chamber and $F \in \Sigma(\mathcal{H})$ is any face, then FC is also a chamber, called the *Tits projection* of F onto C . Also, $CF = C$.

We say that \mathcal{H} is *locally finite* if every $\lambda \in V$ has a neighborhood that intersects only finitely many hyperplanes in \mathcal{H} . In this case, all pairs of faces of \mathcal{H} are multiplicable, and the Tits product gives $\Sigma(\mathcal{H})$ the structure of a semigroup. Moreover, the set $\mathcal{C}(\mathcal{H})$ is a two-sided ideal of $\Sigma(\mathcal{H})$, with the right action being trivial. In particular, all this holds if \mathcal{H} is finite.

If the intersection of all hyperplanes in \mathcal{H} is nonempty, this intersection is a face which is the unit for the Tits product, and $\Sigma(\mathcal{H})$ is in this case a monoid. This holds in particular if all hyperplanes in \mathcal{H} are linear.

For more details, see [1, Sections 1.4 and 10.1], [9, Appendix A], or [10, Section 2].

1.2. The celestial sphere. A *ray* r in V is a subset of the form $r = f(\mathbb{R}_+)$, where $f : \mathbb{R} \rightarrow V$ is an affine transformation. The *base* of the ray is $r_0 := f(0)$. Two rays r and s are *parallel* if there exists $\lambda \in V$ such that $r = \lambda + s$ (equality of subsets). A *celestial point* is a parallelism class of rays. The *celestial sphere* in V is the set $S_\infty(V)$ of celestial points of V . Let $[r] \in S_\infty(V)$ denote the parallelism class of a ray r in V . See Figure 2.

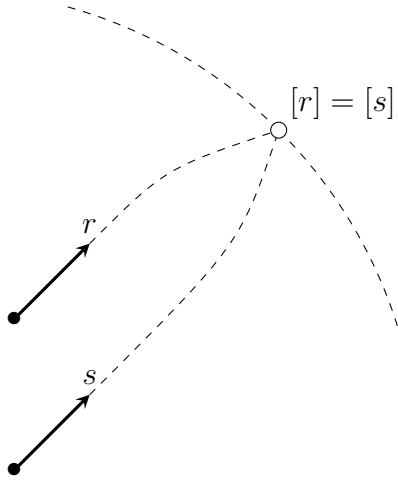


FIGURE 2. A celestial point.

Let H be a hyperplane in V . For each $\sigma \in \{+, -, 0\}$, let

$$H_\infty^\sigma := \{[r] \in S_\infty(V) \mid r \text{ is a ray with } r_0 \in H \text{ and } r \subseteq H^\sigma\}.$$

In other words, H_∞^σ consists of those celestial points that can be represented by a ray based on H and contained in the halfspace H^σ (hyperplane H , if $\sigma = 0$). We may refer to H_∞^+ and H_∞^- as the positive and negative *halfspheres* determined by H .

Let \mathcal{H} be a hyperplane arrangement in V . A *celestial face* is a nonempty subset of $S_\infty(V)$ of the form

$$F = \bigcap_{H \in \mathcal{H}} H_\infty^{\sigma_H},$$

where $(\sigma_H)_{H \in \mathcal{H}}$ is a sign sequence. Let $\Sigma_\infty(\mathcal{H})$ denote the collection of celestial faces. It partitions the celestial sphere. When warranted, we refer to the faces in $\Sigma(\mathcal{H})$ as *mundane*.

The product of two celestial faces F and G is defined as follows. Choose rays r and s with a common base such that $[r] \in F$ and $[s] \in G$. Assume first that r and s are neither equal nor opposite. Let $f : \mathbb{C} \rightarrow V$ be an affine map that sends the positive real axis to r and the positive imaginary axis to s . The product FG is the celestial face that contains the celestial points $[f(e^{it})]$ for all $0 < t < \epsilon$, for some $\epsilon > 0$. (More plainly, we rotate r towards s a small positive angle. Of the two directions of rotation, we choose the one that stays in the convex sector determined by r and s .) If r and s are equal or opposite, we stay put and $FG = F$. For this to be well-defined, we require that once all hyperplanes in \mathcal{H} are translated to the common base of r and s , only finitely many of them intersect the sector $\{f(re^{it}) \mid 0 < t < \epsilon\}$.

Equivalently, FG is the first celestial face entered by walking a small positive distance from a celestial point in F to one in G , along a maximal celestial circle. See Figure 3.

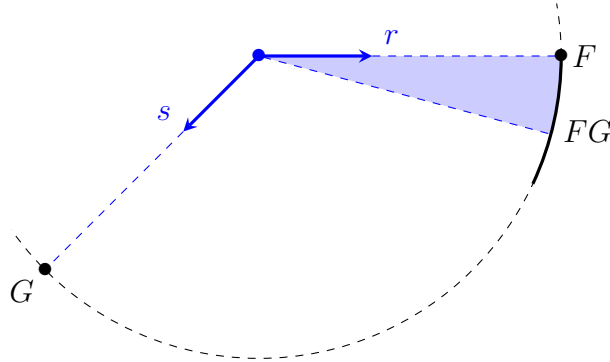


FIGURE 3. The product of two celestial faces.

It is also possible to multiply a mundane face with a celestial face, in that order. Let $F \in \Sigma(\mathcal{H})$ and $G \in \Sigma_\infty(\mathcal{H})$. The product $FG \in \Sigma(\mathcal{H})$ is the mundane face entered upon traveling a small positive distance starting from a point in F and heading in the direction of a celestial point contained in G . A local condition similar to the preceding ones guarantees the existence of this product. See Figure 4.

Proposition 1. *Assume that \mathcal{H} is locally finite and has finitely many parallelism classes. Then the preceding products are globally defined, the set $\Sigma_\infty(\mathcal{H})$ is a semigroup, and the set $\Sigma(\mathcal{H})$ is a right module over it.*

Proof. We resort to the *cone* on \mathcal{H} . This is the linear arrangement $\widehat{\mathcal{H}}$ in $\widehat{V} := \mathbb{R} \oplus V$ consisting of the linear hyperplane $H_0 := \{(0, v) \mid v \in V\}$ together with the linear hyperplanes $\widehat{H} := \ker(\widehat{f}_H)$, for $H \in \mathcal{H}$, where \widehat{f}_H is the unique linear functional on \widehat{V} such that $\widehat{f}_H(1, v) = f_H(v)$.

The *induced* linear arrangement $\widehat{\mathcal{H}}_0 := \{H_0 \cap \widehat{H} \mid H \in \mathcal{H}\}$ (in the space V) is finite, since two parallel hyperplanes in \mathcal{H} have the same intersection with H_0 . Hence $\Sigma(\widehat{\mathcal{H}}_0)$ is a monoid.

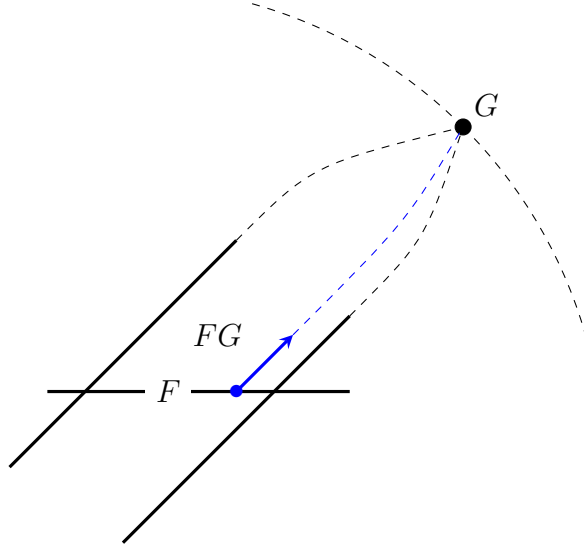


FIGURE 4. The product of a mundane face and a celestial face.

Now, save for its central face, the set $\Sigma(\widehat{\mathcal{H}}_0)$ is in bijection with $\Sigma_\infty(\mathcal{H})$, in such a way that the Tits product corresponds to the product of celestial faces. Thus, $\Sigma_\infty(\mathcal{H})$ is a semigroup.

Note that $\Sigma(\widehat{\mathcal{H}}_0)$ consists of the faces of $\widehat{\mathcal{H}}$ contained in H_0 . On the other hand, let $\Sigma_+(\widehat{\mathcal{H}})$ denote the subset of $\Sigma(\widehat{\mathcal{H}})$ consisting of those faces contained in the positive halfspace of H_0 . It is in bijection with $\Sigma(\mathcal{H})$. The arrangement $\widehat{\mathcal{H}}$ need not be locally finite, but the product of a face in $\Sigma_+(\widehat{\mathcal{H}})$ with a face in $\Sigma(\widehat{\mathcal{H}}_0)$ (in that order) is well-defined, since \mathcal{H} is locally finite. Moreover, the two bijections in question intertwine the right action of $\Sigma(\widehat{\mathcal{H}}_0)$ on $\Sigma_+(\widehat{\mathcal{H}})$ with the right action of $\Sigma_\infty(\mathcal{H})$ on $\Sigma(\mathcal{H})$. Since the former is associative (as an instance of the Tits product in $\Sigma(\widehat{\mathcal{H}})$), $\Sigma(\mathcal{H})$ is a right module over $\Sigma_\infty(\mathcal{H})$. \square

Figure 5 illustrates the discussion in Proposition 1 in the case \mathcal{H} is an arrangement of points in a one-dimensional space.

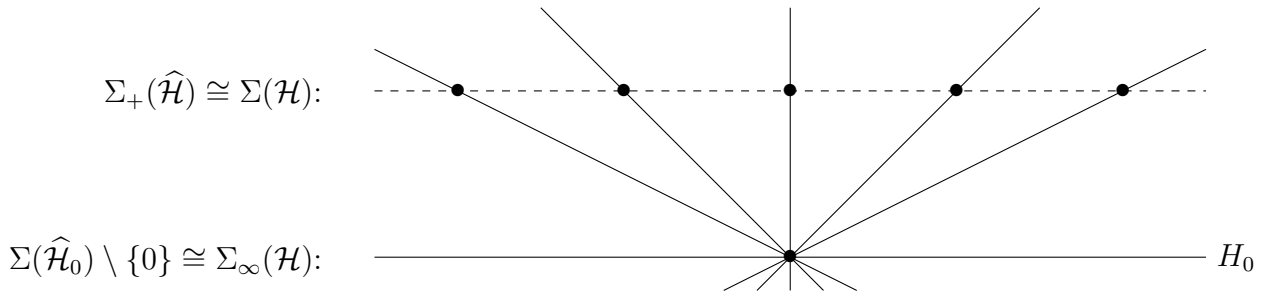


FIGURE 5. The cone of a rank 1 arrangement.

The product GF of a celestial face $G \in \Sigma_\infty(\mathcal{H})$ and a mundane face $F \in \Sigma(\mathcal{H})$, in that order, generally is not defined. Note that the corresponding pair of faces in $\Sigma(\widehat{\mathcal{H}})$ are not multiplicable, since any neighborhood of G intersects infinitely many hyperplanes in $\widehat{\mathcal{H}}$. See Figure 5.

2. ROOT SYSTEMS, COXETER COMPLEXES AND STEINBERG TORI

We turn to hyperplane arrangements associated to root systems. Our ultimate goal in this section is to arrive at a right module structure on the set of faces of the Steinberg torus (of an irreducible crystallographic root system) over the monoid of faces of the finite Coxeter arrangement. These notions are reviewed along the way; additional background may be found in [1, 12, 16].

Throughout, V denotes a Euclidean vector space, with inner product $\langle \cdot, \cdot \rangle$.

2.1. The finite Coxeter arrangement. Let Φ be a root system in V , in the sense of [16, Section 1.2] (a generalized root system in the sense of [1, Definition 1.5]). Let Δ be a set of simple roots. Every root $\beta \in \Phi$ belongs either to the nonnegative span of Δ and is designated *positive*, or to the nonpositive span of Δ and is designated *negative*. We write $\beta > 0$ or $\beta < 0$ accordingly. Let $\Pi = \{\beta \in \Phi \mid \beta > 0\}$ denote the set of positive roots.

For any root $\beta \in \Phi$, let

$$H_\beta := \{\lambda \in V \mid \langle \lambda, \beta \rangle = 0\}$$

be the hyperplane orthogonal to β . The set of hyperplanes

$$\mathcal{H}(\Phi) := \{H_\beta \mid \beta \in \Phi\}$$

is the *Coxeter arrangement* associated to Φ . Since $H_\beta = H_{-\beta}$, we have $\mathcal{H}(\Phi) := \{H_\beta \mid \beta \in \Pi\}$.

Let $\Sigma := \Sigma(\mathcal{H}(\Phi))$ be the set of faces of the arrangement $\mathcal{H}(\Phi)$, and $\mathcal{C} := \mathcal{C}(\mathcal{H}(\Phi))$ the subset of chambers. Since $\mathcal{H}(\Phi)$ is finite and linear, Σ is a monoid under the Tits product. The set \mathcal{C} is a two-sided ideal, with trivial right action.

The *dominant* or *fundamental* chamber is

$$(2) \quad C_\emptyset := \{\lambda \in V \mid \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta\}.$$

The faces of C_\emptyset are the sets of the form

$$(3) \quad C_J := \{\lambda \in V \mid \langle \lambda, \alpha \rangle = 0 \text{ for } \alpha \in J, \langle \lambda, \alpha \rangle > 0 \text{ for } \alpha \in \Delta \setminus J\},$$

where J is a subset of Δ .

Figure 6 illustrates the situation for the root system $\Phi = A_2$ and Figure 7 for $\Phi = C_2$.

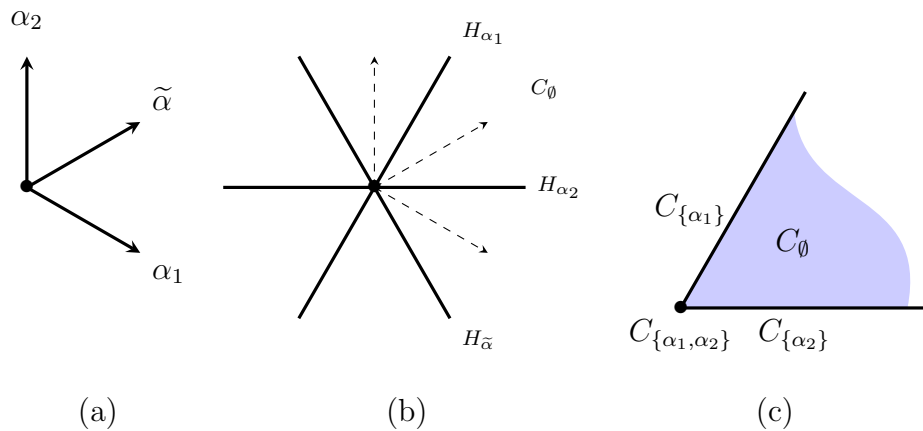


FIGURE 6. The root system A_2 . (a) $\Delta = \{\alpha_1, \alpha_2\}$ and $\Pi = \Delta \cup \{\tilde{\alpha}\}$. (b) The arrangement $\mathcal{H}(A_2)$. (c) The faces of the fundamental chamber.

