THE HEISENBERG PRODUCT OF SYMMETRIC FUNCTIONS AND RELATED ALGEBRAIC STRUCTURES

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Abstract. Many related products and coproducts (e.g. Hadamard, Cauchy, Kronecker, induction, internal, external, Solomon, composition, Malvenuto–Reutenauer, convolution, etc.) have been defined in the following objects : species, representations of the symmetric groups, symmetric functions, endomorphisms of graded connected Hopf algebras, permutations, non–commutative symmetric functions, quasi-symmetric functions, etc. With the purpose of simplifying and unifying this diversity we introduce yet, another –non graded– product the *Heisenberg product*, that for the highest and lowest degrees produces the classical external and internal products (and their namesakes in different contexts). In order to define it, we start from the two opposite more general extremes: species in the "commutative context", and endomorphisms of Hopf algebras in the "non–commutative" environment. Both specialize to the space of commutative symmetric functions where the definitions coincide. We also deal with the different coproducts that these objects carry –to which we add the *Heisenberg coproduct* for quasi-symmetric functions–, and study their Hopf algebra compatibility particularly for symmetric and non-commutative symmetric functions. We obtain combinatorial formulas for the structure constants of the new product that extend, generalize and unify results due to Garsia, Remmel, Reutenauer and Solomon. In the space of quasi-symmetric functions, we describe explicitly the new operations in terms of alphabets.

Contents

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Introduction

0.1. **General description of the paper.** Figure 1 displays the spaces and categories that enter our work. A central place is reserved for the space of symmetric functions. Table 1 lists various products that exist on these spaces and that are frequently treated in the literature, including the internal and external products of symmetric functions. In these spaces we introduce a new operation, that we call the *Heisenberg product* (except in the space of quasi-symmetric functions, where a dual structure is defined). The focus of our work is on the construction of this operation and on the connections among its different incarnations across these spaces.

The spaces under consideration carry a natural grading. The internal and external products (or their namesakes as depicted in Table 1) interact with the grading as follows: the former operates among elements of a fixed degree, while degrees add up under the latter. By contrast, the Heisenberg product of two homogeneous elements is a sum of elements of various degrees. The component of highest degree turns out to coincide with the external product, and that of lowest degree with the internal product (when the factors are of the same degree). This is what we call the *interpolation property* of the Heisenberg product. It holds in each of the spaces in Figure 1 (in dual form in the space of quasi-symmetric functions).

The name we choose for the new product originates in the theory of (graded) Hopf algebras. The latter enter in the top right corner in Figure 1.

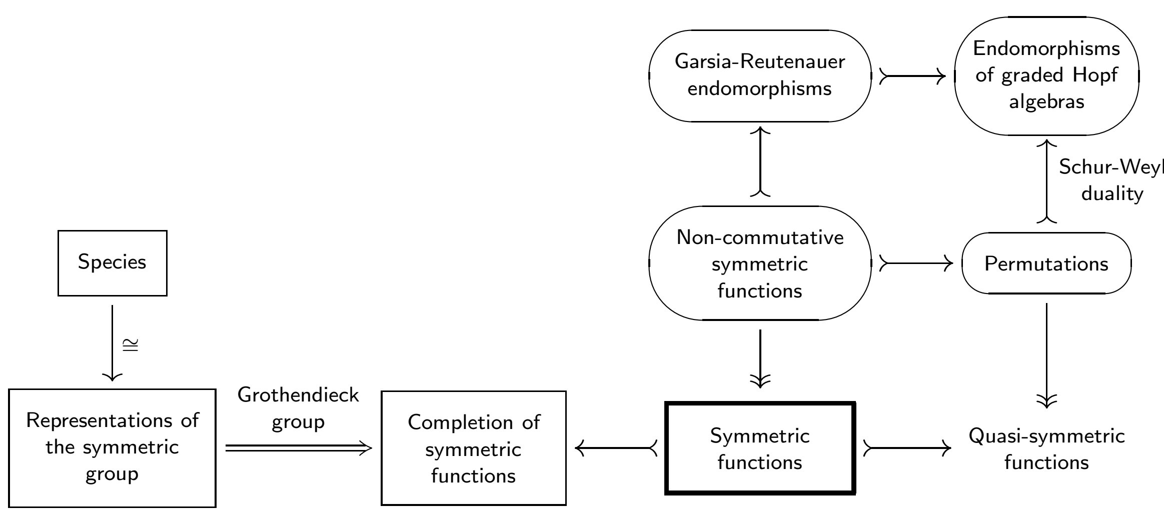

FIGURE 1. Context for the Heinsenberg product. F:spaces

The spaces are organized into three contexts, indicated by different kinds of boxes in Figure 1. The context consisting of the spaces in square boxes is studied first. We refer to it, somewhat loosely, as the commutative context. We start by constructing the Heisenberg product in the category of species, or equivalently of representations of the symmetric groups, and from there we transport it to the space of symmetric functions. We use the Grothendieck group functor and the Frobenius characteristic map to transition from species to symmetric functions. This material comprises Sections 1 through 3. We give an explicit combinatorial formula for the Heisenberg product on the basis of complete homogeneous symmetric functions. This formula is employed later, in Section 8, to bridge betweeen the commutative and non-commutative contexts. We also provide an explicit formula for the structure constants of the Heisenberg product on the basis of power sums. The problem of finding a combinatorial description for the structure constants on the basis of Schur functions arises naturally. We do not undertake this question. These constants contain as extreme cases the Littlewood-Richardson coefficients and the Kronecker coefficients. Finding a combinatorial description for the latter is a notorious open problem.

Speak of the Rota-Stein Cliffordization process.

The non-commutative context comprises the oval boxes. We begin its analysis in Section 4 by considering the Heisenberg product in the space of linear endomorphisms of a graded Hopf algebra. This product is well-known in the theory of Hopf algebras. By specializing to the tensor Hopf algebra of a generic vector space and employing Schur-Weyl duality, a new product on the vector space freely generated by permutations results. We carry out this construction in Section 6. By extending a construction of Garsia and Reutenauer, we introduce in Section 5 a certain canonical subspace of the space of endomorphisms, called the space of descents (or Garsia–Reutenauer) endomorphisms of a Hopf algebra. This subspace is closed under the Heisenberg product. These two constructions are confluent when restricted to the space of non-commutative symmetric functions, as verified in Section 7. From this perspective, the closure of non-commutative symmetric functions under the Heisenberg product of permutations becomes a result in Hopf algebra theory. We present a second proof of this fact, based upon an explicit calculation of the Heisenberg product on the basis X_α of complete homogeneous non-commutative symmetric functions. The index α is an integer composition. By the interpolation property, the combinatorial formula thus obtained contains as special cases explicit descriptions for the product in Solomon's descent algebra and for the external product of non-commutative symmetric functions X_α. One readily verifies that the former is precisely the well-known rule of Garsia, Remmel, Reutenauer, and Solomon, and the latter is simply the formula giving the external product in terms of concatenation of compositions. In this manner, two different products –Solomon's and the external product–, become connected by the introduction of the new Heisenberg product.

The two contexts have as common ground the bold square box consisting of the space of symmetric functions. We show in Section 8 that abelianization turns the Heisenberg product of non-commutative symmetric functions into that of symmetric functions.

The spaces of non-commutative and commutative symmetric functions carry a well-known coalgebra structure, compatible with the external product. We show that this structure remains compatible with the Heisenberg product, resulting in each case in a (non-graded) Hopf algebra structure. This work is done in Sections 9 and 10.

The remaining corner in Table 1 consists of the space of quasi-symmetric functions, which is dual to the space of non-commutative symmetric functions. In Section 11, we construct by means of this duality the Heisenberg coproduct of quasi-symmetric functions. This coproduct admits a description in terms of alphabets which agains bring forth the connection with the Cliffordization process.

In Section 12, we present the proofs of three technical lemmas.

0.2. **Terminology and notation.** The spaces we consider carry at least two well-known products. Although they are closely related by the inclusions, projections, and isomorphisms in Figure 1, mathematical developments have given them non consistent names in many cases. Table 1 summarizes the standard nomenclature employed in the literature.

	Species	Representations	Symmetric functions	Non-commutative symmetric functions	Permutations	Endomorphisms of graded Hopf algebras
internal product	Hadamard $(\times)$	Kronecker $(*)$	internal $(*)$	Solomon $(*)$	composition $(\circ)$	composition $(\circ)$
external product	Cauchy $(\cdot)$	induction $(\cdot)$	external $(\cdot)$	external $(\cdot)$	Malvenuto-Reutenauer $(\star)$	convolution $(\star)$

ble-of-names

Table 1. Standard terminology and notation for the internal and external product.

The new product we introduce in this article will be called the *Heisenberg product*, and denoted by $\#$, in all the different contexts.

When dealing with results from *Hopf algebra* theory, we adopt the usual notations in the area as presented for example in [22] or [25]. We write $\Delta(h) = \sum h_1 \otimes h_2$ for the coproduct and $S(h)$ for the antipode of a Hopf algebra H. An element h is said to be primitive if $\Delta(h) = h \otimes 1 + 1 \otimes h$. The space of primitive elements is denoted by $\mathrm{Prim}(H)$.

For basic notions from the theory of *symmetric group representantions* and *symmetric functions*, some of the standard references are [19, 28, 31, 35]. For the theory of *species*, we refer to [2, 5, 16]; for the theory of *non-commutative symmetric functions*, to [11, 33], and for quasi-symmetric functions to [12, 20, 31].

The set $\{1, \dots, n\}$ is denoted by $[n]$ and the group of the permutations of $[n]$ by S_n. A permutation $\sigma \in S_n$ will be frequently given through the list of its values $\sigma(1), \dots, \sigma(n)$.

We write $(a_1, \dots, a_r) \models n$ to indicate that the a_i are positive integers that add up to n, and say that the sequence $(a_1, \dots, a_r)$ is a composition of n.

We write $I = S \sqcup T$ when S and T are disjoint subsets of a set I whose union is I:

$$I = S \sqcup T \iff I = S \cup T \quad \text{and} \quad S \cap T = \emptyset.$$

We work over a field of characteristic zero, denoted by $\Bbbk$.

S:species

1. The Heisenberg product of species

We introduce the notion of *Heisenberg product* of two species, generalizing the familiar Cauchy and Hadamard products. We cover the necessary background; for additional information see [2, 3, 5, 16].

1.1. **Cauchy, Hadamard, and Heisenberg.** Let $\mathbf{set}^\times$ be the category of finite sets with bijections as morphisms, and $\mathbf{Vec}$ the category of vector spaces over $\Bbbk$ with linear transformations as morphisms.

The *category of species*, denoted by $\mathbf{Sp}$, is the category of functors from $\mathbf{set}^\times$ to $\mathbf{Vec}$, with natural transformations as morphisms.

An object in $\mathbf{Sp}$ (a *species*) is a functor $\mathsf{p} : \mathbf{set}^\times \to \mathbf{Vec}$. The image of a finite set I under the species p is denoted by $\mathsf{p}[I]$, and that of a bijection σ by $\mathsf{p}[\sigma]$. The former is a vector space and the latter an (invertible) linear map. The image of the set $[n]$ is denoted by $\mathsf{p}[n]$. A morphism $f : \mathsf{p} \to \mathsf{p}'$ consists of a family of linear maps $f_I : \mathsf{p}[I] \to \mathsf{p}'[I]$, natural with respect to bijections. The category **Sp** inherits the abelian structure of **Vec**.

The following are some of the simplest species.

(1) The species $\mathbf{1}$: $\mathbf{1}[\emptyset] = \Bbbk$ and $\mathbf{1}[I] = 0$ for $I \neq \emptyset$.
(2) The *exponential* species $\mathbf{E}$: $\mathbf{E}[I] = \Bbbk$ for all I.

These species map all bijections to identities.

Definition 1.1. The *Heisenberg product* of two species p and q is the species $\mathsf{p} \# \mathsf{q}$ whose value on a finite set I is

$$(\mathsf{p} \# \mathsf{q})[I] = \bigoplus_{I = S \cup T} \mathsf{p}[S] \otimes \mathsf{q}[T]. \tag{1}$$

D:smash-spec

The direct sum is over all ordered pairs (S, T) of subsets of I such that $I = S \cup T$. The subsets are not required to be nonempty or disjoint from each other.

To complete the definition, we must specify the value of $\mathsf{p} \# \mathsf{q}$ on bijections. Given two finite sets I and J and a bijection $\sigma : I \to I'$, we have a bijection $(S, T) \mapsto \big(\sigma(S), \sigma(T)\big)$ between pairs (S, T) with $I = S \cup T$ and pairs (S', T') with $I' = S' \cup T'$. The map $(\mathsf{p} \# \mathsf{q})[\sigma] : (\mathsf{p} \# \mathsf{q})[I] \to (\mathsf{p} \# \mathsf{q})[I']$ is defined to be the direct sum of the maps

$$\mathsf{p}[\sigma|_S] \otimes \mathsf{q}[\sigma|_T] : \mathsf{p}[S] \otimes \mathsf{q}[T] \to \mathsf{p}[S'] \otimes \mathsf{q}[T'].$$

The Heisenberg product is also defined on morphisms of species. Given morphisms $f : \mathsf{p} \to \mathsf{p}'$ and $g : \mathsf{q} \to \mathsf{q}'$, the map $(f \# g)_I : (\mathsf{p} \# \mathsf{q})[I] \to (\mathsf{p}' \# \mathsf{q}')[I]$ is the direct sum of the maps

$$f_S \otimes g_T : \mathsf{p}[S] \otimes q[T] \to \mathsf{p}'[S] \otimes \mathsf{q}'[T].$$

We thus obtain a functor $\# : \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$.

In the sum (1), we may group terms according to the intersection $S \cap T$, which ranges from the empty set to the whole set I. The two extremes are

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{\substack{I = S \cup T \\ \emptyset = S \cap T}} \mathsf{p}[S] \otimes \mathsf{q}[T] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T],$$

$$(\mathsf{p} \times \mathsf{q})[I] = \bigoplus_{\substack{I = S \cup T \\ S \cap T = I}} \mathsf{p}[S] \otimes \mathsf{q}[T] = \mathsf{p}[I] \otimes \mathsf{q}[I].$$

These are the *Cauchy* and *Hadamard* product of species, respectively. Thus, we have

$$(\mathsf{p} \# \mathsf{q})[I] = (\mathsf{p} \times \mathsf{q})[I] + \bigoplus_{\substack{I = S \cup T \\ \emptyset \neq S \cap T \neq I}} \mathsf{p}[S] \otimes \mathsf{q}[T] + (\mathsf{p} \cdot \mathsf{q})[I].$$

In this sense, the Heisenberg product interpolates between the Cauchy and Hadamard products.

The Cauchy product endows the category **Sp** with a monoidal structure, for which the unit object is the species $\mathbf{1}$. The same is true of the Hadamard product, this time the unit object is $\mathbf{E}$.

Proposition 1.2. *The Heisenberg product turns the category* **Sp** *into a monoidal category, for which the unit object is* $\mathbf{1}$.

The verification of associativity reduces to the observation that one may define the triple product of species p, q and r by

$$(\mathsf{p} \,\#\, \mathsf{q} \,\#\, \mathsf{r})[I] = \bigoplus_{I = R \cup S \cup T} \mathsf{p}[R] \otimes \mathsf{q}[S] \otimes \mathsf{r}[T],$$

and that these identifies naturally with both $\big((\mathsf{p} \,\#\, \mathsf{q}) \,\#\, \mathsf{r}\big)[I]$ and $\big(\mathsf{p} \,\#(\mathsf{q} \,\#\, \mathsf{r})\big)[I]$.

The monoidal and abelian structures of $\mathbf{Sp}$ are compatible.

The Cauchy, Hadamard and Heisenberg products are related by a natural isomorphism:

$$(\mathsf{p} \cdot \mathbf{E}) \times (\mathsf{q} \cdot \mathbf{E}) \cong (\mathsf{p} \,\#\, \mathsf{q}) \cdot \mathbf{E}. \tag{2}$$

E:isomorphis

To see this, evaluate both sides on a finite set I

$$\big((\mathsf{p} \cdot \mathbf{E}) \times (\mathsf{q} \cdot \mathbf{E})\big)[I] = \Big(\bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \Bbbk\Big) \otimes \Big(\bigoplus_{I = S' \sqcup T'} \mathsf{q}[S'] \otimes \Bbbk\Big),$$

$$\big((\mathsf{p} \,\#\, \mathsf{q}) \cdot \mathbf{E}\big)[I] = \bigoplus_{I = J \sqcup K} (\mathsf{p} \,\#\, \mathsf{q})[J] \otimes \mathbf{E}[K] = \bigoplus_{J \subseteq I} \Big(\bigoplus_{J = S \cup S'} \mathsf{p}[S] \otimes \mathsf{q}[S']\Big) \otimes \Bbbk,$$

and observe that both spaces are naturally isomorphic.

1.2. **The generating function of the Heisenberg product of two species.** The *generating function* associated to a species p is the formal power series

$$F_{\mathsf{p}}(x) = \sum_{n \geq 0} \dim_{\Bbbk} \mathsf{p}[n] \, \frac{x^n}{n!}.$$

The generating series associated to the Cauchy and Hadamard products of two species p and q are as follows. If $F_{\mathsf{p}}(x) = \sum_{n \geq 0} a_n x^n / n!$ and $F_{\mathsf{q}}(x) = \sum_{n \geq 0} b_n x^n / n!$, then

$$F_{\mathsf{p} \cdot \mathsf{q}}(x) = \sum_{n \geq 0} \Big(\sum_{i+j=n} \binom{n}{i} a_i b_j \Big) \frac{x^n}{n!} \quad \text{and} \quad F_{\mathsf{p} \times \mathsf{q}}(x) = \sum_{n \geq 0} a_n b_n \, \frac{x^n}{n!}.$$

Proposition 1.3. *The generating function of the Heisenberg product of two species* p *and* q *is*

$$F_{\mathsf{p} \,\#\, \mathsf{q}}(x) = \sum_{n \geq 0} \Bigg(\sum_{\substack{i,j \leq n \\ n \leq i+j}} \binom{n}{n-i,\, n-j,\, i+j-n} a_i b_j \Bigg) \frac{x^n}{n!}, \tag{3}$$

E:gfHeis

where

$$\binom{n}{n-i,\, n-j,\, i+j-n} = \frac{n!}{(n-i)!(n-j)!(i+j-n)!}.$$

Proof. In (1), the pairs (S, T) with

$$[n] = S \cup T, \ \# S = i, \quad \text{and} \quad \# T = j,$$

are in bijection with the triples (U, V, W) such that

$$[n] = U \sqcup W \sqcup V, \ \# U + \# W = i \quad \text{and} \quad \# W + \# V = j.$$

Indeed, we may take $U = S \setminus T$, $V = T \setminus S$, and $W = S \cap T$. We then have $\# U = n - j$, $\# V = n - i$, and $\# W = i + j - n$. The multinomial coefficient in (3) accounts for the number of such triples U, V, W. $\square$

S:symrep

2. The Heisenberg product of symmetric group representations

The category of species is equivalent to that of representations of the family of symmetric groups. We define the Heisenberg product of symmetric group representations and show that it corresponds to the Heisenberg product of species under this equivalence.

s:parabolic

2.1. **Parabolic embeddings.** Let p and q be non-negative integers. Given permutations $\sigma \in S_p$ and $\tau \in S_q$, let $\sigma \times \tau \in S_{p+q}$ be the permutation such that

$$(\sigma \times \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \le i \le p, \\ \tau(i-p)+p & \text{if } p+1 \le i \le p+q. \end{cases}$$

The *parabolic embedding* is the map

$$S_p \times S_q \hookrightarrow S_{p+q}, \quad (\sigma, \tau) \mapsto \sigma \times \tau. \tag{4}$$ D:definitio

It is an injective morphism of groups. We employ the following variants. Given an integer n with $\max(p,q) \le n \le p+q$, let $S_p \times_n S_q$ denote the group

$$S_p \times_n S_q = S_{n-q} \times S_{p+q-n} \times S_{n-p}.$$

There are group embeddings

$$S_p \times_n S_q \hookrightarrow S_n, \quad (\sigma, \rho, \tau) \mapsto \sigma \times \rho \times \tau, \tag{5}$$ E:embed-time

$$S_p \times_n S_q \hookrightarrow S_p \times S_q, \quad (\sigma, \rho, \tau) \mapsto (\sigma \times \rho, \rho \times \tau). \tag{6}$$ E:embed-time

When $n = p+q$, (5) reduces to (4) and (6) is the identity of $S_p \times S_q$. On the other hand, when $n = p = q$, (5) is the identity of S_p and (6) is the diagonal embedding $S_p \hookrightarrow S_p \times S_p$.

