ZONOTOPES, BRAIDS AND QUANTUM GROUPS

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ABSTRACT. Various results about the action of the *binomial braids* and other *braid analogs* [A2] on some particular higher dimensional representations of the braid groups are presented. These representations are constructed from a fixed integer square matrix A. The common nullspace of the binomial braids is studied in some detail. This space is graded over $\mathbb{N}^r$, where r is the size of A. Our main results state that the non-trivial components occur only on the lattice points of a certain hypersurface in r-space that is canonically associated to A, and that when A is the symmetrization of a Cartan matrix C of finite type, these lattice points are closely related to the vertices of the zonotope associated to C (the precise relationship is given in theorem 5.3). The same action is used to construct a quantum group $U_q^0(A)$ from an arbitrary integer square matrix A. The simplest choices of A yield the usual polynomial and Eulerian Hopf algebras of Joni and Rota (in the corresponding representations, the binomial braids become the usual binomial or q-binomial coefficients). The other choice we consider is that when A is the symmetrization of a symmetrizable Cartan matrix C. Some of the previous results are used to prove that in this case $U_q^0(A)$ coincides with the usual quantum group of Drinfeld and Jimbo. The quantum group is actually defined in a more general setting involving Hopf algebras and crossed bimodules. This paper is a continuation of [A2].

1. INTRODUCTION

This paper studies the combinatorics of a certain action of the braid groups on the tensor powers of a vector space, defined in terms of a given matrix, and its relation to quantum groups.

Let B_n denote the braid group in n strands and kB_n the group algebra over an arbitrary field k. In [A2], elements $b_i^{(n)} \in kB_n$ where defined and shown to satisfy properties analogous to those of the ordinary or q-binomial coefficients $\begin{bmatrix} n \\ i \end{bmatrix}$. *Braid analogs* of classical identities of Pascal, Vandermonde, Cauchy and several others where presented there.

The identities among braids in fact specialize to the q-identities after passing to the one-dimensional representation of B_n where every canonical generator acts by multiplication by a fixed scalar $q \in k$; in particular, the case of the usual identities corresponds to the choice of the trivial representation ($q = 1$). Higher dimensional representations yield new realizations of these identities, where numbers or q-numbers are now replaced by matrices.

The study of the actions of the binomial braids $b_i^{(n)} \in kB_n$ on some particular higher dimensional representations is relevant to the definition of the quantum groups of Drinfeld [D] and Jimbo [Jim] (these are certain q-analogues of the universal enveloping algebras of simple Lie algebras), as explained in [A1] and [A2]. A similar observation had been made before by Schauenburg in [Sch].

In this paper we derive some results on these higher dimensional representations and use them to define a quantum group $U_q^+(A)$ from any integer square matrix A, that coincides with the quantum group of Drinfeld and Jimbo when A is a symmetric Cartan matrix (or, more generally, when A is the symmetrization of a symmetrizable Cartan matrix C).

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The contents of the paper are as follows. We start in section 2 by recalling some definitions and braid identities from [A2], as well as the language of monoidal categories and some basic Hopf algebraic notions that are useful in dealing with higher dimensional representations of the braid groups.

In section 3.2 we define an algebra $U_q^0(A)$ associated to any integer square matrix A and scalar $q \in k$, and also a companion Hopf algebra $U_q^+(A)$. We show that the simplest choices of A ($A = [0]$ and $A = [1]$) yield respectively the usual *binomial* and *Eulerian* Hopf algebras of Joni and Rota. This comes to no surprise since it is the basic observation of [A2] that in the corresponding representations of the braid groups, the binomial braids act as the usual binomial or q-binomial coefficients, and these are the section coefficients for these Hopf algebras (in the sense of Joni and Rota).

It will be shown in section 5.1 that when A is a symmetric Cartan matrix (or, more generally, the symmetrization of a symmetrizable Cartan matrix C) $U_q^0(A)$ and $U_q^+(A)$ coincide respectively with the q-analogs of the universal enveloping algebras of the nilpotent and Borel subalgebra of the simple Lie algebra corresponding to C, as defined by Drinfeld and Jimbo.

If the size of A is r, $U_q^0(A)$ has r generators and the relations among them are defined in terms of the binomial braids $b_i^{(n)}$ and a representation of the braid groups defined by means of A. The construction of the quantum group is carried out in section 3.1 in a more general setting where the initial data are a Hopf algebra H and a crossed H-bimodule instead of a matrix A and a scalar q, although the latter is the only case to be considered elsewhere in the paper.

In section 4 we study the higher dimensional representations of the braid groups defined by A, with the goal of describing the relations of the algebra $U_q^0(A)$ in some detail (corollaries 4.12 and 4.13). By definition, the ideal of relations is generated by the common nullspace of the binomial braids on these representations, and this space is graded over $\mathbb{N}^r$. We prove that the only non-trivial components occur on the locus of a certain hypersurface in k^r that is canonically associated to A (proposition 4.4). In particular we deduce that, if A is symmetric and positive-definite, then $U_q^0(A)$ is finitely-related (corollary 4.7). The proofs of this type of results are combinatorial, in the sense that they are based on some of the combinatorial identities obtained in [A2].

The special case when A is the symmetrization of a symmetrizable Cartan matrix C is treated in section 5. The basic facts on Cartan matrices, root systems and their zonotopes are reviewed as they are needed for the exposition.

In section 5.1 we prove that, for the most general case of a symmetrizable Cartan matrix C, the ideal of relations is generated by the so-called quantum Serre relations. This means that our definition of the quantum groups associated to C coincides with the usual one of Drinfeld and Jimbo. Part of the proof of this result is combinatorial, but another part relies on a somewhat deep result of Lusztig about the definition of $U_q^0(C)$ (which is obtained through representation theory) and Schauenburg's observation about the occurrence of the factorial braid in Lusztig's definition. A pure combinatorial proof would be desirable.

If the Cartan matrix is of finite type (i.e. positive definite), we can describe the nullspace of the binomial braids (that is, the relations of the quantum group) more explicitly in terms of the the root system of C and the corresponding zonotope. This is done in section 5.2, independently of the results in 5.1. In this case, the common nullspace of the binomial braids is naturally graded over the root lattice of C, and the only non-trivial components occur on the sphere through the origin, centered at the lowest weight of the root system. Theorem 5.3 states that among these lattice points, those that do not lie on the walls of the root system are precisely the vertices of the zonotope associated to C. This is one of the main results of the paper.

In section 6 we list some additional results and a few open questions that are motivated by the previous work.

This paper is a continuation of [A2]. Some of the results in section 4 already appeared in [A1, section 9.8.5]. The relation to quantum groups was studied from a different point of view in [A1, section 9.8].

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2. Braid identities and other prerequisites

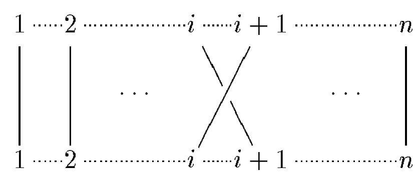
2.1. **Braid groups and braid analogs.** The group B_n of *braids in n strands* has $n-1$ generators $s_1^{(n)}, \dots, s_{n-1}^{(n)}$ subject to the relations

$$s_i^{(n)} s_j^{(n)} = s_j^{(n)} s_i^{(n)} \qquad \text{if } |i-j| \geq 2, \tag{A1}$$

$$s_i^{(n)} s_{i+1}^{(n)} s_i^{(n)} = s_{i+1}^{(n)} s_i^{(n)} s_{i+1}^{(n)} \qquad \text{if } 1 \leq i \leq n-2. \tag{A2}$$

The generator $s_i^{(n)}$ is represented by the following picture, and the product st of two braids s and t in B_n is obtained by putting the picture of s on top of that of t. The identity of B_n is represented by the picture with n vertical strands; the inverse of s is obtained by reflecting its picture across a horizontal line, without leaving the plane of the picture.