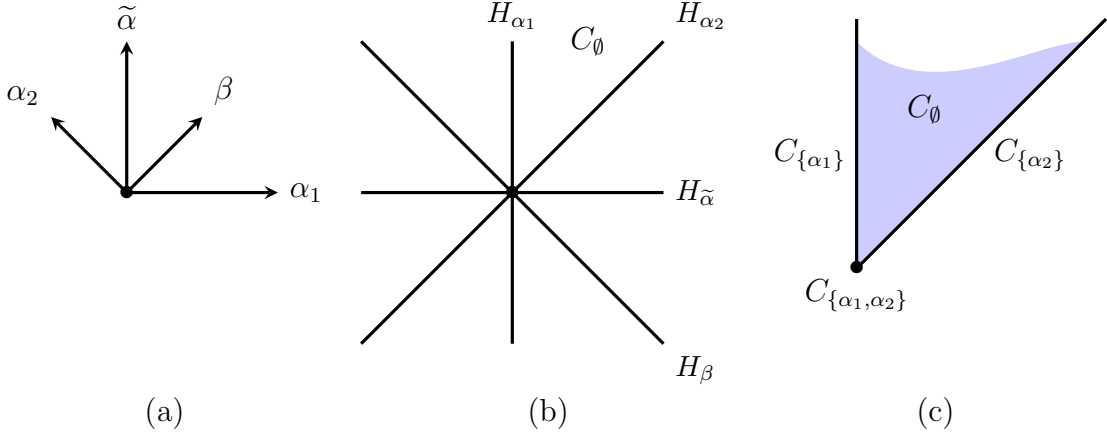


FIGURE 7. The root system C_2 . (a) $\Delta = \{\alpha_1, \alpha_2\}$ and $\Pi = \Delta \cup \{\beta, \tilde{\alpha}\}$. (b) The arrangement $\mathcal{H}(C_2)$. (c) The faces of the fundamental chamber.

For each $\beta \in \Phi$, let s_β denote the orthogonal reflection through H_β . Let W be the subgroup of $\text{GL}(V)$ generated by these reflections. This is the *Coxeter group* of Φ . It permutes the hyperplanes in $\mathcal{H}(\Phi)$ and therefore acts on the set Σ . We let $w \cdot F$ denote the action of $w \in W$ on $F \in \Sigma$.

The closure of the fundamental chamber is a strict fundamental domain for the action of W on V [1, Theorem 1.104]. Therefore, every face F in Σ is in the orbit of a unique face C_J of C_\emptyset . We let

$$\text{col}(F) := \Delta \setminus J$$

denote the complement of the unique subset of Δ such that

$$(4) \quad F = w \cdot C_J$$

for some $w \in W$. We call $\text{col}(F)$ the *color set* of F . Proposition 2 below provides information on the uniqueness of w .

The action of W on \mathcal{C} is simply-transitive [1, Theorem 1.69]. Therefore, given a face F there is a unique $w \in W$ such that

$$(5) \quad w \cdot C_\emptyset = FC_\emptyset.$$

We let

$$w_F := w$$

denote this element of W . In other words, acting with the group element w_F on the fundamental chamber has the same effect as projecting the face F onto that chamber.

A face $F \in \Sigma$ has sign vector $\sigma(F) = (\sigma_\beta(F))_{\beta \in \Pi}$, where $\sigma_\beta(F) \in \{+, -, 0\}$ is the sign of $\langle \lambda, \beta \rangle$, λ being any point in F .

The face C_Δ has sign vector $(0, 0, \dots, 0)$. It is the intersection of all the hyperplanes, and hence the unit for the Tits product. The fundamental chamber C_\emptyset has sign vector $(+, +, \dots, +)$. More generally, for a positive root β and $\lambda \in C_J$, we have $\langle w(\lambda), \beta \rangle =$

$\langle \lambda, w^{-1}(\beta) \rangle$, and hence

$$(6) \quad \sigma_\beta(w \cdot C_J) = \begin{cases} 0 & \text{if } w^{-1}\beta \in \text{span}\{\alpha \mid \alpha \in J\}, \\ + & \text{if } w^{-1}\beta \in \Pi \setminus \text{span}\{\alpha \mid \alpha \in J\}, \\ - & \text{if } -w^{-1}\beta \in \Pi \setminus \text{span}\{\alpha \mid \alpha \in J\}. \end{cases}$$

Let

$$D(w) = \{\alpha \in \Delta \mid w(\alpha) < 0\}.$$

This is the *descent set* of w , which will be discussed at length in Section 3.

Proposition 2. *Let $F \in \Sigma$ be a face. The element w_F is the unique $w \in W$ such that*

$$(7) \quad F = w \cdot C_J \quad \text{and} \quad D(w) \subseteq \text{col}(F).$$

where $J = \Delta \setminus \text{col}(F)$.

Proof. We first show that w_F fulfills (7). Since F is a face of $FC_\emptyset = w_F \cdot C_\emptyset$, there exists $I \subseteq \Delta$ such that $F = w_F \cdot C_I$. By the uniqueness in (4), we must have $I = J$. Now suppose there exists $\alpha \in D(w_F) \cap J$. Since $\alpha \in D(w_F)$, we have $w_F\alpha = -\beta$ for some positive root β . Since $\alpha \in J$, we have from (6) that

$$\sigma_\beta(F) = \sigma_\beta(w_F \cdot C_J) = 0.$$

Hence, from (1), we have

$$\sigma_\beta(FC_\emptyset) = \sigma_\beta(C_\emptyset) = +.$$

On the other hand, again from (6),

$$\sigma_\beta(w_F \cdot C_\emptyset) = -.$$

This contradicts $FC_\emptyset = w_F \cdot C_\emptyset$. Thus, $D(w_F) \subseteq \Delta \setminus J$.

It remains to establish uniqueness. Suppose $w \in W$ satisfies (7). Since $D(w) \subseteq \text{col}(F)$, we may apply [9, Proposition 4] to the chambers C_\emptyset and $w \cdot C_\emptyset$ to conclude that $FC_\emptyset = w \cdot C_\emptyset$. Hence $w = w_F$ by (5). \square

The rank 1 faces of Σ are of the form $w \cdot C_{\Delta \setminus \{\alpha\}}$ where $w \in W$ and $\alpha \in \Delta$. If we assign color α to all such faces, we obtain a *balanced coloring* of Σ ; i.e., every chamber of Σ has exactly one rank 1 face of each color [1, Proposition 1.128]. The set $\text{col}(F)$ defined by (4) is the set of colors of the rank 1 faces of a face F . Thus, the face $w \cdot C_J$ has color set $\Delta \setminus J$.

2.2. Coxeter complexes. Let $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of simple reflections. The group W is generated by S and in fact (W, S) is a *Coxeter system*.

Let $W_J := \langle s \mid s \in J \rangle$ denote the *parabolic* subgroup of W generated by the subset $J \subseteq S$. The Coxeter complex of (W, S) is the abstract simplicial complex whose faces are cosets of parabolic subgroups,

$$\Sigma(W, S) := \{wW_J \mid w \in W, J \subseteq S\},$$

with inclusion of faces given by containment of subsets of W , i.e.,

$$wW_J \leq vW_K \Leftrightarrow wW_J \supseteq vW_K.$$

In particular, $W_S = W$ corresponds to the empty face, as it contains all cosets. The facets (maximal faces) of $\Sigma(W, S)$ correspond to the singletons $wW_\emptyset = \{w\}$, and are thus indexed by elements of W .

We have that

$$w \cdot C_J \subseteq v \cdot C_K \iff wW_J \supseteq vW_K.$$

This defines an order-preserving bijection $\Sigma \leftrightarrow \Sigma(W, S)$ between Σ and the Coxeter complex $\Sigma(W, S)$ [1, Theorem 1.111]. Chambers in Σ correspond to facets in $\Sigma(W, S)$, and rank 1 faces correspond to vertices. From now on, we refer to Σ as the Coxeter complex.

For more details on finite Coxeter arrangements and complexes, see [1, Sections 1.5 and 1.6].

2.3. The affine Coxeter arrangement. We assume from now on that the root system Φ is crystallographic (so W is a Weyl group) and irreducible. For definitions, see [1, Appendix B] or [16, Section 2.9].

Such a system has an associated *affine Weyl group* \widetilde{W} . This is the group generated by the reflections $s_{\beta, k}$ through the affine hyperplanes

$$(8) \quad H_{\beta, k} := \{\lambda \in V : \langle \lambda, \beta \rangle = k\} \quad (\beta \in \Phi, k \in \mathbb{Z}).$$

Let Φ^\vee be the set of *coroots*

$$\beta^\vee := 2\beta / \langle \beta, \beta \rangle$$

($\beta \in \Phi$). The additive subgroup of V it generates is the *coroot lattice* $\mathbb{Z}\Phi^\vee$. Composing two reflections $s_{\beta, k}$ corresponding to the same β results in a translation by a vector in $\mathbb{Z}\Phi^\vee$; see Figure 8. Thus, \widetilde{W} contains $\mathbb{Z}\Phi^\vee$. It also contains the finite Weyl group W , as this consists of the reflections across the hyperplanes H_β . The crystallographic condition guarantees that the action of W on V stabilizes $\mathbb{Z}\Phi^\vee$, and \widetilde{W} identifies with the semidirect product $\mathbb{Z}\Phi^\vee \rtimes W$. The product in the latter group is

$$(\mu, w) \cdot (\mu', w') = (\mu + w(\mu'), ww').$$

The action of \widetilde{W} on V extends the action of W by linear reflections and the action of $\mathbb{Z}\Phi^\vee$ by translations:

$$(\mu, w) \cdot \lambda = \mu + w(\lambda),$$

for $\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, and $\lambda \in V$.

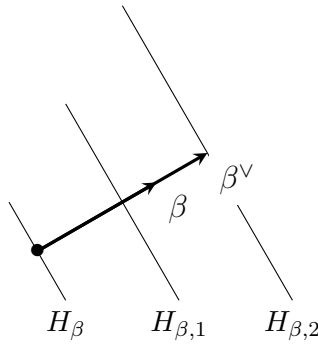


FIGURE 8. A root and its coroot.

Moreover, Φ has a unique *highest root* $\tilde{\alpha}$, the group \widetilde{W} is generated by $\tilde{S} := S \cup \{s_{\tilde{\alpha}, 1}\}$, and $(\widetilde{W}, \tilde{S})$ is an irreducible Coxeter system [16, Sections 4.3 and 4.6]. For the root systems A_2 and C_2 , the coroots corresponding to the positive roots are shown in Figures 9(a) and 10(a).

For the root system A_n , we have $\alpha^\vee = \alpha$ for all $\alpha \in A_n$. For the root system C_n , some coroots are half the size of the corresponding root, others are equal.

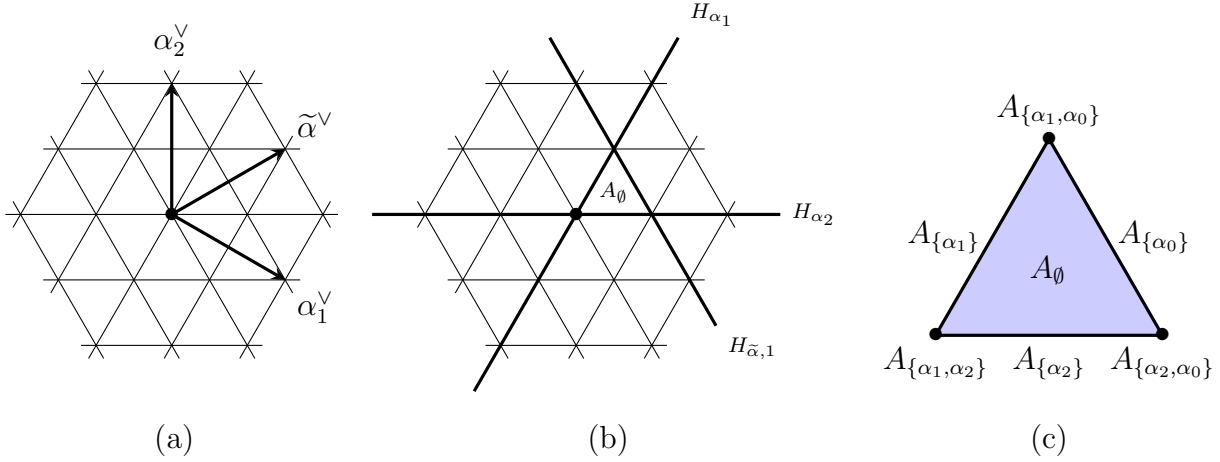


FIGURE 9. The affine arrangement $\tilde{\mathcal{H}}(A_2)$. (a) Positive (co)roots. (b) Affine hyperplanes and the fundamental alcove. (c) The faces of the fundamental alcove.

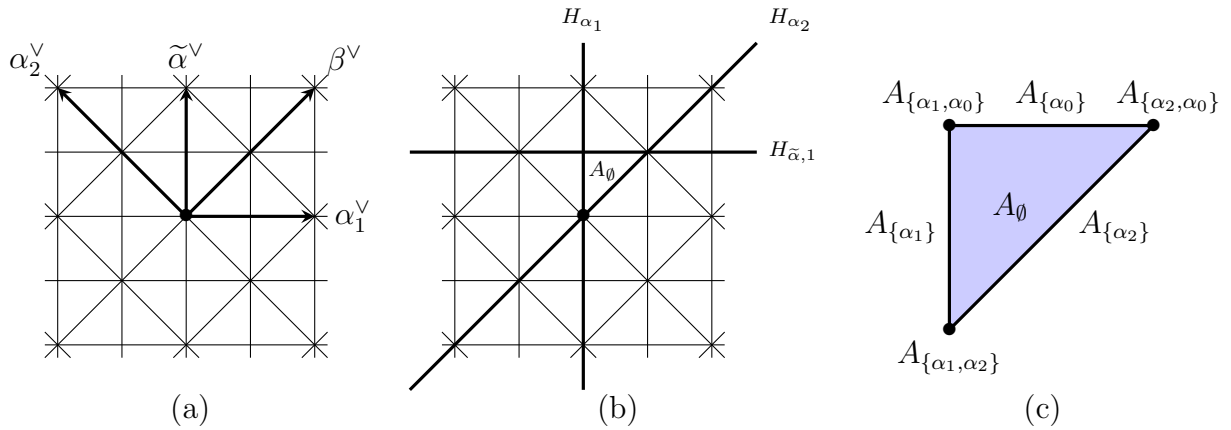


FIGURE 10. The affine arrangement $\tilde{\mathcal{H}}(C_2)$. (a) Positive coroots: $\alpha_1^\vee = \frac{1}{2}\alpha_1$, $\tilde{\alpha}^\vee = \frac{1}{2}\tilde{\alpha}$, $\alpha_2^\vee = \alpha_2$, $\beta^\vee = \beta$. (b) Affine hyperplanes and the fundamental alcove. (c) The faces of the fundamental alcove.

The *affine Coxeter arrangement* is

$$\tilde{\mathcal{H}}(\Phi) := \{H_{\beta,k} \mid \beta \in \Pi, k \in \mathbb{Z}\}.$$

Two examples are shown in Figures 9(b) and 10(b).