2.2. **The Heisenberg product in terms of induction and restriction.** Let $\mathrm{Rep}(S_n)$ be the category whose objects are $\Bbbk$-linear representations of S_n and whose morphisms are homomorphisms of $\Bbbk S_n$-modules. Let

$$\mathbf{R} = \bigoplus_{n \ge 0} \mathrm{Rep}(S_n)$$

be the direct sum of these abelian categories. An object is a finite sequence of symmetric group representations. Each category $\mathrm{Rep}(S_p)$ embeds in $\mathbf{R}$ by viewing $V \in \mathrm{Rep}(S_p)$ as the sequence $(V_n)_{n\ge 0}$ with $V_p = V$ and $V_n = 0$ for all $n \ne p$. In this manner, a finite sequence $(V_n)_{n \ge 0} \in \mathbf{R}$ identifies with $\bigoplus_{n\ge 0} V_n$.

Definition 2.1. Let $V \in \mathrm{Rep}(S_p)$ and $W \in \mathrm{Rep}(S_q)$. Their *Heisenberg product* is the object $V \# W \in \mathbf{R}$ given by

$$V \# W = \bigoplus_{n=\max(p,q)}^{p+q} \mathrm{Ind}_{S_p \times_n S_q}^{S_n} \mathrm{Res}_{S_p \times_n S_q}^{S_p \times S_q} (V \otimes W). \tag{7}$$ E:definitio

Here, $V \otimes W$ is first viewed as a representation of the group $S_p \times S_q$ by means of

$$(\sigma, \tau) \cdot v \otimes w = \sigma \cdot v \otimes \tau \cdot w.$$

We then apply the induction and restriction functors along the embeddings (5) and (6), respectively.

In view of the remarks in Section 2.1, we have that the highest component of $V \# W$ is

$$(V \# W)_{p+q} = \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} (V \otimes W).$$

This is the familiar induction product of symmetric group representations. Similarly, when $p = q$, the lowest component (of degree $n = p = q$) is

$$(V \# W)_n = \mathrm{Res}^{S_n \times S_n}_{S_n}(V \otimes W).$$

This is the classical Kronecker product of S_n-representations.

The Heisenberg product contains terms of intermediate degree n between $\max(p, q)$ and $p + q$; in this sense it interpolates between the Kronecker and induction products.

The operation $V \# W$ can be defined for V and $W \in \mathbf{R}$ by extending (7) by biadditivity. It can also be defined on morphisms of representations. The result is a functor $\# : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$.

:rep-finite **Proposition 2.2.** *The Heisenberg product turns* $\mathbf{R}$ *into a monoidal category. The unit object is* $\Bbbk \in \mathrm{Rep}(S_0)$.

In this situation, associativity of the Heisenberg product is not straightforward. It follows from Theorem 2.3 below.

2.3. **Species and representations.** Let

$$\widehat{\mathbf{R}} = \prod_{n \geq 0} \mathrm{Rep}(S_n)$$

be the product of the categories of symmetric group representations. An object is a sequence of symmetric group representations. Note that $\mathbf{R}$ is a full subcategory of $\widehat{\mathbf{R}}$.

Let p be a species. Each space $\mathsf{p}[n]$ is an S_n-representation, by means of

$$\sigma \cdot v = \mathsf{p}[\sigma](v)$$

for $\sigma \in S_n$ and $v \in \mathsf{p}[n]$. In this manner, the species p gives rise to a sequence of representations $(\mathsf{p}[n])_{n \geq 0}$, and we obtain a functor

$$\mathcal{F} : \mathbf{Sp} \to \widehat{\mathbf{R}}, \quad \mathcal{F}(\mathsf{p}) = (\mathsf{p}[n])_{n \geq 0}.$$

This is in fact an equivalence, since $\mathbf{Sp}$ is the category of presheafs on $\mathbf{set}^{\times}$, and $\widehat{\mathbf{R}}$ may be seen as the category of presheafs on a skeleton of $\mathbf{set}^{\times}$.

The next theorem states that the operations (1) and (7) correspond to each other under the equivalence $\mathcal{F}$.

resentations **Theorem 2.3.** *For any species* p *and* q,

$$\mathcal{F}(\mathsf{p} \# \mathsf{q}) \cong \mathcal{F}(\mathsf{p}) \# \mathcal{F}(\mathsf{q}).$$

In addition, $\mathcal{F}(\mathbf{1}) \cong \Bbbk$.

It follows from here that the Heisenberg product turns $\widehat{\mathbf{R}}$ into a monoidal category with unit objet $\Bbbk$, and that the equivalence $\mathcal{F}$ is strong monoidal. Proposition 2.2 also follows, and a fortiori $\mathbf{R}$ is a full monoidal subcategory of $\widehat{\mathbf{R}}$.

Proof. Fix i, j, and n, three non-negative integers such that $\max(i, j) \leq n \leq i + j$. Our task is to build an isomorphism of S_n-representations

$$\bigoplus_{\substack{[n] = S \cup T \\ \# S = i,\, \# T = j}} \mathsf{p}[S] \otimes \mathsf{q}[T] \cong \mathrm{Ind}^{S_n}_{S_i \times_n S_j} \mathrm{Res}^{S_i \times S_j}_{S_i \times_n S_j} \big(\mathsf{p}[i] \otimes \mathsf{q}[j]\big). \qquad (8)$$ E:iso-specie

Let A and B be finite totally ordered sets. Given decompositions $A = A_1 \sqcup \cdots \sqcup A_n$ and $B = B_1 \sqcup \cdots \sqcup B_n$ with $\# A_i = \# B_i$ for $i = 1, \ldots, n$, there is a unique bijection $f : A \to B$ such that $f(A_i) = B_i$ and f is increasing on each A_i. We say that f the *canonical* bijection induced by the decompositions.

Let S and T be a pair of sets such that $[n] = S \cup T$, $\# S = i$ and $\# T = j$. Let $S' = S \setminus T$ and $T' = T \setminus S$. Then $\# S' = n - j$, $\# T' = n - i$, and $\# S \cap T = i + j - n$. We may then consider the canonical bijection $f_{S,T} : [n] \to [n]$ induced by the following decompositions of the set $[n]$ (ordered in the standard manner):

$$S' \sqcup (S \cap T) \sqcup T' \quad \text{and} \quad [n-j] \sqcup [n-j+1, i] \sqcup [i+1, n].$$

Let $f_{S'}$, $f_{S\cap T}$ and $f_{T'}$ denote the restrictions of $f_{S,T}$ to the corresponding subsets.

Let f_S and f_T denote the restrictions of $f_{S,T}$ to S and T, respectively. Note that f_S maps S onto $[i]$ and f_T maps T onto $[n-j+1, n]$.

We consider the standard identification of the induction module $\mathrm{Ind}_H^G(V)$ with the tensor product $\Bbbk G \otimes_{\Bbbk H} V$. Let $u \in \mathsf{p}[S]$ and $v \in \mathsf{q}[T]$, and define the map

$$\begin{aligned} \mathsf{p}[S] \otimes \mathsf{q}[T] &\xrightarrow{\psi} \mathrm{Ind}_{S_i \times_n S_j}^{S_n} \mathrm{Res}_{S_i \times_n S_j}^{S_i \times S_j} \big(\mathsf{p}[i] \otimes \mathsf{q}[j]\big) \\ u \otimes v &\longmapsto f_{S,T}^{-1} \otimes \big(\mathsf{p}[f_{S'} \sqcup f_{S\cap T}](u) \otimes \mathsf{q}[f_{S\cap T} \sqcup f_{T'}](v)\big) \end{aligned} \tag{9}$$ E:def-of-psi

and extend it to the direct sum in (8).

For a permutation $\sigma \in S_n$, the action of σ in $u \otimes v$ is

$$\sigma \cdot (u \otimes v) = \mathsf{p}[\sigma|_S](u) \otimes \mathsf{q}[\sigma|_T](v). \tag{10}$$ E:Sn_acts_te

Observe that $\sigma \cdot (u \otimes v) \in \mathsf{p}\big[\sigma(S)\big] \otimes \mathsf{q}\big[\sigma(T)\big]$. The application of the map ψ yields

$$\psi\big(\sigma \cdot (u \otimes v)\big) = f_{\sigma(S),\sigma(T)}^{-1} \otimes \big(\alpha(u) \otimes \beta(v)\big) \tag{11}$$ E:psi_of_the

where $\alpha = \mathsf{p}[f_{\sigma(S')} \sqcup f_{\sigma(S)\cap\sigma(T)}]\mathsf{p}[\sigma|_S]$ and $\beta = \mathsf{q}[f_{\sigma(S)\cap\sigma(T)} \sqcup f_{\sigma(T')}]\mathsf{q}[\sigma|_T]$. Since we can decompose $\sigma|_S$ into $\sigma|_{S'} \sqcup \sigma|_{S\cap T}$, then by the functoriality of p we get that

$$\alpha = \mathsf{p}\big[(f_{\sigma(S')}\sigma|_{S'}) \sqcup (f_{\sigma(S\cap T)}\sigma|_{S\cap T})\big].$$

Let $\tilde{\sigma}_{S'}$ and $\tilde{\sigma}_{S\cap T}$ be the only bijections such that the following diagrams commute

$$\begin{tikzcd} S' \arrow[r, "f_{S'}"] \arrow[d, "\sigma|_{S'}"'] & {[n-j]} \arrow[d, "\tilde{\sigma}_{S'}"] \\ \sigma(S') \arrow[r, "f_{\sigma(S')}"] & {[n-j]} \end{tikzcd} \qquad \begin{tikzcd} S \cap T \arrow[r, "f_{S\cap T}"] \arrow[d, "\sigma|_{S\cap T}"'] & {[n-j+1, i]} \arrow[d, "\tilde{\sigma}_{S\cap T}"] \\ \sigma(S\cap T) \arrow[r, "f_{\sigma(S\cap T)}"] & {[n-j+1, i]} \end{tikzcd} \tag{12}$$ E:canonical_

We conclude that α can be rewritten as

$$\alpha = \mathsf{p}\big[(\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T})(f_{S'} \sqcup f_{S\cap T})\big] = \mathsf{p}[\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T}]\, \mathsf{p}[f_{S'} \sqcup f_{S\cap T})],$$

and we proceed similarly with β.
In accordance with (10) we deduce that:

$$\alpha(u) \otimes \beta(v) = (\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T} \sqcup \tilde{\sigma}_{T'}) \cdot \big(\mathsf{p}[f_{S'} \sqcup f_{S\cap T}](u) \otimes \mathsf{q}[f_{S\cap T} \sqcup f_{T'}](v)\big).$$

Note that the permutation $\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T} \sqcup \tilde{\sigma}_{T'}$ clearly belongs to $S_i \times_n S_j$. In equation (11), since the tensor product of $f_{\sigma(S),\sigma(T)}^{-1}$ with $\alpha(u) \otimes \beta(v)$ is performed with respect to this subgroup, we can move the permutation to the left factor where we get $f_{\sigma(S),\sigma(T)}^{-1}(\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T} \sqcup \tilde{\sigma}_{T'}) = \sigma f_{S,T}^{-1}$. This equality results again from the diagrams (12). This is precisely the definition of the action of σ on the image of the map ψ.

The map ψ is invertible, since for any element $\sigma \otimes (x \otimes y)$, we can decompose $\sigma = \xi(\alpha \times \beta \times \gamma)$, where $\alpha \times \beta \times \gamma \in S_{n-j} \times S_{i+j-n} \times S_{n-i} = S_i \times_n S_j$ and ξ is increasing in the intervals $[n-j]$, $[n-j+1, i]$, and $[i+1, n]$. Define the disjoint sets

$$A = \xi\big([n-j]\big), \quad B = \xi\big([n-j+1, i]\big), \quad C = \xi\big([i+1, n]\big).$$

Then, let $S = A \sqcup B$ and $T = B \sqcup C$. It is straightforward to find $u \otimes v$ in $\mathsf{p}[S] \otimes \mathsf{q}[T]$ such that $\psi(u \otimes v) = \sigma \otimes (x \otimes y)$. Similarly, this process applied to the image of ψ in (9) yields back $u \otimes v$. $\square$

3. The Heisenberg product of symmetric functions

3.1. Species, representations of symmetric groups and symmetric functions. In this subsection we recall some basic facts about the relations between species –viewed as such or as representations of the family of all symmetric groups– and the space of symmetric functions.

Let $\mathrm{K}(S_n)$ be the Grothendieck group or representation group of the category of finite dimensional S_n-modules, and call $\mathcal{K}$ and $\widehat{\mathcal{K}}$ the groups:

$$\mathcal{K} = \bigoplus_{n \geq 0} \mathrm{K}(S_n) \quad \subseteq \quad \widehat{\mathcal{K}} = \prod_{n \geq 0} \mathrm{K}(S_n).$$

Consider the ring of polynomials $\Bbbk[x_1, \ldots, x_n]$ in n variables in which the symmetric group S_n acts by permuting the variables. Call Λ_n^k the subring consisting of the homogeneous polynomials of degree k which are invariant under the action of S_n. When $m \geq n$, Λ_m^k proyects naturally onto Λ_n^k via the homomorphism $\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$ which maps the first n variables to themselves, and the other variables to 0. The space Λ^k is defined as the inverse limit of the system considered above.

For Λ^k the *space of symmetric functions* of degree k, define

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k \quad \subseteq \quad \widehat{\Lambda} = \prod_{k \geq 0} \Lambda^k,$$

called the *space of symmetric functions* and its *completion*, respectively (see [19]).

Observe that Λ and $\widehat{\Lambda}$ are subspaces of $\Bbbk[x_1, x_2, \ldots]$ and $\Bbbk[\![x_1, x_2, \ldots]\!]$ respectively.

We recall the following definition of special elements in Λ.

(1) *The elementary symmetric functions* are defined by the generating series:

$$\sum_{r \geq 0} e_r(x_1, x_2, \ldots) t^r = \prod_{i \geq 1} (1 + x_i t),$$

(2) *The complete homogeneous symmetric functions* are defined by:

$$\sum_{r \geq 0} h_r(x_1, x_2, \ldots) t^r = \prod_{i \geq 1} (1 - x_i t)^{-1},$$

(3) *The power sums* are defined by:

$$\sum_{r \geq 0} p_r(x_1, x_2, \ldots) t^r = \sum_{i \geq 1} x_i (1 - x_i t)^{-1}.$$

The above defined functions are elements of $\Bbbk[\![x_1, x_2, \dots]\!]$, if we want to consider the corresponding elements in $\Bbbk[x_1, x_2, \dots x_n]$ we simply set $0 = x_{n+1} = x_{n+2} = \cdots$.

These, can in turn be defined in terms of the *monoidal symmetric functions*.

A *partition* –finite or almost finite– $\alpha = (a_1, a_2, \dots)$ with $a_1 \geq a_2 \geq \cdots \geq 0$ determines a monomial $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots$.

The monomial symmetric function associated to α and denoted by m_α is

$$m_\alpha = \sum_{\{\widehat{\alpha}\,:\,\widehat{\alpha} \rightsquigarrow \alpha\}} x^\alpha,$$

where $\widehat{\alpha}$ stands for a composition and the symbol $\widehat{\alpha} \rightsquigarrow \alpha$ means that the mentioned composition produces the given partition α by permutation of the entries. For example $m_{(21)} = \sum_{i \neq j} x_i^2 x_j$.

We have the following equalites:

Elementary symmetric functions: $e_r = m_{(1^r)}$ where (1^r) is the partition of r formed only by 1's.

Complete homogenous symmetric functions: $h_r = \sum_{\{\alpha : |\alpha| = r\}} m_\alpha$ where the sum is taken over all the partitions of r.

Power sums: $p_r = m_{(r)}$, where (r) is the partition $(r, 0, 0 \dots)$.

For an arbitrary partition $\alpha = (a_1, a_2, \dots)$ we define: $e_\alpha = e_{a_1} e_{a_2} \cdots$, $h_\alpha = h_{a_1} h_{a_2} \cdots$, $p_\alpha = p_{a_1} p_{a_2} \cdots$.

When α runs over all partitions, the set of all functions m_α form a $\mathbb{Z}$–basis of Λ, and the same happens with the sets of the e_α's or the set of the h_α's. The set of the p_α's form a $\mathbb{Q}$–basis of $\Lambda_{\mathbb{Q}}$.

The Frobenius characteristic map is the linear isomorphism

$$\mathrm{ch} : \widehat{\mathcal{K}} \otimes_{\mathbb{Z}} \Bbbk \to \widehat{\Lambda}, \quad \mathrm{ch}(V) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\mathrm{V}}(\sigma) p_{\mathrm{cycle}(\sigma)},$$

where V is a representation of S_n, χ_{V} its character and $p_{\mathrm{cycle}(\sigma)}$ is the power sum associated the partition of n defined by to the cycle-type of σ. The map ch restricts to an isomorphisms of $\mathcal{K}$ and Λ. See [19, Proposition I.7.3] for proofs of the isomorphisms.

The above result, yields another perspective regarding the complete homogeneous symmetric functions.

Indeed, if $\alpha = (a_1, \dots, a_r)$ is a composition of n and

$$S_\alpha = S_{a_1} \times \cdots \times S_{a_r},$$

it can be viewed as a subgroup of S_n by iterating (4). These are the so called *parabolic subgroups* of S_n. Let $\mathfrak{h}_\alpha$ denote the permutation representation of S_n corresponding to the action by multiplication on the quotient S_n/S_α. The isomorphism class of $\mathfrak{h}_\alpha$ does not depend on the order of the parts of α, hence we will consider the representations $\mathfrak{h}_\alpha$ for α running over the partitions of n.

If we denote the trivial S_α-module by $\mathbf{1}$ (we omit the dependence on α for clarity), then the representation $\mathfrak{h}_\alpha$ can also be expressed as

$$\mathfrak{h}_\alpha = \mathrm{Ind}_{S_\alpha}^{S_n}(\mathbf{1}). \tag{13}$$

E:def-of-h-a

The following equality holds $\mathrm{ch}(\mathfrak{h}_\alpha) = h_\alpha$ (see [19, Proposition I.7.3]).

homogeneous

3.2. **The Heisenberg product of complete homogeneous symmetric functions.** In Section 1 and in Subsection 3.1 we have established a path between the objects described in the diagram below:

$$\mathbf{Sp} \xrightarrow{\mathcal{F}} \widehat{\mathbf{R}} \Longrightarrow \widehat{\mathcal{K}} \supset \mathcal{K} \xrightarrow{\otimes_{\mathbb{Z}}\Bbbk} \mathcal{K}_{\Bbbk} \xrightarrow{\mathrm{ch}} \Lambda,$$

where the double arrow means the application of the Grothendieck functor.

Using the universal property of the Grothendick group functor, it is clear that in order to translate the Heisenberg product from $\widehat{\mathbf{R}}$ to $\widehat{\mathcal{K}}$ it is enough to verify that it is compatible with direct sums.

It is easy to make this verification in the category of species, where the colimits are defined pointwisely as the colimits of vector spaces. Then, the distributive property of the tensor product with respect to direct sums shows that $(\mathsf{p} \oplus \mathsf{q}) \# \mathsf{r} = (\mathsf{p} \# \mathsf{r}) \oplus (\mathsf{q} \# r)$ for $\mathsf{p}\,,\mathsf{q}\,,\mathsf{r} \in \mathbf{R}$.