$s_i^{(n)} =$



Let kB_n denote the group algebra of B_n over an arbitrary fixed field k. In [A2, sections 3 and 5], elements $[n]$, $b_i^{(n)}$ and $f^{(n)} \in kB_n$ were defined and called natural, binomial and factorial braids respectively. From the several identities involving them that were obtained in [A2], we will only need the following:

$$f^{(n)} = f^{(j)} \otimes f^{(n-j)} \cdot b_j^{(n)} \tag{1}$$

$$1^{(i)} \otimes b_{j-i}^{(n-i)} \cdot b_i^{(n)} = b_i^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)} \tag{2}$$

$$b_p^{(m+n)} = \sum_{k=0}^{p} 1^{(k)} \otimes \beta_{m-k,p-k} \otimes 1^{(n-p+k)} \cdot b_k^{(m)} \otimes b_{p-k}^{(n)} \tag{3}$$

$$f^{(n)} = 1^{(n-1)} \otimes [1] \cdot 1^{(n-2)} \otimes [2] \cdot \ldots \cdot 1 \otimes [n-1] \cdot [n] \tag{4}$$

$$\sum_{k=0}^{n} \mu^{(k)} \otimes 1^{(n-k)} \cdot b_k^{(n)} = 0 \;\forall\; n > 0 \tag{5}$$

$$\mu^{(n)} b_i^{(n)} {\mu^{(n)}}^{-1} = b_{n-i}^{(n)} \; . \tag{6}$$

Here $\beta_{m,n} \in B_{m+n}$ is *the braiding* [A2, section 2.4] and $\mu^{(k)} \in kB_k$ is the *Möbius* braid of [A2, sections 6.2 and 2.2], whose definitions will be recalled below. Formulas (1), (2), (3), (4), (5) and (6) are respectively formulas (21), (20), (14), (15) , (25) and (12) from [A2]. Each of these identities generalizes a well-known q-identity. For instance, (5) is the braid analog of

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q = 0 \;\forall\; n > 0 \;,$$

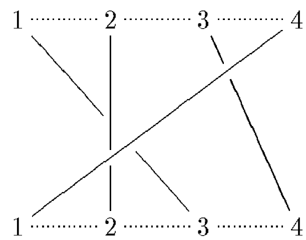
while (3) is the braid analog of Vandermonde's identity

$$\left[\begin{array}{c} m+n \\ p \end{array}\right] = \sum_{k=0}^{p} q^{(m-k)(p-k)} \left[\begin{array}{c} m \\ k \end{array}\right] \left[\begin{array}{c} n \\ p-k \end{array}\right]$$

and (6) is simply the analog of

$$\left[\begin{array}{c} n \\ i \end{array}\right] = \left[\begin{array}{c} n \\ n-i \end{array}\right] .$$

We also need to recall the braid $s_\sigma^{(n)} \in B_n$ associated to a permutation $\sigma \in S_n$. The picture of $s_\sigma^{(n)}$ is obtained by drawing a straight line from 1 in the bottom to $\sigma(1)$ in the top, then *under* that a straight line from 2 to $\sigma(2)$, etc. For instance if $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{smallmatrix}\right)$ then

$s_\sigma^{(4)} =$ 

An explicit expression for $s_\sigma^{(n)}$ in terms of the generators $s_i^{(n)}$ was given in [A2, section 5.1]. The map $S_n \to B_n$, $\sigma \mapsto s_\sigma^{(n)}$, is a section of the canonical projection $B_n \to S_n$. Moreover, it was shown in [A2, lemma 5.2] that

(7) if $\sigma = \tau\rho$ and $\mathsf{length}(\sigma) = \mathsf{length}(\tau) + \mathsf{length}(\rho)$, then $s_\sigma^{(n)} = s_\tau^{(n)} s_\rho^{(n)}$.

Here the length of a permutation is the minimum number of elementary transpositions required to write it as a product of such. We will make use of this fact in section 4.3.

The braiding and Möbius braids are defined as follows. If

$$\sigma = \left(\begin{smallmatrix} 1 & 2 & \dots & m & m+1 & m+2 & \dots & m+n \\ n+1 & n+2 & \dots & n+m & 1 & 2 & \dots & n \end{smallmatrix}\right) ,$$

then $\beta_{m,n} = s_\sigma^{(m+n)}$. If $\sigma = \left(\begin{smallmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{smallmatrix}\right)$ (the permutation with the longest length), then $\mu^{(n)} = (-1)^n s_\sigma^{(n)}$.

2.2. **The braid category and its representations.** The collection $\mathfrak{B} = \coprod_{n \geq 0} B_n$ of all braid groups forms a category, where the objects are the natural numbers, B_n is the set of endomorphisms of n, and there are no morphisms between distinct objects. This category is monoidal, in the sense that there is a functor

$$\mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}, \ (s,t) \mapsto s \otimes t$$

that is associative and unital. Explicitly, this multiplication consists of morphisms of groups $B_n \times B_m \to B_{n+m}$ that are defined on the generators by $s_i^{(n)} \otimes s_j^{(m)} = s_i^{(n+m)} s_{n+j}^{(n+m)}$. In terms of pictures, $s \otimes t$ is obtained by putting t to the right of s.

For more details on the basics on monoidal categories and the category of braids the reader is referred to [Kas, X.6, XI.2 and XIII.2] or [JS].

We are interested in *monoidal representations* of the category $\mathfrak{B}$. Explicitly, such a representation consists of a vector space X, such that, for every n, the braid group B_n acts on the tensor power $X^{\otimes n}$, with the property that

$$s \otimes t \cdot x \otimes y = (s \cdot x) \otimes (t \cdot y) \ \forall \ s \in B_n, \ t \in B_m, \ x \in X^{\otimes n}, \ y \in X^{\otimes m} \ .$$

Since $s_i^{(n)} = 1^{(i-1)}\otimes s_1^{(2)}\otimes 1^{(n-i+1)}$, this property implies that the action of B_n on $X^{\otimes n}$ is uniquely determined by the action of $s_1^{(2)}$ on $X\otimes X$. Moreover, a linear operator $R : X\otimes X \to X\otimes X$ defines a monoidal representation of $\mathfrak{B}$ if and only if it is invertible and satisfies the *Yang-Baxter equation*:

$$(R\otimes \mathsf{id}_X)\circ(\mathsf{id}_X\otimes R)\circ(R\otimes \mathsf{id}_X) = (\mathsf{id}_X\otimes R)\circ(R\otimes \mathsf{id}_X)\circ(\mathsf{id}_X\otimes R)\ .$$

This is a consequence of (A2).

If X is one-dimensional, then any invertible operator $R : X \to X$ satisfies this equation. R is necessarily given by multiplication by some non-zero scalar $q \in k$. In this case, $s_i^{(n)}$ acts by multiplication by q for every $n \geq 2$, $1 \leq i \leq n-1$ and, as was shown in [A2], $b_i^{(n)}$ and $f^{(n)}$ act on $X^{\otimes n}$ by multiplication by the q-binomial coefficient $\left[\begin{array}{c} n \\ i \end{array}\right]_q$ and the q-factorial $[n]!_q$ respectively (so it this simplest choice that produces the classical q-identities from the identities for braids).

An equivalent way to describe monoidal representations of the braid category is by means of the following fact: $\mathfrak{B}$ is the free braided monoidal strict category on one object (the object $1 \in \mathbb{N}$). This says that given any object X of a braided monoidal category $\mathfrak{C}$, there is a unique functor $F : \mathfrak{B} \to \mathfrak{C}$ that preserves the monoidal structures and the braidings and such that $F(1) = X$ [Kas, lemma XIII.3.5]. This highlights the fundamental role of the braiding $\beta_{m,n}$. If $\mathfrak{C}$ carries in addition a k-linear structure (compatible with the rest of the structure), then F extends to $F : k\mathfrak{B} \to \mathfrak{C}$. Usually $\mathfrak{C}$ consists of vector k-spaces with some additional structure, and then the vector space X becomes a monoidal representation of $\mathfrak{B}$, as defined above. An example of such a category is $\mathfrak{C} = \mathfrak{D}_G$, the category of crossed G-bimodules, for any group G. An object of $\mathfrak{D}_G$ is a k-space X equipped with a linear action of G and a linear G-grading, i.e. a decomposition $X = \oplus_{g\in G} X_g$ into subspaces, such that the action of $h \in G$ carries X_g to $X_{hgh^{-1}}$. In this context, one usually writes $|x| = g$ when $x \in X_g$, so that the condition just mentioned becomes $|h\cdot x| = h|x|h^{-1}$. This category is braided monoidal under the usual tensor product of k-spaces, where $X\otimes Y$ is equipped with the G-action $g\cdot(x,y) = (g\cdot x, g\cdot y)$ and the G-grading $|(x,y)| = |x||y|$, and the braiding is

$$\beta_{X,Y} : X\otimes Y \to Y\otimes X,\quad \beta_{X,Y}(x,y) = (|x|\cdot y)\otimes x.$$

This construction can in fact be carried out for any Hopf algebra H in place of G: there is a category $\mathfrak{D}_H$ of crossed H-bimodules [Kas, definition IX.5.1], which is braided monoidal. Crossed G-bimodules as defined above correspond to the choice $H = kG$, the group algebra of G. Crossed bimodules are also called Yetter-Drinfeld modules in the literature.