Let $\tilde{\Sigma}$ denote the set of faces of $\tilde{\mathcal{H}}(\Phi)$. Since $\tilde{\mathcal{H}}(\Phi)$ is locally finite, $\tilde{\Sigma}$ is a semigroup under the Tits product.

In the poset $\tilde{\Sigma}$ (with faces ordered by inclusion of their closures), every vertex is a minimal element. Adding in a smallest element turns $\tilde{\Sigma}$ into a simplicial complex, isomorphic to the

Coxeter complex of \widetilde{W} [1, Proposition 10.13]. The Tits product can be extended as well, by letting the smallest element act as a unit. This turns the semigroup $\widetilde{\Sigma}$ into a monoid.

Since in $\widetilde{\mathcal{H}}(\Phi)$ there is one parallelism class for each linear hyperplane in $\mathcal{H}(\Phi)$, the set of celestial faces of $\widetilde{\mathcal{H}}(\Phi)$ identifies with the finite Coxeter complex Σ minus its central face. Proposition 1 then yields a right Σ -module structure on $\widetilde{\Sigma}$. (The cone built on the affine arrangement as in the proof of the proposition is in this context known as the *Tits cone*.)

The maximal faces of $\widetilde{\Sigma}$ are called *alcoves*, *chambers* being reserved for the maximal faces of Σ . The set $\widetilde{\mathcal{C}}$ of alcoves is a two-sided ideal in the semigroup $\widetilde{\Sigma}$. The right action of Σ on $\widetilde{\mathcal{C}}$ is trivial. On the other hand, the right action of a chamber $C \in \mathcal{C}$ on a face $F \in \widetilde{\Sigma}$ results in an alcove $FC \in \widetilde{\mathcal{C}}$.

Let $\alpha_0 = -\tilde{\alpha}$ be the *lowest* root and $\widetilde{\Delta} := \Delta \cup \{\alpha_0\}$. The highest root is positive; in particular, $\alpha_0 \notin \Delta$. Moreover, the angle between any two distinct elements of $\widetilde{\Delta}$ is obtuse and this may be used to code the extended Dynkin diagram of $(\widetilde{W}, \widetilde{S})$ [16, Section 4.7].

Recall the fundamental chamber C_\emptyset from (2). The *fundamental* alcove is

$$A_\emptyset := C_\emptyset \cap \{\lambda \in V \mid \langle \lambda, \tilde{\alpha} \rangle < 1\}.$$

The faces of A_\emptyset are the sets of the form

$$(9) \quad A_J := \begin{cases} C_J \cap \{\lambda \in V \mid \langle \lambda, \tilde{\alpha} \rangle < 1\} & \text{if } \alpha_0 \notin J, \\ C_{J \setminus \{\alpha_0\}} \cap \{\lambda \in V \mid \langle \lambda, \tilde{\alpha} \rangle = 1\} & \text{if } \alpha_0 \in J, \end{cases}$$

where J is a proper subset of $\widetilde{\Delta}$ and C_J is as in (3). See Figures 9(c) and 10(c).

The closure of A_\emptyset is a fundamental domain for the action of \widetilde{W} on V . Hence, each face in $\widetilde{\Sigma}$ is of the form $\mu + w \cdot A_J$, where $\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, and J is a proper subset of $\widetilde{\Delta}$.

We may let $J = \widetilde{\Delta}$ in (9). Then $A_J = \emptyset$, and we may think of this *empty face* as the smallest element that when added to $\widetilde{\Sigma}$ turns it into a simplicial complex (and a monoid).

The vertices of $\widetilde{\Sigma}$ are of the form $\mu + w \cdot A_{\widetilde{\Delta} \setminus \{\alpha\}}$, where μ and w are as before, and $\alpha \in \widetilde{\Delta}$. If we assign color α to all such vertices, we obtain a balanced coloring of $\widetilde{\Sigma}$. The face $F = \mu + w \cdot A_J$ receives color set $\widetilde{\Delta} \setminus J$, which we denote $\text{col}(F)$, just as for faces of the Coxeter complex.

A face $F \in \widetilde{\Sigma}$ can be encoded by an infinite sign vector $\sigma(F)$ that records whether the face is “above”, “below”, or “on” a particular hyperplane [1, Section 2.7]. We have

$$(10) \quad \sigma(F) = (\sigma_{\beta,k}(F))_{\beta \in \Pi, k \in \mathbb{Z}},$$

where $\sigma_{\beta,k}(F) \in \{+, -, 0\}$ is the sign of $\langle \lambda, \beta \rangle - k$ for points $\lambda \in F$. Fix $\beta \in \Pi$. The face F will be contained in a stripe between two consecutive hyperplanes of the form $H_{\beta,k}$. More precisely, there is a unique j such that

$$(11) \quad \sigma_{\beta,k}(F) = \begin{cases} + & \text{if } k < j, \\ - & \text{if } k > j, \\ + \text{ or } 0 & \text{if } k = j. \end{cases}$$

Let $k_\beta(F)$ denote this integer j . Equivalently, $k_\beta(F) = j$ means $j \leq \langle \lambda, \beta \rangle < j + 1$ for points $\lambda \in F$, and $\sigma_{\beta,j}(F) = 0$ or $+$ according to whether $\langle \lambda, \beta \rangle$ is equal to or greater than j . In

view of (11), to gain full knowledge of the sign vector of F , we need no more than the pairs $(k_\beta, \sigma_{\beta, k_\beta}(F))$ for all $\beta \in \Pi$. Thus, let us write instead

$$(12) \quad \sigma(F) = (k_\beta(F), \sigma_{\beta, k_\beta(F)}(F))_{\beta \in \Pi}.$$

We refer to (10) and (12) as the *expanded* and *compact* sign vector of F , respectively.

For example, if $\Phi = A_2$ and F is the edge given by

$$\{\lambda \in \mathbb{R}^2 \mid \langle \lambda, \alpha_1 \rangle = 1, -1 < \langle \lambda, \alpha_2 \rangle < 0, 0 < \langle \lambda, \tilde{\alpha} \rangle < 1\}$$

(see Figure 11), the entries of its compact sign vector are

$$\sigma_{\alpha_1}(F) = (1, 0), \quad \sigma_{\alpha_2}(F) = (-1, +), \quad \text{and} \quad \sigma_{\tilde{\alpha}}(F) = (0, +).$$

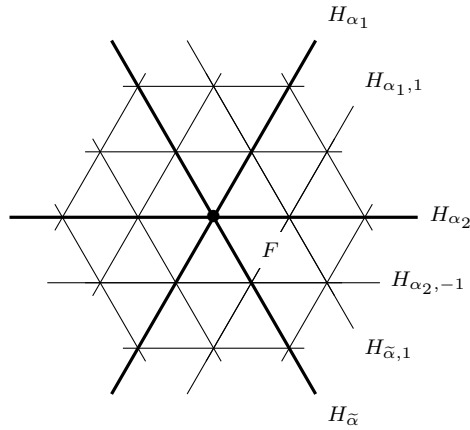


FIGURE 11. An edge in the affine arrangement $\tilde{\mathcal{H}}(A_2)$.

The compact sign vector of the fundamental alcove A_\emptyset has

$$\sigma_\beta(A_\emptyset) = (0, +)$$

for every $\beta \in \Pi$.

For $w \in W$, let

$$\tilde{D}(w) = \{\alpha \in \tilde{\Delta} \mid w(\alpha) < 0\},$$

which we call the *affine descent set* of w . This set is discussed in more detail in Section 3.2.

Arguments similar to those in Proposition 2 lead to the following result.

Proposition 3. *Given a face $F \in \tilde{\Sigma}$, there are unique $\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, and a proper subset J of $\tilde{\Delta}$ such that*

$$(13) \quad F = \mu + w \cdot A_J \quad \text{and} \quad \tilde{D}(w) \subseteq \text{col}(F).$$

The elements μ and w are determined by

$$\mu + w \cdot A_\emptyset = FC_\emptyset,$$

where the latter is the right action of the fundamental chamber $C_\emptyset \in \Sigma$ on the face $F \in \tilde{\Sigma}$. The existence and uniqueness of μ and w follow from the fact that the action of \tilde{W} on the set of alcoves is simply transitive. Letting

$$w_F := (\mu, w) \in \mathbb{Z}\Phi^\vee \times W = \tilde{W},$$

the condition defining this element of the affine Weyl group can be rewritten as

$$(14) \quad w_F \cdot A_\emptyset = FC_\emptyset.$$

Compare with (5).

2.4. The Steinberg torus. Consider the action of the coroot lattice $\mathbb{Z}\Phi^\vee$ by translations on the ambient space V . This action preserves the arrangement $\tilde{\mathcal{H}}(\Phi)$ (see Figure 9), and hence also the set of faces $\tilde{\Sigma}$. The *Steinberg torus* [12] is the set of orbits for the action of the coroot lattice on the set of faces:

$$\bar{\Sigma} := \tilde{\Sigma}/\mathbb{Z}\Phi^\vee.$$

As $\tilde{\Sigma}$ partitions V , $\bar{\Sigma}$ partitions the geometric torus $V/\mathbb{Z}\Phi^\vee$. This cell decomposition of the torus is not simplicial, as different faces may share the same vertex set, but it does have the property that the poset of subfaces of a face is isomorphic to the poset of nonempty subsets of the vertex set of the face. Let $\bar{\mathcal{C}}$ denote the set of maximal faces of $\bar{\Sigma}$.

The Steinberg torus may also be obtained as a quotient of the convex polytope

$$P_\Phi = \{\lambda \in V \mid -1 \leq \langle \lambda, \alpha \rangle \leq 1 \text{ for all } \alpha \in \Phi\}.$$

A point λ on the boundary of P_Φ satisfies $\langle \lambda, \beta \rangle = -1$ for some root $\beta \in \Phi$. To obtain the torus, each such point λ is identified with the point $\lambda' := \lambda + \beta^\vee$, which satisfies $\langle \lambda', \beta \rangle = 1$ and also lies on the boundary. See Figure 12. The polytope P_Φ is the union of the closures of the alcoves of the form $w \cdot A_\emptyset$, $w \in W$. It is an *alcoved polytope* in the sense of [17], see particularly [18, Section 4].

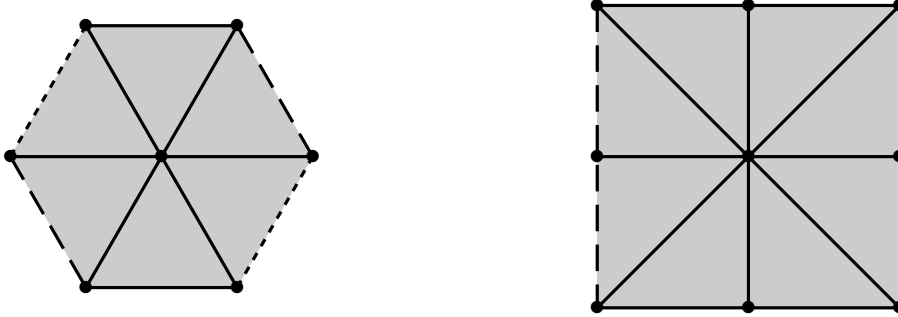


FIGURE 12. The polytopes P_{A_2} and P_{C_2} . The Steinberg tori are obtained by identifying points on the boundary.

The geometric construction of the product of faces (Figure 4) shows that translations commute with the right action of celestial faces on the set of mundane faces. We thus have the following.

Proposition 4. *Let $F \in \tilde{\Sigma}$, $G \in \Sigma$, and $\mu \in \mathbb{Z}\Phi^\vee$. Then $(\mu + F)G = \mu + FG$.*

Figure 13 illustrates this fact.

As an immediate consequence of Proposition 4, we have the following.

Corollary 5. *The set $\bar{\Sigma}$ is a right Σ -module and the canonical quotient map*

$$\tilde{\Sigma} \twoheadrightarrow \bar{\Sigma}$$

is a morphism of Σ -modules.

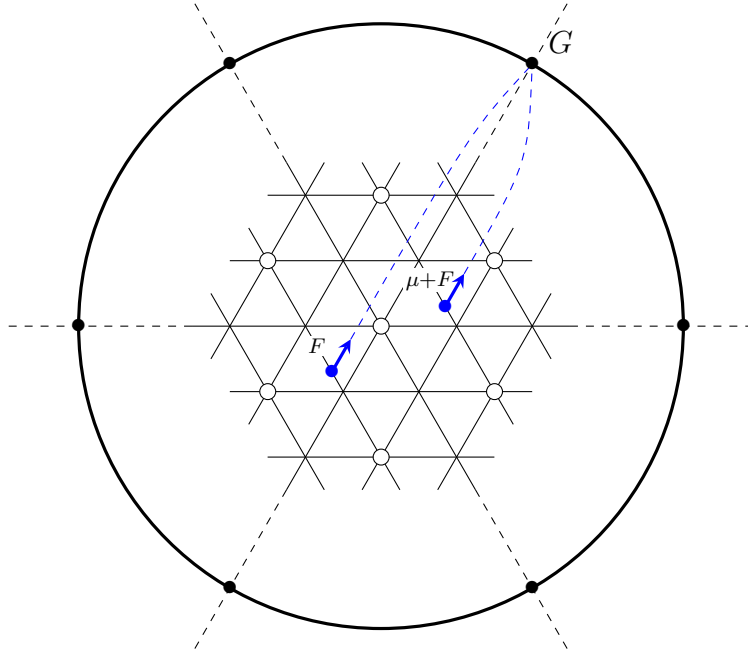


FIGURE 13. Translate first and then walk, or vice versa. Elements of the coroot lattice are indicated with white circles.

Let $\overline{F} \in \overline{\Sigma}$ denote the $\mathbb{Z}\Phi^\vee$ -orbit of a face $F \in \tilde{\Sigma}$. Given $G \in \Sigma$, we let $\overline{F}G \in \overline{\Sigma}$ denote the right action of G on \overline{F} .

Note, however, that the product of translates of two faces F and G of $\tilde{\Sigma}$ is in general *not* a translate of FG . For this reason, the semigroup structure of $\tilde{\Sigma}$ does not descend to $\overline{\Sigma}$.

The finite Weyl group W acts on $\overline{\Sigma}$. This is explained in more detail in Section 3; see the discussion around (21). (Another way to see this is by noting that the polytope P_Φ is stable under the action of W on V and that this action preserves the identifications.) We denote the action of $w \in W$ on $\overline{F} \in \overline{\Sigma}$ by

$$w \cdot \overline{F} := \overline{w \cdot F}.$$

The action of W on $\overline{\mathcal{C}}$ is simply-transitive.

The faces of $\overline{\Sigma}$ can be identified with cosets of the subgroups

$$W_J := \langle s_\alpha \mid \alpha \in J \rangle,$$

where J is a proper subset of $\tilde{\Delta}$. Namely,

$$\overline{\Sigma} \cong \{wW_J \mid w \in W, J \subsetneq \tilde{\Delta}\},$$

with inclusion of faces given by reverse inclusion of cosets. See [12].

The action of the affine Weyl group \widetilde{W} on $\tilde{\Sigma}$ is color-preserving, and hence so is the action of the subgroup $\mathbb{Z}\Phi^\vee$. Therefore, the complex $\overline{\Sigma}$ inherits the balanced coloring of $\tilde{\Sigma}$, for which the faces of the form $F = w \cdot \overline{A_J}$ receive color set $\tilde{\Delta} \setminus J$, which we again denote $\text{col}(\overline{F})$. See Figure 14.