Hence the operation $\#$ can be defined in $\widehat{\mathcal{K}}$, and the subgroup $\mathcal{K}$ is clearly $\#$– closed since the definition of the Heisenberg product involves only a finite number of summands.

Now, by composition with the Frobenius characteristic isomorphism $\mathrm{ch} : \mathcal{K}_{\Bbbk} \to \Lambda$ we obtain an associative product on symmetric functions, which we call *Heisenberg product of symmetric functions.*

The next theorem gives an explicit formula for the Heisenberg product in the linear basis of Λ formed by the complete homogenous symmetric functions. This theorem, besides providing a combinatorial rule useful for computations, will later be used to make the connection with the Heisenberg product of non-commutative symmetric functions in Section 8.

In order to express the coefficients of the Heisenberg product of two complete homogenous symmetric functions, we need to define a particular set of plane partitions as follows. Let $\alpha = (a_1, \dots, a_r) \vDash p$ and $\beta = (b_1, \dots, b_s) \vDash q$ be two compositions and n an integer with $\max(p, q) \leq n \leq p + q$. Let $a_0 = n - p$, $b_0 = n - q$, and let $\mathcal{M}^n_{\alpha,\beta}$ be the set of all $(s+1) \times (r+1)$-matrices

$$M = (m_{ij})_{\substack{0 \leq i \leq s \\ 0 \leq j \leq r}}$$

with non-negative integer entries and such that

- the sequence of column sums is $(a_0, a_1, \dots, a_r)$,
- the sequence of row sums is $(b_0, b_1, \dots, b_s)$,
- the first entry is $m_{00} = 0$.

We illustrate these conditions as follows:

$$\begin{array}{cccc|c}
0 & m_{01} & \cdots & m_{0r} & n-q \\
m_{10} & m_{11} & \cdots & m_{1r} & b_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{s0} & m_{s1} & \cdots & m_{sr} & b_s \\
\hline
n-p & a_1 & \cdots & a_r &
\end{array}$$

Let $p(M)$ be the partition of n whose parts are the non-zero m_{ij}.

ombinatorial **Theorem 3.1.** *There is an associative product $\#$ in Λ, interpolating between the internal and external products, which can be expressed on the basis (h_α) of complete homogeneous functions as*

$$h_\alpha \# h_\beta = \sum_{n=\max(p,q)}^{p+q} \sum_{M\in\mathcal{M}^n_{\alpha,\beta}} h_{p(M)}. \tag{14}$$

E:smash-comp

For example, using such theorem we get

$$h_{(2,1)} \# h_3 = h_{(2,1)} + h_{(1,1,1,1)} + h_{(2,1,1)} + h_{(2,2,1)} + h_{(2,1,1,1)} + h_{(3,2,1)},$$

where the external product is recognized in the last term and the internal product in the first one, together with additional terms of degrees four and five.

The existence of this operation poses the problem of finding an explicit description for its structure constants on the basis of Schur functions. The answer would contain as extreme cases the Littlewood-Richardson rule and (a still unknown) rule for the Kronecker coefficients.

Proof of Theorem 3.1. We prove that the following formula holds in the category $\mathbf{R}$:

$$\mathfrak{h}_\alpha \# \mathfrak{h}_\beta = \bigoplus_{n=\max(p,q)}^{p+q} \bigoplus_{M\in\mathcal{M}^n_{\alpha,\beta}} \mathfrak{h}_{p(M)},$$

where the representations $\mathfrak{h}_\alpha$ are the induced representations defined in (13). The application of the Grothendieck group functor and the Frobenius characteristic immediately yields (14).

We fix n in the range $\max(p,q) \le n \le p+q$. The n-summand of $\mathfrak{h}_\alpha \# \mathfrak{h}_\beta$ is, according to (7),

$$(\mathfrak{h}_\alpha \# \mathfrak{h}_\beta)_n = \mathrm{Ind}^{S_n}_{S_p\times_n S_q} \mathrm{Res}^{S_p\times S_q}_{S_p\times_n S_q}(\mathfrak{h}_\alpha\otimes\mathfrak{h}_\beta) = \mathrm{Ind}^{S_n}_{S_p\times_n S_q} \mathrm{Res}^{S_p\times S_q}_{S_p\times_n S_q} \mathrm{Ind}^{S_p\times S_q}_{S_\alpha\times S_\beta}(\mathbf{1}). \tag{15}$$

E:nth-term-c

Consider the composition of the first two functors $\mathrm{Res}^{S_p\times S_q}_{S_p\times_n S_q} \mathrm{Ind}^{S_p\times S_q}_{S_\alpha\times S_\beta}$ in the right hand side of (7). We use Mackey's formula to interchange them (see [34]), as follows.

Let $\Upsilon \subset S_p \times S_q$ be a complete set of representatives of the family of double cosets $(S_p \times_n S_q) \setminus (S_p\times S_q) / (S_\alpha\times S_\beta)$.
For each $\upsilon\in\Upsilon$, define

$${}^{\upsilon}(S_\alpha\times S_\beta) = \upsilon^{-1}(S_\alpha\times S_\beta)\upsilon \quad\text{and}\quad S_\alpha \times^\upsilon_n S_\beta = (S_p\times_n S_q)\cap{}^{\upsilon}(S_\alpha\times S_\beta). \tag{16}$$

E:def-v-para

The following diagram ilustrates the relative position of these groups and subgroups

$$\begin{tikzcd}
S_n & & S_p\times S_q & \\
& S_p\times_n S_q \arrow[ul, tail] \arrow[ur, tail] & & S_\alpha\times S_\beta \arrow[ul, tail] \\
& & S_\alpha\times^\upsilon_n S_\beta \arrow[ul, tail] \arrow[ur, tail] &
\end{tikzcd}$$

In this situation Mackey's formula reads as the equality

$$\mathrm{Res}^{S_p\times S_q}_{S_p\times_n S_q} \mathrm{Ind}^{S_p\times S_q}_{S_\alpha\times S_\beta}(\mathbf{1}) = \bigoplus_{\upsilon\in\Upsilon} \mathrm{Ind}^{S_p\times_n S_q}_{S_\alpha\times^\upsilon_n S_\beta} \mathrm{Res}^{S_\alpha\times S_\beta}_{S_\alpha\times^\upsilon_n S_\beta}(\mathbf{1}).$$

Using the transitivity of the induction functor and the property that it commutes with coproducts we deduce that (15) can be written as

$$(\mathfrak{h}_\alpha \,\#\, \mathfrak{h}_\beta)_n = \operatorname{Ind}_{S_p \times_n S_q}^{S_n} \operatorname{Res}_{S_p \times_n S_q}^{S_p \times S_q}(\mathfrak{h}_\alpha \otimes \mathfrak{h}_\beta) = \bigoplus_{\upsilon \in \Upsilon} \operatorname{Ind}_{S_\alpha \times_n^\upsilon S_\beta}^{S_n}(\mathbf{1}). \tag{17}$$

E:sum-over-c

In Lemma 3.2 we construct a bijection $\upsilon \mapsto M_\upsilon$ between Υ and $\mathcal{M}_{\alpha,\beta}^n$ with the property that $S_{p(M_\upsilon)} = S_\alpha \times_n^\upsilon S_\beta$. Then (17) becomes

$$(\mathfrak{h}_\alpha \,\#\, \mathfrak{h}_\beta)_n = \bigoplus_{\upsilon \in \Upsilon} \operatorname{Ind}_{S_\alpha \times_n^\upsilon S_\beta}^{S_n}(\mathbf{1}) = \bigoplus_{\upsilon \in \Upsilon} \operatorname{Ind}_{S_{p(M_\upsilon)}}^{S_n}(\mathbf{1}) = \bigoplus_{M \in \mathcal{M}_{\alpha,\beta}^n} \mathfrak{h}_{p(M)},$$

proving the theorem. □

ha-smash-hb

Lemma 3.2. *In the notations of Theorem 3.1, there is a bijection* $\Upsilon \cong \mathcal{M}_{\alpha,\beta}^n$ *given by* $\upsilon \mapsto M_\upsilon$*, such that* $S_{p(M_\upsilon)} = S_\alpha \times_n^\upsilon S_\beta$.

Proof. The proof of this rather technical lemma is postponed until the appendix 12.1. □

3.3. **Relation with Rota–Stein Cliffordization process.** In the articles [27], [26] the authors define what they call the *Cliffordization* process, that was first carried out to produce the Clifford algebra from the exterior algebra. This process is applied to a general bicommutative Hopf algebra H endowed with an associative operation $(\,|\,) : H \otimes H \to H$ that satisfies the following compatibility conditions –that are known under different names in the literature–:

I:laplace
I:frobenius

(1) $(xy|z) = \sum (x|z_1)(y|z_2)$, $(x|yz) = \sum (x_1|y)(x_2|z)$;
(2) $\sum (x|y)_1 \otimes (x|y)_2 = \sum (x_1|y_1) \otimes (x_2|y_2)$,

valid for all $x, y \in H$.

In this context, if we call m –$m(x,y) = xy$– the original product in H, the mentioned authors define the *circle product* as the convolution $\circledast = (\,|\,) \star m$ in the space $\operatorname{End}(H \otimes H, H)$.

In explicit terms: $x \circledast y = \sum (x_1|y_1)x_2y_2$ or $x \circledast y = \sum x_1(x_2|y_1)y_2$ –as it is written in [27] and [26]–. Recall that the original product and coproduct are commutative. In the above papers the authors prove that $\circledast$ is an associative product that is compatible with the original comultiplication Δ –compare with Theorem 9.2 where the same compatibility is proved for the Heisenberg product of non-commutative symmetric functions, and hence also for the commutative situation–.

A more general construction along similar lines appears in [8]. The authors construct the so called *hash products*, working in the context of *Frobenius Laplace pairings* that are defined in the paper by properties related to the conditions (1) and (2) above. In the particular case of the space Λ of commutative symmetric functions, they use the hash (or circle) product to formulate classical character formulæ in a unified framework. In the combinatorial applications they deal for example with the *reduced characters* of Murnaghan and Littlewood or with what they call *Thibon characters* : c.f. [17],[18],[24],[32] and compare also with [6].

In the case of the symmetric functions, the circle product and the smash product are the same.

More precisely, recall that the symmmetric functions admit a natural coproduct –dual to the external product– that can be defined on the generators of the basis of complete homogeneous symmetric functions as:

$$\Delta(h_a) = \sum_{i+j=a} h_i \otimes h_j. \tag{18}$$ E:coproducto

Clearly, this coproduct is compatible with the external product. Moreover the corresponding "external" Hopf algebra structure in Λ, together with the *internal* product are in the hypothesis of the Cliffordization process as one can easily check working on the basis h_α for α a partition.

The identity that follows –that states that the Heisenberg product and the circle product coincide in the above context– was suggested to the authors by A. Zelevinski and does not hold for the space of non-commutative symmetric funcions (see comment after Theorem (7.4)). We present a combinatorial proof of this identity.

ki-identity **Lemma 3.3** (A. Zelevinski)**.** *Assume that $f, g \in \Lambda$, then:*

$$f \# g = \sum f_1 \cdot (f_2 * g_1) \cdot g_2 = f \circledast g, \tag{19}$$ E:zelevinski

where $\Delta(f) = \sum f_1 \otimes f_2$ and $\Delta(g) = \sum g_1 \otimes g_2$.

Proof. The identity (19) follows from formula (14), by collecting the first row and first column of the matrix M as $(h_\alpha)_1$ and $(h_\beta)_2$, respectively, and the remaining submatrix of M is precisely the internal product of the second tensorand $(h_\alpha)_2$ of the coproduct of h_α with the first tensorand $(h_\beta)_1$ of the coproduct of h_β. $\square$

S:powersums 3.4. **The Heisenberg product of power sums.** The power sums $(p_\lambda)_{\lambda\vdash n, n\geq 0}$ form a linear basis of Λ over $\mathbb{Q}$. In this subsection we give an explicit formula for the Heisenberg product in this basis.

Given two partitions λ and μ, denote by $\lambda\mu$ the concatenation and reordering of λ and μ. For example, if $\lambda = (3, 2, 1, 1)$ and $\mu = (2, 2, 1)$, then $\lambda\mu = (3, 2, 2, 2, 1, 1, 1)$.

Theorem 3.4. *The Heisenberg product on the basis of power sums can be expressed as*

$$p_\lambda \# p_\mu = \sum_{\substack{\alpha\gamma=\lambda \\ \gamma\beta=\mu}} z(\gamma)\, p_{\alpha\gamma\beta}, \tag{20}$$ E:smash-powe

where $z(\gamma)$ is the order of the stabilizer of the conjugacy class of a permutation of cycle-type γ:

$$z(\gamma) = \prod_r r^{m_r} m_r!, \tag{21}$$ E:number-z

being m_r the number of times r occurs in γ.

Proof. The coproduct on the basis of power sums is determined by requiring the functions p_n, with n a non-negative integer, to be primitive elements: $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$. More explicity,

$$\Delta(p_\lambda) = \sum_{\alpha\beta=\lambda} p_\alpha \otimes p_\beta.$$

Then, formula (19) reads

$$p_\lambda \# p_\mu = \sum_{\substack{\alpha_1\alpha_2=\lambda \\ \beta_1\beta_2=\mu}} p_{\alpha_1} \cdot (p_{\alpha_2} * p_{\beta_1}) \cdot p_{\beta_2}.$$

But $p_{\alpha_2} * p_{\beta_1} = z(\alpha_2)\delta_{\alpha_2,\beta_1} p_{\alpha_2}$ –see [19, Chapter I (7.12)]–. Since the external product of power sums is done by concatenating the partitions, we obtain the result of the theorem. $\square$

As a particular case, assume that λ and μ are partitions of n. Note that there is a term in degree n only when $\lambda = \mu$, otherwise γ would never be the empty partition and the degree of $p_{\alpha\gamma\beta}$ would be strictly greater than n. Therefore, the only term in degree n is

$$\begin{cases} z(\lambda)\, p_\lambda, & \text{if } \lambda = \mu; \\ 0, & \text{otherwise;} \end{cases}$$

which is the expression of the internal product on the basis of power sums.

On the other hand for any partitions λ and μ, when γ is the empty partition, we obtain the term of largest degree, namely $p_{\alpha\beta}$, since $z(\gamma) = 1$ in this case. This gives the external product $p_\lambda \cdot p_\mu = p_{\lambda\mu}$.

Note that the coefficients of Formula (20) in the basis of power sums are not necessarily the numbers $z(\gamma)$. Indeed, the partition λ may be decomposed, in general, in more than one way as $\lambda = \alpha\gamma$, since the operation of concatenation of partitions involves a reordering of the final result. For example, let (1^n) be the partitions with n parts equal to 1. Then,

$$p_{(1^u)} \# p_{(1^v)} = \sum_{n=\max(u,v)}^{u+v} \binom{u}{n-v}\binom{v}{n-u}(u+v-n)!\, p_{(1^n)}. \tag{22}$$ E:smash-p_1-

In this case, the partitions of Formula (20) are $\alpha = (1^{n-v})$, $\beta = (1^{n-u})$, and $\gamma = (1^{u+v-n})$. The number of possible decompositions of (1^u) into two partitions of length $n-v$ and $u+v-n$ is $\binom{u}{n-v}$, and the same argument for (1^v) yields the second binomial coefficient. The remaining factor of the coefficient is $z(\gamma) = z\big((1^{u+v-n})\big) = (u+v-n)!$, according to Formula (21).

From the explicit expression $h_{(n)}(x) = \sum x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ –where $(a_1, \ldots, a_k)$ ranges over all possible permutations of the parts of $\alpha = (\ell_1, \ldots, \ell_k)$ for all partitions α of n–, it is clear that $h_{(1^u)} = p_{(1^u)}$. Hence, Formula (22) can also be deduced from Theorem (3.1). We use this method in (41) for non-commutative symmetric functions.

domorphisms

4. The Heisenberg product of linear endomorphisms

Let $(H, m, \Delta, \iota, \varepsilon, S)$ be an arbitrary Hopf algebra, where $m : H \otimes H \to H$ is the product, $\Delta : H \to H \otimes H$ is the coproduct, $\iota : \Bbbk \to H$ is the unit, $\varepsilon : H \to \Bbbk$ is the counit, and $S : H \to H$ is the antipode. The space $\operatorname{End}(H)$ of linear endomorphisms of H carries several associative products. Let $f, g \in \operatorname{End}(H)$. Composition and convolution are respectively defined by the diagrams:

$$\begin{tikzcd} & H \arrow[dr, "g"] & \\ H \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[r, "f \otimes g"] & H \otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[r, "f \star g"'] & H \end{tikzcd} \tag{23}$$ D:compositi

Definition 4.1. The *Heisenberg product of endomorphisms* –denoted by $f\,\#\,g$, for $f, g \in \operatorname{End}(H)$– is defined by the diagram:

$$\begin{tikzcd} & H^{\otimes 3} \arrow[r, "\text{cyclic}"] & H^{\otimes 3} \arrow[dr, "1\otimes m"] & \\ H^{\otimes 2} \arrow[ur, "\Delta\otimes 1"] & & & H^{\otimes 2} \arrow[d, "1\otimes g"] \\ H^{\otimes 2} \arrow[u, "f\otimes 1"] & & & H^{\otimes 2} \arrow[dl, "m"] \\ & H \arrow[ul, "\Delta"] \arrow[r, "f\,\#\,g"'] & H & \end{tikzcd} \tag{24}$$ D:smash-diag

where the map cyclic : $H^{\otimes 3} \to H^{\otimes 3}$ is $x \otimes y \otimes z \mapsto y \otimes z \otimes x$. The associativity of the Heisenberg product follows from the Hopf algebra axioms and its unit is the map $\iota\varepsilon$.

In explicit terms one has:

$$(f\,\#\,g)(h) = \sum f(h_1)_2 g\big(h_2 f(h_1)_1\big). \tag{25}$$ D:smash-expl

We call $\operatorname{End}_f(H)$ the subspace of $\operatorname{End}(H)$ of finite rank linear homomorphisms, i.e. the image of the canonical inclusion $H^* \otimes H \hookrightarrow \operatorname{End}(H)$. It is clear that the three above operations restrict to $\operatorname{End}_f(H)$ –observe that for the linear generators of $\operatorname{End}_f(H)$ the Heisenberg product takes the following form: $\alpha|h\#\beta|\ell = \sum \alpha(h_1 \rightharpoonup \beta)|h_2\ell$, where $(\alpha|h)(h') = \alpha(h')h$ for $\alpha \in H^*$, $h,\ h' \in H$–. It is also clear that if H is finite dimensional $\operatorname{End}_f(H) = \operatorname{End}(H)$, is endowed with a coproduct given by the tensor products of the coproducts in H^* and H. This coproduct, is compatible with the convolution product but not with the others.

Remark 4.2. The Heisenberg product appears in the literature in different settings (see for example [22]). Given a Hopf algebra H and a H-module algebra A, the Heisenberg product is defined as the operation on the space $A \otimes H$ given as:

$$(a \otimes h)\,\#(b \otimes k) = \sum a(h_1 \cdot b) \otimes h_2 k. \tag{26}$$ E:smash-usua

If $A = H^*$ and H acts on A by translations then (26) corresponds to (24) via the canonical inclusion $H^* \otimes H \hookrightarrow \operatorname{End}(H)$. Note that in the definition of the Heisenberg product in $\operatorname{End}(H)$ there are no restrictions about the dimensions.

4.1. **The case of K–equivariant endomorphisms.** We need an equivariant version of the above construction.

Assume that K is another bialgebra and that H is a K–module bialgebra : H is endowed with a left action of the algebra K –if $k \in K$ on $h \in H$ the action is denoted by $(k, h) \mapsto k \cdot h : K \times H \to H$–. We have that for all $k \in K$ and $h, \ell \in H$, $\Delta(k \cdot h) = \sum k_1 \cdot h_1 \otimes k_2 \cdot h_2$ and $k \cdot (h\ell) = \sum (k_1 \cdot h)(k_2 \cdot \ell)$.