The following statement summarizes the part of these results we are mostly interested in.

Proposition 2.1. *Let H be a Hopf algebra and X a crossed H-bimodule. Then the braid group B_n acts on $X^{\otimes n}$ for every $n \geq 0$, by morphisms of crossed H-bimodules and with the property that*

$$s\otimes t\cdot x\otimes y = (s\cdot x)\otimes(t\cdot y)\ \forall\ s\in B_n,\ t\in B_m,\ x\in X^{\otimes n},\ y\in X^{\otimes m}\ .$$

Proof. □

2.3. **Hopf algebras in categories.** There is a notion of Hopf algebra in an arbitrary braided monoidal category [M, section 10.5], that we review next.

Let $\mathfrak{C}$ be a monoidal category and I its unit object. An *algebra in* $\mathfrak{C}$ is a triple (A,μ,u) where A is an object of $\mathfrak{C}$ and $\mu : A\otimes A \to A$ and $u : I \to A$ are maps in $\mathfrak{C}$, subject to the obvious associativity and unitality axioms. Coalgebras in $\mathfrak{C}$ are defined similarly. If $\mathfrak{C}$ is also braided, then the notion of bialgebras and Hopf algebras in $\mathfrak{C}$ are also defined. First, one defines an algebra structure on the tensor product of two algebras A and B via

$$\mu_{A\otimes B} : (A\otimes B)\otimes(A\otimes B) \xrightarrow{id_A\otimes\beta_{B,A}\otimes id_B} (A\otimes A)\otimes(B\otimes B) \xrightarrow{\mu_A\otimes\mu_B} A\otimes B\ .$$

A bialgebra in $\mathfrak{C}$ is a object A that is an algebra and a coalgebra in such a way that $\Delta : A \to A \otimes A$ and $\epsilon : A \to I$ are morphisms of algebras in $\mathfrak{C}$. A Hopf algebra in $\mathfrak{C}$ is a bialgebra in $\mathfrak{C}$ for which $id : A \to A$ has a convolution inverse $S : A \to A$, called the antipode. When $\mathfrak{C}$ is the category of vector spaces, equipped with the trivial braiding $x \otimes y \mapsto y \otimes x$, the above notions boil down to the usual notions of algebras, coalgebras, bialgebras and Hopf algebras.

We are mainly interested in Hopf algebras in the category $\mathfrak{D}_H$ of crossed H-bimodules. The basic example of such an object is provided by the ordinary tensor algebra $T(X)$ of a crossed H-bimodule X (proposition 3.1).

If A is a Hopf algebra in $\mathfrak{D}_H$, then the tensor product $A \otimes H$ carries a structure of ordinary Hopf algebra, called the *biproduct* of A and H and denoted $A \star H$ [M, theorem 10.6.5]. This construction of a Hopf algebra from a Hopf algebra in a braided monoidal category is sometimes called *bosonization.*

3. The quantum group associated to a crossed bimodule

3.1. **Definition of the quantum group.** Let H be a Hopf k-algebra, X a crossed H-bimodule and

$$T(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n} \ ,$$

the tensor algebra of the vector space X. Schauenburg proved that $T(X)$ is a bialgebra in $\mathfrak{D}_H$ [Sch, corollary 2.4 and theorem 2.7]. We present a different proof next, based on the combinatorial identities for the braid analogs, and show that $T(X)$ is actually a Hopf algebra in $\mathfrak{D}_H$. Recall that by proposition 2.1, B_n acts on $X^{\otimes n}$ $\forall n \geq 0$.

Proposition 3.1. . *Let X be a crossed H-bimodule and $T(X)$ the tensor algebra of the underlying vector space X. Then $T(X)$ is a Hopf algebra in $\mathfrak{D}_H$ with comultiplication, counit and antipode defined on $\mathbf{x} \in X^{\otimes n}$ by*

$$\Delta(\mathbf{x}) = \sum_{i=0}^{n} (b_i^{(n)}\mathbf{x})_{(i)} \otimes (b_i^{(n)}\mathbf{x})_{(n-i)} \ , \ \epsilon(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = 1 \\ 0 & \text{if } n > 0 \end{cases} \ , \ S(\mathbf{x}) = \mu^{(n)}\mathbf{x} \ .$$

The notation above stands for the natural identifications $X^{\otimes n} \xrightarrow{\cong} X^{\otimes i} \otimes X^{\otimes (n-i)}$, $\mathbf{y} \mapsto \mathbf{y}_{(i)} \otimes \mathbf{y}_{(n-i)}$.

Proof. We compute

$$(id \otimes \Delta)\Delta(\mathbf{x}) = \sum_{i=0}^{n} \sum_{h=0}^{n-i} \left(b_i^{(n)}\mathbf{x}\right)_{(i)} \otimes \left(b_h^{(n-i)}(b_i^{(n)}\mathbf{x})_{(n-i)}\right)_{(h)} \otimes \left(b_h^{(n-i)}(b_i^{(n)}\mathbf{x})_{(n-i)}\right)_{(n-i-h)}$$

$$= \sum_{i=0}^{n} \sum_{h=0}^{n-i} \left(1^{(i)} \otimes b_h^{(n-i)} \cdot b_i^{(n)}\mathbf{x}\right)_{(i)} \otimes \left(1^{(i)} \otimes b_h^{(n-i)} \cdot b_i^{(n)}\mathbf{x}\right)_{(h)} \otimes \left(1^{(i)} \otimes b_h^{(n-i)} \cdot b_i^{(n)}\mathbf{x}\right)_{(n-i-h)}$$

and

$$(\Delta \otimes id)\Delta(\mathbf{x}) = \sum_{j=0}^{n} \sum_{i=0}^{j} \left(b_i^{(j)}(b_j^{(n)}\mathbf{x})_{(j)}\right)_{(i)} \otimes \left(b_i^{(j)}(b_j^{(n)}\mathbf{x})_{(j)}\right)_{(j-i)} \otimes \left(b_j^{(n)}\mathbf{x}\right)_{(n-j)}$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{j} \left(b_i^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)}\mathbf{x}\right)_{(i)} \otimes \left(b_i^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)}\mathbf{x}\right)_{(j-i)} \otimes \left(b_i^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)}\mathbf{x}\right)_{(n-j)} \ .$$

Coassociativity thus follows from formula (2): $1^{(i)} \otimes b_h^{(n-i)} \cdot b_i^{(n)} = b_i^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)}$ if $h = j - i$.

Counitality boils down to the fact that $b_0^{(n)} = b_n^{(n)} = 1$, which holds by definition of the binomial braids.

Take $\mathbf{x} \in X^{\otimes n}$ and $\mathbf{y} \in X^{\otimes m}$. Then,

$$\Delta(\mathbf{x}\mathbf{y}) = \sum_{p=0}^{n+m} (b_p^{(n+m)}\mathbf{x}\otimes\mathbf{y})_{(p)}\otimes(b_p^{(n+m)}\mathbf{x}\otimes\mathbf{y})_{(n+m-p)} \ .$$

On the other hand, by definition of multiplication in $T(X)\otimes T(X)$, we have

$$\Delta(\mathbf{x})\Delta(\mathbf{y}) =$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(1^{(i)}\otimes\beta_{n-i,j}\otimes 1^{(m-j)}\cdot b_i^{(n)}\mathbf{x}\otimes b_j^{(m)}\mathbf{y}\Big)_{(i+j)}\otimes\Big(\text{ same element }\Big)_{(n+m-i-j)} \ .$$

Multiplicativity for Δ thus follows from Vandermonde's formula (3). It is clear then that Δ and ϵ are morphisms of algebras in $\mathfrak{D}_H$.