The following is a consequence of Proposition 3.

Corollary 6. *Given a face $\overline{F} \in \overline{\Sigma}$, there is a unique $w \in W$ and a unique proper subset J of $\widetilde{\Delta}$ such that*

$$(15) \quad \overline{F} = w \cdot \overline{A}_J \quad \text{and} \quad \widetilde{D}(w) \subseteq \text{col}(\overline{F}).$$

The element w is determined by

$$(16) \quad w \cdot \overline{A}_\emptyset = \overline{F}C_\emptyset.$$

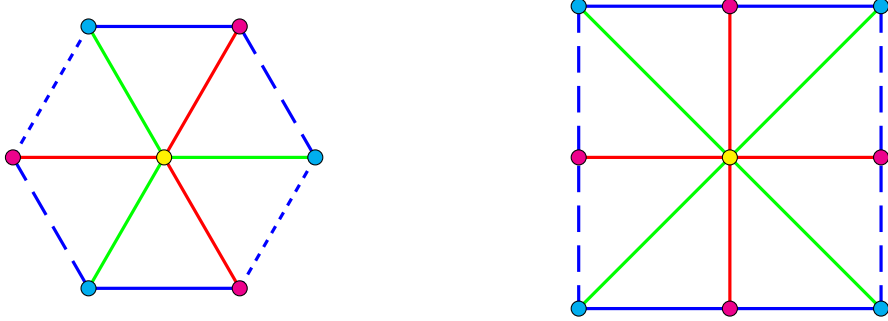


FIGURE 14. The Steinberg tori for A_2 and C_2 with faces colored according to W -orbits.

3. SOLOMON'S DESCENT RING AND THE MODULE OF AFFINE DESCENT CLASSES

We review Solomon's definition of the descent ring $\text{Sol}(\Phi)$ and introduce an analogous object $\overline{\text{Sol}}(\Phi)$ in terms of Cellini's notion of affine descents. For the latter, the root system is assumed to be irreducible and crystallographic.

We then review the approach of Tits to the descent ring from [26], following Bidigare [6] and Brown [9, Section 4.8], and adapt it to the case of affine descents. Via the constructions of Section 2, this leads to a left module structure on $\overline{\text{Sol}}(\Phi)$ over $\text{Sol}(\Phi)$.

3.1. Descents. In this section, Φ is allowed to be an arbitrary root system.

The Coxeter group W is generated by the set S of simple reflections. Let ℓ denote the corresponding length function. According to [7, Proposition 4.4.6], for any positive root β and any $w \in W$,

$$(17) \quad \ell(w) > \ell(ws_\beta) \iff w(\beta) < 0.$$

For $w \in W$, the set of *right descents* of w is

$$(18) \quad D(w) := \{\alpha \in \Delta \mid \ell(w) > \ell(ws_\alpha)\} = \{\alpha \in \Delta \mid w(\alpha) < 0\}.$$

For any $J \subseteq \Delta$, let

$$x_J := \sum_{D(w) \subseteq J} w$$

denote the sum, in the group ring $\mathbb{Z}W$, of all elements of W whose descent set is contained in J . As J runs over the subsets of Δ , the elements x_J span a subring of $\mathbb{Z}W$, called *Solomon's*

descent ring [24, Theorem 1]. We denote it by $\text{Sol}(\Phi)$. There is another standard basis of $\text{Sol}(\Phi)$ given by the sums of elements with a fixed descent set,

$$y_J := \sum_{D(w)=J} w.$$

The descent algebra was introduced by Solomon in [24] and has been the object of many subsequent works including [2, 3, 4, 5, 8, 11, 14, 15, 19, 21, 23].

3.2. Affine descents. We turn to *affine descent sets*, a notion introduced by Cellini [11, Section 2], and further studied in [12, 13, 22].

We assume that root system Φ is irreducible and crystallographic, as in Sections 2.3 and 2.4. Thus, W is a Weyl group. Recall that $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$. The *affine descent set* of an element $w \in W$ is

$$(19) \quad \tilde{D}(w) := \{\alpha \in \tilde{\Delta} \mid w(\alpha) < 0\}.$$

Thus, $D(w) \subseteq \tilde{D}(w)$, and the only difference occurs when w does not take α_0 to a positive root. Note that the set $\tilde{D}(w)$ is defined only for elements w of the finite Weyl group W , and not for general elements of the affine Weyl group \tilde{W} .

Remark 7. One may wonder about other generalized notions of descents arising from different choices of subsets of roots. The above choice is singled out by its connection with the Steinberg torus, a fact discovered in [12]. In this regard, see also [12, Remark 3.8] and Remark 10 below.

For each subset J of $\tilde{\Delta}$, let

$$\bar{x}_J := \sum_{\tilde{D}(w) \subseteq J} w \quad \text{and} \quad \bar{y}_J := \sum_{\tilde{D}(w)=J} w.$$

We have

$$\bar{x}_J = \sum_{I \subseteq J} \bar{y}_I,$$

and by inclusion-exclusion

$$\bar{y}_J = \sum_{I \subseteq J} (-1)^{|J-I|} \bar{x}_I.$$

This is a refinement of the basis for Solomon's descent algebra in the sense that for $J \subseteq \Delta$,

$$y_J = \bar{y}_J + \bar{y}_{J \cup \{\alpha_0\}}.$$

While the elements \bar{x}_J (or equivalently, \bar{y}_J) do not span a subring of $\mathbb{Z}W$, we show below (Theorem 11) that they span a left module over $\text{Sol}(\Phi)$, which we denote by $\overline{\text{Sol}}(\Phi)$.

Recall that α_0 is a negative linear combination of simple roots. If an element w sends each simple root to a negative (positive) root, then it must send α_0 to a positive (negative) one. Thus, $\tilde{D}(w)$ is always a proper nonempty subset of $\tilde{\Delta}$, and we have

$$\bar{x}_\emptyset = \bar{y}_\emptyset = 0, \quad \bar{y}_{\tilde{\Delta}} = 0 \quad \text{and} \quad \sum_{J \subseteq \tilde{\Delta}} (-1)^{|\tilde{\Delta}-J|} \bar{x}_J = 0.$$

For each proper nonempty subset J of $\tilde{\Delta}$, there exists at least one element w with $\tilde{D}(w) = J$, so \bar{y}_J is nonzero. Moreover, being sums over nonempty disjoint classes, the elements \bar{y}_J (for

$J \neq \emptyset, \tilde{\Delta}$) are linearly independent and thus form a basis of $\overline{\text{Sol}}(\Phi)$. Therefore, the elements \bar{x}_J for J as before also constitute a basis of $\overline{\text{Sol}}(\Phi)$.

Remark 8. Root systems with isomorphic Coxeter groups (such as B_n and C_n) have isomorphic descent rings. On the other hand, such systems may have nonisomorphic modules of affine descents. The module $\overline{\text{Sol}}(\Phi)$ depends on the root system and not just on the Weyl group.

Remark 9. Cellini [11, Proposition 1.2] constructs a certain commutative subring of $\mathbb{Z}W$. It follows from [11, Lemma 2.4] that this subring lies inside $\overline{\text{Sol}}(\Phi)$. We do not pursue any further connections with Cellini's work in this paper.

Remark 10. In [20], Moszkowski defines a family of modules over Solomon's descent ring. This family contains the module $\overline{\text{Sol}}(\Phi)$, as we now explain.

Let $T = \{wsw^{-1} \mid s \in S, w \in W\}$ denote the set of *reflections* in W . Fix a subset R of T . For any element $w \in W$, define the R -descent set of w to be:

$$(20) \quad D_R(w) := \{r \in R \mid \ell(w) > \ell(wr)\}.$$

Given $J \subseteq R$, let

$$x_J^R := \sum_{D_R(w) \subseteq J} w \quad \text{and} \quad y_J^R := \sum_{D_R(w) = J} w$$

denote the sum of elements whose R -descent set is contained in, or, respectively, equal to J . By arguments similar to Solomon's, Moszkowski shows that the subspace of $\mathbb{Z}W$ spanned by the elements x_J^R (or equivalently, by the elements y_J^R), as J runs over the subsets of R , is a left module over $\text{Sol}(\Phi)$ [20, Théorème 1]. (Moszkowski, as Solomon, indexes the spanning elements by the complement of the subsets J above.)

Now let s_0 denote the linear reflection through the linear hyperplane orthogonal to α_0 , i.e., the hyperplane $\{\lambda \in V \mid \langle \lambda, \alpha_0 \rangle = 0\}$, and let $R = S \cup \{s_0\}$. Applying (17) to $\tilde{\alpha}$, we have that

$$\ell(w) > \ell(ws_0) \iff w(\alpha_0) > 0.$$

Comparing (19) to (20) we see that

$$D_R(w) = \begin{cases} \tilde{D}(w) \cup \{\alpha_0\} & \text{if } w(\alpha_0) > 0, \\ \tilde{D}(w) \setminus \{\alpha_0\} & \text{if } w(\alpha_0) < 0. \end{cases}$$

It follows that, for $J \subseteq \Delta$,

$$y_J^R = \bar{y}_{J \cup \{\alpha_0\}} \quad \text{and} \quad \bar{y}_J = y_{J \cup \{\alpha_0\}}^R.$$

Thus, $\overline{\text{Sol}}(\Phi)$ is Moszkowski's module for $R = S \cup \{s_0\}$.

3.3. The geometric approach to descents. We continue to assume that Φ is an irreducible crystallographic root system, W is the associated Weyl group, and $\Sigma, \tilde{\Sigma}$ and $\bar{\Sigma}$ are the complexes discussed in Section 2.

Let $\mathbb{Z}\Sigma$ denote the monoid ring of Σ and consider the subring $(\mathbb{Z}\Sigma)^W$ of W -invariants. The latter is anti-isomorphic to Solomon's descent ring. Bidigare obtained this result in [6]; it also appears somewhat implicitly in earlier work Tits [26, Section 4]. We follow here the proof of this fact by Brown [9, Section 9.6], and obtain counterparts for $\tilde{\Sigma}$ and $\bar{\Sigma}$.

As discussed in Section 2, the Tits product turns Σ into a monoid, and $\tilde{\Sigma}$ and $\bar{\Sigma}$ are right Σ -modules, with the projection $\pi : \tilde{\Sigma} \rightarrow \bar{\Sigma}$ being a morphism of right Σ -modules.

The Weyl group W acts on both $\mathbb{Z}\Phi^\vee$ and $\tilde{\Sigma}$, and these actions and the action of $\mathbb{Z}\Phi^\vee$ on $\tilde{\Sigma}$ are related by the *semilinearity* condition

$$(21) \quad w \cdot (\mu + F) = w \cdot \mu + w \cdot F$$

for $w \in W$, $\mu \in \mathbb{Z}\Phi^\vee$, and $F \in \tilde{\Sigma}$. It follows that W acts on $\bar{\Sigma}$ and that π is a morphism of left W -modules.

The Weyl group W also acts on the monoid Σ and we have

$$(22) \quad w \cdot (FG) = (w \cdot F)(w \cdot G)$$

for $w \in W$, G in Σ and F in either Σ , $\tilde{\Sigma}$, or $\bar{\Sigma}$.

We linearize the sets $\tilde{\Sigma}$ and $\bar{\Sigma}$, obtaining (free) abelian groups $\mathbb{Z}\tilde{\Sigma}$ and $\mathbb{Z}\bar{\Sigma}$. We emphasize that $\mathbb{Z}\tilde{\Sigma}$ consists of *finite* linear combinations of elements of $\tilde{\Sigma}$. For this reason, 0 is the only element of $\mathbb{Z}\tilde{\Sigma}$ invariant under the linear action of the (infinite) group \tilde{W} . We consider the linear action of the finite Weyl group W on the abelian groups $\mathbb{Z}\tilde{\Sigma}$ and $\mathbb{Z}\bar{\Sigma}$, and the corresponding subgroups of W -invariant elements. It follows from (22) that $(\mathbb{Z}\tilde{\Sigma})^W$ and $(\mathbb{Z}\bar{\Sigma})^W$ are right modules over the ring $(\mathbb{Z}\Sigma)^W$, and also that the map $\pi : \tilde{\Sigma} \rightarrow \bar{\Sigma}$ gives rise to a morphism of right $(\mathbb{Z}\Sigma)^W$ -modules $\pi : (\mathbb{Z}\tilde{\Sigma})^W \rightarrow (\mathbb{Z}\bar{\Sigma})^W$.

Recall again from Section 2 that the set of chambers \mathcal{C} is a left ideal of the monoid Σ : the product of a face of Σ and a chamber of Σ is another chamber of Σ . Similarly, the right action of a chamber of Σ on a face of $\tilde{\Sigma}$ (respectively, of $\bar{\Sigma}$) results in an alcove of $\tilde{\Sigma}$ (respectively, a maximal face of $\bar{\Sigma}$). This gives rise to three maps

$$(23) \quad \mathbb{Z}\Sigma \rightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}) \quad \mathbb{Z}\tilde{\Sigma} \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}}) \quad \mathbb{Z}\bar{\Sigma} \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}})$$

denoted in every case by Θ and given by

$$\Theta(F)(C) := FC$$

(and extended by \mathbb{Z} -linearity). Here, F denotes a face of either Σ , $\tilde{\Sigma}$, or $\bar{\Sigma}$, according to the case, while C denotes a chamber of Σ in every case.

The abelian group $\text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$ is a ring under composition, while both $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}})$ and $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}})$ are right $\text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$ -modules in the same manner. Associativity for the product of Σ (or for the right action of Σ on $\tilde{\Sigma}$, or on $\bar{\Sigma}$) translates into the fact that $\Theta(FG) = \Theta(F) \circ \Theta(G)$ for $G \in \Sigma$ and F in either Σ , $\tilde{\Sigma}$, or $\bar{\Sigma}$. This says that the first map in (23) is a morphism of rings, while the other two maps are morphisms of right Σ -modules, where $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}})$ and $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}})$ are viewed as right $\mathbb{Z}\Sigma$ -modules by restriction via $\Theta : \mathbb{Z}\Sigma \rightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$.

The sets \mathcal{C} , $\tilde{\mathcal{C}}$, and $\bar{\mathcal{C}}$ are stable under the action of W , and hence the groups $\text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$, $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}})$, and $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}})$ are acted upon by W from the left. The action is

$$(w \cdot f)(C) = w \cdot f(w^{-1} \cdot C)$$

for $w \in W$, $C \in \mathcal{C}$, and f in either $\text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$, $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}})$, or $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}})$. Equation (22) implies that

$$\Theta(w \cdot F) = w \cdot \Theta(F)$$

for $w \in W$ and F in either Σ , $\tilde{\Sigma}$, or $\bar{\Sigma}$. It follows that each map Θ restricts as follows:

$$(\mathbb{Z}\Sigma)^W \rightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})^W, \quad (\mathbb{Z}\tilde{\Sigma})^W \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}})^W, \quad (\mathbb{Z}\bar{\Sigma})^W \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}})^W.$$

These maps are still denoted by Θ . The first one is a morphism of rings and the other two are morphisms of right $(\mathbb{Z}\Sigma)^W$ -modules.