The action of K on H, induces a *right* action of K on H^* via the formula: $\alpha \in H^*$, $h \in H$, $k \in K$, $(\alpha \leftharpoonup k)(h) = \alpha(k \cdot h)$. With respect to this action and if H is finite dimensional, H^* becomes a *right* K–module bialgebra. In general H^* is only a K–module algebra.

Definition 4.3. Assume that H is a K–module bialgebra for a certain Hopf algebra K. We define an action of K on $\operatorname{End}(H)$ as follows: if $k \in K$ and $f \in \operatorname{End}(H)$,

$(k\cdot f)(h) = \sum k_1 \cdot f(S(k_2)h)$, $-S : K \to K$ denotes the antipode–. Explictly, the action on the generators of $\mathrm{End}_f(H)$ is the following: for $\alpha|h \in \mathrm{End}_f(H)$ and $k \in K$: $k\cdot(\alpha|h) = \sum(\alpha \leftharpoonup Sk_2)|k_1\cdot h$.

equivariant

Remark 4.4.

(1) The K-invariant elements for this action, i.e. the elements $f \in \mathrm{End}(H)$ such that for all $k \in K$, $k\cdot f = \varepsilon(k)f$, are the K–equivariant homomorhisms. They form a vector subspace of $\mathrm{End}(H)$ denoted by $\mathrm{End}_K(H)$. Clearly, the K–action is compatible with composition.

(2) In the case that K is cocommutative, the K–action is also compatible with the convolution product. Then $\mathrm{End}(H)$ and $\mathrm{End}_f(H)$ are K–module algebras with convolution and $\mathrm{End}_K(H)$, $\mathrm{End}_{f,K}(H)$ are subalgebras.

(3) In the case that H is finite dimensional and K is cocommutative, the coproduct of $\mathrm{End}(H)$ is compatible with the action of K. Indeed, if $\alpha\in H^*$, $h \in H$, $k\in K$, we have that $\Delta(k\cdot(\alpha|h)) = \Delta((\alpha \leftharpoonup Sk_2)|k_1\cdot h) = \sum(\alpha \leftharpoonup Sk_2)_1|(k_1\cdot h)_1 \otimes (\alpha \leftharpoonup Sk_2)_2|(k_1\cdot h)_2 = \sum(\alpha_1 \leftharpoonup Sk_4)|(k_1\cdot h_1)\otimes(\alpha_2 \leftharpoonup Sk_3)|(k_2\cdot h_2) = \sum(\alpha_1 \leftharpoonup Sk_2)|(k_1\cdot h_1)\otimes(\alpha_2 \leftharpoonup Sk_4)|(k_3\cdot h_2) = \sum k_1\cdot(\alpha_1|h_1)\otimes k_2\cdot(\alpha_2|h_2)$.

emm:subalg1

Lemma 4.5.

(1) In the situation above, if K is cocommutative and H is a K–module bialgebra, then $\mathrm{End}(H)$ endowed with the Heisenberg product is a K–module algebra and $\mathrm{End}_f(H)$ is a K–subalgebra.

(2) Moreover, $\mathrm{End}_K(H)$ and $\mathrm{End}_{f,K}(H)$ are $\#$–subalgebras of $\mathrm{End}(H)$.

Proof. We prove only the assertion concerning the Heisenberg product in $\mathrm{End}_f(H)$ as is the one we use in the applications. The rest of the proof is left to the reader. Consider $\alpha,\beta\in H^*$, $h,\ell \in H$ and $k\in K$.

We first compute:

$$
\begin{gathered}
k\cdot(\alpha|h\#\beta|\ell) = \sum k\cdot(\alpha(h_1 \rightharpoonup \beta)|h_2\ell) = \\
\sum(\alpha(h_1 \rightharpoonup \beta)) \leftharpoonup Sk_2|k_1\cdot(h_2\ell) = \sum(\alpha(h_1 \rightharpoonup \beta)) \leftharpoonup Sk_3|(k_1\cdot h_2)(k_2\cdot\ell) = \\
\sum(\alpha \leftharpoonup Sk_4)((h_1 \rightharpoonup \beta) \leftharpoonup Sk_3)|(k_1\cdot h_2)(k_2\cdot \ell),
\end{gathered}
\tag{27}
$$

E:equiv1

next:

$$
\begin{aligned}
\sum k_1\cdot(\alpha|h)\#k_2\cdot(\beta|\ell) = \sum(\alpha \leftharpoonup Sk_2)|k_1\cdot h)\#(\beta \leftharpoonup Sk_4)|k_3\cdot\ell = \\
\sum(\alpha \leftharpoonup Sk_2)((k_1\cdot h)_1 \rightharpoonup (\beta \leftharpoonup Sk_4))|(k_1\cdot h)_2k_3\cdot \ell = \\
\sum(\alpha \leftharpoonup Sk_3)(k_1\cdot h_1 \rightharpoonup (\beta \leftharpoonup Sk_5))|(k_2\cdot h_2)(k_4\cdot\ell) = \\
\sum(\alpha \leftharpoonup Sk_3)(k_4\cdot h_1 \rightharpoonup (\beta \leftharpoonup Sk_5))|(k_1\cdot h_2)(k_2\cdot\ell)\,.
\end{aligned}
\tag{28}
$$

E:equiv2

Now, the equality of (27) and (28) can be deduced from the following calculation: take $\beta\in H^*$, $h,r\in H$ and $k \in K$,

$\sum(k_1\cdot h \rightharpoonup (\beta \leftharpoonup Sk_2))(r) = \sum(\beta \leftharpoonup Sk_2)(rk_1\cdot h) = \sum \beta((Sk_2)\cdot(r(k_1\cdot h))) = \sum\beta((Sk_3\cdot r)(Sk_2k_1\cdot h)) = \sum\beta((Sk\cdot r)h) = ((h \rightharpoonup \beta) \leftharpoonup Sk)(r)$.

$\square$

4.2. **The case of endomorphisms of graded Hopf algebras.** Assume that $H = \bigoplus_{n\geq 0} H_n$ is a graded connected bialgebra, i.e. for all $n, m \in \mathbb{N}$, $H_nH_m \subset H_{n+m}$, $\Delta(H_n) \subseteq \bigoplus_{p+q=n} H_p \otimes H_q$ and $H_0 = \Bbbk$ –conectivity condition–. It is well-known that in this situation H is a Hopf algebra, and that the antipode preserves the degree.

domorphisms **Definition 4.6.** Consider the following chain of linear subspaces of $\mathrm{End}(H)$, where $\mathrm{End}_{\mathrm{gr}}(H)$ is the subspace of the linear endomorphisms of H that preserve the degree: $\mathrm{End}(H) \supseteq \mathrm{End}_{\mathrm{gr}}(H) = \prod_n \mathrm{End}(H_n) \supseteq \bigoplus_n \mathrm{End}(H_n) := \mathrm{end}(H) \supseteq \bigoplus_n \mathrm{End}_f(H_n) := \mathrm{end}_f(H)$.

In the case that each H_n is finite dimensional, $\mathrm{end}(H) = \mathrm{end}_f(H) = \bigoplus_n (H_n^* \otimes H_n)$, that can be endowed with with a coproduct defined as below.

coprod-grad **Definition 4.7.** In the situation above, take $\alpha|h \in H_n^* \otimes H_n$, if $\Delta(\alpha) = \sum_{p+q=n} \alpha_p \otimes \alpha_q$ and $\Delta(h) = \sum_{r+s=n} h_r \otimes h_s$, with $\alpha_p \in H_p^*$, $\alpha_q \in H_q^*$, $h_r \in H_r$, $h_s \in H_s$; then $\Delta(\alpha|h) = \sum_{a+b=n} \alpha_a|h_a \otimes \alpha_b|h_b$.

R:compatigr **Remark 4.8.** It is clear that the composition and convolution product defined in $\mathrm{End}(H)$ restricts to the chain of subspaces considered above. Moreover, in the case that the H_n are finite dimensional, $\mathrm{end}(H)$ endowed with the convolution product and the above defined coproduct is a graded bialgebra.

The behaviour of the Heisenberg product in the graded case is described in the proposition that follows, that plays a central role in our constructions.

terpolation **Proposition 4.9.** *(1) The Heisenberg product of* $\mathrm{End}(H)$ *restricts to* $\mathrm{end}(H)$*. Moreover, if* $f \in \mathrm{End}(H_p)$ *and* $g \in \mathrm{End}(H_q)$ *then*

$$f \# g \in \bigoplus_{n=\max(p,q)}^{p+q} \mathrm{End}(H_n) \tag{29}$$ E:smash-degr

and the top and bottom components of $f \# g$ *are*

$$(f \# g)_{p+q} = f \star g \quad \text{and, if } p = q, \quad (f \# g)_p = g \circ f. \tag{30}$$ E:smash-endo

(2) In the case that $f = \alpha|k \in \mathrm{End}(H_p)$ *and* $g = \beta|\ell \in \mathrm{End}(H_q)$*, we have:*

$$\alpha|k\#\beta|\ell = \sum_{0\leq n\leq \min(p,q)} \alpha(k_n \rightharpoonup \beta)|k_{p-n}\ell$$

, if $\Delta(k) = \sum_n k_n \otimes k_{p-n}$ *where* $k_n \in H_n$ *and* $k_{p-n} \in H_{p-n}$*. Hence the Heisenberg product of* $\mathrm{End}(H)$ *also restricts to* $\mathrm{end}_f(H)$*.*

Proof. (1) Let $h \in H_n$. The coproduct of h is

$$\Delta(h) = \sum_{a+b=n} h_a \otimes h_b,$$

with $h_a \in H_a$ and $h_b \in H_b$. Using the formula (25) we obtain:

$$(f \# g)(h) = \sum_{a+b=n} f(h_a)_2 g\big(h_b f(h_a)_1\big). \tag{31}$$ E:smash-endo

Suppose that f and g belong to $\mathrm{end}(H)$. The computation of the degree of every term in the sum yields

$$\begin{aligned}\deg\Big(f(h_a)_2 g\big(h_b f(h_a)_1\big)\Big) &= \deg\big(f(h_a)_2\big) + \deg\Big(g\big(h_b f(h_a)_1\big)\Big)\\ &= \deg\big(f(h_a)_2\big) + \deg\big(h_b f(h_a)_1\big)\\ &= \deg\big(f(h_a)_2\big) + \deg(h_b) + \deg\big(f(h_a)_1\big)\\ &= \deg\big(f(h_a)\big) + \deg(h_b)\\ &= a + b = n,\end{aligned}$$

proving that $f \# g$ is in $\mathrm{end}(H)$.

We can refine the previous analysis as follows. Assume that $f \in \mathrm{End}(H_p)$ and $g \in \mathrm{End}(H_q)$. Then, Expression (31) is zero unless

$$a = p \quad \text{and} \quad b + \deg\big(f(h_a)_1\big) = q. \tag{32}$$ E:non-null-t

Adding these two equations we get that $n = a + b \leq p + q$. On the other hand, $p = a \leq a + b = n$ and $q = b + \deg\big(f(h_a)_1\big) \leq b + a = n$, hence $\max(p, q) \leq n$. This proves (29).

If we set $n = p + q$ in (32) then we get $\deg\big(f(h_a)_1\big) = 0$, and (31) reduces to the convolution diagram in (23). If we set $n = p = q$, then $\deg(h_b) = \deg\big(f(h_a)_2\big) = 0$, and (31) reduces to $g\big(f(h)\big) = (g \circ f)(h)$, which is the composition product.

(2) The equality $\alpha|k\#\beta|\ell = \sum_{0\leq n\leq \min(p,q)} \alpha(k_n \rightharpoonup \beta)|k_{p-n}\ell$ where $\Delta(k) = \sum k_n \otimes k_{p-n}$ follows immediately from the explicit formulæ (31) and from the considerations of (1) with the corresponding bounds for the degrees. $\square$

Thus and as expected, the Heisenberg product interpolates between the composition and convolution products. The analogous interpolation property at all other non-commutative levels (permutations and non-commutative symmetric functions) is a consequence of this general result.

terpolation **Remark 4.10.**

(1) Assume that that K is a commutative Hopf algebra and that H is graded connected K–module Hopf algebra as above. Assume also that the action of K preserves the grading. In this situation one can consider the chain of subspaces of $\mathrm{End}(H)$ that follows:

$$\begin{array}{ccccccc}\mathrm{End}(H) & \supseteq & \mathrm{End}_{\mathrm{gr}}(H) & \supseteq & \mathrm{end}(H) & \supseteq & \mathrm{end}_f(H)\\ \cup\!| & & \cup\!| & & \cup\!| & & \cup\!|\\ \mathrm{End}_K(H) & \supseteq & \mathrm{End}_{\mathrm{gr},K}(H) & \supseteq & \mathrm{end}_K(H) & \supseteq & \mathrm{end}_{f,K}(H).\end{array}$$

(2) In this context is clear that all the subspacs considered above are closed under the composition, convolution and Heisenberg products. In particular the following holds: if $f \in \mathrm{End}_K(H_p)$ and $g \in \mathrm{End}_K(H_q)$ then

$$f \# g \in \bigoplus_{n=\max(p,q)}^{p+q} \mathrm{End}_K(H_n). \tag{33}$$ E:smash-degr

S:descents

5. The Heisenberg product of Garsia-Reutenauer endomorphisms

In this section we define certain distinguished subspace of endomorphisms of the Hopf algebra H, that we call the *Garsia-Reutenauer endomorphisms*. Then we show that the Heisenberg product in $\operatorname{End}(H)$ (Section 4) can be restricted to this special subspace.

These endomorphisms are characterized in terms of their action on products of primitive elements of H.

The motivation for the definition is that in the case that H is the tensor algebra of a vector space, an important result of Garsia and Reutenauer –see [9]– relates this subspace with the space of non-commutative symmetric functions via Schur-Weyl duality (Lemma 6.1 and Theorem 7.2).

scents-hopf

Definition 5.1. Let H be an arbitrary Hopf algebra. If $h_1, \ldots, h_n \in H$, define

$$G(h_1, \ldots, h_n) = \operatorname{Span}(h_{\sigma(1)} \cdots h_{\sigma(n)} \mid \sigma \in S_n),$$

or in other words, $G(h_1, \ldots, h_n)$ is the subspace generated by the products of the form $h_{\sigma(1)} \cdots h_{\sigma(n)}$ for $\sigma \in S_n$.

For later use, we record the explicit expressions of the comultiplication in elements that are products of primitives $h_1 \cdots h_n$.

We consider the set of (p,q)-shuffles –that is denoted by $\operatorname{Sh}(p,q)$–. A (p,q)-*shuffle* is a permutation $\xi \in S_{p+q}$ such that

$$\xi(1) < \cdots < \xi(p) \quad \text{and} \quad \xi(p+1) < \cdots < \xi(p+q)\,.$$

The comultiplication of $h_1 \cdots h_n$ is given as:

$$\Delta(h_1 \cdots h_n) = \sum_{p+q=n} \sum_{\xi \in \operatorname{Sh}(p,q)} h_{\xi(1)} \cdots h_{\xi(p)} \otimes h_{\xi(p+1)} \cdots h_{\xi(p+q)}\,. \tag{34}$$ E:explicit-c

The following lemma lists some of the basic properties of the subspaces $G(h_1, \ldots, h_n)$.

-primitives

Lemma 5.2. *For any $h_1, \ldots, h_n \in H$ we have:*

(i) If $a \in G(h_1, \ldots, h_k)$ and $b \in G(h_{k+1}, \ldots, h_n)$, then $ab \in G(h_1, \ldots h_n)$.
(ii) If $a \in G(h_1, \ldots, h_n)$ and $h_1, \ldots, h_n \in \operatorname{Prim}(H)$, then

$$\Delta(a) = \sum_{\substack{k+\ell=n \\ \xi \in \operatorname{Sh}(k,\ell)}} a_\xi^{(1)} \otimes a_\xi^{(2)},$$

where $a_\xi^{(1)} \in G(h_{\xi(1)}, \ldots, h_{\xi(k)})$ and $a_\xi^{(2)} \in G(h_{\xi(k+1)}, \ldots, h_{\xi(n)})$.

-Reutenauer

Definition 5.3. Let H be an arbitrary Hopf algebra. The space of *Garsia-Reutenauer endomorphisms* of H –denoted by $\Sigma(H)$– is:

$$\Sigma(H) = \bigl\{ f \in \operatorname{End}(H) \bigm| f\bigl(G(h_1, \ldots, h_n)\bigr) \subseteq G(h_1, \ldots, h_n) \text{ for all } h_1, \ldots, h_n \in \operatorname{Prim}(H) \bigr\}.$$

The subspace considered above, plays for $\operatorname{End}(H)$ the same role that the subspace of descents plays for the algebra of permutations.

:sigma-hopf

Theorem 5.4. *If H is a Hopf algebra, the space $\Sigma(H)$ of Garsia-Reutenauer endomorphisms is a subalgebra of $\operatorname{End}(H)$ with respect to the Heisenberg product.*

Proof. Given a primitive element h, we have $\iota\varepsilon(h) = 0$, hence the unit of the Heisenberg product is in $\Sigma(H)$.

Take two endomorphisms f and g in $\Sigma(H)$, and let $h_1, \dots, h_n \in \mathrm{Prim}(H)$. Then, we have by definition –see (34), (24) and (25)–:

$$(f \# g)(h_1 \cdots h_n) = \sum_{\substack{k+\ell=n \\ \xi \in \mathrm{Sh}(k,\ell)}} \big(f(h_{\xi(1)} \cdots h_{\xi(k)})\big)_2 g\Big(h_{\xi(k+1)} \cdots h_{\xi(n)} \big(f(h_{\xi(1)} \cdots h_{\xi(k)})\big)_1\Big). \tag{35}$$

As $f(h_{\xi(1)} \cdots h_{\xi(k)}) \in G(h_{\xi(1)} \cdots h_{\xi(k)})$, it follows from Lemma 5.2 that

$$\Delta\big(f(h_{\xi(1)} \cdots h_{\xi(k)})\big) = \sum_{\substack{r+s=k \\ \eta \in \mathrm{Sh}(r,s)}} a_\eta^{(1)} \otimes a_\eta^{(2)},$$

with $a_\eta^{(1)} \in G(h_{\xi\eta(1)}, \dots, h_{\xi\eta(r)})$ and $a_\eta^{(2)} \in G(h_{\xi\eta(r+1)}, \dots, h_{\xi\eta(k)})$. Hence, we rewrite (35) as

$$(f \# g)(h_1 \cdots h_n) = \sum_{\substack{k+\ell=n \\ \xi \in \mathrm{Sh}(k,\ell)}} a_\eta^{(2)}\, g(h_{\xi(k+1)} \cdots h_{\xi(n)}\, a_\eta^{(1)}).$$

But the argument of g belongs to $G(h_{\xi(k+1)}, \dots, h_{\xi(n)}, h_{\xi\eta(1)}, \dots, h_{\xi\eta(r)})$. Using that $g \in \Sigma(H)$ and using part (1) of Lemma 5.2 we obtain that

$$\begin{aligned}(f \# g)(h_1 \cdots h_n) \in G(h_{\xi\eta(r+1)}, \dots, h_{\xi\eta(k)}, h_{\xi(k+1)}, \dots, h_{\xi(n)}, h_{\xi\eta(1)}, \dots, h_{\xi\eta(r)})& \\ \subseteq G(h_1, \dots, h_n)&,\end{aligned}$$

proving that $f \# g \in \Sigma(H)$. $\square$

Remark 5.5. It is easy to show that $\Sigma(H)$ is also a subalgebra of $\mathrm{End}(H)$ with respect to the composition and convolution products.

In the situation that H is a graded connected bialgebra, we can produce an homogeneous and equivariant version of the above results.