Finally,

$$m(S\otimes id)\Delta(\mathbf{x}) = \sum_{i=0}^{n}\mu^{(i)}(b_i^{(n)}\mathbf{x})_{(i)}\otimes(b_i^{(n)}\mathbf{x})_{(n-i)} = \sum_{i=0}^{n}\mu^{(i)}\otimes 1^{(n-i)}\cdot b_i^{(n)}\mathbf{x} = \epsilon(\mathbf{x})1$$

precisely by formula (5). The other axiom for the antipode follows from a similar identity (equation (26) in [A2]). □

Let $K^{(0)} = K^{(1)} = 0$ and, for each $n \geq 2$,

$$K^{(n)} = \bigcap_{i=1}^{n-1}\mathsf{Ker}(b_i^{(n)} : X^{\otimes n} \to X^{\otimes n}),\ K = \bigoplus_{n=0}^{\infty} K^{(n)} \subseteq T(X)$$

and let $\bar{K}$ denote the ideal generated by K in $T(X)$. Define an algebra

$$U_H^0(X) := T(X)/\bar{K} \ .$$

Thus, $U_H^0(X)$ is generated by X subject to the relations K.

Since B_n acts on $X^{\otimes n}$ by morphisms of crossed H-bimodules (proposition 2.1), K is a crossed subbimodule of $T(X)$. Therefore, so is $\bar{K}$, and hence the quotient $U_H^0(X)$ is also an algebra in $\mathfrak{D}_H$. Moreover:

Proposition 3.2. *In the Hopf algebra $T(X)$ in $\mathfrak{D}_H$, K is a coideal stable under the antipode. Therefore, $U_H^0(X)$ is a Hopf algebra in $\mathfrak{D}_H$.*

Proof. If $\mathbf{x} \in K^{(n)}$ then

$$\Delta(\mathbf{x}) = \mathbf{x}\otimes 1 + 1\otimes\mathbf{x} \in K\otimes T(X) + T(X)\otimes K$$

and, for $1 \leq i \leq n-1$,

$$b_i^{(n)}S(\mathbf{x}) = b_i^{(n)}\mu^{(n)}(\mathbf{x}) \overset{(6)}{=} \mu^{(n)}b_{n-i}^{(n)}(\mathbf{x}) = 0 \ ,$$

so K is a coideal stable under the antipode. Hence $\bar{K}$ is a Hopf ideal of $T(X)$ and the quotient is a Hopf algebra in $\mathfrak{D}_H$. □

Remark 3.1. One could also derive this result from the fact that K consists precisely of those *primitive* elements of $T(X)$ of degree at least 2 (with respect to the natural grading of $T(X)$).

As explained in section 2.3, the above result allows us to construct an ordinary Hopf algebra as the biproduct

$$U_H^+(X) := U_H^0(X) \star H \ .$$

$U_H^0(X)$ and $U_H^+(X)$ are the quantum groups whose construction was announced in the introduction.

Example 3.1. Take $H = k\mathbb{Z}_m$, the group algebra of the cyclic group of order $m > 1$, q a root of unity of order m and $X = k\{x\}$, the one-dimensional crossed $\mathbb{Z}_m$-bimodule defined by

$$|x| = 1 \text{ and } n \cdot x = q^n x \ \forall\ n \in \mathbb{Z}_m \ .$$

The action of the binomial braids is given in this case by the q-binomial coefficients:

$$b_i^{(n)} x^n = \begin{bmatrix} n \\ i \end{bmatrix}_q x^n \ .$$

If m does not divide n, then $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n - 1}{q-1} \neq 0$, so $K^{(n)} = 0$. If m divides n, then, for $1 \leq i \leq n-1$,

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-i+1} - 1)}{(q^i - 1)(q^{i-1} - 1) \cdots (q-1)} = 0 \ ,$$

so $K^{(n)} = k\{x^n\}$. Therefore, $\bar{K} = (x^m)$, the ideal generated by x^m, and $U^0_{k\mathbb{Z}_m}(X) = k[x]/(x^m)$ is an m-dimensional Hopf algebra in $\mathfrak{D}_{\mathbb{Z}_m}$ with structure

$$\Delta(x^n) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i \otimes x^{n-i} \text{ and } S(x^n) = (-1)^n q^{\binom{n}{2}} x^n \ .$$

Notice that $S^k(x^n) = (-1)^{kn} q^{k\binom{n}{2}} x^n$, so the antipode has order $2m$. The biproduct

$$U^+_{k\mathbb{Z}_m}(X) = k[x]/(x^m) \star k\mathbb{Z}_m$$

is an ordinary Hopf algebra of dimension m^2 with antipode of order $2m$. This Hopf algebra was originally constructed by Taft [T] (and by Sweedler for $q = -1$).

We close this section with two simple but important observations about the common nullspace K of the binomial braids K. A related space is the nullspace F of the factorial braids, defined as follows:

$$F = \bigoplus_{n=0}^{\infty} F^{(n)} \subseteq T(X) \text{ where } F^{(n)} = \mathsf{Ker}(f^{(n)} : X^{\otimes n} \to X^{\otimes n}) \ .$$

Lemma 3.3. *F is an ideal of $T(X)$, and $\bar{K} \subseteq F$.*

Proof. Formula (1)

$$f^{(i)} \otimes f^{(n-i)} \cdot b_i^{(n)} = f^{(n)}$$

shows that $\mathsf{Ker} b_i^{(n)} \subseteq \mathsf{Ker} f^{(n)} \ \forall\ i$; in particular, $K^{(n)} \subseteq F^{(n)}$, so $K \subseteq F$. On the other hand, applying the *horizontal symmetry* operator $*$ of [A2, section 2.3] to formula (1) one obtains

$$b_i^{(n)*} \cdot (f^{(i)} \otimes f^{(n-i)}) = f^{(n)} \ ,$$

since this operator preserves tensor products, reverses compositions and fixes the factorial braids (formula (17) in [A2]). This immediately implies that F is an ideal of $T(X)$. Since it contains K, it must also contain $\bar{K}$. ☐

Lemma 3.4. *$K^{(n)} \subseteq \mathsf{Ker}(\mu^{(n)} + 1)$.*

Proof. This is an immediate consequence of identity (5), since $b_0^{(n)} = b_n^{(n)} = 1$. ☐

3.2. **The quantum group associated to a matrix.** Let $A = [a_{hk}] \in M_r(\mathbb{Z})$ be an integer square matrix of size r and $q \in k^*$ a fixed scalar. We will always assume that q is not a root of unity, $q^{1/2} \in k$ and $\mathsf{char} k = 0$.

Let $G = \mathbb{Z}^r$, the free abelian group of rank r, and X the vector space with basis $\{x_1, \ldots, x_r\}$, viewed as crossed G-bimodule with

$$|x_h| = (a_{1h}, \ldots, a_{rh}) \in \mathbb{Z}^r, \quad (n_1, \ldots, n_r) \cdot x_k = q^{n_k} x_k \ \forall\ (n_1, \ldots, n_r) \in \mathbb{Z}^r .$$

The algebras $U^0_{kG}(X)$ and $U^+_{kG}(X)$ will be denoted by $U^0_q(A)$ and $U^+_q(A)$ respectively. They will be studied in sections 4 and 5, where, in particular, it will be shown that when A is the symmetrization of a symmetrizable Cartan matrix C, the quantum group $U^+_q(A)$ defined above coincides with the usual one defined by Drinfeld and Jimbo (thus justifying the choice of notation).

According to the discussion in section 2.2, the action of $s_1^{(2)} \in B_2$ on $X \otimes X$ is given in this case by

$$x_h \otimes x_k \mapsto q^{a_{kh}} x_k \otimes x_h .$$

It follows that the action of $s_i^{(n)} = 1^{(i-1)} \otimes s_1^{(2)} \otimes 1^{(n-1)} \in B_n$ on $X^{\otimes n}$ is given by

$$x_{h_1} \otimes \ldots \otimes x_{h_n} \mapsto q^{a_{h_{i+1}, h_i}} x_{h_1} \otimes \ldots \otimes x_{h_{i+1}} \otimes x_{h_i} \otimes \ldots \otimes x_{h_n} . \tag{8}$$

We close this section by considering the simplest instances of this construction, namely those corresponding to $A = [0]$ and $A = [1]$.

Examples 3.2.