Since the action of W on \mathcal{C} is free and transitive, we have isomorphisms

$$\begin{aligned} \text{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})^W &= \text{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}) \cong \mathbb{Z}\mathcal{C}, & \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}})^W &= \text{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}}) \cong \mathbb{Z}\tilde{\mathcal{C}}, \\ \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}})^W &= \text{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}}) \cong \mathbb{Z}\bar{\mathcal{C}}, \end{aligned}$$

given in every case by $f \mapsto f(C_\emptyset)$, where C_\emptyset is the fundamental chamber of Σ . Let ϕ denote these maps, so that $\phi(f) = f(C_\emptyset)$. Notice in particular that $\phi(\Theta(F)) = \Theta(F)(C_\emptyset) = FC_\emptyset$, where F is a face of either Σ , $\tilde{\Sigma}$, or $\bar{\Sigma}$.

We may further identify W with \mathcal{C} by means of $w \leftrightarrow w \cdot C_\emptyset$, where C_\emptyset is the fundamental chamber of Σ . Consider the composite isomorphism of abelian groups

$$(24) \quad \text{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}) \cong \mathbb{Z}W.$$

A group element $u \in W$ corresponds to the endomorphism f such that $f(C_\emptyset) = u \cdot C_\emptyset$. If another element $v \in W$ corresponds to the endomorphism g , then

$$(f \circ g)(C_\emptyset) = f(v \cdot C_\emptyset) = v \cdot f(C_\emptyset) = vu \cdot C_\emptyset,$$

so $f \circ g$ corresponds to vu . Therefore, the map (24) is an anti-isomorphism of rings.

Similarly, we have $\tilde{\mathcal{C}} \cong \tilde{W}$ and $\bar{\mathcal{C}} \cong W$ via the actions of these groups on the fundamental alcoves of $\tilde{\Sigma}$ and $\bar{\Sigma}$. In each case, let ι denote the identification between chambers and group elements. Notice in particular that $\iota(FC_\emptyset) = w_F$, where if F is a face in \mathcal{C} then w_F is in W (see Proposition 2), if F is a face in $\tilde{\mathcal{C}}$ then w_F is in \tilde{W} (see Proposition 3), and if F is a face in $\bar{\mathcal{C}}$ then w_F is in W (see Corollary 6).

This gives rise to isomorphisms of right $\text{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C})$ -modules

$$\text{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\tilde{\mathcal{C}}) \cong \mathbb{Z}\tilde{W} \quad \text{and} \quad \text{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\bar{\mathcal{C}}) \cong \mathbb{Z}W$$

where now $\mathbb{Z}\tilde{W}$ and $\mathbb{Z}W$ are first viewed as left $\mathbb{Z}W$ -modules by multiplication, and then as right $\text{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C})$ -modules via the anti-morphism (24).

Composing the maps Θ with the preceding (anti)isomorphisms we obtain three maps

$$(25) \quad (\mathbb{Z}\Sigma)^W \rightarrow \mathbb{Z}W, \quad (\mathbb{Z}\tilde{\Sigma})^W \rightarrow \mathbb{Z}\tilde{W}, \quad (\mathbb{Z}\bar{\Sigma})^W \rightarrow \mathbb{Z}W,$$

denoted in every case by Ψ and given by

$$\Psi \left(\sum_F a_F F \right) = \sum_F a_F w_F.$$

Indeed,

$$\sum a_F F \xrightarrow{\Theta} \sum a_F \Theta(F) \xrightarrow{\phi} \sum a_F FC_\emptyset \xrightarrow{\iota} \sum a_F w_F.$$

In each case, $\sum_F a_F F$ stands for a W -invariant element of $\mathbb{Z}\Sigma$, $\mathbb{Z}\tilde{\Sigma}$, or $\mathbb{Z}\bar{\Sigma}$, and w_F is the element determined by (5), (14) or (16). Note that in the second case $w_F \in \tilde{W}$ is an element of the affine Weyl group, while in the other cases it is an element of the finite Weyl group W .

The first map in (25) is an anti-morphism of rings and the other two are morphisms of right $(\mathbb{Z}\Sigma)^W$ -modules, where $\mathbb{Z}\widetilde{W}$ and $\mathbb{Z}W$ are first viewed as left $\mathbb{Z}W$ -modules by multiplication, and then as right $(\mathbb{Z}\Sigma)^W$ -modules via the anti-morphism $\Psi : (\mathbb{Z}\Sigma)^W \rightarrow \mathbb{Z}W$.

Let us analyze the subgroup $(\mathbb{Z}\Sigma)^W$ of W -invariants. In the action of W on Σ there is one orbit for each subset J of the set of simple roots Δ ; namely, the orbit of the face $C_{\Delta \setminus J}$, which consists of all the faces of color set J . Let us denote it by Σ_J . Then, the group $(\mathbb{Z}\Sigma)^W$ is freely generated by the elements

$$\sigma_J := \sum_{F \in \Sigma_J} F,$$

where J runs over the subsets of Δ . Define elements $x_J \in \mathbb{Z}W$ by

$$x_J := \sum_{w \in W: D(w) \subseteq J} w.$$

It follows from Proposition 2 that there is a bijection

$$\{w \in W \mid D(w) \subseteq J\} \leftrightarrow \Sigma_J$$

that sends w to $w \cdot C_{\Delta \setminus J}$ and returns F to w_F as in (5). Therefore, the map $\Psi : (\mathbb{Z}\Sigma)^W \rightarrow \mathbb{Z}W$ satisfies

$$\Psi(\sigma_J) = x_J.$$

Let

$$\text{Sol}(\Phi) := \mathbb{Z}\{x_J \mid J \subseteq \Delta\}$$

denote the subgroup of $\mathbb{Z}W$ generated by the x_J , $J \subseteq \Delta$. Thus, $\text{Sol}(\Phi)$ is the image of Ψ . As the latter is an anti-morphism of rings, $\text{Sol}(\Phi)$ a subring of $\mathbb{Z}W$. The latter fact is a result of Solomon [24, Theorem 1]. In addition, as J varies, the sets $\{w \in W \mid D(w) = J\}$ are disjoint, and therefore the x_J are linearly independent. It follows that the ring $\text{Sol}(\Phi)$ is anti-isomorphic to $(\mathbb{Z}\Sigma)^W$ via Ψ . The anti-isomorphism is due to Bidigare [6] and Tits [26, Section 4]; the approach followed above is due to Brown [9, Section 9.6].

A similar analysis applies to $(\mathbb{Z}\widetilde{\Sigma})^W$ and $(\mathbb{Z}\overline{\Sigma})^W$. Given a nonempty subset J of $\widetilde{\Delta}$ and $\mu \in \mathbb{Z}\Phi^\vee$, let $\widetilde{\Sigma}_{J,\mu}$ denote the W -orbit of the face $\mu + A_{\widetilde{\Delta} \setminus J}$ in $\widetilde{\Sigma}$, and let $\overline{\Sigma}_J$ denote the W -orbit of the face $\overline{A_{\widetilde{\Delta} \setminus J}}$ in $\overline{\Sigma}$. The groups $(\mathbb{Z}\widetilde{\Sigma})^W$ and $(\mathbb{Z}\overline{\Sigma})^W$ are then freely generated by the elements

$$\widetilde{\sigma}_{J,\mu} := \sum_{F \in \widetilde{\Sigma}_{J,\mu}} F \quad \text{and} \quad \overline{\sigma}_J := \sum_{F \in \overline{\Sigma}_J} F,$$

respectively.

For J and μ as above, define elements $\widetilde{x}_{J,\mu} \in \mathbb{Z}\widetilde{W}$ and $\overline{x}_J \in \mathbb{Z}W$ by

$$\widetilde{x}_{J,\mu} := \sum_{w \in W: \widetilde{D}(w) \subseteq J} (\mu, w), \quad \text{and} \quad \overline{x}_J := \sum_{w \in W: \overline{D}(w) \subseteq J} w.$$

Invoking Proposition 3 and Corollary 6, we obtain that the maps $\Psi : (\mathbb{Z}\widetilde{\Sigma})^W \rightarrow \mathbb{Z}\widetilde{W}$ and $\Psi : (\mathbb{Z}\overline{\Sigma})^W \rightarrow \mathbb{Z}W$ satisfy

$$\Psi(\widetilde{\sigma}_{J,\mu}) = \widetilde{x}_{J,\mu} \quad \text{and} \quad \Psi(\overline{\sigma}_J) = \overline{x}_J.$$

Recall from Section 3.2 that there is precisely one linear relation among the elements \bar{x}_I , as I varies over the nonempty subsets of $\tilde{\Delta}$. The same is true of the elements $\tilde{x}_{I,\mu}$ for fixed μ . These relations are, respectively,

$$\sum_{\emptyset \neq I \subseteq \tilde{\Delta}} (-1)^{|\tilde{\Delta}-I|} \tilde{x}_{I,\mu} = 0 \quad \text{and} \quad \sum_{\emptyset \neq I \subseteq \tilde{\Delta}} (-1)^{|\tilde{\Delta}-I|} \bar{x}_I = 0.$$

Thus the kernels of Ψ on $(\mathbb{Z}\tilde{\Sigma})^W$ and $(\mathbb{Z}\bar{\Sigma})^W$ are free abelian groups, spanned by

$$\left\{ \sum_{\emptyset \neq I \subseteq \tilde{\Delta}} (-1)^{|\tilde{\Delta}-I|} \tilde{\sigma}_{I,\mu} \mid \mu \in \mathbb{Z}\Phi^\vee \right\}$$

in the former case, and by the element

$$\sum_{\emptyset \neq I \subseteq \tilde{\Delta}} (-1)^{|\tilde{\Delta}-I|} \tilde{\sigma}_I$$

in the latter. Define

$$\widetilde{\text{Sol}}(\Phi) := \mathbb{Z}\{\tilde{x}_{J,\mu} \mid \emptyset \neq J \subseteq \tilde{\Delta}, \mu \in \mathbb{Z}\Phi^\vee\}$$

and

$$\overline{\text{Sol}}(\Phi) := \mathbb{Z}\{\bar{x}_J \mid \emptyset \neq J \subseteq \tilde{\Delta}\} = \mathbb{Z}\{\bar{y}_J \mid \emptyset \neq J \subseteq \tilde{\Delta}\}.$$

The preceding shows that these groups are respectively isomorphic to the quotients

$$(\mathbb{Z}\tilde{\Sigma})^W / \ker_{\tilde{\Sigma}}(\Psi) \quad \text{and} \quad (\mathbb{Z}\bar{\Sigma})^W / \ker_{\bar{\Sigma}}(\Psi),$$

with the obvious notation for the kernels of the maps Ψ . Taking into account the multiplicative properties of Ψ , we arrive at our main result.

Theorem 11. *Let W be a Weyl group.*

- (i) $\widetilde{\text{Sol}}(\Phi)$ is a left module over the ring $\text{Sol}(\Phi)$. More precisely, it is a submodule of $\mathbb{Z}\widetilde{W}$, where the latter is a left module over $\mathbb{Z}W$ by multiplication and then over the subring $\text{Sol}(\Phi)$ by restriction.
- (ii) $\overline{\text{Sol}}(\Phi)$ is a left module over the ring $\text{Sol}(\Phi)$. More precisely, it is a submodule of $\mathbb{Z}W$, where the latter is a left module over the subring $\text{Sol}(\Phi)$ by multiplication.
- (iii) The map Ψ in its three versions

$$(\mathbb{Z}\Sigma)^W \rightarrow \text{Sol}(\Phi),$$

$$(\mathbb{Z}\tilde{\Sigma})^W / \ker_{\tilde{\Sigma}}(\Psi) \rightarrow \widetilde{\text{Sol}}(\Phi),$$

and

$$(\mathbb{Z}\bar{\Sigma})^W / \ker_{\bar{\Sigma}}(\Psi) \rightarrow \overline{\text{Sol}}(\Phi)$$

constitutes an anti-isomorphism of rings (first version) and a isomorphism of right $(\mathbb{Z}\Sigma)^W$ -modules (second and third versions), where $\widetilde{\text{Sol}}(\Phi)$ and $\overline{\text{Sol}}(\Phi)$ are viewed first as left $\text{Sol}(\Phi)$ -modules and then as right $(\mathbb{Z}\Sigma)^W$ -modules via $\Psi : (\mathbb{Z}\Sigma)^W \rightarrow \text{Sol}(\Phi)$.

Example 12. We illustrate the isomorphism between (the quotient of) the module of invariant faces of the Steinberg torus and the module of affine descents with an explicit computation.

Consider the root system A_2 (Figures 6 and 9). There are 3 rays of color $\{\alpha_2\}$ in the finite Coxeter complex Σ . They constitute the W -orbit $\Sigma_{\{\alpha_2\}}$ of the ray $C_{\{\alpha_1\}}$, and thus $\sigma_{\{\alpha_2\}}$ is the sum of these 3 rays. In the Steinberg torus, there are 9 edges in total, and 3 of them

constitute the W -orbit $\overline{\Sigma}_{\{\alpha_1, \alpha_2\}}$ of the edge $A_{\{\alpha_0\}}$. Thus, $\overline{\sigma}_{\{\alpha_1, \alpha_2\}}$ is the sum of these 3 edges. See Figure 15.

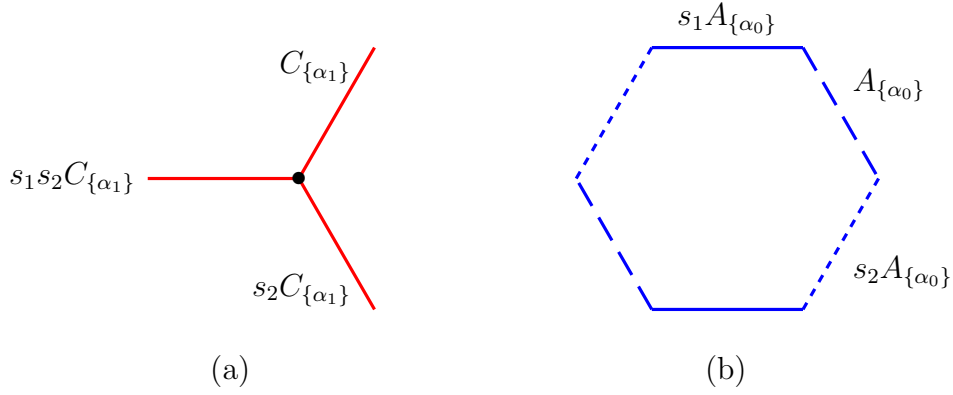


FIGURE 15. (a) The rays of color α_2 . (b) The edges of color $\{\alpha_1, \alpha_2\}$.