We define the following chain of subspaces of $\Sigma(H)$:

$$\Sigma_{\mathrm{gr}}(H) := \Sigma(H) \cap \mathrm{End}_{\mathrm{gr}}(H) \supseteq \sigma(H) := \Sigma(H) \cap \mathrm{end}(H),$$

and in the case that K is a commutative bialgebra that acts on H, by homogenous bialgebra endomorphisms –see Section 4– we define:

$$\Sigma(H) \cap \mathrm{End}_K(H) = \Sigma_K(H) \supseteq \Sigma_{\mathrm{gr},K}(H) \supseteq \sigma_K(H).$$

As all the objects described above are defined as intersections, it is clear that they are closed under composition, convolution and Heisenberg product.

6. The Heisenberg product of permutations

In order to translate the Heisenberg product from endomorphisms of Hopf algebras to permutations we specialize the constructions of Section 4 and apply the methods related to the Schur-Weyl duality theorem –see [15]–.

Let

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

be the tensor algebra of a finite dimensional vector space V. It is a graded connected Hopf algebra with product defined by concatenation and with coproduct uniquely determined by the condition that the algebra generators –the elements $v \in V$– are primitive:

$$\Delta : v \mapsto 1 \otimes v + v \otimes 1 \quad \text{for } v \in V. \tag{36}$$

D:coproduct-

As $v_1 \otimes \cdots \otimes v_n = v_1 \cdots v_n$, we omit the tensors when writing elements of $T(V)$.

The general linear group $\mathrm{GL}(V)$ acts on V and hence on each $V^{\otimes n}$ diagonally. Schur-Weyl duality –as presented for example in [7] or [15]– guarantees that the only endomorphisms of $T(V)$ which commute with the action of $\mathrm{GL}(V)$ are (linear combinations of) permutations.

Let

$$\mathcal{S} = \bigoplus_{n \geq 0} \Bbbk S_n$$

be the direct sum of all symmetric group algebras. The product in $\mathcal{S}$ is defined on permutations as the usual composition –denoted by $\sigma \circ \tau$ or $\sigma\tau$– when σ and τ belong to the same homogeneous component of $\mathcal{S}$, and is 0 in any other case. The identity in S_n is denoted by Id_n.

eyl-duality

Lemma 6.1 (Schur-Weyl duality)**.** *In the notations above, let Ψ be the map*

$$\Psi : \mathcal{S} \to \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big),$$

defined by sending $\sigma \in S_n$ to the endomorphism $\Psi(\sigma)$ of $T(V)$, which in degree n is given by the right action of σ on $V^{\otimes n}$:

$$v_1 \cdots v_n \overset{\Psi(\sigma)}{\longmapsto} v_{\sigma(1)} \cdots v_{\sigma(n)}$$

and is 0 in the other homogeneous components. Then, Ψ is an homogeneous isomorphism of vector spaces.

senberg_per

Definition 6.2. The *Heisenberg product of permutations* $\# : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is defined by the commutativity of the diagram below:

$$\begin{tikzcd}
\mathcal{S} \times \mathcal{S} \arrow[r, "\Psi \times \Psi"] \arrow[d, "\#"'] & \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big) \times \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big) \arrow[d, "\#"] \\
\mathcal{S} \arrow[r, "\Psi"'] & \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big)
\end{tikzcd}$$

Compare the above definition with the results by Malvenuto and Reutenauer in [20] where the authors deal with the convolution product in $\mathcal{S}$. The considerations of Section 4, guarantee that the same methods can be applied in the situation treated in Definition 6.2 for the Heisenberg product.

This method, presented in [20] and used above, is important because it could be applied to other dualities than Schur-Weyl, i.e. to centralizer algebras of other groups (or Hopf algebras) acting on a tensor algebra.

It also can be applied to other products of endomorphisms, a remarkable case being that of the *Drinfel'd* product, which is studied in [23].

Next we exhibit an explicit formula for the Heisenberg product of two permutations, that at the lowest and highest degree, yield the usual formulæ for the composition and the Malvenuto–Reutenauer products respectively –see the table appearing in Figure 1–.

We establish the following notation for the (p,q) shuffle of maximal length:

$$\beta_{p,q} = \begin{pmatrix} 1 & 2 & \cdots & p & p+1 & p+2 & \cdots & p+q \\ q+1 & q+2 & \cdots & q+p & 1 & 2 & \cdots & q \end{pmatrix}.$$

Notice in particular that $\beta_{p,q} = \beta_{q,p}^{-1}$.

Theorem 6.3. *(1) Let $\sigma \in S_p$ and $\tau \in S_q$. Then, the Heisenberg product in $\mathcal{S}$ can be expressed as*

$$\sigma \# \tau = \sum_{n=\max(p,q)}^{p+q} \sum_{\substack{\xi\in \mathrm{Sh}(p,n-p)\\ \eta \in \mathrm{Sh}(p+q-n,n-q)}} \xi\big((\sigma\eta)\times \mathrm{Id}_{n-p}\big)\beta_{2n-p-q,p+q-n}(\mathrm{Id}_{n-q}\times\tau). \qquad (37)$$

T:smash-perm

(2) When $n = p+q$:

$$(\sigma \# \tau)_n = \sigma \star \tau = \sum_{\xi\in\mathrm{Sh}(p,q)} \xi(\sigma\times\tau),$$

where $\sigma\times\tau \in S_{p+q}$ via the standard inclusion:

$$(\sigma\times\tau)(i) = \begin{cases} \sigma(i) & \text{if } 1\le i\le p\,,\\ p+\tau(i-p) & \text{if } p+1\le i\le p+q\,. \end{cases}$$

When $n = p = q$:

$$(\sigma \# \tau)_n = \sigma\tau.$$

Proof. (1) Using (34) and (35) for the endomorphisms $\Psi(\sigma)$ and $\Psi(\tau)$ induced by the permutations $\sigma\in S_p$ and $\tau\in S_q$, respectively, we obtain:

$$\begin{aligned}\big(\Psi(\sigma)\#\Psi(\tau)\big)&(v_1\cdots v_n)\\ &= \sum_{\substack{r+s=n\\ \xi\in\mathrm{Sh}(r,s)}} \big(\Psi(\sigma)(v_{\xi(1)}\cdots v_{\xi(r)})\big)_2\,\Psi(\tau)\Big(v_{\xi_{r+1}}\cdots v_{\xi_n}\,\big(\Psi(\sigma)(v_{\xi(1)}\cdots v_{\xi(r)})\big)_1\Big).\end{aligned}$$

The only non-zero terms occur when $r=p$, hence

$$\begin{aligned}\big(\Psi(\sigma)\#\Psi(\tau)\big)&(v_1\cdots v_n)\\ &= \sum_{\xi\in\mathrm{Sh}(p,n-p)} \big(\Psi(\sigma)(v_{\xi(1)}\cdots v_{\xi(p)})\big)_2\,\Psi(\tau)\Big(v_{\xi(p+1)}\cdots v_{\xi(n)}\,\big(\Psi(\sigma)(v_{\xi(1)}\cdots v_{\xi(p)})\big)_1\Big)\\ &= \sum_{\substack{\xi\in\mathrm{Sh}(p,n-p)\\ u+v=p\\ \eta\in\mathrm{Sh}(u,v)}} v_{\xi\sigma\eta(u+1)}\cdots v_{\xi\sigma\eta(p)}\Psi(\tau)\big(v_{\xi(p+1)}\cdots v_{\xi(n)}v_{\xi\sigma\eta(1)}\cdots v_{\xi\sigma\eta(u)}\big)\\ &= \sum_{\substack{\xi\in\mathrm{Sh}(p,n-p)\\ \eta\in\mathrm{Sh}(p+q-n,n-q)}} v_{\xi\sigma\eta(p+q-n+1)}\cdots v_{\xi\sigma\eta(p)}v_{\xi\tau(p+1)}\cdots v_{\xi\tau(n)}v_{\xi\sigma\eta\tau(1)}\cdots v_{\xi\sigma\eta\tau(p+q-n)}\\ &= \sum_{\substack{\xi\in\mathrm{Sh}(p,n-p)\\ \eta\in\mathrm{Sh}(p+q-n,n-q)}} \Psi\Big[\xi\big((\sigma\eta)\times\mathrm{Id}_{n-p}\big)\beta_{2n-p-q,p+q-n}(\mathrm{Id}_{n-q}\times\tau)\Big](v_1\cdots v_n),\end{aligned}$$

which proves the first part of the theorem.

(2) This part follows directly. Observe that for the case $n = p + q$ we obtain the product of permutations as defined by Malvenuto-Reutenauer in [21, 20].
In the case $n = p = q$ and since the action of S_n on $V^{\otimes n}$ is from the right, the composition of permutations corresponds to composition of endomorphisms in the opposite order –compare with the results of Proposition 4.9–.

□

For example, writing the permutations in word format we obtain that:

$$\begin{aligned} 12 \,\#\, 132 = {} & 132 + 231 + 321 \\ & + \mathrm{Id}_4 + 1243 + 1324 + 2134 + 2143 + 2314 \\ & + 3124 + 3142 + 3214 + 4123 + 4132 + 4213 \\ & + 12354 + 13254 + 14253 + 15243 + 23154 \\ & + 24153 + 25143 + 34152 + 35142 + 45132. \end{aligned}$$

Notice that the red terms correspond to the Malvenuto–Reutenauer product.

ncsymmetric

7. The Heisenberg product of non-commutative symmetric functions

The *descent set* of a permutation $\sigma \in S_n$ is the subset of $[n-1]$ defined by

$$\mathrm{Des}(\sigma) = \big\{ i \in [n-1] \mid \sigma(i) > \sigma(i+1) \big\}.$$

Given $J \subseteq [n-1]$, define $\mathcal{B}_J$ as the set of permutations $\sigma \in S_n$ with $\mathrm{Des}(\sigma) \subseteq J$, and consider the following elements of $\Bbbk S_n$:

$$X_J = \sum_{\sigma \in \mathcal{B}_J} \sigma. \tag{38}$$ E:def-X

The family of subsets of $[n-1]$ is in bijective correspondence with the set of compositions of n. Recally that $\alpha = (a_1, \dots, a_r)$ is a composition of n if all the a_i are positive integres and $\sum_{i=1}^{r} a_i = n$, in this situation we write $\alpha = (a_1, \dots, a_r) \models n$.

The bijection between compositions and subsets is:

$$(a_1, a_2, \dots, a_r) \longleftrightarrow \{a_1, a_1 + a_2, \dots, a_1 + \cdots + a_{r-1}\}.$$

For instance, if $n = 9$, then $X_{(1,2,4,2)} = X_{\{1,3,7\}}$ and $X_{(2,4,2,1)} = X_{\{2,6,8\}}$.

efidescents

Definition 7.1. Let $\Sigma_n = \Bbbk\{X_\alpha : \alpha \models n\} \subseteq \Bbbk S_n$, be the subspace *linearly spanned* by X_α with α composition of n and

$$\Sigma = \bigoplus_{n \geq 0} \Sigma_n.$$

An important result of Garsia and Reutenauer characterizes the elements of $\mathcal{S}$ whose images by Ψ –in the notations of Lemma 6.1– belong to $\sigma_{\mathrm{GL}(V)}\big(T(V)\big)$. in terms of their action on the tensor algebra. Recall the Definition 5.3 of $\sigma_{\mathrm{GL}(V)}\big(T(V)\big)$.

T:sigma-GR

Theorem 7.2 (Garsia–Reutenauer, [9]). *Let Ψ be isomorphism defined in Lemma 6.1, then the following diagram commutes:*

$$\begin{tikzcd} \Sigma \arrow[d, "\Psi|_\Sigma"'] & \subseteq & \mathcal{S} \arrow[d, "\Psi"] \\ \sigma_{\mathrm{GL}(V)}\big(T(V)\big) & \subseteq & \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big), \end{tikzcd}$$

and the map $\Psi|_{\Sigma}$ *is surjective.*

A fundamental result of Solomon [30] states that Σ_n is a subalgebra of the symmetric group algebra $\Bbbk S_n$ with the composition product. This is *Solomon's descent algebra*. It is also well-known that Σ is closed under the external product [11, 12, 20]; in fact,

$$X_{(a_1,\dots,a_r)} \cdot X_{(b_1,\dots,b_s)} = X_{(a_1,\dots,a_r,b_1,\dots,b_s)}. \tag{39}$$ E:con-X

The space Σ with the external product is the algebra of *non-commutative symmetric functions*.

The following theorem generalizes these two situations in view of the interpolation property of the Heisenberg product of permutations.

smash-sigma **Theorem 7.3.** *The subspace $\Sigma \subseteq \mathcal{S}$ is closed under the Heisenberg product.*

Proof. This a direct result from Theorem 7.2, Theorem 5.4, and Schur-Weyl duality (Lemma 6.1). □

The next theorem gives another version of the same result with an explicit description of the value of $X_\alpha \# X_\beta$ for $\alpha \models p$ and $\beta \models q$. The structure coefficients of $X_\alpha \# X_\beta$ are expressed in terms of the matrices $\mathcal{M}^n_{\alpha,\beta}$ defined in Subsection 3.2.

Another combinatorial proof appears in [23]; this extends a proof of Schocker in [29] for the composition product.

Yet another proof of Theorem 7.3 can be obtained by extending the Heisenberg product to the Coxeter complex of the symmetric group (that is, the faces of the permutahedron) [1].

Recall the following definition: take $\alpha = (a_1,\dots,a_r) \models p$, $\beta = (b_1,\dots,b_s) \models q$ two compositions and let n be an integer, $\max(p,q) \le n \le p+q$. Let $a_0 := n-p$, $b_0 := n-q$, and $\mathcal{M}^n_{\alpha,\beta}$ be the set of all integral $(s+1)\times(r+1)$–matrices with non-negative entries $M = (m_{ij})_{\substack{0\le i\le s\\ 0\le j\le r}}$ such that: the sequence of column sums is $(a_0, a_1, \dots, a_r)$, the sequence of row sums is $(b_0, b_1, \dots, b_s)$ and the first entry is $m_{00} = 0$. To visualize these conditions we write the diagram:

$$\begin{array}{cccc|c}
0 & m_{01} & \cdots & m_{0r} & n-q \\
m_{10} & m_{11} & \cdots & m_{1r} & b_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{s0} & m_{s1} & \cdots & m_{sr} & b_s \\
\hline
n-p & a_1 & \cdots & a_r &
\end{array}$$

T:smash-X **Theorem 7.4.** *Let $\alpha \models p$ and $\beta \models q$ be two compositions. Then*

$$X_\alpha \# X_\beta = \sum_{n=\max(p,q)}^{p+q} \sum_{M \in \mathcal{M}^n_{\alpha,\beta}} X_{c(M)} \tag{40}$$ E:smash-X

where $c(M)$ is the composition whose parts are the non-zero entries of M, read from left to right and from top to bottom.

Observe that even though this formula is similar to the one we had for symmetric functions (14), the occurrence of the compositions as indices of the basis makes the connection between the Heisenberg product and the external and Solomon products considerably subtler than in the commutatative context. In particular, a formula like (19) does not longer hold. Moreover, the above expression will allow us to make the connection with the Heisenberg product of the representations of the symmetric group. This point is taken up in Section 8.

By the interpolation property of the Heisenberg product, Theorem 7.4 contains as special cases rules for the product in Solomon's descent algebra and for the external product of two basis elements of Σ. One readily verifies that the former is precisely the well-known rule of Garsia, Remmel, Reutenauer, and Solomon as given in [9, Proposition 1.1], while the latter is the one appearing in (39).

As an example we have the following formula

$$X_{(1^u)} \# X_{(1^v)} = \sum_{n=\max(u,v)}^{u+v} \binom{u}{n-v}\binom{v}{n-u}(u+v-n)!\, X_{(1^n)}, \tag{41}$$

E:smash-X_1-

where (1^n) is the composition of n with n parts equal to 1. The coefficients arise by inspection of all the possible ways to fill out the entries of the matrix M in a diagram as below with row and column sums as prescribed.

$$\begin{array}{cccc|c} 0 & ? & \cdots & ? & n-v \\ ? & ? & \cdots & ? & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ? & ? & \cdots & ? & 1 \\ \hline n-u & 1 & \cdots & 1 & \end{array}$$

Similarly, one verifies that:

$$X_{(1)}^{\#(n)} = \sum_{k=1}^{n} S(n,k) X_{(1^k)},$$

where the $S(n,k)$ are the Stirling numbers of the second kind.

Proof of Theorem 7.4. Let us take a fixed integer n between $\max(p,q)$ and $p+q$. To the compositions $\alpha = (a_1,\dots,a_r)$ and $\beta = (b_1,\dots,b_s)$ we associate the following sets:

$$\begin{aligned} E_0^n &= [p+1,n], & F_0^n &= [1,n-q], \\ E_1^n &= [1,a_1], & F_1^n &= n-q+[1,b_1], \\ E_2^n &= [a_1+1,a_1+a_2], & F_2^n &= n-q+[b_1+1,b_1+b_2], \\ &\ \vdots & &\ \vdots \\ E_r^n &= [a_1+\cdots+a_{r-1}+1,p], & F_s^n &= n-q+[b_1+\cdots+b_{s-1}+1,q]. \end{aligned}$$

Observe that the family of intervals $\{E_j^n\}_{j\in\{0,\dots,r\}}$ and $\{F_i^n\}_{i\in\{0,\dots,s\}}$ are partitions of $[1,n]$. It is also clear that $\sigma \in \mathcal{B}_\alpha$ if and only if $\sigma \times \mathrm{Id}_{n-p}$ is increasing in E_j^n for all $j \in \{0,\dots,r\}$. Similarly, $\tau \in \mathcal{B}_\beta$ if and only if $\mathrm{Id}_{n-q} \times \tau$ is increasing in F_i^n for all $i \in \{0,\dots,s\}$. Observe, also, that $\# E_j^n$ is the j-th coordinate of the pseudo-composition $(n-p,a_1,\dots,a_r)$, and $\# F_i^n$ is the i-th coordinate of $(n-q,b_1,\dots,b_s)$.

Given $\eta \in \mathrm{Sh}(p+q-n, n-q)$ and $\tau \in \mathcal{B}_\beta$, call $\varphi_{\eta,\tau} = (\eta \times \mathrm{Id}_{n-p})\beta_0(\mathrm{Id}_{n-q} \times \tau)$ and define the matrix

$$M_{\eta,\tau} = \big\{\#(F_i^n \cap \varphi_{\eta,\tau}^{-1} E_j^n)\big\}_{\substack{0\leq i\leq s,\\ 0\leq j\leq r}}$$

where we have abbreviated $\beta_{2n-p-q,p+q-n} = \beta_0$. In this situation $M_{\eta,\tau} \in \mathcal{M}^n_{\alpha,\beta}$. Indeed, if we call $m_{ij} = \#(F_i^n \cap \varphi_{\eta,\tau}^{-1} E_j^n)$, for $i = j = 0$ we have that

$$\varphi_{\eta,\tau}[1, n-q] = \eta[p+q-n+1, p] \subseteq [1,p],$$

which shows that the intersection $F_0^n \cap \varphi_{\eta,\tau}^{-1} E_0^n$ is empty, and then $m_{00} = 0$. The sum $m_{0j} + \cdots + m_{sj}$ equals the number of elements of E_j^n, which is, as noted before, the j-th entry of the composition $(n-p, a_1, \ldots, a_r)$. The same argument applies to the sum of the rows. In this manner, sending $\tau \mapsto M_{\eta,\tau}$ we define a map $\mathcal{B}_\beta \to \mathcal{M}^n_{\alpha,\beta}$. Take $M = \{m_{ij}\} \in \mathcal{M}^n_{\alpha,\beta}$ and let $\mathcal{B}^{\eta,n}_\beta(M)$ the corresponding fiber of this map:

$$\mathcal{B}^{\eta,n}_\beta(M) = \big\{\tau \in \mathcal{B}_\beta \;\big|\; \#(F_i^n \cap \varphi_{\eta,\tau}^{-1} E_j^n) = m_{ij} \text{ for all } j \in \{0,\ldots,r\},\, i \in \{0,\ldots,s\}\big\}.$$

Therefore, we have a partition of $\mathcal{B}_\beta = \bigcup_{M\in\mathcal{M}^n_{\alpha,\beta}} \mathcal{B}^{\eta,n}_\beta(M)$.