1. Consider the case $A = [0]$. $X^{\otimes n}$ is then one-dimensional and, as recalled in section 2.2, $b_i^{(n)}$ acts by multiplication by the ordinary binomial coefficient $\binom{n}{i}$. It follows that $K^{(n)} = 0$, since $\mathsf{char} k = 0$. Therefore, $U^0_q(A)$ is just the polynomial algebra $k[x]$ with its usual Hopf algebra structure
$$\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i} \text{ and } S(x^n) = (-1)^n x^n$$
(the binomial Hopf algebra).
2. When $A = [1]$, $X^{\otimes n}$ is one-dimensional and $b_i^{(n)}$ acts by multiplication by the q-binomial coefficient $\left[\begin{smallmatrix} n \\ i \end{smallmatrix} \right]$. Since $\mathsf{char} k = 0$ and q is not a root of unity, this operator is injective. Hence $U^0_q(A)$ is still the polynomial algebra $k[x]$, but the Hopf algebra structure is
$$\Delta(x^n) = \sum_{i=0}^{n} \left[\begin{matrix} n \\ i \end{matrix} \right] x^i \otimes x^{n-i} \text{ and } S(x^n) = (-1)^n q^{\binom{n}{2}} x^n .$$
This is the Eulerian Hopf algebra of [JR].

4. The braid action defined by a matrix

Let A, q and X be as above. In this section we study the nullspaces K and F of the binomial and factorial braids acting on the tensor powers of X by means of A as in section 3.2. It will be shown that these spaces are naturally graded over $\mathbb{N}^r$ and that the non-trivial components of K occur only on the locus of a certain hypersurface in k^r that is canonically associated to A. These results will be applied to decide when the quantum group $U^+_q(A)$ is finitely related.

4.1. **The grading on K and F.** For each n and $r \in \mathbb{N}$ let $\mathcal{F}(n,r)$ denote the set of all functions $\{1,\dots,n\} \to \{1,\dots,r\}$, and $\mathcal{C}(n,r) = \{(\eta_1,\dots,\eta_r) \in \mathbb{N}^r \ / \ \eta_1+\dots+\eta_r = n\}$. For any $\eta = (\eta_1,\dots,\eta_r) \in \mathcal{C}(n,r)$ let

$$\mathcal{S}(\eta) = \{f \in \mathcal{F}(n,r) \ / \ \#f^{-1}(1) = \eta_1, \#f^{-1}(2) = \eta_2, \dots, \#f^{-1}(r) = \eta_r\}.$$

For each $f \in \mathcal{F}(n,r)$, consider the tensor

$$\mathbf{x}_f := x_{f(1)} \otimes \dots \otimes x_{f(n)} \in X^{\otimes n} \ .$$

The set

$$\{\mathbf{x}_f \ / \ f \in \mathcal{F}(n,r)\}$$

is a k-basis of the vector space $X^{\otimes n}$. It follows from (8) that, for each $\eta \in \mathcal{C}(n,r)$, the subspace $X^{(\eta)}$ of $X^{\otimes n}$ spanned by $\{\mathbf{x}_f \ / \ f \in \mathcal{S}(\eta)\}$ is invariant under the action of B_n. In other words, the distinguished basis of X determines a grading of $T(X)$ over the monoid $\mathbb{N}^r$, and this grading is preserved by the action of B_n. Therefore, the nullspaces K and F inherit a grading over $\mathbb{N}^r$ as follows

$$K = \bigoplus_{\eta \in \mathbb{N}^r}^{\infty} K^{(\eta)} \quad \text{where} \quad K^{(\eta)} = \bigcap_{i=1}^{n-1} \mathsf{Ker}(b_i^{(n)} : X^{(\eta)} \to X^{(\eta)}) \ ,$$

and

$$F = \bigoplus_{\eta \in \mathbb{N}^r}^{\infty} F^{(\eta)} \quad \text{where} \quad F^{(\eta)} = \mathsf{Ker}(f^{(n)} : X^{(\eta)} \to X^{(\eta)}) \ (\text{and} \quad n = \eta_1 + \dots + \eta_r) \ .$$

4.2. **The action of the braids $s_\sigma^{(n)}$.** The action of the braids $s_\sigma^{(n)} \in B_n$ on the above basis of $X^{\otimes n}$ can be easily described in terms of the natural action of S_n on $\mathcal{F}(n,r)$, $\sigma \cdot f = f \circ \sigma^{-1}$.

Recall that the *set of inversions* of a permutation $\sigma \in S_n$ is

$$\mathsf{Inv}(\sigma) = \{(i,j) \ / \ 1 \le i < j \le n \text{ and } \sigma(i) > \sigma(j)\},$$

and the *inversion index* of σ is $\mathsf{inv}(\sigma) = \#\mathsf{Inv}(\sigma)$. We will make use of the well-known fact that $\mathsf{inv} = \mathsf{length}$.

Proposition 4.1. *Let $\sigma \in S_n$ and $f \in \mathcal{F}(n,r)$, Then*

$$s_\sigma^{(n)} \cdot \mathbf{x}_f = q^{\sum_{(i,j)\in \mathsf{Inv}(\sigma)} a_{f(j)f(i)}} \mathbf{x}_{\sigma \cdot f} \ .$$

Proof. We proceed by induction on $\mathsf{length}(\sigma)$. If $\mathsf{length}(\sigma) = 1$, that is σ is the transposition $(i, i+1)$ for some i, then $s_\sigma^{(n)} = s_i^{(n)}$ and $\mathsf{Inv}(\sigma) = \{(i,i+1)\}$, so the result holds precisely by equation (8).

Thus, it suffices to show that if $\sigma = \tau \cdot (i,i+1)$, $\mathsf{length}(\sigma) = \mathsf{length}(\tau) + 1$ and the result holds for τ, then it also holds for σ.

In this case, $\sigma = \left(\begin{smallmatrix} 1 & \dots & i-1 & i & i+1 & i+2 & \dots & n \\ \tau(1) & \dots & \tau(i-1) & \tau(i+1) & \tau(i) & \tau(i+2) & \dots & \tau(n) \end{smallmatrix}\right)$. We claim that

$$\mathsf{Inv}(\sigma) = \mathsf{Inv}(\tau) \cup \{(i,i+1)\} \ .$$

This is clearly equivalent to: $\tau(i) < \tau(i+1)$. If we had $\tau(i) > \tau(i+1)$, it would be $(i,i+1) \in \mathsf{Inv}(\tau)$ and $(i,i+1) \notin \mathsf{Inv}(\sigma)$, from where it would follow that $\mathsf{length}(\sigma) = \mathsf{inv}(\sigma) = \mathsf{inv}(\tau) - 1 = \mathsf{length}(\tau) - 1$, against our hypothesis.

We calculate

$$s_\sigma^{(n)} \cdot \mathbf{x}_f \overset{(7)}{=} s_\tau^{(n)} s_i^{(n)} \cdot \mathbf{x}_f = q^{a_{f(i+1)f(i)}} s_\tau^{(n)} \cdot \mathbf{x}_{(i,i+1)\cdot f} \ .$$

Let $g = (i, i+1)\cdot f$. Since $(i, i+1) \notin \mathsf{Inv}(\tau)$, we have $(g(j), g(i)) = (f(j), f(i))$ for all $(i, j) \in \mathsf{Inv}(\tau)$. Therefore, by induction,

$$s_\sigma^{(n)} \cdot \mathbf{x}_f = q^{a_{f(i+1)f(i)}} q^{\sum_{(i,j)\in\mathsf{Inv}(\tau)} a_{g(j)g(i)}} \mathbf{x}_{\tau\cdot g} = q^{\sum_{(i,j)\in\mathsf{Inv}(\sigma)} a_{f(j)f(i)}} \mathbf{x}_{\sigma\cdot f} \ .$$

□

4.3. **The action of the Möbius braid.** It turns out that the action of $\mu^{(n)}$ on $X^{\otimes n}$ diagonalizes and thus can be completely described. It is in this description that a certain hypersurface in k^r naturally arises. To define it, consider first the quadratic form associated to A, namely

$$Q_A(x) = \sum_{h=1}^{r} a_{hh} x_h^2 + \sum_{1\le h<k\le r} (a_{hk} + a_{kh}) x_h x_k \ ,$$

where $x = (x_1, x_2, \ldots, x_r)$ is a vector in k^r, and also the linear form

$$D_A(x) = \sum_{h=1}^{r} a_{hh} x_h \ .$$

We are interested in the hypersurface

$$\boxed{(H): \quad Q_A(x) = D_A(x)}$$

in particular in points with non-negative coordinates $x \in \mathbb{N}^r$.