The product of one of these edges by each one of the three rays is computed in Figure 16(a). The result is the edge itself plus the 2 adjacent chambers. Performing this operation for each of the 3 edges we obtain the 3 edges back plus 6 chambers. Modulo translations, the chambers tile the torus, as shown in Figure 16(b). The 6 chambers constitute the orbit $\overline{\sigma}_{\{\alpha_0, \alpha_1, \alpha_2\}}$. In conclusion,

$$(26) \quad \overline{\sigma}_{\{\alpha_1, \alpha_2\}} \cdot \sigma_{\{\alpha_2\}} = \overline{\sigma}_{\{\alpha_1, \alpha_2\}} + \overline{\sigma}_{\{\alpha_0, \alpha_1, \alpha_2\}}.$$

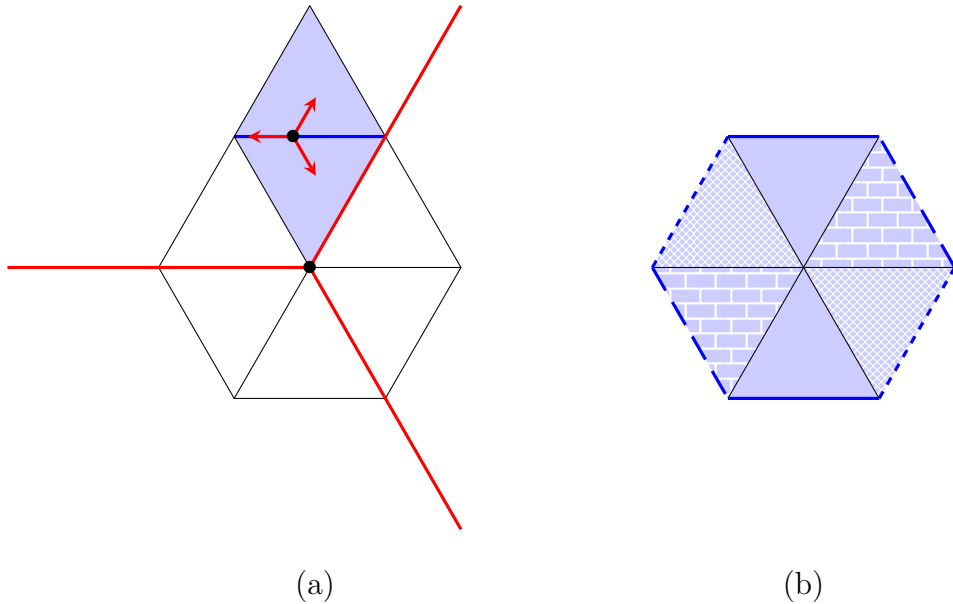


FIGURE 16. (a) Walking from an edge along 3 rays. (b) 3 edges and 6 chambers.

We turn to the Weyl group. Let s_1 and s_2 denote the reflections corresponding to the simple roots α_1 and α_2 . The affine descent set $\widetilde{D}(w)$ of each group element $w \in W$ is shown below. The descent set is determined by $D(w) = \widetilde{D}(w) \setminus \{\alpha_0\}$.

w	$\tilde{D}(w)$
id	α_0
s_1	α_0, α_1
s_2	α_0, α_2
$s_1 s_2$	α_2
$s_2 s_1$	α_1
$s_1 s_2 s_1$	α_1, α_2

It follows that

$$x_{\{\alpha_2\}} = \text{id} + s_2 + s_1 s_2 \quad \text{and} \quad \bar{x}_{\{\alpha_1, \alpha_2\}} = s_1 s_2 + s_2 s_1 + s_1 s_2 s_1.$$

Multiplying out in the Weyl group ring we find that

$$(27) \quad x_{\{\alpha_2\}} \cdot \bar{x}_{\{\alpha_1, \alpha_2\}} = s_1 s_2 + s_2 s_1 + s_1 s_2 s_1 \\ + \text{id} + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1 = \bar{x}_{\{\alpha_1, \alpha_2\}} + \bar{x}_{\{\alpha_0, \alpha_1, \alpha_2\}}.$$

A comparison of (26) and (27) witnesses the isomorphism $(\mathbb{Z}\bar{\Sigma})^W / \ker_{\bar{\Sigma}}(\Psi) \cong \overline{\text{Sol}}(\Phi)$.

4. COMBINATORIAL MODELS

We analyze the constructions of the preceding sections for the case of the root systems of types A and C . We review the combinatorial description of the finite Coxeter complex of type A in terms of set compositions and provide a similar model for the Steinberg torus in terms of *spin necklaces*. We describe the module structure of the latter in these terms. We also discuss the situation for the root systems of type C , more briefly.

Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ denote the canonical basis of \mathbb{R}^n . We equip this space with the standard inner product, for which the ε_i are orthonormal.

4.1. The root system A_{n-1} . The root system in question is

$$A_{n-1} := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}.$$

For the ambient space we take $V_n := \{x \in \mathbb{R}^n \mid \sum x_i = 0\}$, the subspace spanned by A_{n-1} .

The Weyl group of A_{n-1} may be identified with S_n , the symmetric group of permutations of $[n]$, via its action on the canonical basis of \mathbb{R}^n . The reflection associated to the root $\varepsilon_i - \varepsilon_j$ exchanges ε_i for ε_j ; it thus corresponds to the transposition (i, j) . See Figure 17.

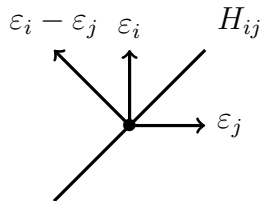


FIGURE 17. A root of type A .

For the simple roots, we choose

$$\alpha_i := \varepsilon_{i+1} - \varepsilon_i,$$

with $1 \leq i \leq n-1$. Let s_i denote the associated simple reflection. It corresponds to the elementary transposition $(i, i+1)$. The positive roots are then $\varepsilon_j - \varepsilon_i$ for $i < j$, and the lowest root $\alpha_0 = -\tilde{\alpha}$ is $\varepsilon_1 - \varepsilon_n$. See Figure 6 for an illustration.

We let w denote both an element of the Weyl group and the corresponding permutation. We write permutations in one-line form $w = w_1 w_2 \cdots w_n$, where $w_i = w(i)$, and adopt the convention that other subscripts are taken modulo n , so that $w_0 = w_n$ and $w_{n+1} = w_1$. In these terms, we have that $w(\alpha_i) = \varepsilon_{w_{i+1}} - \varepsilon_{w_i}$, and therefore

$$w(\alpha_i) < 0 \iff w_{i+1} < w_i,$$

for $i = 0, \dots, n-1$. We identify Δ with $\{1, \dots, n-1\}$ and $\tilde{\Delta}$ with $\{1, \dots, n-1, n\}$ via

$$\alpha_0 \leftrightarrow n \quad \text{and} \quad \alpha_i \leftrightarrow i$$

for $i = 1, \dots, n-1$. Then, the descent sets (ordinary and affine) of w are

$$D(w) = \{i \mid 1 \leq i \leq n-1, w_i > w_{i+1}\} \quad \text{and} \quad \tilde{D}(w) = \{i \mid 1 \leq i \leq n, w_i > w_{i+1}\}.$$

For example, $D(25413) = \{2, 3\}$ and $\tilde{D}(25413) = \{2, 3, 5\}$.

The hyperplane orthogonal to $\varepsilon_i - \varepsilon_j$ is $H_{ij} := \{x \in V_n \mid x_i = x_j\}$. The Coxeter arrangement

$$\mathcal{H}(A_{n-1}) = \{H_{ij} \mid 1 \leq i < j \leq n\}$$

is the *braid arrangement*. The affine Coxeter arrangement is

$$\tilde{\mathcal{H}}(A_{n-1}) = \{H_{ij,k} \mid 1 \leq i < j \leq n, k \in \mathbb{Z}\},$$

where $H_{ij,k} = \{x \in V_n \mid x_j - x_i = k\}$.

For any root $\alpha \in A_{n-1}$, we have $\langle \alpha, \alpha \rangle = 2$. Therefore, $\Phi^\vee = \Phi$. The coroot lattice is thus spanned by the (simple) roots, and we have $\mathbb{Z}\Phi^\vee = \{x \in \mathbb{Z}^n \mid \sum x_i = 0\}$.

4.2. Faces of the finite Coxeter complex of type A . A *composition* of a set I is a sequence (S_1, \dots, S_k) of disjoint nonempty subsets of I such that $S_1 \cup \cdots \cup S_k = I$. Each S_i is a *block* of the composition. We represent such a composition by means of a string as follows

$$\textcircled{S_1} - \textcircled{S_2} - \cdots - \textcircled{S_k}.$$

The faces of the Coxeter complex $\Sigma(A_{n-1})$ are in correspondence with compositions of the set $[n]$. A composition corresponds to the face determined by the (in)equalities

$$\begin{aligned} x_i &= x_j \text{ if } i \text{ and } j \text{ belong to the same block,} \\ x_i &< x_j \text{ if the block of } i \text{ precedes that of } j. \end{aligned}$$

The faces in $\Sigma(A_2)$, labeled with set compositions, are shown in Figure 18. The picture shows the lines in $\mathcal{H}(A_2)$ drawn in $V_3 \cong \mathbb{R}^2$.

The partial order on faces (given by inclusion of closures) corresponds to refinement of set compositions. The finest set compositions are those for which all blocks are singletons. The chambers thus correspond to linear orders on $[n]$, which in turn are identified with permutations of $[n]$.

With the preceding choices, the fundamental chamber corresponds to the identity permutation, and the action of the Weyl group $W(A_{n-1})$ on chambers corresponds to the action of S_n on itself by left multiplication. More generally, the action of the Weyl group on faces

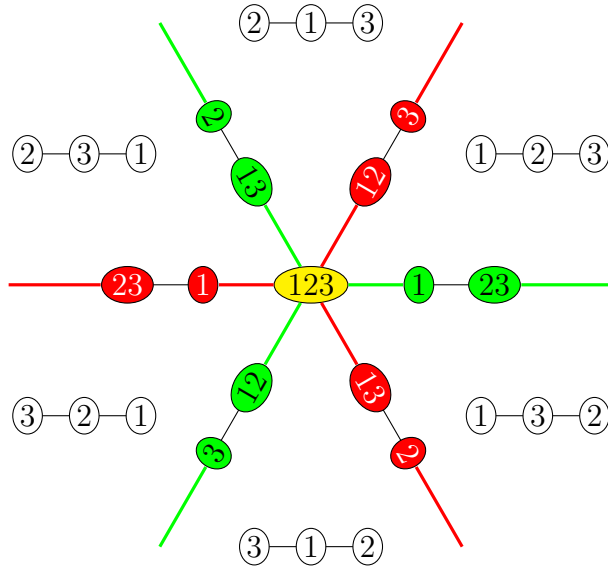


FIGURE 18. The faces in $\Sigma(A_2)$, with colors corresponding to W -orbits.

translates as follows: if the face F corresponds to the composition (S_1, \dots, S_k) , then $w \cdot F$ corresponds to $(w(S_1), \dots, w(S_k))$. The color set of F is

$$\text{col}(F) = \{a_1, a_1 + a_2, \dots, a_1 + a_2 + \dots + a_{k-1}\} \subseteq \{1, \dots, n-1\},$$

where $a_i := |S_i|$. The faces of color J constitute the W -orbit Σ_J and correspond to the set compositions for which the partial sums of the block sizes yield the elements of J . Figure 18 shows the orbits in $\Sigma(A_2)$. For example, the rays in red constitute the orbit of color set $\{2\}$, corresponding to the set compositions with block sizes $(2, 1)$.

The permutation w_F associated to the face F as in (5) (or as in Proposition 2) is obtained by listing the blocks in the given order, and writing the elements in each block in increasing order. For example, if $F \in \Sigma(A_4)$ corresponds to the composition $(134) \text{---} (5) \text{---} (2)$, then $w_F = 13452 \in S_5$.

The sign vector of a face F has entries

$$(28) \quad \sigma_{ij}(F) = \begin{cases} 0 & \text{if } i \text{ and } j \text{ are in the same block,} \\ + & \text{if the block of } i \text{ precedes that of } j, \\ - & \text{if the block of } i \text{ succeeds that of } j. \end{cases}$$

4.3. Faces of the Steinberg torus of type A . Let I be a finite set. A *spin necklace on I* consists of a partition of the set I (into disjoint nonempty blocks), a cyclic order on the set of blocks, and a labeling of the edges of the cycle with integers from 1 to $n = |I|$ satisfying the following condition. Let B be a block of the partition and i and j the labels of the edges of the cycle incident to B , with i coming before j when we read the labels according to the cyclic order. Then

$$(29) \quad j \equiv i + |B| \pmod{n}.$$

In our pictures, the cyclic order is always represented clockwise. For instance, the following is a spin necklace on the set $[6]$:



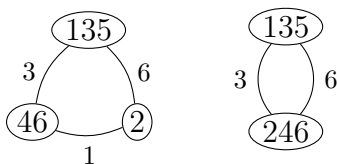
The following is a spin necklace which differs from the previous one only in the labeling:



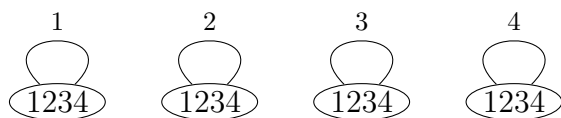
Given a partition and a cyclic order on its blocks, a spin necklace is uniquely determined by the choice of one edge label (the remaining labels being determined by (29)), and this choice is arbitrary. Note also that all edge labels in a spin necklace are distinct modulo n . This is because as we traverse the cycle from one edge to another, the labels increase by the total size of the intermediate blocks, which is an integer strictly between 0 and n .

A spin necklace has a distinguished block, the *clasp*, defined as follows. Choose the edge labels from $\{1, \dots, n\}$. Since the edge labels increase cyclically, the maximum label occurs immediately before the minimum label somewhere along the necklace. This intermediate block is the clasp of the necklace. In example (30), the clasp is the block $\{1, 3, 5\}$. In example (31), the clasp is $\{2\}$. The clasp may occur at a block of any size.

Given a spin necklace, we can contract any edge (and remove its label) to obtain another spin necklace. If the edge connected blocks B and C , then in the new spin necklace there is one new block $B \cup C$. Note that condition (29) is verified by the new necklace. For example, the spin necklace on the right below is obtained by contracting one edge from the one on the left.

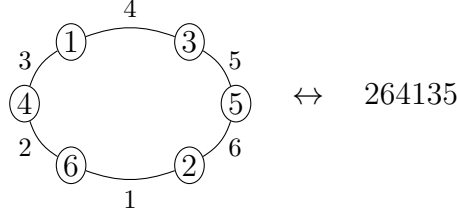


There is a partial order on the set of spin necklaces on I in which a spin necklace precedes another if the former can be obtained from the latter by contracting some of its edges. This is such that given a spin necklace, the elements below it form a poset isomorphic to the poset of nonempty subsets of its edge set. The minimal elements of this partial order are the spin necklaces with one block; there are n of them. For example, when $n = 4$ they are:



The maximal elements are the spin necklaces for which each block is a singleton. There are then n edges and each integer from 1 to n occurs as a label. If we read the elements in the blocks starting at the clasp and proceeding cyclically, we obtain a bijection between the set

of maximal elements and permutations of $[n]$.

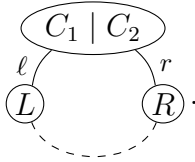


The faces of the Steinberg torus $\overline{\Sigma}(A_{n-1})$ are in correspondence with spin necklaces on the set $[n]$, by means of the following procedure.