For $\xi \in \mathrm{Sh}(p, n-p)$ and $\eta \in \mathrm{Sh}(p+q-n, n-q)$, let us denote $g^n_{\xi,\eta}(\sigma,\tau) = \xi\big((\sigma\eta) \times \mathrm{Id}_{n-p}\big)\beta_0(\mathrm{Id}_{n-q} \times \tau)$, the n-term in the sum (37). The function $g^n_{\xi,\eta}$ is bilinear, and we can write

$$X_\alpha \# X_\beta = \sum_n \sum_{\xi,\eta} g^n_{\xi,\eta}(X_\alpha, X_\beta).$$

From now on as n is fixed we will omit it in the notations of the sets and the functions. Next we show that

$$\sum_{\xi,\eta} g_{\xi,\eta}(X_\alpha, X_\beta) = \sum_{M\in\mathcal{M}_{\alpha,\beta}} X_{c(M)}.$$

For this, we write

$$\begin{aligned}\sum_{\xi,\eta} g_{\xi,\eta}(X_\alpha, X_\beta) &= \sum_{\xi,\eta} g_{\xi,\eta}\Big(\sum_{\sigma\in\mathcal{B}_\alpha}\sigma, \sum_{M\in\mathcal{M}_{\alpha,\beta}}\sum_{\tau\in\mathcal{B}^\eta_\beta(M)}\tau\Big)\\ &= \sum_{M\in\mathcal{M}_{\alpha,\beta}}\sum_{\xi,\eta}\sum_{\sigma\in\mathcal{B}_\alpha}\sum_{\tau\in\mathcal{B}^\eta_\beta(M)} g_{\xi,\eta}(\sigma,\tau). \qquad (42)\end{aligned}$$ E:sums_of_g

If we denote by $S_{\alpha,\beta}(M)$ the set of elements (ξ,η,σ,τ) such that $\xi \in \mathrm{Sh}(p, n-p)$, $\eta \in \mathrm{Sh}(p+q-n, n-q)$, $\sigma \in \mathcal{B}_\alpha$ and $\tau \in \mathcal{B}^\eta_\alpha(M)$; then the map $\psi : S_{\alpha,\beta}(M) \to \mathcal{B}_{c(M)}$ given by $\psi(\xi,\eta,\sigma,\tau) = g_{\xi,\eta}(\sigma,\tau)$ is a bijection. We prove this fact in Lemma 7.6. In this situation, if we group together the last three sums of (42) we obtain

$$\sum_{\xi,\eta} g_{\xi,\eta}(X_\alpha, X_\beta) = \sum_{M\in\mathcal{M}_{\alpha,\beta}} X_{c(M)},$$

which concludes the proof of the theorem. □

In the following two lemmas we assume the notations of the previous theorem. Their proofs, being rather technical are presented in the Appendix. See: 12.2 and 12.3.

L:interval **Lemma 7.5.** *For $\eta \in \mathrm{Sh}(p+q-n, n-q)$, $\tau \in \mathcal{B}_\beta$ and for all $i = 0,\ldots,s$ and $j = 0,\ldots,r$, the sets*

$$F_i \cap \varphi_{\eta,\tau}^{-1} E_j$$

are disjoint intervals. Moreover, in each of these intervals the function $\varphi_{\eta,\tau}$ is increasing and has image either contained in $[1,p]$ or contained in $[p+1,n]$.

L:bijection **Lemma 7.6.** *For $M \in \mathcal{M}_{\alpha,\beta}$, the map $\psi : S_{\alpha,\beta}(M) \to \mathcal{B}_{c(M)}$, which sends (ξ,η,σ,τ) into $g_{\xi,\eta}(\sigma,\tau)$, is a bijection.*

commutative 8. From non-commutative to commutative symmetric functions

In the previous four sections of *Part 2: Non–commutative context*; we constructed the following commutative diagram of algebras –endowed with their respective Heisenberg products– (omitting the part that includes the space Λ):

$$\begin{tikzcd}
\Lambda & \Sigma \arrow[l, dotted, "\pi"'] \arrow[d, "\Psi|_\Sigma"'] & \subseteq & \mathcal{S} \arrow[d, "\Psi"] & & \\
& \sigma_{\mathrm{GL}(V)}\big(T(V)\big) & \subseteq & \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big) & \subseteq & \mathrm{end}\big(T(V)\big).
\end{tikzcd}$$

In the part of the above diagram that excludes Λ, all the Heisenberg products are induced by the one defined in the larger space $\mathrm{end}\big(T(V)\big)$. Recall also that the vertical isomorphisms are due to Schur–Weyl duality and to the results of [9] as mentioned in Section 7, Theorem 7.2.

We want to incorporate into this diagram the space of symmetric functions Λ equipped also with the Heisenberg product, this is expressed in the dotted arrow that is to be defined.

For each $n \geq 0$ we define the linear map $\pi_n : \Sigma_n \to \Lambda_n$ by its values on the basis $\{X_\alpha : \alpha \text{ composition of } n\}$ as:

$$\pi_n(X_\alpha) = h_{\widetilde{\alpha}},$$

where $\widetilde{\alpha}$ is the partition of n obtained by reordering the entries of the composition α. Let us denote by $\pi : \Sigma \twoheadrightarrow \Lambda$ the map induced in the direct sums.

It is well-known that π is a morphism if we endow Σ with the Solomon product and Λ with the internal product, it is also a morphism when we endow both spaces with the external products.

The theorem that follows generalizes these compatibilities by proving that the map $\pi : \Sigma \twoheadrightarrow \Lambda$ is a morphism with respect to the Heisenberg products in Λ and Σ, as constructed in Sections 3 and 7, respectively.

Theorem 8.1. *For any pair of compositions α and β:*

$$\pi(X_\alpha \# X_\beta) = h_{\widetilde{\alpha}} \# h_{\widetilde{\beta}}.$$

Proof. Comparing the explicit formulas –see equations (14) and (40)– it is clear that it is enough to construct for each n a bijection $\psi : \mathcal{M}^n_{\alpha,\beta} \to \mathcal{M}^n_{\widetilde{\alpha},\widetilde{\beta}}$, such that for all $M \in \mathcal{M}^n_{\alpha,\beta}$:

$$p\big(\psi(M)\big) = \widetilde{c(M)} \tag{43}$$ E:reordering

Let σ and τ be two permutations which reorder into partitions the compositions $\alpha = (a_1,\dots,a_r)$ and $\beta = (b_1,\dots,b_s)$, respectively, i.e.:

$$\widetilde{\alpha} = (a_{\sigma(1)},\dots,a_{\sigma(r)}), \quad \widetilde{\beta} = (b_{\tau(1)},\dots,b_{\tau(r)}).$$

Define $\psi(M)$ as the matrix obtained from M by permuting its columns with the permutation $\mathrm{Id}_1 \times \sigma$ and its rows with the permutation $\mathrm{Id}_1 \times \tau$. Clearly $\psi(M)$ is in $\mathcal{M}^n_{\widetilde{\alpha},\widetilde{\beta}}$ and the map ψ is a bijection.

Moreover, since the (unordered) entries of M and $\psi(M)$ are the same, we get Equation (43). $\square$

S:hopf

9. Compatibility of the coproduct with the Heisenberg product

In various of the spaces depicted in the table in Figure 1, coproducts can be introduced, and frequently they are compatible with the products we are considering. For brevity we concentrate in the consideration of the compatibility of the coproducts with the Heisenberg product at the level of permutations, of non-commutative and of commutative symmetric functions. We start by considering a coproduct that is called the convolution (or external) coproduct.

We start by defining this coproduct at the level of $\mathcal{S}$.

Let n be a positive integer and decompose it as $n = p + q$. Then, as $\mathrm{Sh}^{-1}(p,q)$ is a set of representatives for the left cosets of $S_p \times S_q \subseteq S_n$, given $\sigma \in S_n$ there is a unique triple $(\xi, \sigma_p, \sigma'_q)$ such that: $\xi \in \mathrm{Sh}(p,q)$, $\sigma_p \in S_p$, $\sigma'_q \in S_q$, and

$$\sigma = (\sigma_p \times \sigma'_q)\xi^{-1}\,. \tag{44}$$

E:parabolic

The coproduct $\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ is defined on $\sigma \in S_n$ as

$$\Delta(\sigma) = \sum_{p=0}^{n} \sigma_p \otimes \sigma'_q, \tag{45}$$

E:coprodper

where σ_p and σ'_q are as in (44). For instance : $\Delta(52413) = (\ ) \otimes (52413) + (1) \otimes (4132) + (21) \otimes (321) + (231) \otimes (21) + (2413) \otimes (1) + (52413) \otimes (\ )$.

If follows directly from the formula (45) above (see [21]) that:

$$\Delta(X_{(a_1,\dots,a_r)}) = \sum_{\substack{b_i + c_i = a_i \\ 0 \le b_i, c_i \le a_i}} X_{(b_1,\dots,b_r)^{\frown}} \otimes X_{(c_1,\dots,c_r)^{\frown}}, \tag{46}$$

D:coproduct-

where for a pseudopartition α, $\alpha^{\frown}$ indicates that parts equal to zero have been omitted.

It is then clear that the comultiplication Δ can be restricted to the space of descents: i.e. that

$$\Delta(\Sigma) \subseteq \Sigma \otimes \Sigma.$$

roduct-in-S

Remark 9.1. In [21, Thèoréme 5.3, Remarque 5.15] it is proved that equipped with the convolution product and the above coproduct, $\mathcal{S}$ becomes a graded connected Hopf algebra and it is also shown that Δ is not compatible with the composition of permutations. Also in [4] a more recent and detalied study of $(\mathcal{S}, \star, \Delta)$ is presented.

Taking into account that the Heisenberg product, interpolates between the convolution (or Malvenuto–Reutenauer) and the composition product at the level of the permutations, we cannot expect it to be compatible with the coproduct considered above.

These operations are better behaved if we restrict our attention to the non-commutative symmetric functions.

Next we prove that Δ is compatible with the Heisenberg product in Σ, and in particular this implies that for descents the composition product *is* compatible with the comultiplication.

-smash-hopf

Theorem 9.2. *The space $(\Sigma, \#, \Delta)$ is a cocommutative Hopf algebra.*

Proof. It is enough to prove the Heisenberg–multiplicativity of Δ on elements of the form X_α with α a composition of p for different p's.

Let α and β be compositions of p and q, respectively. We use Formula (40) to compute

$$\Delta(X_\alpha \# X_\beta) = \sum_n \sum_{M\in\mathcal{M}^n_{\alpha,\beta}} \Delta(X_{c(M)}) = \sum_{n=\max(p,q)}^{p+q} \sum_{M\in\mathcal{M}^n_{\alpha,\beta}} \sum_{\gamma+\gamma'=c(M)} X_\gamma \otimes X_{\gamma'}. \tag{47}$$

E:coproduct-

On the other hand,

$$\begin{aligned}\Delta(X_\alpha)\#\Delta(X_\beta) &= \Big(\sum_{\gamma+\gamma'=\alpha} X_\gamma\otimes X_{\gamma'}\Big)\#\Big(\sum_{\delta+\delta'=\beta} X_\delta\otimes X_{\delta'}\Big)\\ &= \sum_{\substack{\gamma+\gamma'=\alpha\\ \delta+\delta'=\beta}} (X_\gamma \# X_\delta)\otimes(X_{\gamma'}\# X_{\delta'})\\ &= \sum_{\substack{\gamma+\gamma'=\alpha\\ \delta+\delta'=\beta}} \sum_{n,n'} \sum_{\substack{M\in\mathcal{M}^n_{\delta,\gamma}\\ M'\in\mathcal{M}^{n'}_{\delta',\gamma'}}} X_{c(M)}\otimes X_{c(M')}.\end{aligned} \tag{48}$$

E:smash-of-

We show that the sums (47) and (48) are the same as follows: take an octuple of indices corresponding to the sum (48): $(\gamma,\gamma',\delta,\delta',n,n',M,M')$ and construct the quadruple $\big(n+n', M+M', c(M), c(M')\big)$. Denote by $\operatorname{col}(M)$ ($\operatorname{row}(M)$) the vector whose entries are the sum of the columns (rows) of the matrix M. Since

$$\operatorname{col}(M+M') = \operatorname{col}(M)+\operatorname{col}(M') = \big(n-|\gamma|\big)\gamma + \big(n'-|\gamma'|\big)\gamma' = (n+n'-p)\alpha,$$

where $|\zeta|$ is the sum of the parts of a composition ζ, and similarly with $\operatorname{row}(M+M') = (n+n'-q)\beta$, we see that $M+M'\in\mathcal{M}^n_{\alpha,\beta}$. As $c(M)+c(M') = c(M+M')$, if we set

$$(\widetilde{n},\widetilde{M},\widetilde{\gamma},\widetilde{\gamma}') = \big(n+n', M+M', c(M), c(M')\big)$$

it is clear that $(\widetilde{n},\widetilde{M},\widetilde{\gamma},\widetilde{\gamma}')$ is a quadruple of indices appearing in the sum (47) and that the corresponding summands of (47) and (48) are the same.

Moreover, it is clear that the above correspondence between the indices of the sums is bijective. $\square$

Consider now the space of symmetric functions Λ and the proyection $\pi:\Sigma\to\Lambda$. In Subsections 3.2 and 3.4, equations (18) and (20) we defined a coproduct on Λ similar to the one defined above and dual to the external product. We then gave its expression on the natural basis of complete homogeneous functions and power sums.

In particular we have that –see the notations of Subsection 3.2 and 3.4–: $\Delta(h_a) = \sum_{i+j=a} h_i\otimes h_j$ and $\Delta(p_n) = 1\otimes p_n + p_n\otimes 1$.

Being $\pi(X_\alpha) = h_\alpha$, and using the first of the above equalities and Theorem 9.2, we conclude the following result.

Corollary 9.3. *The space of symmetric functions Λ equipped with the operations $(\#,\Delta)$ is a cocommutative Hopf algebra and the map $\pi:\Sigma\to\Lambda$ is a morphism of Hopf algebras.*

Proof. We have that:

$$\Delta(\pi(X_\alpha)) = \Delta(h_\alpha) = \sum h_{\alpha_1}\otimes h_{\alpha_2} = \sum \pi(X_{\alpha_1})\otimes\pi(X_{\alpha_2}) = (\pi\otimes\pi)(\Delta(X_\alpha)).$$

Since π is Heisenberg multiplicative, we conclude that the coproduct and the Heisenberg product are compatible in the space Σ. The compatibility of the coproduct and the Heisenberg product in Σ induces the compatibility in Λ. $\square$

This theorem generalizes known results on $(\Lambda, \cdot, \Delta)$ and $(\Lambda, *, \Delta)$ –see for example [10]–.

10. Isomorphisms between Heisenberg, convolution, and composition products

somorphisms

We have mentioned until now, two kinds of relations between the three products we have been dealing with: one is the interpolation connection and the other the formula appearing in Lemma 3.3.

In the spaces of symmetric functions there are further relations between the Heisenberg, internal and external products.

First we show that the external and Heisenberg products are isomorphic (in the commutative and non–commutative situations), but the isomorphism is not degree-preserving.

Similarly we prove that the Heisenberg and internal products are isomorphic in the commutative context –degrees not preserved–, but the isomorphism is only valid in the completion of the space.

T:iso-sigma **Theorem 10.1.** *The map $\psi : (\Sigma, \cdot, \Delta) \to (\Sigma, \#, \Delta)$ given by*

$$\psi(X_{(a_1,\dots,a_r)}) = X_{(a_1)} \# \cdots \# X_{(a_r)} \tag{49}$$

E:iso-sigma

is an isomorphism of Hopf algebras (which does not preserve the gradings).

Proof. Since the Heisenberg product has the external product as the only term in the upper degree, the matrix of the linear map ψ in the basis of the X_α's is triangular with 1 in the diagonal. Hence ψ is invertible and it is multiplicative because the external product on the basis of the X_α's is the concatenation of the compositions.

We finish by proving that ψ is comultiplicative:

$$\Delta\big(\psi(X_\alpha) \# \psi(X_\beta)\big) = (\psi \otimes \psi)\Delta(X_\alpha \cdot X_\beta). \tag{50}$$

E:psi-comult

Clearly, it is enough to prove (50) on the algebra generators $X_{(a)}$ for all non-negative integers a. For the right hand side of (50) we have:

$$\Delta(X_{(a_1)} \cdot X_{(a_2)}) = \Delta(X_{(a_1,a_2)}) = \sum_{\substack{a+b=a_1\\ a'+b'=a_2}} X_{(a,a')^{\wedge}} \otimes X_{(b,b')^{\wedge}}.$$

Applying the map $\psi \otimes \psi$ and using formula (40) to compute $\psi(X_{(a,a')}) = X_{(a)} \# X_{(a')}$ and $\psi(X_{(b,b')}) = X_{(b)} \# X_{(b')}$ (note that we assume $\psi(X_{(0)})$ to be the identity) we get

$$(\psi \otimes \psi)\Delta(X_{(a_1,a_2)^{\wedge}}) = \sum_{\substack{a+b=a_1\\ a'+b'=a_2}} \sum_{n,m} X_{(n-a',n-a,a+a'-n)^{\wedge}} \otimes X_{(m-b',m-b,b+b'-m)^{\wedge}}. \tag{51}$$

E:psi-comult

On the other hand, taking into account that $\psi(X_\alpha) = X_\alpha$ for partitions with only one part, the left hand side of (50) is:

$$\Delta\big(X_{(a_1)} \# X_{(a_2)}\big) = \sum_k \sum_{\substack{c_1+c_1'=k-a_2 \\ c_2+c_2'=k-a_1 \\ c_3+c_3'=a_1+a_2-k}} X_{(c_1,c_2,c_3)^\frown} \otimes X_{(c_1',c_2',c_3')^\frown}, \tag{52}$$

E:psi-comult

By collecting together the terms in (51) with $n+m=k$ and interchanging the sums, it is easy to see that (51) and (52) are the same expression. $\square$

C:iso-sigma

Corollary 10.2. *The map* $(\Lambda, \cdot, \Delta) \to (\Lambda, \#, \Delta)$ *given by*

$$h_{(a_1,\dots,a_r)} \mapsto h_{(a_1)} \# \cdots \# h_{(a_r)} \tag{53}$$

E:iso-sigma-

is an isomorphism of Hopf algebras (which does not preserve the gradings).

The Heisenberg and internal products are also isomorphic at the level of $\widehat{\Lambda}$.

:smash-comp

Theorem 10.3. *The map* $(\widehat{\Lambda}, \#) \to (\widehat{\Lambda}, *)$ *given by*

$$f \mapsto f \cdot \sum_{n\geq 0} h_{(n)} \tag{54}$$

E:smash-comp

is an isomorphism of algebras.

Proof. This isomorphism follows from the isomorphism (2) in the category of species. Note that the species $\mathbf{E}$ corresponds to the object $(\mathbf{1}_0, \mathbf{1}_1, \dots)$ in the category $\widehat{\mathbf{R}}$, where $\mathbf{1}_n$ is the trivial S_n-module. Applying the Grothendieck group construction and then the Frobenius map ch, we deduce that $\mathbf{E}$ maps into the element $\sum_{n\geq 0} h_{(n)}$ in $\widehat{\Lambda}$. $\square$

-smash-comp

Remark 10.4. We give negative answers to three questions on possible extensions of the above results.

(1) The isomorphism $(\Sigma, \cdot) \cong (\Sigma, \#)$ of Theorem 10.1 does *not* extend to an isomorphism between $(\widehat{\Sigma}, \cdot)$ and $(\widehat{\Sigma}, \#)$. Indeed, in case it did extend:

$$\psi(X_{(1)} + X_{(1,1)} + X_{(1,1,1)} + \cdots) = X_{(1)} + X_{(1)} \# X_{(1)} + X_{(1)} \# X_{(1)} \# X_{(1)} + \cdots, \tag{55}$$

E:nonextensi

each of the terms in the infinite sum appearing in the right hand side of (55), contributes with a factor of degree 1 (namely, $X_{(1)}$); this infinite sum is not a well-defined element of $\widehat{\Sigma}$.