Lemma 4.2. *Let $f \in \mathcal{F}(n, r)$ and $\eta = (\#f^{-1}(1), \#f^{-1}(2), \ldots, \#f^{-1}(r)) \in \mathcal{C}(n, r)$. Then*

$$\sum_{1\le i\ne j\le n} a_{f(i)f(j)} = Q_A(\eta) - D_A(\eta) \ .$$

Proof. For each h and $k \in \{1, 2, \ldots, r\}$, let $A_{hk} = \{(i, j) \ / \ 1 \le i \ne j \le n \text{ and } f(i) = h, \ f(j) = k\}$. If $h \ne k$ then $A_{hk} = \{(i, j) \ / \ f(i) = h, \ f(j) = k\}$, so $\#A_{hk} = \eta_h \eta_k$. On the other hand, $A_{hh} = \{(i, j) \ / \ i \ne j \text{ and } f(i) = f(j) = h\}$, so $\#A_{hh} = \eta_h(\eta_h - 1)$. Therefore,

$$\begin{aligned}
\sum_{1\le i\ne j\le n} a_{f(i)f(j)} &= \sum_{h=1}^{r} \sum_{(i,j)\in A_{hh}} a_{f(i)f(j)} + \sum_{1\le h\ne k\le r} \sum_{(i,j)\in A_{hk}} a_{f(i)f(j)} \\
&= \sum_{h=1}^{r} a_{hh} \eta_h(\eta_h - 1) + \sum_{1\le h\ne k\le r} a_{hk}\eta_h\eta_k = \sum_{h=1}^{r} a_{hh}\eta_h^2 + \sum_{1\le h<k\le r} (a_{hk} + a_{kh})\eta_h\eta_k - \sum_{h=1}^{r} a_{hh}\eta_h \\
&= Q_A(\eta) - D_A(\eta) \ .
\end{aligned}$$

□

The diagonalization of $\mu^{(n)}$ on $X^{\otimes n}$ will be described in detail in the proof of the following proposition. For simplicity, its statement only provides the essential information about its eigenvalues.

Proposition 4.3. *Let $\eta \in \mathcal{C}(n, r)$. Then $\mu^{(n)} : X^{(\eta)} \to X^{(\eta)}$ diagonalizes. Moreover, $\mu^{(n)}$ only has two possible eigenvalues:*

$$q^{\frac{1}{2}[Q_A(\eta) - D_A(\eta)]} \ \ \textit{and} \ \ -q^{\frac{1}{2}[Q_A(\eta) - D_A(\eta)]} \ .$$

Proof. We need to stablish some notation. For each $f \in \mathcal{F}(n, r)$, let $\tilde{f} \in \mathcal{F}(n, r)$ be $\tilde{f}(i) = f(n+1-i)$. If $f \in \mathcal{S}(\eta)$ then $\tilde{f} \in \mathcal{S}(\eta)$ too. Let $\mathcal{S}_0 = \{f \in \mathcal{S}(\eta) \ / \ f = \tilde{f}\}$ and fix a disjoint decomposition

$$\mathcal{S}(\eta) = \mathcal{S}_0 \cup \bigcup_{f\in\mathcal{S}_1} \{f, \tilde{f}\} \ .$$

Correspondingly, $X^{(\eta)}$ splits as a direct sum

$$X^{(\eta)} = \bigoplus_{f \in \mathcal{S}_0} k\{\mathbf{x}_f\} \oplus \bigoplus_{f \in \mathcal{S}_1} k\{\mathbf{x}_f,\ \mathbf{x}_{\tilde{f}}\}\ .$$

Recall that $\mu^{(n)} = (-1)^n s_\sigma^{(n)}$ where $\sigma = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{smallmatrix}\right)$. Notice that $\sigma \cdot f = \tilde{f}$ and that the set of inversions of this permutation is $\mathsf{Inv}(\sigma) = \{(i,j)\ /\ 1 \le i < j \le n\}$. Therefore, by proposition 4.1,

$$\mu^{(n)} \cdot \mathbf{x}_f = (-1)^n q^{\alpha_f} \mathbf{x}_{\tilde{f}}\ ,$$

where $\alpha_f = \sum_{1 \le i < j \le n} a_{f(j)f(i)}$.

Therefore, each subspace $k\{\mathbf{x}_f\}$ with $f \in \mathcal{S}_0$ or $k\{\mathbf{x}_f, \mathbf{x}_{\tilde{f}}\}$ with $f \in \mathcal{S}_1$ is invariant under $\mu^{(n)}$. On $k\{\mathbf{x}_f\}$ with $f \in \mathcal{S}_0$, $\mu^{(n)}$ diagonalizes with eigenvector $\mathbf{x}_f$ and eigenvalue $(-1)^n q^{\alpha_f}$. It also follows readily that, on $k\{\mathbf{x}_f, \mathbf{x}_{\tilde{f}}\}$ with $f \in \mathcal{S}_1$, $\mu^{(n)}$ diagonalizes with eigenvectors

$$q^{\frac{1}{2}\alpha_{\tilde{f}}} \mathbf{x}_f + q^{\frac{1}{2}\alpha_f} \mathbf{x}_{\tilde{f}} \quad \text{and} \quad q^{\frac{1}{2}\alpha_{\tilde{f}}} \mathbf{x}_f - q^{\frac{1}{2}\alpha_f} \mathbf{x}_{\tilde{f}}$$

and respective eigenvalues

$$(-1)^n q^{\frac{1}{2}(\alpha_f + \alpha_{\tilde{f}})} \quad \text{and} \quad -(-1)^n q^{\frac{1}{2}(\alpha_f + \alpha_{\tilde{f}})}\ .$$

But notice that

$$\alpha_f + \alpha_{\tilde{f}} = \sum_{1 \le i < j \le n} a_{f(j)f(i)} + \sum_{1 \le i < j \le n} a_{f(n+1-j)f(n+1-i)}$$
$$= \sum_{1 \le i \ne j \le n} a_{f(i)f(j)} = Q_A(\eta) - D_A(\eta)\ ,$$

by lemma 4.2.

In particular, if $f \in \mathcal{S}_0$, then

$$\alpha_f = \frac{1}{2}(\alpha_f + \alpha_{\tilde{f}}) = \frac{1}{2}[Q_A(\eta) - D_A(\eta)]\ .$$

This shows that $\mu^{(n)}$ diagonalizes and that the eigenvalues are as indicated. □

We can now derive the main result on the nullspace of the binomial braids announced at the beginning of section 4.

Proposition 4.4. *The only non-trivial components $K^{(\eta)}$ of K occur when $\eta \in \mathbb{N}^r$ lies on the hypersurface (H). In other words, if $K^{(\eta)} \ne 0$, then $Q_A(\eta) = D_A(\eta)$.*

Proof. By lemma 3.4, if $K^{(\eta)} \ne 0$ then -1 is an eigenvalue of $\mu^{(n)}$. By proposition 4.3, this happens only if $Q_A(\eta) - D_A(\eta) = 0$. □

4.4. **The hypersurface associated to A.** We assume that $k = \mathbb{R}$ for the results of this section. The first couple of results hold for arbitrary real matrices $A = [a_{hk}]$ of size r (not necessarily integer).

In this section we obtain some basic information about the hypersurface (H) associated to A by means of the equation

$$(H) \qquad Q_A(x) = D_A(x)\ .$$

This study is motivated by the result of proposition 4.4. In fact, we will obtain as a corollary the finite generation of the ideal of relations of $U_q^0(A)$ when A is symmetric and positive-definite.