Given a spin necklace, first locate its clasp C . Let ℓ and r be the incoming and outgoing labels, respectively (with respect to the cyclic order). Then, by (29), we have

$$\ell + |C| = r + n.$$

Split C into two blocks C_1 and C_2 , consisting respectively of the first $n - \ell$ elements of C and of the last r elements. (The elements of C are ordered by the standard order of $[n]$.) Note that C_1 may be empty (since ℓ may be n), but C_2 may not. Schematically,



We then turn the necklace into a string by listing the blocks in the given cyclic order, starting with C_2 and ending with C_1 (and dropping the edge labels):

$$\textcircled{C_2} - \textcircled{R} - \cdots - \textcircled{L} - \textcircled{C_1}.$$

We refer to this list as the *split necklace*. If C_1 is nonempty, the split necklace is a composition of $[n]$. If C_1 is empty, we keep track of this event by displaying the list as follows:

$$\textcircled{C_2} - \textcircled{R} - \cdots - \textcircled{L} - \text{---}.$$

Note that the spin necklace can be reconstructed from the split necklace.

The given spin necklace corresponds to the $\mathbb{Z}\Phi^\vee$ -orbit of the affine face determined by the (in)equalities

$$\begin{aligned} x_i &= x_j \text{ if } i \text{ and } j \text{ belong to the same block,} \\ x_i &< x_j \text{ if the block of } i \text{ precedes that of } j, \\ x_i &= x_j + 1 \text{ if } i \text{ belongs to } C_1 \text{ and } j \text{ to } C_2, \\ x_i &< x_j + 1 \text{ if } C_1 = \emptyset, i \text{ belongs to } L \text{ and } j \text{ to } C_2. \end{aligned}$$

The blocks in question and their relative order are those of the split necklace.

For the spin necklaces (30) and (31), the split necklaces are

$$\textcircled{5} - \textcircled{2} - \textcircled{46} - \textcircled{13} \quad \text{and} \quad \textcircled{2} - \textcircled{46} - \textcircled{135} - \text{---},$$

and the conditions defining the faces are

$$x_5 < x_2 < x_4 = x_6 < x_1 = x_3 = x_5 + 1$$

and

$$x_2 < x_4 = x_6 < x_1 = x_3 = x_5 < x_2 + 1,$$

respectively.

The faces in $\overline{\Sigma}(A_2)$, labeled with spin necklaces, are shown in Figure 19.

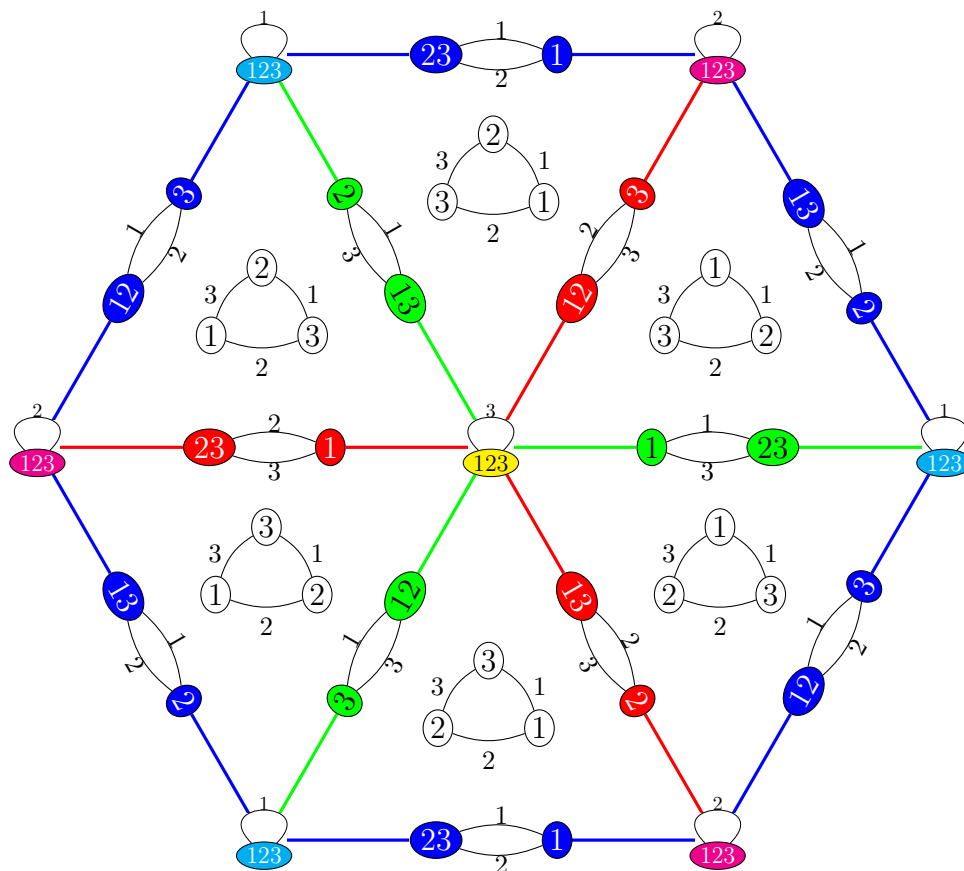


FIGURE 19. The faces of the Steinberg torus $\overline{\Sigma}(A_2)$, with colors corresponding to W -orbits. Note the identifications along the boundary.

The partial order on faces (given by inclusion of closures) corresponds to the partial order on spin necklaces given by edge contraction. On the other hand, sliding consecutive beads (blocks) past each other in the necklace corresponds to walking between adjacent chambers.

A permutation w acts on a spin necklace by changing each block B into $w(B)$, and keeping the edge labels. This corresponds to the action of the Weyl group on faces of the torus, and the set of edge labels of the necklace corresponds to the color set of the face (under the identification between $\tilde{\Delta}$ and $\{1, \dots, n\}$.) The orbits are thus parametrized by nonempty subsets of $\{1, \dots, n\}$, with the orbit $\overline{\Sigma}_J$ consisting of the spin necklaces with edge label set J . Figure 19 shows the orbits in $\overline{\Sigma}(A_2)$. For example, the edges in red constitute the orbit of color set $\{2, 3\}$.

The permutation associated to the torus face as in (16) (or as in Corollary 6) is obtained by listing the blocks in the split necklace from left to right, and writing the elements in each block in increasing order. For example, the permutations associated to the faces (spin necklaces) (30) and (31) are 524613 and 246135, respectively.

Recall an affine face F has both an *expanded* and a *compact* sign vector (see Equations (10) and (12)). For each pair $i < j$, there is a critical value $k = k_{ij}(F)$ (see (11)), and the compact sign vector consists of the values $k_{ij}(F)$ along with the signs relative to the corresponding hyperplanes, which are either $+$ or 0 . In terms of spin necklaces these signs are simple:

$$(32) \quad \sigma_{ij,k}(F) = \begin{cases} 0 & \text{if } i \text{ and } j \text{ are in the same block,} \\ + & \text{if } i \text{ and } j \text{ are in different blocks.} \end{cases}$$

4.4. Products of faces in type A. We turn to a combinatorial description for the right module structure of $\overline{\Sigma}(A_{n-1})$ over $\Sigma(A_{n-1})$.

First, recall that the product of two faces in the finite Coxeter complex admits the following description. Let F and $G \in \Sigma(A_{n-1})$ be the faces corresponding to the compositions $(S_1) \cdots (S_p)$ and $(T_1) \cdots (T_q)$ of $[n]$. Then the Tits product of $FG \in \Sigma(A_{n-1})$ corresponds to the composition of $[n]$ whose blocks are the pairwise intersections $S_i \cap T_j$, arranged lexicographically:

$$(S_1 \cap T_1) \cdots (S_1 \cap T_q) \cdots (S_p \cap T_1) \cdots (S_p \cap T_q),$$

with the understanding that empty intersections are removed from the string.

For example,

$$(3567) \text{---} (4) \text{---} (12) \cdot (26) \text{---} (35) \text{---} (17) \text{---} (4) = (6) \text{---} (35) \text{---} (7) \text{---} (4) \text{---} (2) \text{---} (1).$$

The agreement between the geometric definition of the Tits product and the combinatorial procedure is illustrated in Figure 20.

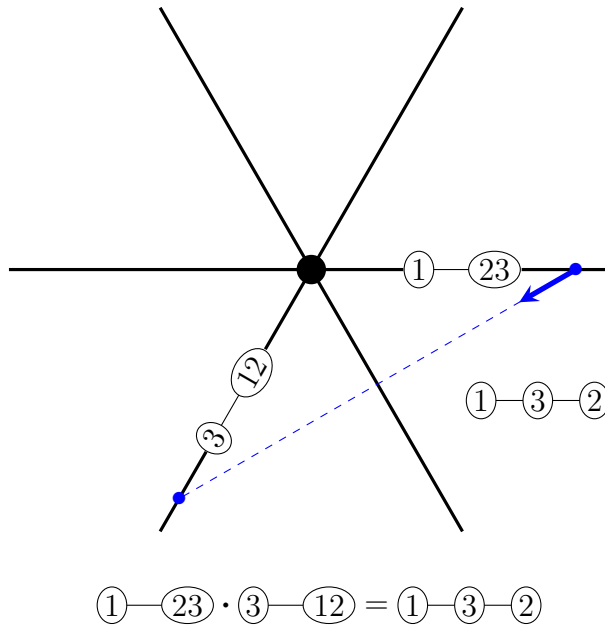


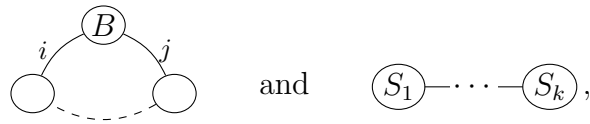
FIGURE 20. The product of two faces of $\Sigma(A_2)$.

The product of a face of the Steinberg torus by a face of the finite Coxeter complex admits a similar description.

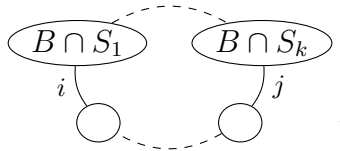
Proposition 13. *Let $F \in \overline{\Sigma}(A_{n-1})$ be a face of the Steinberg torus and $G \in \Sigma(A_{n-1})$ a face of the Coxeter complex. Let (S_1, \dots, S_k) be the composition of $[n]$ corresponding to G . To obtain the spin necklace on $[n]$ corresponding to the face $FG \in \overline{\Sigma}(A_{n-1})$, replace each block B in the spin necklace of F by the string of intersections $(B \cap S_1, \dots, B \cap S_k)$, removing those that are empty. The incoming edge to B is now incoming to $B \cap S_1$, and the outgoing edge from B is now outgoing from $B \cap S_k$. The label of the intermediate edges is uniquely determined by (29).*

This proposition can be proved by considering the effect of the Tits product on sign vectors (Equation (1)) and employing (28) and (32) for translating set compositions and spin necklaces into sign vectors.

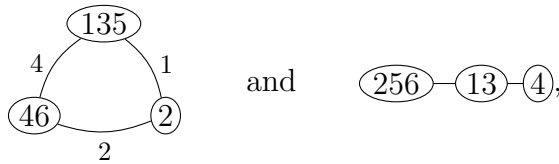
Schematically, if F and G correspond respectively to



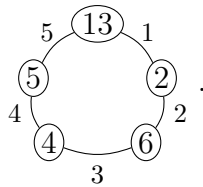
then FG corresponds to



For a concrete example, let $F \in \overline{\Sigma}(A_4)$ and $G \in \Sigma(A_4)$ be the faces corresponding to



respectively. Then the face $FG \in \overline{\Sigma}(A_4)$ corresponds to



The agreement between the geometric definition and the combinatorial procedure is illustrated in Figure 21.

4.5. The root system C_n .

$$C_n := \{\pm 2\varepsilon_i \mid 1 \leq i \leq n\} \cup \{a\varepsilon_i + b\varepsilon_j \mid 1 \leq i \neq j \leq n, a, b = \pm 1\}.$$

It contains $2n^2$ vectors. The ambient space is \mathbb{R}^n .

Let $[-n, n]$ denote the interval $\{-n, \dots, -1, 0, 1, \dots, n\}$. The Weyl group of C_n may be identified with the wreath product $S_2 \wr S_n$, or with the group of permutations w of $[-n, n]$ such that $w(-i) = -w(i)$ for all $i \in [-n, n]$. Note that such a permutation has $w(0) = 0$

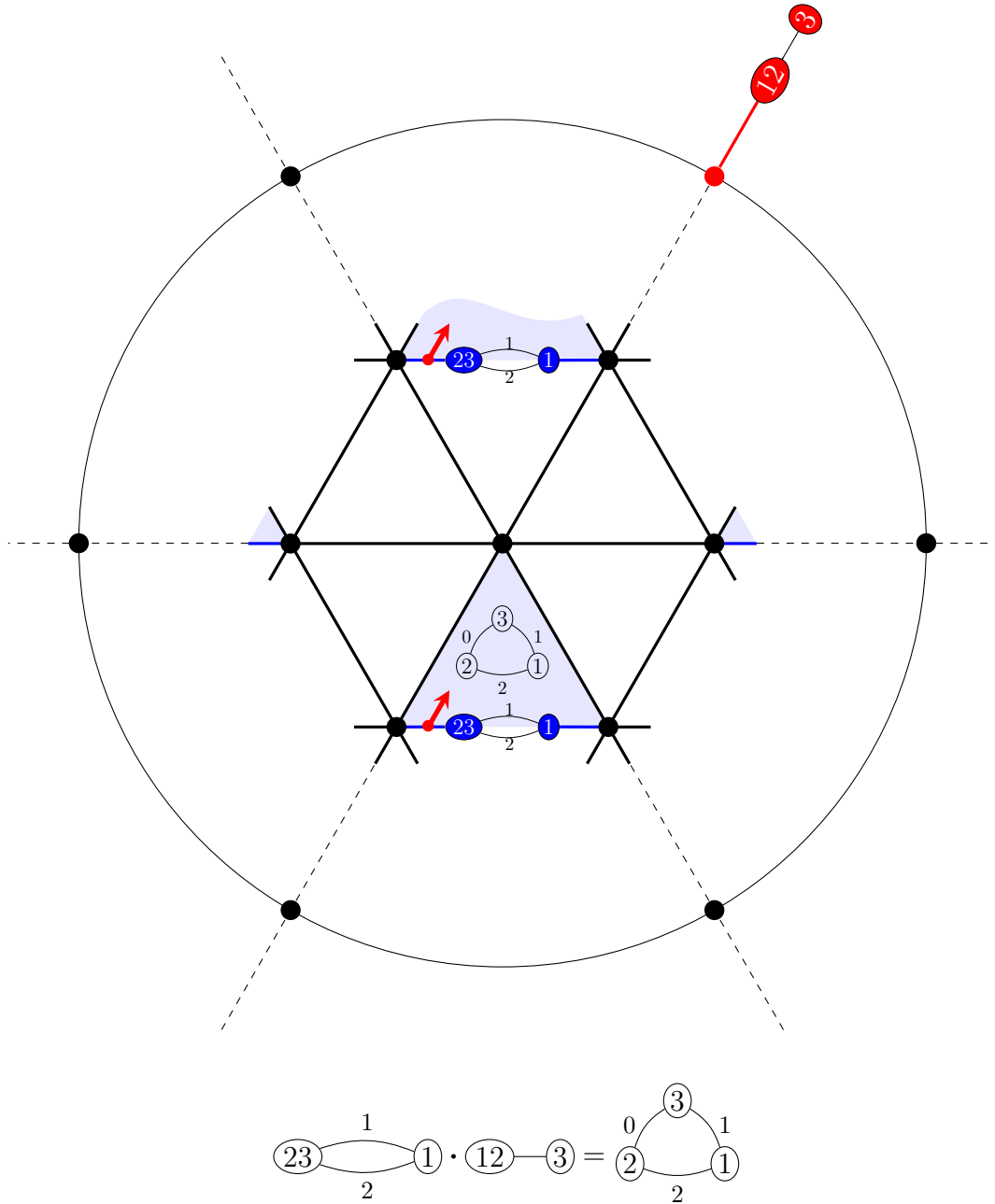


FIGURE 21. The product of a face of $\overline{\Sigma}(A_2)$ with a face of $\Sigma(A_2)$.

and is determined by the list of values $w_i = w(i)$, $i = 1, \dots, n$. In examples, we represent negatives with bars, so that $w = 25\bar{1}43$ stands for the following permutation of $[-5, 5]$:

$$\begin{pmatrix} -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ -3 & 4 & 1 & -5 & -2 & 0 & 2 & 5 & -1 & -4 & 3 \end{pmatrix}.$$

We also adopt the convention that $w_{n+1} = w_0 = 0$.