(2) The isomorphism $(\widehat{\Lambda}, \#) \cong (\widehat{\Lambda}, *)$ of Theorem 10.3 does not restrict to an isomorphism between $(\Lambda, \#)$ and $(\Lambda, *)$. Indeed, the element $1 \in \Lambda$ maps to $\sum_{n\geq 0} h_{(n)}$ which is in $\widehat{\Lambda}$ but not in Λ.

(3) A similar isomorphism to (54) cannot be established at the level of $\widehat{\Sigma} = \prod_{n\geq 0} \Sigma_n$. The maps $\varphi : f \mapsto f \cdot \sum_{n\geq 0} X_{(n)}$ and $\psi : f \mapsto \sum_{n\geq 0} X_{(n)} \cdot f$ are not isomorphisms between $(\widehat{\Sigma}, \#)$ and $(\widehat{\Sigma}, *)$ because they are not not multiplicative.

Indeed, using the rule (40) we obtain that:

$$X_{(3)} \# X_{(3)} = X_{(3)} + X_{(1,1,2)} + X_{(2,2,1)} + X_{(3,3)}.$$

Then:

$$\varphi(X_{(3)} \# X_{(3)}) = \sum_{n\geq 0} X_{(3,n)^\frown} + \sum_{n\geq 0} X_{(1,1,2,n)^\frown} + \sum_{n\geq 0} X_{(2,2,1,n)^\frown} + \sum_{n\geq 0} X_{(3,3,n)^\frown}. \tag{56}$$

E:counterexa

On the other hand, computing $\varphi(X_{(3)}) * \varphi(X_{(3)})$ using Solomon's rule, gives:

$$\begin{aligned}\varphi(X_{(3)}) * \varphi(X_{(3)}) &= \sum_{n,m} X_{(3,n)^\frown} * X_{(3,m)^\frown} = \sum_{n} X_{(3,n)^\frown} * X_{(3,n)^\frown} \\ &= \sum_{n\geq 0} X_{(3,n)^\frown} + X_{(2,1,1,n)^\frown} + X_{(1,2,2,n)^\frown} + X_{(3,3,n)^\frown}. \qquad (57)\end{aligned}$$

E:counterexa

We can see that (56) and (57) are different since, for example, the term $X_{(2,1,1,n)^\frown}$ appears in (57) but there is no term in (56) whose index is a composition starting with $2, 1, 1$. A similar argument can be applied to show that ψ is also not multiplicative.

i-symmetric

11. The Heisenberg coproduct of quasi-symmetric functions

In this section we consider space of quasi-symmetric functions $\mathcal{Q}$, dual to the non-commutative symmetric functions. In preparation for the introduction of the *Heisenberg coproduct*, we recall the definitions of the internal and external coproducts.

11.1. **The internal and external coproduct of quasi-symmetric functions.** In Sections 7, 8 and 9 we dealt with the following diagram:

$$\begin{tikzcd} \Sigma \arrow[d, two heads, "\pi"'] & \subseteq & \mathcal{S} \\ \Lambda & & , \end{tikzcd}$$

and endowed the different spaces with $\#$, the Heisenberg product and Δ, the external coproduct. In this part we introduce other coproducts, and to avoid confusions we rename the external coproduct as $\Delta_{\cdot}$ (it was called simply Δ when considered in (45) and (18)). We proved the compatibility of $(\#, \Delta_{\cdot})$ at the level of Σ and Λ. This compatibility is consistent with the known results about of the external (or Malvenuto–Reutenauer) product and the coproduct at the three levels of the above diagram, and about the compatibility with the internal (and Solomon) product at the levels of Λ and Σ (compare with Remark 9.1).

Of the three Hopf algebras: $(\Sigma, \cdot, \Delta_{\cdot})$, $(\mathcal{S}, \star, \Delta_{\cdot})$ and $(\Lambda, \cdot, \Delta_{\cdot})$ the second and third are self dual (see for example [11, 12, 13, 14, 20]).

Hence, we can complete the picture adding the space $\mathcal{Q}$ that is the graded dual of Σ, and that will fit into the following commutative diagram of Hopf algebras:

$$\begin{tikzcd} \Sigma \arrow[d, two heads, "\pi"'] & \subseteq & \mathcal{S} \arrow[d, two heads, "F"] \\ \Lambda & \overset{\pi^*}{\subseteq} & \mathcal{Q} \end{tikzcd}$$

The diagram is self dual with respect to the antidiagonal. The new maps F and π^* are the duals of the inclusion of Σ in $\mathcal{S}$ and of the projection of Σ onto Λ, respectively. The map F is described in detail in [4] and will not be used here. Note that the space $\mathcal{S}$ is a Hopf algebra with respect to the Malvenuto-Reutenauer product (as noted in Remark 9.1) and not with respect to the composition product.

Next we recall the definition of $\mathcal{Q}$. Let $\mathbf{X} = \{x_1, x_2, \dots\}$ be an *alphabet*, i.e. a countable set, totally ordered by $x_1 < x_2 < \cdots$. Let $\Bbbk[[\mathbf{X}]]$ be the algebra of formal power series on $\mathbf{X}$ and $\mathcal{Q} = \mathcal{Q}(\mathbf{X})$ the subspace linearly spanned by the elements

$$M_\alpha = \sum_{i_1 < \cdots < i_r} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r} \tag{58}$$

E:def-M

as $\alpha = (a_1, \ldots, a_r)$ runs over all compositions of n, for $n \geq 0$. The space $\mathcal{Q}$ is a graded subalgebra of $\Bbbk[[\mathbf{X}]]$ known as the algebra of *quasi-symmetric functions* (see [12]). It is clear that any symmetric function is quasi-symmetric, hence we have the inclusion of algebras $\Lambda \subseteq \mathcal{Q}$. In [20] it is proved that this map is π^* defined above as the dual of the projection $\pi : \Sigma \twoheadrightarrow \Lambda$ (see Section 8).

The algebra $\mathcal{Q}$ carries two coproducts Δ_* and $\Delta.$ which are defined via evaluation of quasi-symmetric functions on alphabets. Let $\mathbf{Y}$ be another alphabet. We can view the disjoint union $\mathbf{X}+\mathbf{Y}$ and the Cartesian product $\mathbf{X}\times\mathbf{Y}$ as alphabets as follows: on $\mathbf{X}+\mathbf{Y}$ we keep the ordering among the variables of $\mathbf{X}$ and among the variables of $\mathbf{Y}$, and we require that every variable of $\mathbf{X}$ precede every variable of $\mathbf{Y}$. On $\mathbf{X} \times \mathbf{Y}$ we impose the reverse lexicographic order:

$$(x_h, y_i) \leq (x_j, y_k) \qquad \text{means} \qquad y_i < y_k \text{ or } (y_i = y_k \text{ and } x_h < x_j).$$

The coproducts are defined by the formulas

$$\Delta_*\big(f(\mathbf{X})\big) = f(\mathbf{X} \times \mathbf{Y}) \quad \text{and} \quad \Delta.\big(f(\mathbf{X})\big) = f(\mathbf{X} + \mathbf{Y}),$$

together with the identification $\mathcal{Q}(\mathbf{X}, \mathbf{Y}) \cong \mathcal{Q}(\mathbf{X}) \otimes \mathcal{Q}(\mathbf{X})$ (separation of variables).

Consider the following pairing between the homogeneous components of degree n of $\mathcal{Q}$ and Σ:

$$\langle M_\alpha, X_\beta \rangle = \delta_{\alpha,\beta}. \tag{59}$$

E:pairing

It is known [11, 12, 20] that this pairing identifies the product of quasi-symmetric functions with the coproduct (46) of Σ, and the coproducts Δ_* and $\Delta.$ with the internal and external products of Σ. In other words,

$$\langle fg, u \rangle = \langle f \otimes g, \Delta(u) \rangle, \quad \langle \Delta_* f, u \otimes v \rangle = \langle f, uv \rangle, \quad \langle \Delta. f, u \otimes v \rangle = \langle f, u \cdot v \rangle,$$

for any $f, g \in \mathcal{Q}$ and $u, v \in \Sigma$. Here we set $\langle f \otimes g, u \otimes v \rangle = \langle f, u \rangle \langle g, v \rangle$.

11.2. **The Heisenberg coproduct of quasi-symmetric functions.** Let $\Delta_\#$ be the coproduct of $\mathcal{Q}$ dual to the Heisenberg product of Σ:

$$\langle \Delta_\# f, u \otimes v \rangle = \langle f, u \# v \rangle.$$

Since the Heisenberg product is a sum of terms of various degrees (29), the Heisenberg coproduct is a finite sum of the form

$$\Delta_\#(f) = \sum_i f_i \otimes f_i'$$

with $0 \leq \deg(f_i)$ and $\deg(f_i') \leq \deg(f) \leq \deg(f_i) + \deg(f_i')$. The terms corresponding to $\deg(f) = \deg(f_i) = \deg(f_i')$ and to $\deg(f) = \deg(f_i) + \deg(f_i')$ are the coproducts $\Delta_*(f)$ and $\Delta.(f)$, respectively.

Let $\mathbf{1} + \mathbf{X}$ denote the alphabet $\mathbf{X}$ together with a new variable x_0 smaller than all the others and with the property $x_0^k = x_0$ for any natural k. Let

$$(\mathbf{1} + \mathbf{X}) \times (\mathbf{1} + \mathbf{Y}) - \mathbf{1}$$

be the Cartesian product of the alphabets $\mathbf{1}+\mathbf{X}$ and $\mathbf{1}+\mathbf{Y}$ with reverse lexicographic ordering and with the variable (x_0,y_0) removed. We can suggestively denote $(\mathbf{1}+\mathbf{X})\times(\mathbf{1}+\mathbf{Y})-\mathbf{1}$ by $\mathbf{X}+\mathbf{Y}+\mathbf{XY}$, although the order is given properly by the former expression.

The following result was obtained in conversation with Arun Ram.

T:coprod-Q

Theorem 11.1. *For any* $f\in\mathcal{Q}$,

$$\Delta_{\#}\big(f(\mathbf{X})\big)=f(\mathbf{X}+\mathbf{Y}+\mathbf{XY}).$$

Proof. We have to show that, with respect to the pairing (59),

$$\langle M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY}),X_\alpha\otimes X_\beta\rangle=\langle M_\gamma,X_\alpha\,\#\,X_\beta\rangle \tag{60}$$

E:pairing-sm

for all γ, α and β compositions of n, p and q, respectively. Let us fix a composition γ of n and let k the length of γ. Denote the set of indices of $M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY})$ by

$$\mathcal{Y}=\big\{\big((i_1,j_1),\dots,(i_k,j_k)\big)\;\big|\;(i_1,j_1)<\cdots<(i_k,j_k)\big\}.$$

Consider the set $\mathcal{A}_{\alpha,\beta}=\big\{M\in\mathcal{M}^n_{\alpha,\beta}\mid w(M)=\gamma\big\}$ and define the map

$$\psi:\mathcal{Y}\to\bigcup_{\alpha,\beta}\mathcal{A}_{\alpha,\beta}$$

as follows: given $(i_1,j_1)<\cdots<(i_k,j_k)$, let $\widetilde{M}=(\widetilde{m}_{ij})$ be a matrix of zeros big enough to set $\widetilde{m}_{j_\ell i_\ell}=\gamma_\ell$ (as usual in these proofs, we start the indices in 0). Then, remove all zero rows and columns, except those with index 0; let us call M to the result. Since $(0,0)$ is not a possible index, we have $m_{00}=0$. Thus, $M\in\mathcal{M}^n_{\alpha,\beta}$ where α is the composition obtained by adding all the rows of M but the first, and analogously with β and the rows of M.

The map ψ is surjective, since, given some $M\in\mathcal{A}_{\alpha,\beta}$, we can build a sequence of indices in $\mathcal{Y}$ by reading the nonzero entries of M, say m_{uv}, and considering the pairs (v,u) lexicographically ordered. Therefore, we can write

$$M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY})=\sum_{q\in\mathcal{Y}}(xy)^\gamma_q=\sum_{\alpha,\beta}\sum_{M\in\mathcal{A}_{\alpha,\beta}}\sum_{q\in\psi^{-1}(M)}(xy)^\gamma_q$$

where $(xy)^\gamma_q$ denotes the monomial $(x_{i_1}y_{i_1})^{\gamma_1}\cdots(x_{i_k}y_{i_k})^{\gamma_k}$ for $q=\big((i_1,j_1),\dots,(i_k,j_k)\big)$. Collecting together the x's and y's establishes a bijection between the terms of the last sum indexed over $\psi^{-1}(M)$ and the terms of $M_\alpha(\mathbf{X})M_\beta(\mathbf{Y})$. Indeed, take a term from this product, given by indices $i_{r_1}<\cdots<i_{r_k}$ and $j_{s_1}<\cdots<j_{s_\ell}$, and build the pairs (j_{s_u},i_{r_v}) such that $m_{v,u}\neq 0$. We also have to consider the pairs $(0,i_{r_v})$ and $(j_{s_u},0)$ according to nonzero entries in the first row and column of M. Ordering these indices it is clear that they belong to $\psi^{-1}(M)$ and this is the inverse process of grouping x's and y's.

Then, we can write

$$M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY})=\sum_{\alpha,\beta}\sum_{M\in\mathcal{A}_{\alpha,\beta}}M_\alpha(\mathbf{X})M_\beta(\mathbf{Y})=\sum_{\alpha,\beta}\#\,\mathcal{A}_{\alpha,\beta}M_\alpha(\mathbf{X})M_\beta(\mathbf{Y}).$$

which obviously implies the equation (60). $\square$

We can express the dual of the isomorphism in Theorem 10.1 in term of alphabets in the full dual of Σ, which is $\widehat{\mathcal{Q}}=\prod_{n\geq 0}\mathcal{Q}_n$. The pairing $\langle\,,\rangle:\Sigma\times\widehat{\mathcal{Q}}\to\Bbbk$ is defined by

$$\langle f,g\rangle=\sum_n\langle f_n,g_n\rangle_n$$

where f_n and g_n are the restrictions of f and g to the homogeneous components of degree n, and $\langle\,,\rangle_n$ is the pairing defined in (59).

For this, given an alphabet $\mathbf{X}$ we define its exponential, $\mathbf{e}(\mathbf{X})$, by

$$\mathbf{e}(\mathbf{X}) = \mathbf{X} + \mathbf{X}^{(2)} + \mathbf{X}^{(3)} + \cdots$$

where the *divided power* $\mathbf{X}^{(n)}$ is the set

$$\mathbf{X}^{(n)} = \big\{(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in \mathbf{X}^n \;\big|\; x_{i_1} < x_{i_2} < \cdots < x_{i_n}\big\}. \tag{61}$$

E:definitio

We endow $\mathbf{e}(\mathbf{X})$ with the reverse lexicographic order. With this notations the following equation holds:

$$\mathbf{e}(\mathbf{X}+\mathbf{Y}) = \big(1+\mathbf{e}(\mathbf{X})\big)\big(1+\mathbf{e}(\mathbf{Y})\big) - 1$$

where the equality is considered as ordered sets. Indeed, denote by $(x)_k$ the monomial $x_{i_1}\cdots x_{i_k}$ with $i_1 < \cdots < i_k$. Then, given $(x)_k(y)_\ell < (x')_{k'}(y')_{\ell'}$ in $\mathbf{e}(\mathbf{X}+\mathbf{Y})$, it is immediate to see that either $(y)_\ell < (y')_{\ell'}$ or $(y)_\ell = (y')_{\ell'}$ and $(x)_k < (x')_{k'}$, which is the definition of the order in the left hand side. Clearly, the same argument applies in the other direction.

l-iso-sigma

Theorem 11.2. *The dual of the isomorphism ψ from $(\Sigma, \cdot, \Delta_.)$ to $(\Sigma, \#, \Delta_.)$ of Theorem 10.1 with respect to the pairing $\langle\,,\rangle$, is the isomorphism ψ^* from $(\widehat{\mathcal{Q}}, \cdot, \Delta_\#)$ to $(\widehat{\mathcal{Q}}, \cdot, \Delta_.)$ given by*

$$\psi^*(f) = f\big(\mathbf{e}(\mathbf{X})\big).$$

Proof. We have to show that $\big\langle \psi(X_\gamma), f\big\rangle = \big\langle X_\gamma, \psi^*(f)\big\rangle$. Observe that, from the definition of the pairing, it is enough to prove this equation for each degree. Moreover, it is enough to prove it for the generators of the algebra $(\Sigma, \cdot)$ since, for g and g' generators

$$\begin{aligned}
\big\langle \psi(g\cdot g'), f\big\rangle &= \big\langle \psi(g) \# \psi(g'), f\big\rangle \\
&= \big\langle \psi(g)\otimes\psi(g'), f(\mathbf{X}+\mathbf{Y}+\mathbf{X}\mathbf{Y})\big\rangle \\
&= \sum_i \big\langle \psi(g), f_i(\mathbf{X})\big\rangle\, \big\langle \psi(g'), f_i'(\mathbf{Y})\big\rangle \\
&= \sum_i \big\langle g, f_i\big(\,\mathbf{e}(\mathbf{X})\big)\big\rangle\, \big\langle g', f_i'\big(\,\mathbf{e}(\mathbf{Y})\big)\big\rangle \\
&= \big\langle g\otimes g', f\big(\mathbf{e}(\mathbf{X}) + \mathbf{e}(\mathbf{Y}) + \mathbf{e}(\mathbf{X})\,\mathbf{e}(\mathbf{Y})\big)\big\rangle \\
&= \big\langle g\otimes g', f\big(\mathbf{e}(\mathbf{X}+\mathbf{Y})\big)\big\rangle \\
&= \big\langle g\otimes g', \Delta\big(f(\mathbf{e}(\mathbf{X}))\big)\big\rangle \\
&= \big\langle g\cdot g', f\big(\mathbf{e}(\mathbf{X})\big)\big\rangle.
\end{aligned}$$

Next, we prove the duality for the set of generators $X_{(n)}$ for $n \geq 0$, and for $f = M_\alpha$ where α is a composition of n. In this case we have $\psi(X_{(n)}) = X_{(n)}$ and the equation $\big\langle X_{(n)}, M_\alpha\big\rangle = \big\langle X_{(n)}, M_\alpha\big(\mathbf{e}(\mathbf{X})\big)\big\rangle = \delta_{(n),\alpha}$ is immediately verified. □

Endowed with the coproduct $\Delta_\#$, the algebra $\mathcal{Q}$ is a graded connected Hopf algebra, in duality with the graded connected Hopf algebra $(\Sigma, \#, \Delta)$.

We finish this section by expressing the antipode of this Hopf algebra in terms of the alphabets.

First, define the evaluation of quasi-symmetric functions on the the opposite of an alphabet $\mathbf{X}$ by the equation

$$M_\alpha(-\mathbf{X}) = (-1)^r \sum_{i_1 \geq \cdots \geq i_r} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}, \tag{62}$$

for any composition $\alpha = (a_1, \ldots, a_r)$ (compare with the definition of M_α in (58)).

Next, we define the alphabet

$$\mathbf{X}^* = \mathbf{X} + \mathbf{X}^2 + \mathbf{X}^3 + \cdots \tag{63}$$

as the disjoint union of the Cartesian powers $\mathbf{X}^n$ under reverse lexicographic order. For instance $(x_3, x_1, x_2) < (x_2, x_2) < (x_1, x_3, x_2)$.

Theorem 11.3. *The antipode of the Hopf algebra of quasi-symmetric functions* $(\mathcal{Q}, \cdot, \Delta_\#)$ *is:*

$$S_\#(f) = f\big((-\mathbf{X})^*\big).$$

Proof. By Theorem 11.1, it is enough to prove that $M_\alpha\big(\mathbf{X} + (-\mathbf{X})^* + \mathbf{X}(-\mathbf{X})^*\big) = 0$ for any alphabet $\mathbf{X}$ and for any composition α. We ilustrate the argument for a composition with only one part: $\alpha = (a)$, the argument for a composition with several parts is essentially the same.