Recall that

$$Q_A(x) = \langle Ax, x \rangle \text{ and } D_A(x) = \langle d_A, x \rangle\ ,$$

where $d_A = (a_{11}, \ldots, a_{rr}) \in \mathbb{R}^r$ and $\langle x, y\rangle = \sum_{i=1}^r x_i y_i$ denotes the standard inner product in $\mathbb{R}^r$. Clearly,

$$Q_A = Q_{\frac{A+A^t}{2}} \text{ and } D_A = D_{\frac{A+A^t}{2}} \ ,$$

so there is no loss of generality in assuming that A is symmetric. In this case, the inner product and norm in $\mathbb{R}^r$ associated to A are simply

$$\langle x, y\rangle_A = \langle Ax, y\rangle \text{ and } ||x||_A = Q_A(x)^{1/2} \ .$$

Proposition 4.5. *Suppose that A is symmetric and positive-definite. Then, with respect to the norm defined by A on $\mathbb{R}^r$, (H) is the equation of the sphere of center $\frac{1}{2}A^{-1}d_A$ that goes through the origin.*

Proof. We compute

$$\begin{aligned} ||x - \frac{1}{2}A^{-1}d_A||_A^2 &= ||x||_A^2 - \langle A^{-1}d_A, x\rangle_A + \frac{1}{4}||A^{-1}d_A||^2 \\ &= Q_A(x) - D_A(x) + \frac{1}{4}||A^{-1}d_A||^2 \ , \end{aligned}$$

so (H) is equivalent to

$$||x - \frac{1}{2}A^{-1}d_A||_A = \frac{1}{2}||A^{-1}d_A|| \ ,$$

which is the equation of a sphere as stated. □

Corollary 4.6. *Let $u = (1, 1, \ldots, 1) \in \mathbb{R}^r$. If A is symmetric and positive-definite, and*

$$n > \frac{1}{2}\langle A^{-1}d_A, u\rangle + \frac{1}{2}\sqrt{\langle A^{-1}u, u\rangle\langle A^{-1}d_A, d_A\rangle} \ ,$$

then there are no solutions x to equation (H) satisfying at the same time

$$x_1 + x_2 + \ldots + x_r = n \ .$$

Proof. Let H_n be the hyperplane in $\mathbb{R}^r$ with equation

$$x_1 + x_2 + \ldots + x_r = n \ .$$

Let $n_0 \in \mathbb{R}$ be such that H_{n_0} is tangent to the sphere of proposition 4.5 (cf. figure 1).

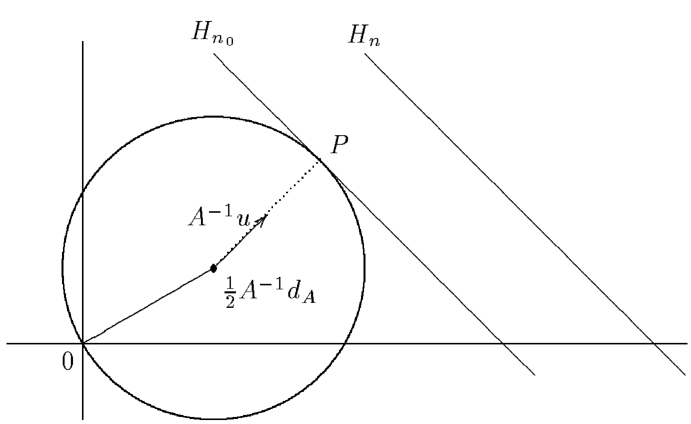


FIGURE 1. The sphere and the tangent plane of corollary 4.6.

If $n > n_0$, H_n does not intersect the sphere. So the result will follow once we prove that

$$n_0 = \frac{1}{2}\langle A^{-1}d_A, u\rangle + \frac{1}{2}\sqrt{\langle A^{-1}u, u\rangle\langle A^{-1}d_A, d_A\rangle} \ .$$

Let P be the point of tangency between H_{n_0} and the sphere. The radial vector to P must be perpendicular to H_{n_0}, with respect to the inner product defined by A. But the vector $A^{-1}u$ is clearly normal to H_{n_0} with respect to this inner product, so it follows that

$$P = \frac{1}{2}A^{-1}d_A + \lambda A^{-1}u$$

for some $\lambda \in \mathbb{R}$. We can determine λ by imposing the fact that P lies on the sphere:

$$\|\frac{1}{2}A^{-1}d_A + \lambda A^{-1}u - \frac{1}{2}A^{-1}d_A\|_A = \frac{1}{2}\|A^{-1}d_A\|_A \ ;$$

it follows that

$$\lambda = \frac{1}{2}\frac{\|A^{-1}d_A\|_A}{\|A^{-1}u\|_A} = \frac{1}{2}\frac{\langle A^{-1}d_A, d_A\rangle^{1/2}}{\langle A^{-1}u, u\rangle^{1/2}}.$$

Now, since n_0 is the sum of the coordinates of P, we have that

$$n_0 = \langle\frac{1}{2}A^{-1}d_A + \lambda A^{-1}u, u\rangle = \frac{1}{2}\langle A^{-1}d_A, u\rangle + \lambda\langle A^{-1}u, u\rangle = \frac{1}{2}\langle A^{-1}d_A, u\rangle + \frac{1}{2}\sqrt{\langle A^{-1}u, u\rangle\langle A^{-1}d_A, d_A\rangle}$$

and the proof is complete. □

Assume now that A is an integer matrix. Consider the action of the braid groups B_n on $X^{\otimes n}$ defined by A and the ideal $\bar{K}$ of $T(X)$ defined by means of the binomial braids (the ideal of relations) as in sections 3.1 and 3.2.

Corollary 4.7. *If A is symmetric and positive-definite, and*

$$n > \frac{1}{2}\langle A^{-1}d_A, u\rangle + \frac{1}{2}\sqrt{\langle A^{-1}u, u\rangle\langle A^{-1}d_A, d_A\rangle} \ ,$$

then $K^{(n)} = 0$. In particular, the ideal $\bar{K}$ is finitely generated.

Proof. Suppose $K^{(n)} \neq 0$. Since $K^{(n)} = \bigoplus_{\eta\in\mathcal{C}(n,r)} K^{(\eta)}$, there must be some $\eta \in \mathcal{C}(n,r)$ such that $K^{(\eta)} \neq 0$. By proposition 4.4, η lies in (H), and it also satisfies $\eta_1 + \eta_2 + \ldots + \eta_r = n$ by definition of $\mathcal{C}(n,r)$. This contradicts corollary 4.6. □

Remarks 4.1.

If the matrix A is not positive definite, then (H) does not represent a compact hypersurface anymore, and there may be infinitely many solutions in $\mathbb{N}^r$. For instance, consider the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \ .$$

(A is the Cartan matrix of the root system $A_1^{(1)}$). In this case equation (H) becomes

$$x^2 + y^2 - 2xy = x + y \ .$$

The solutions $(x, y) \in \mathbb{N}^2$ to this equation are precisely

$$x = \frac{n^2+n}{2},\ y = \frac{n^2-n}{2} \quad \text{and} \quad x = \frac{n^2-n}{2},\ y = \frac{n^2+n}{2} \quad \text{for every } n \geq 0 \ .$$

This equation represents a parabola in $\mathbb{R}^2$, cf. figure 2.

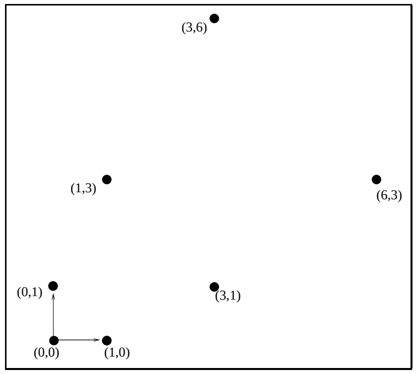


FIGURE 2. The solutions for $A = A_1^{(1)}$.

4.5. **The quantum Serre relations.** In this section we study the nullspaces $K^{(\eta)}$ for some simple choices of $\eta \in \mathbb{N}^r$.

Let $\{\epsilon_1, \dots, \epsilon_r\}$ be the canonical basis of $\mathbb{N}^r$. The simplest choice is

$$\eta = n\epsilon_h = (0, \dots, 0, n, 0, \dots, 0) \in \mathbb{N}^r ,$$

where $n \in \mathbb{N}$ and $h \in \{1, 2, \dots, r\}$ are arbitrary. In this case there is only one function $f \in \mathcal{S}(\eta)$, namely $f \equiv h$, so the space $X^{(\eta)}$ is one-dimensional, spanned by $\mathbf{x}_f = x_h^{\otimes n}$. Now, from equation (8), every generator $s_i^{(n)}$ acts on this space by multiplication by $q^{a_{hh}}$; therefore, as recalled in section 2, $b_i^{(n)}$ acts by multiplication by the binomial coefficient $\left[\begin{smallmatrix} n \\ i \end{smallmatrix} \right]_{q^{a_{hh}}}$. But since q is not a root of unity, this element is non-zero, so we conclude that $K^{(\eta)} = 0$ in this case.