For the simple roots, we choose $\Delta = \{\alpha_1, \dots, \alpha_n\}$ where

$$\alpha_1 := 2\varepsilon_1 \quad \text{and} \quad \alpha_i := \varepsilon_i - \varepsilon_{i-1}$$

for $2 \leq i \leq n$. The set of positive roots is then

$$\Pi = \{2\varepsilon_i \mid 1 \leq i \leq n\} \cup \{\varepsilon_i \pm \varepsilon_j \mid i > j\}.$$

The highest root is $\tilde{\alpha} = \alpha_1 + 2 \sum_{i=2}^n \alpha_i = 2\varepsilon_n$. See Figure 7 for an illustration.

We identify Δ with $\{0, 1, \dots, n-1\}$ and $\tilde{\Delta}$ with $\{0, 1, \dots, n\}$ via

$$\alpha_0 \leftrightarrow n \quad \text{and} \quad \alpha_i \leftrightarrow i-1$$

for $i = 1, \dots, n$. With these choices, the descent sets (ordinary and affine) of w are

$$D(w) = \{i \mid 0 \leq i \leq n-1, w_i > w_{i+1}\} \quad \text{and} \quad \tilde{D}(w) = \{i \mid 0 \leq i \leq n, w_i > w_{i+1}\}.$$

The integers in $[-n, n]$ are ordered in the standard fashion. For example, $\tilde{D}(25\bar{1}43) = \{2, 3, 5\}$ and $\tilde{D}(25\bar{1}43) = \{0, 2, 3, 5\}$.

The Coxeter arrangement $\mathcal{H}(C_n)$ consists of the coordinate hyperplanes $\{x \in \mathbb{R}^n \mid x_i = 0\}$, $i = 1, \dots, n$, and the hyperplanes $\{x \in \mathbb{R}^n \mid x_i = \varepsilon x_j\}$, where $1 \leq i < j \leq n$ and $\varepsilon = \pm 1$.

The set C_n^\vee of coroots is the root system B_n .

4.6. Faces of the finite Coxeter complex of type C . The faces of the Coxeter complex $\Sigma(C_n)$ are in correspondence with compositions of the set $[-n, n]$ with the following property: reversing the order of the list of blocks has the same effect as replacing each block B by its negative $-B := \{-i \mid i \in B\}$. Such a composition has an odd number of blocks with the middle block being equal to its negative and containing 0. For example,

$$\textcircled{5\bar{1}3} - \textcircled{\bar{4}} - \textcircled{202} - \textcircled{4} - \textcircled{\bar{3}15}.$$

is such a composition of $[-5, 5]$.

Given an integer i , write

$$x_i := \varepsilon(i)x_{|i|},$$

where $\varepsilon(i) = \pm 1$ and $|i|$ denote the sign and the absolute value of i . With this convention, a composition as above corresponds to the face in $\Sigma(C_n)$ determined by the (in)equalities

$$\begin{aligned} x_i &= x_j \text{ if } i \text{ and } j \text{ belong to the same block,} \\ x_i &< x_j \text{ if the block of } i \text{ precedes that of } j. \end{aligned}$$

The faces in $\Sigma(C_2)$, labeled with compositions, are shown in Figure 22.

Let $F \in \Sigma(C_n)$ be a face. The color set of F is

$$\text{col}(F) = \{a_0, a_0 + a_1, \dots, a_0 + a_1 + \dots + a_{k-1}\} \subseteq \{0, 1, \dots, n-1\},$$

where a_0 is the number of positive elements in the block of 0, a_i is the size of the i -th block as we proceed outwards (either to the right or to the left), and the $(k-1)$ -th is the penultimate block we encounter. Figure 22 shows the orbits in $\Sigma(C_2)$. For example, the rays in red constitute the orbit of color set $\{1\}$ and the rays in green that of color set $\{0\}$. The permutation w_F (as in (5), or as in Proposition 2) is obtained by first listing the positive elements in the block of 0 in increasing order, then proceeding to the right and listing all the elements in each block in increasing order.

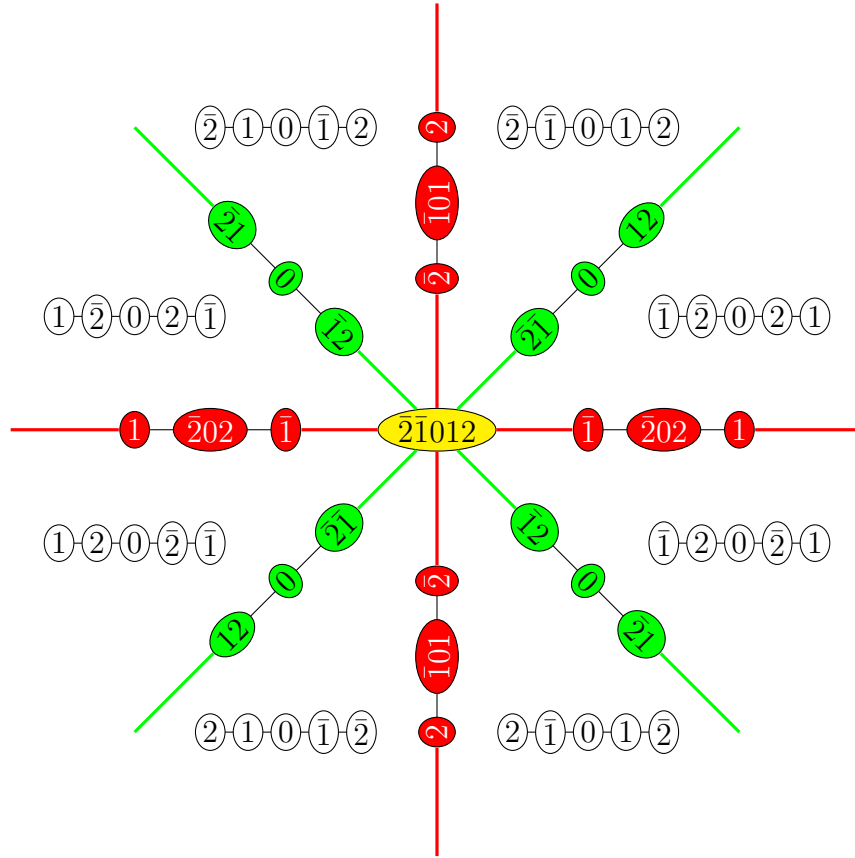


FIGURE 22. The faces in $\Sigma(C_2)$, with colors corresponding to W -orbits.

4.7. **Faces of the Steinberg torus of type C .** Consider a partition of the set $[-n, n]$ and a cyclic order on the set of blocks, with the property that replacing each block by its negative has the same effect as reversing the order. In such a partition the block containing 0 is equal to its negative. We represent such structures by drawing each block as a vertex of a regular polygon and proceeding clockwise according to the cyclic order. The result is a necklace with the property that flipping it across the diameter through 0 has the same effect as replacing each block by its negative. Here is an example, for $n = 5$:



The faces of the Steinberg torus $\bar{\Sigma}(C_n)$ are in correspondence with such necklaces. We leave to the reader the task of describing the equations of (an affine representative) determining a face. The faces in $\bar{\Sigma}(C_2)$, labeled with such necklaces, are shown in Figure 23. Inclusion of faces corresponds to edge contractions, with the understanding that if you contract an edge, you must also contract its mirror image.

Let $F \in \bar{\Sigma}(C_n)$ be a face. The color set of F is

$$\text{col}(F) = \{a_0, a_0 + a_1, \dots, a_0 + a_1 + \dots + a_{k-1}\} \subseteq \{0, 1, \dots, n\},$$

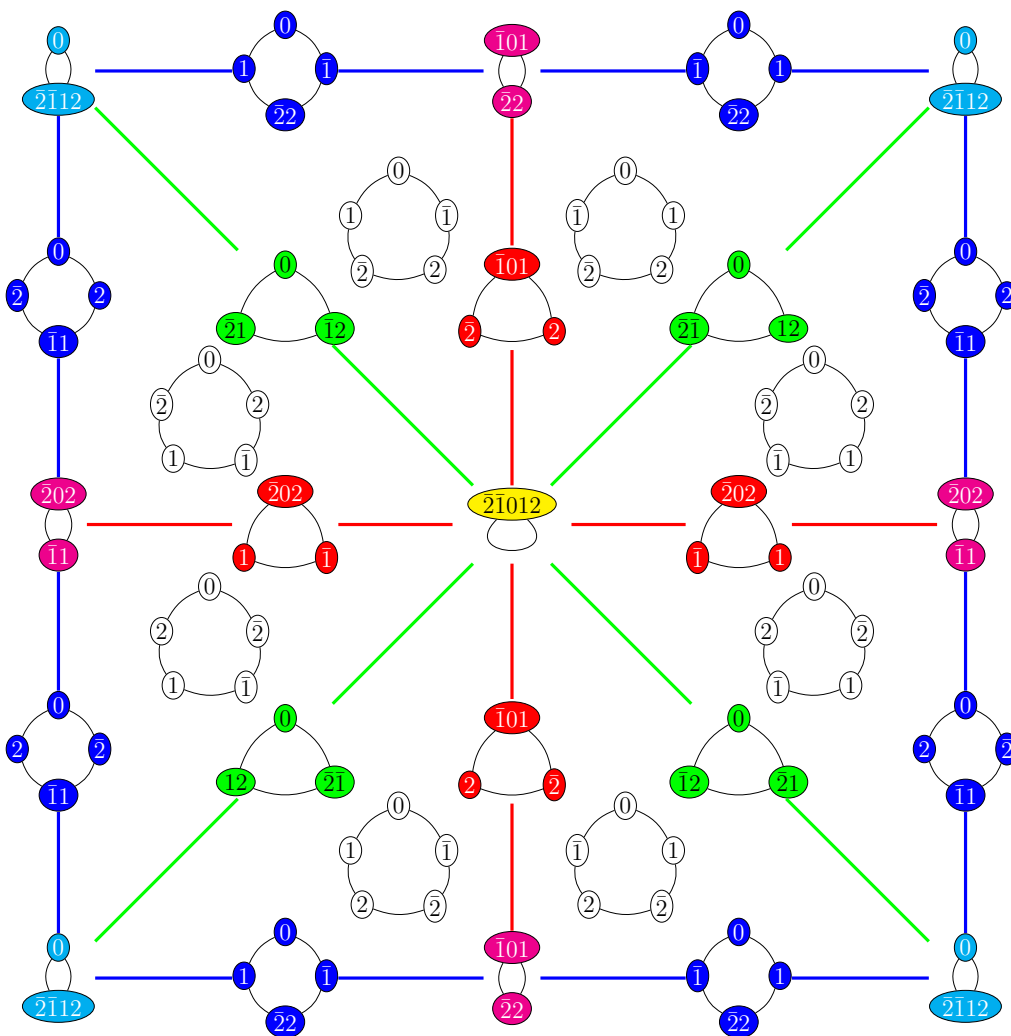


FIGURE 23. The faces of the Steinberg torus $\overline{\Sigma}(C_2)$, with colors corresponding to W -orbits. Note the identifications along the boundary.

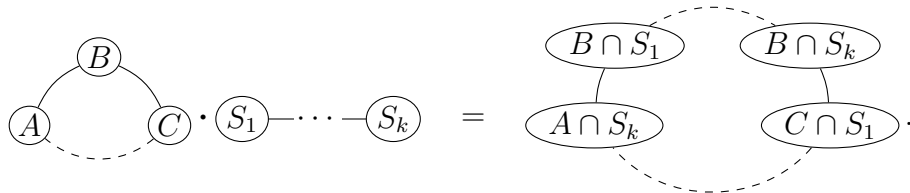
where a_0 is the number of positive elements in the block of 0, a_i is the size of the i -th block as we proceed around the corresponding necklace (either clockwise or counterclockwise), and the $(k - 1)$ -th is the last block we encounter before reaching the point diametrically opposed to the block of 0. Figure 23 shows the orbits in $\Sigma(C_2)$. For example, the edges in red constitute the orbit of color set $\{1, 2\}$ and the edges in green that of color set $\{0, 2\}$. The permutation w_F (as in (16), or as in Corollary 6) is obtained by first listing the positive elements in the block of 0 in increasing order, then proceeding clockwise and listing all the elements in each block in increasing order, stopping at the point diametrically opposed to the block of 0. If there is a block at that location, we list only its negative elements, in increasing order. For example, if F corresponds to the necklace (33), then $\text{col}(F) = \{1, 2, 4\}$ and $w_F = 24\bar{3}15$.

4.8. Products of faces in type C . The product of two faces in the finite Coxeter complex of type C admits the same description as in type A : we list the nonempty intersections of

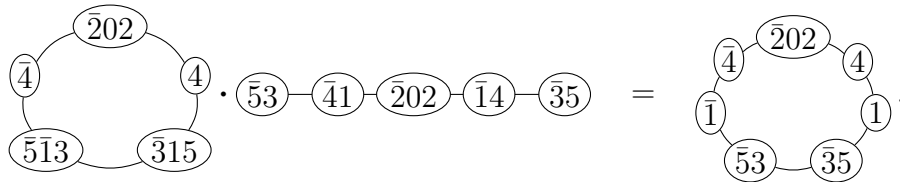
the blocks lexicographically. For example,

$$\overline{2135} - \overline{404} - \overline{5\bar{3}\bar{1}2} \cdot \overline{135} - \overline{4\bar{2}024} - \overline{5\bar{3}\bar{1}} = \overline{15} - \overline{2} - \overline{3} - \overline{404} - \overline{3} - \overline{2} - \overline{5\bar{1}}.$$

The product of a face of the Steinberg torus by a face of the finite Coxeter complex admits the following description, similar to that in type A : we replace each block in the necklace representing a torus face by the string of intersections with the blocks of the composition representing a face in the Coxeter complex. Schematically,



For example,



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