By selecting variables from each of the three alphabets $\mathbf{X}$, $(-\mathbf{X})^*$, and $\mathbf{X}(-\mathbf{X})^*$, we can write

$$M_{(a)}\big(\mathbf{X} + (-\mathbf{X})^* + \mathbf{X}(-\mathbf{X})^*\big)$$
$$= \sum x_i^a + \sum_r (-1)^r \sum (x_{i_1} \cdots x_{i_r})^a + \sum_r (-1)^r \sum x_j^a (x_{i_1} \cdots x_{i_r})^a.$$

It is easy to see that the first sum cancel with the terms with $r = 1$ of the second sum, while the remaining terms of the second sum cancel with the last sum. □

12. Appendix: proofs of three lemmas

In this Appendix we provide the postponed proofs of the technical lemmas used in the paper.

12.1. **Proof of Lemma 3.2.**

Proof. To define the bijection $\Upsilon \to \mathcal{M}^n_{\alpha,\beta}$, we start by splitting the interals $[1, p]$ and $[1, q]$ as below:

$$\begin{aligned}
E_1 &= [1, a_1], & F_1 &= [1, b_1],\\
E_2 &= [a_1 + 1, a_1 + a_2], & F_2 &= [b_1 + 1, b_1 + b_2],\\
&\ \vdots & &\ \vdots\\
E_k &= [a_1 + \cdots + a_{k-1} + 1, p], & F_s &= [b_1 + \cdots + b_{s-1} + 1, q],
\end{aligned}$$

where $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_s)$. Given an element $\upsilon = \sigma \times \tau \in S_p \times S_q$ we consider the shuffles $\zeta_\alpha(\sigma) \in \mathrm{Sh}(\alpha)$ and $\zeta_\beta(\tau) \in \mathrm{Sh}(\beta)$ characterized by the equations

$$\sigma = \zeta_\alpha(\sigma) u, \qquad \tau = \zeta_\beta(\tau) v, \tag{64}$$

with $u \in S_\alpha$ and $v \in S_\beta$. To simplify the notation, we write $\zeta_\alpha = \zeta_\alpha(\sigma)$ and $\zeta_\beta = \zeta_\beta(\tau)$. We further split each interval E_i and F_j as below:

$$E_i = E_i' \sqcup E_i'', \qquad F_j = F_j' \sqcup F_j'',$$

such that

$$\begin{aligned} \zeta_\alpha(E_i') &\subseteq [1, n-q], & \zeta_\beta(F_j') &\subseteq [1, p+q-n], \\ \zeta_\alpha(E_i'') &\subseteq [n-q+1, p], & \zeta_\beta(F_j'') &\subseteq [p+q-n+1, q], \end{aligned}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, s$. Observe that with these definitions we have the decomposition of the interval $[1, n]$ into

$$[1, n-q] = \bigsqcup_{i=1}^{k} \zeta_\alpha(E_i'), \tag{65}$$

E:decomposit

$$[n-q+1, p] = \bigsqcup_{i=1}^{k} \zeta_\alpha(E_i'') = \bigsqcup_{j=1}^{s} \big(n - q + \zeta_\beta(F_j')\big), \tag{66}$$

E:decomposit

$$[p+1, n] = \bigsqcup_{j=1}^{s} \big(n - q + \zeta_\beta(F_j'')\big). \tag{67}$$

E:decomposit

Define the matrix $M_{\sigma\times\tau}$ of dimension $(k+1) \times (s+1)$ whose entries are

$$\begin{aligned} m_{00} &= 0, & & \\ m_{i0} &= \#\, E_i', & & \text{for } i = 1, \ldots, k, \\ m_{0j} &= \#\, F_j'', & & \text{for } j = 1, \ldots, s, \\ m_{ij} &= \#\, \Big[\zeta_\alpha(E_i'') \cap \big(n - q + \zeta_\beta(F_j')\big)\Big] & & \text{otherwise.} \end{aligned}$$

The matrix $M_{\sigma\times\tau}$ belongs to $\mathcal{M}^n_{\alpha,\beta}$. Assume that $i \neq 0$. Since $\zeta_\alpha(E_i'') \subseteq [n-q+1, p] \subseteq \bigsqcup_{j=1}^{s}\big(n - q + \zeta_\beta(F_j'')\big)$, we get

$$\begin{aligned} \sum_{j=0}^{s} m_{ij} &= \#\, E_i' + \sum_{j=1}^{s} \#\, \Big[\zeta_\alpha(E_i'') \cap \big(n - q + \zeta_\beta(F_j')\big)\Big] \\ &= \#\, E_i' + \#\Big[\zeta_\alpha(E_i'') \cap \bigsqcup_{j=1}^{s}\big(n - q + \zeta_\beta(F_j')\big)\Big] \\ &= \#\, E_i' + \#\big(\zeta_\alpha(E_i'')\big) = \#\, E_i = a_i. \end{aligned}$$

On the other hand, if $i = 0$, then, by (67), the sum of m_{0j} for $j = 0, \ldots, s$, coincides with $\#[p+1, n] = n - p$.

Next we show that the matrix $M_{\sigma\times\tau}$ does not depend on the choice of representative of the coset $(S_p \times_n S_q)v$. Let $x \in S_{n-q}$, $y \in S_{p+q-n}$, and $z \in S_{n-p}$, so that $x \times y \times z \in S_p \times_n S_q$. Consider the representative $v' = \sigma' \times \tau'$ where

$$\sigma' = (x \times y)\sigma \quad \text{and} \quad \tau' = (y \times z)\tau.$$

Let ζ_α' and ζ_β' the shuffles associate to v'. As $\zeta_\alpha(E_i') \subseteq [1, n-q]$, then $(x \times y)\big(\zeta_\alpha(E_i')\big) = x\big(\zeta_\alpha(E_i')\big)$. But we also have $x\big(\zeta_\alpha(E_i')\big) = \zeta_\alpha'(E_i) = \zeta_\alpha'(\tilde{E}_i') \sqcup \zeta_\alpha'(\tilde{E}_i'')$, where $E_i = \tilde{E}_i' \sqcup \tilde{E}_i''$ is the decomposition of E_i corresponding to the shuffle ζ_α', that satisfies $\zeta_\alpha'(\tilde{E}_i') \subseteq [1, n-q]$

and $\zeta'_\alpha(\tilde{E}''_i) \subseteq [n-q+1,p]$. In summary, $x\big(\zeta_\alpha(E'_i)\big) \subseteq \zeta'_\alpha(\tilde{E}'_i)$. Interchanging the roles of ζ_α and ζ'_α we obtain an equality, which implies that $m_{i0} = \#\, E'_i = \#\, \tilde{E}'_i = m'_{i0}$, where m'_{ij} are the entries of the matrix $M_{\upsilon'}$. This proves the equality of the first row of the matrices. The argument for the other rows is similar.

The matrix M_υ do not depend on the choice of representative of $\upsilon(S_\alpha \times S_\beta)$, since the shuffles satisfying (64) are the same for all the elements on this coset. In conclusion, the matrix M_υ depends only on the double cosets $(S_p \times_n S_q)\upsilon(S_\alpha \times S_\beta)$.

Next we show that the parabolic subgroup $S_{p(M_\upsilon)}$ is $S_\alpha \times_n^\upsilon S_\beta$. An element of $S_\alpha \times_n^\upsilon S_\beta$ can be written as $x \times y \times z$ where

$$x \times y = \zeta_\alpha(\sigma_{a_1} \times \cdots \times \sigma_{a_k})\zeta_\alpha^{-1},$$
$$y \times z = \zeta_\beta(\tau_{b_1} \times \cdots \times \tau_{b_s})\zeta_\beta^{-1}.$$

Evaluating at $\zeta_\alpha(E'_i)$ we deduce that $\zeta_\alpha\sigma_{a_i}(E'_i) = x(E'_i)$ and conclude that $\sigma_{a_i}(E'_i) = E'_i$. Proceeding in a similar manner with the other decompositions we obtain

$$\sigma_{a_i}(E'_i) = E'_i, \qquad \tau_{b_j}(F'_j) = F'_j, \tag{68}$$

E:first-deco

$$\sigma_{a_i}(E''_i) = E''_i, \qquad \tau_{b_j}(F''_j) = F''_j, \tag{69}$$

for all $i = 1, \dots, k$ and $j = 1, \dots, s$.

This decomposition can be further refined. Evaluating as above at the subsets $X_{ij} = \zeta_\alpha(E''_i) \cap \zeta_\beta(F'_j)$, we obtain the equality

$$\zeta_\alpha\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) = y(X_{ij}) = \zeta_\beta\tau_{b_j}\big(\zeta_\beta^{-1}(X_{ij})\big).$$

Now, $\zeta_\alpha\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) \subseteq \zeta_\alpha(E''_i)$ and also $\zeta_\beta\tau_{b_j}\big(\zeta_\beta^{-1}(X_{ij})\big) \subseteq \zeta_\beta(F'_j)$. From the above equality we conclude that $\zeta_\alpha\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) \subseteq \zeta_\alpha(E''_i) \cap \zeta_\beta(F'_j)$, and then $\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) \subseteq \zeta_\alpha^{-1}(X_{ij})$. This inclusion is actually an equality, since both sets have the same cardinality. Therefore, we get the following refinment of (68)

$$\sigma_{a_i}(E'_i) = E'_i, \qquad \sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) = \zeta_\alpha^{-1}(X_{ij}),$$
$$\tau_{b_j}(F''_j) = F''_j, \qquad \tau_{b_j}\big(\zeta_\beta^{-1}(X_{ij})\big) = \zeta_\beta^{-1}(X_{ij}).$$

Note that $\#\, X_{ij} = m_{ij}$, and thus the previous decomposition shows that $x \times y \times z$ belongs to $S_{p(M)}$.

The map $\upsilon \mapsto M_\upsilon$ is invertible, since from the entries of the matrix M_υ we can recover the shuffles ζ_α and ζ_β, which are in the same double coset as υ. $\square$

P:interval

12.2. **Proof of Lemma 7.5.**

Proof. As η and τ are fixed throughout this lemma, we write $\varphi = \varphi_{\eta,\tau}$. Let $x, y \in F_i \cap \varphi^{-1}E_j$ with $x < y$. Consider z such that $x < z < y$. Therefore, $x, y \in F_i$ and, since F_i is an interval, we conclude that $z \in F_i$.

On the other hand $\varphi(x), \varphi(y) \in E_j$. Since $\tau \in \mathcal{B}_\beta$, then $\mathrm{Id} \times \tau$ is increasing in F_i:

$$(\mathrm{Id} \times \tau)(x) < (\mathrm{Id} \times \tau)(z) < (\mathrm{Id} \times \tau)(y). \tag{70}$$

E:inequality

In order to prove that $\varphi(z)$ also belongs to E_j, we consider the following cases:

(1) Assume that $j = 0$. Then, $\varphi(x), \varphi(y) \in E_0 = [p+1, n]$. Since $(\eta \times \mathrm{Id})$ is the identity on that interval, this implies that $\beta_0(\mathrm{Id} \times \tau)(x)$ and $\beta_0(\mathrm{Id} \times \tau)(y)$ are in $[p+1, n]$. But $\beta_0^{-1}[p+1, n] = [n-q+1, 2n-p-q]$ and β_0 is increasing in that set.

Therefore, the three terms in (70) belong to $[n-q+1, 2n-p-q]$ and, applying $(\eta \times \mathrm{Id})\beta_0$, which is increasing on this set, we obtain that $\varphi(x) < \varphi(z) < \varphi(y)$.

(2) Assume that $j > 0$. Consider the cases:
   (a) Assume $i = 0$. In this case we have $x, z, y \in F_0 = [1, n-q]$. Then, applying $\mathrm{Id} \times \tau|_{F_0} = \mathrm{Id}$ we continue in the same set. The permutation β_0 sends increasingly $[1, n-q]$ into $[p+q-n+1, p]$. In this last interval, η is also increasing. Thus, the inequality (70) implies that $\varphi(x) < \varphi(z) < \varphi(y)$.
   (b) Assume $i > 0$. We have that $x, y, z \in F_j \subset [n-q+1, n]$. Applying $\mathrm{Id} \times \tau$ we have that the terms of (70) are also in $[n-q+1, n]$. If $(\mathrm{Id} \times \tau)(x) \in [n-q+1, 2n-p-q]$, then $\beta_0(\mathrm{Id} \times \tau)(x) \in [p+1, n]$ and $\varphi(x) \in [p+1, n] = E_0$, which contradicts the assumption $j > 0$. Therefore, the terms in (70) belong to $[2n-p-q+1, n]$. The permutation β_0 maps increasingly this interval into $[1, p+q-n]$, and η is also increasing in that image. Thus, we conclude that $\varphi(x) < \varphi(z) < \varphi(y)$.

In all the cases we obtain that $\varphi(x) < \varphi(z) < \varphi(y)$, and since $\varphi(x)$ and $\varphi(y)$ belong to the interval E_j, we deduce that $\varphi(z) \in E_j$. This proves that $F_i \cap \varphi^{-1}E_j$ is an interval.

Notice that along the way we also proved that φ is increasing in the intervals $F_i \cap \varphi^{-1}E_j$ as well as the assertions concerning the images.

The fact that the intervals $F_i \cap \varphi^{-1}E_j$ are disjoint follows immediately from the fact that the sets E_j, for $j = 0, \dots, r$, and the sets F_i, for $i = 0, \dots, s$, are disjoint. This finishes the proof. $\square$

P:bijection

12.3. **Proof of Lemma 7.6.**

Proof. For the matrix $M = \{m_{ij}\}$, denote by s_{ij} the sum of the entries $m_{k\ell}$ of M for $(k, \ell) \le (i, j)$ with respect to the lexicographical order of pairs. We define $R_{00} = [1, s_{00}]$ and $R_{ij} = [s_{k\ell}, s_{ij}]$ where s_{ij} covers $s_{k\ell}$. Observe that some of the intervals R_{ij} may be empty. Also note that $\# R_{ij} = m_{ij}$.

The sequence $(R_{00}, R_{01}, \dots, R_{sr})$ is a pseudo-partition of the interval $[n]$ and $\gamma \in \mathcal{B}_{c(M)}$ if and only if γ is increasing in R_{ij} for all $i \in \{0, \dots, s\}$ and $j \in \{0, \dots, r\}$.

Since $M \in \mathcal{M}_{\alpha,\beta}$ and therefore, $\sum_j \#(R_{ij}) = \sum_j m_{ij} = \# F_i$, it follows that

$$F_i = \bigcup_j R_{ij}. \tag{71}$$

E:Fiunion

Moreover, if $\eta \in \mathrm{Sh}(p+q-n, n-q)$ and $\tau \in \mathcal{B}^{\eta}_{\beta}$, then $F_i \cap \varphi^{-1}_{\eta,\tau} E_j = R_{ij}$. This can be seen from the fact both sets are intervals with the same cardinal and from the following relation:

$$\bigcup_j (F_i \cap \varphi^{-1}_{\eta,\tau} E_j) = F_i = \bigcup_j R_{ij}.$$

In particular, we deduce that $\varphi_{\eta,\tau}$ is increasing in R_{ij}.

Given $(\xi, \eta, \sigma, \tau) \in S_{\alpha,\beta}(M)$, we will show that $g_{\xi,\eta}(\sigma, \tau) \in \mathcal{B}_{c(M)}$. To prove this, since $\varphi_{\eta,\tau}|_{R_{ij}}$ is increasing and $\varphi_{\eta,\tau} R_{ij} \subseteq E_j$, we observe that

$$(\sigma \times \mathrm{Id})(\eta \times \mathrm{Id})\beta_0(\mathrm{Id} \times \tau)|_{R_{ij}}$$

is also increasing. According to Lemma 7.5, the images of R_{ij} under the previous permutation are in $[1, p]$ or $[p+1, n]$, where ξ is increasing. Therefore, left multiplying by ξ we deduce that $g_{\xi,\eta}(\sigma, \tau)$ is increasing in R_{ij}, which proves that it belongs to $\mathcal{B}_{c(M)}$.

We prove now that ψ is bijective. Given $\gamma \in \mathcal{B}_{w(M)}$, we show that there exists a unique quadruple $(\xi, \eta, \sigma, \tau) \in S_{\alpha,\beta}(M)$ such that $\psi(\xi, \eta, \sigma, \tau) = \gamma$.

Assume there exists such a quadruple. Using the fact that $E_j = \bigcup_i \varphi_{\eta,\tau} R_{ij}$, we deduce that

$$\xi(\sigma \times \mathrm{Id})E_j = \gamma\big(\bigcup_i R_{ij}\big). \tag{72}$$

E:construct

This proves the uniqueness of the permutation $\xi(\sigma \times \mathrm{Id})$, in other words, it is the only permutation which maps E_j increasingly into the set on the right side; and this implies the uniqueness of ξ and σ. Therefore, we have that $(\eta \times \mathrm{Id})\beta_0(\mathrm{Id} \times \tau) = (\sigma \times \mathrm{Id})^{-1}\xi^{-1}\gamma$. Thus, η is characterized by the image of $[1, n-q]$ under the permutation on the right, which is $\eta[p+q-n+1, p]$. The uniqueness of τ follows immediately.

Given $\gamma \in \mathcal{B}_{c(M)}$, to construct $(\xi, \eta, \sigma, \tau)$ we note that

$$\#(E_j) = \sum_i m_{ij} = \#\Big(\bigcup_i R_{ij}\Big) = \#\Bigg(\gamma\Big(\bigcup_i R_{ij}\Big)\Bigg), \tag{73}$$

E:inM

and, thus, we can construct a permutation μ such that (72) is verified, increasingly mapping E_j into $\gamma\big(\bigcup_i R_{ij}\big)$. This permutation can be written as $\mu = \xi(\sigma \times \mu')$ with $\xi \in \mathrm{Sh}(p, n-p)$, $\sigma \in S_p$ and $\mu' \in S_{n-p}$. Since μ is increasing on $E_0 = [p+1, n]$ we conclude that $\mu' = \mathrm{Id}_{n-p}$, and from the monotony on E_j with $j > 0$ we deduce that $\sigma \in \mathcal{B}_\alpha$. In the same way as before, we construct η by mapping the interval $[1, n-q]$ and for this, we will show that

$$(\sigma \times \mathrm{Id})^{-1}\xi^{-1}\gamma \text{ is increasing in } F_i \text{ for all } i. \tag{74}$$

E:property

In particular, for $i = 0$, we obtain the desired property to define η. We then consider $\beta_0^{-1}(\eta \times \mathrm{Id})^{-1}(\sigma \times \mathrm{Id})^{-1}\gamma$, which equals $\mathrm{Id} \times \tau$ for some $\tau \in S_p$. Using (74) for $i > 0$ we conclude that $\tau \in \mathcal{B}_\beta$; and it follows from (73) that the constructed τ belongs to $\mathcal{B}^\eta_\beta(M)$.

It remains to prove (74). Take $x_1, x_2 \in F_i$ with $x_1 < x_2$. Then, $x_1 \in R_{ij_1}$ and $x_2 \in R_{ij_2}$ for some $j_1 \leq j_2$. Assume that $j_1 = j_2$, then $\gamma(x_1) < \gamma(x_2)$. In this case, we have $\gamma(x_1) = \xi(\sigma \times \mathrm{Id})(e_1)$ and $\gamma(x_2) = \xi(\sigma \times \mathrm{Id})(e_2)$ with $e_1, e_2 \in E_j$. Since σ is increasing in E_j we obtain that $e_1 < e_2$ as desired.

On the other hand, if $j_1 < j_2$, then $e_1 \in E_{j_1}$ and $e_2 \in E_{j_2}$ and the conclusion follows easily as all the elements of E_{j_1} are smaller than those of E_{j_2}. $\square$

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