The next simplest choice is

$$\eta = n\epsilon_h + \epsilon_k = (0, \dots, 0, n, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^r ,$$

where $n \in \mathbb{N}$ and h and $k \in \{1, 2, \dots, r\}$ are arbitrary but distinct. In this case there are $n+1$ functions in $\mathcal{S}(\eta)$, namely

$$f_i = \left(\begin{smallmatrix} 1 & \dots & i & i+1 & i+2 & \dots & n+1 \\ h & \dots & h & k & h & \dots & h \end{smallmatrix} \right) , \text{ for } i = 0, \dots, n .$$

Therefore, the space $X^{(\eta)}$ is $(n+1)$-dimensional, spanned by $\mathbf{x}_{f_i} := x_h^{\otimes i} \otimes x_k \otimes x_h^{\otimes (n-i)} \in X^{\otimes (n+1)}$.

We will show that, for these η, the nullspaces $K^{(\eta)}$ are either trivial or one-dimensional, spanned by some particular vectors $S_{hk}^n \in X^{\otimes (n+1)}$ to be defined in proposition 4.11, called the *quantum Serre relations*. The first result in this direction is:

Lemma 4.8. *Let $\eta = n\epsilon_h + \epsilon_k$ as above. If $(n-1)a_{hh} + a_{hk} + a_{kh} \neq 0$, then $K^{(\eta)} = 0$.*

Proof. By lemma 3.4, $K^{(\eta)} = 0$ if -1 is not an eigenvalue of $\mu^{(n+1)}$ in $X^{(\eta)}$. By proposition 4.3, if -1 is an eigenvalue of $\mu^{(n+1)}$ then $Q_A(\eta) = D_A(\eta)$. But for this η,

$$Q_A(\eta) = a_{hh}n^2 + a_{kk} + (a_{hk} + a_{kh})n \quad \text{and} \quad D_A(\eta) = a_{hh}n + a_{kk} ;$$

therefore, $Q_A(\eta) = D_A(\eta)$ if and only if $(n-1)a_{hh} + a_{hk} + a_{kh} = 0$. □

The spaces $K^{(\eta)}$ were defined in section 4.1 as the intersection of the nullspaces of the binomial braids $b_j^{(n+1)}$ acting on $X^{(\eta)} \subseteq X^{\otimes (n+1)}$. The following result says that, under certain hypothesis, all these nullspaces coincide.

Lemma 4.9. *If for each $p=1,\dots,n$ the braid $[p]\in kB_p$ is injective on $X^{((p-1)\epsilon_h+\epsilon_k)}\subseteq X^{\otimes p}$, then,*

$$\mathsf{Ker}\Big(b_j^{(n+1)}|_{X^{(n\epsilon_h+\epsilon_k)}}\Big)=K^{(n\epsilon_h+\epsilon_k)}\quad\forall\; j=1,\dots,n\ .$$

Proof. Recall the factorial formulas (1) and (4)

$$f^{(j)}\otimes f^{(n+1-j)}\cdot b_j^{(n+1)}=f^{(n+1)}=1^{(n)}\otimes[1]\cdot 1^{(n-1)}\otimes[2]\cdot\ldots\cdot 1\otimes[n]\cdot[n+1]\ .$$

We claim that $1\otimes[n]$ is injective on the space $X^{(n\epsilon_h+\epsilon_k)}$. In fact, this space splits as the direct sum of the one-dimensional space spanned by $\mathbf{x}_{f_0}=x_k\otimes x_h^{\otimes n}\in X^{\otimes(n+1)}$ and the space $k\{x_h\}\otimes X^{((n-1)\epsilon_h+\epsilon_k)}$, and both of these are invariant under $1\otimes[n]$. On the first, $1\otimes[n]$ acts by multiplication by the q-analog $[n]_{q^{a_{hh}}}$, which is non-zero since q is not a root of unity, while on the second it is injective by hypothesis. Similarly, all the lower factors $1^{(n+1-p)}\otimes[p]$ are injective on $X^{(n\epsilon_h+\epsilon_k)}$, for $p=1,\dots,n$. It follows that $f^{(j)}$ and $f^{(n+1-j)}$ are injective on $X^{(n\epsilon_h+\epsilon_k)}$, for $j=1,\dots,n$, and from here that

$$\mathsf{Ker}\Big(b_j^{(n+1)}|_{X^{(n\epsilon_h+\epsilon_k)}}\Big)=\mathsf{Ker}\Big([n+1]|_{X^{(n\epsilon_h+\epsilon_k)}}\Big)\ \text{ for } j=1,\dots,n\ .$$

Since $K^{(n\epsilon_h+\epsilon_k)}$ is the intersection of these kernels, the result follows. □

In the proofs below, $q^{a_{hh}}$ will be abbreviated by q_h and $q^{a_{hk}}$ by q_{hk}. We will assume, for simplicity, that $a_{hh}\neq 0$ (a hypothesis that is always satisfied by Cartan matrices). Also, $[n]_h$ will denote the q_h-analog of the natural number n and $\left[\begin{smallmatrix} n\\ j\end{smallmatrix}\right]_h$ the q_h-analog of the binomial coefficient $\binom{n}{j}$. The subindices $h\neq k$ remain fixed.

Lemma 4.10. *Assume that $a_{hh}\neq 0$ and let $a=\frac{a_{hk}+a_{kh}}{a_{hh}}$. Then*

$$\sum_{i=0}^{n}(-1)^i\begin{bmatrix} n\\ i\end{bmatrix}_{q^{a_{hh}}}q^{i(a_{kh}+a_{hk})+\binom{i}{2}a_{hh}}=0 \text{ if and only if } -a\in\{0,1,2,\dots,n-1\}\ .$$

Proof. First notice that

$$(*)\qquad q_{hk}q_{kh}=q_h^a\ .$$

Consider the polynomial

$$f(x)=\sum_{i=0}^{n}(-1)^i q_h^{\binom{i}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_h x^{n-i}\in k[x]\ .$$

By one of Cauchy's identities for q-binomials [A2, section 6.2, or GR, corollary 2], $f(x)$ factors as follows:

$$(**)\qquad f(x)=(x-1)(x-q_h)\dots(x-q_h^{n-1})\ .$$

Now,

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i\begin{bmatrix} n\\ i\end{bmatrix}_{q^{a_{hh}}}q^{i(a_{kh}+a_{hk})+\binom{i}{2}a_{hh}}&=\sum_{i=0}^{n}(-1)^i q_{hk}^i q_{kh}^i q_h^{\binom{i}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_h\\ &\overset{(*)}{=}\sum_{i=0}^{n}(-1)^i q_h^{ai}q_h^{\binom{i}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_h=\frac{f(x)}{x^n}|_{x=q_h^{-a}}\\ &\overset{(**)}{=}\frac{(q_h^{-a}-1)(q_h^{-a}-q_h)\dots(q_h^{-a}-q_h^{n-1})}{q_h^{-an}}\ ,\end{aligned}$$

which is zero if and only if $-a\in\{0,1,2,\dots,n-1\}$.

□

Proposition 4.11. *For each $i = 0, \dots, n$, let*

$$\lambda_i = (-1)^i \left[\begin{array}{c} n \\ i \end{array} \right]_{q^{a_{hh}}} q^{i a_{kh} + \binom{i}{2} a_{hh}} \in k \quad \textit{and} \quad S^n_{hk} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_{f_i} \in X^{(n\epsilon_h + \epsilon_k)} \ .$$

Assume that $a_{hh} \neq 0$. Then

$$\mathsf{Ker}\Big(b_1^{(n+1)}|_{X^{(n\epsilon_h+\epsilon_k)}}\Big) = \begin{cases} k\{S^n_{hk}\} & \textit{if } -\frac{a_{hk}+a_{kh}}{a_{hh}} \in \{0,1,2,\dots,n-1\} \\ 0 & \textit{otherwise} \ . \end{cases}$$

Proof. By definition [A2, sections 3 and 7.1], $b_1^{(n+1)} = \sum_{j=1}^{n+1} s^{(n+1)}_{\sigma_j}$, where $\sigma_j \in S_{n+1}$ is the j-cycle $\sigma_j = (1, 2, \dots, j-1, j)$. The set of inversions of this permutation is

$$\mathsf{Inv}(\sigma_j) = \{(1,j), (2,j), \dots, (j-1,j)\}$$

and

$$\sigma_j \cdot f_i = f_i \circ \sigma_j^{-1} = \begin{cases} f_{i+1} & \text{if } 0 \le i \le j-2 \\ f_0 & \text{if } i = j-1 \\ f_i & \text{if } j \le i \le n \ . \end{cases}$$

Hence, by proposition 4.1, the action of $s^{(n+1)}_{\sigma_j}$ on the basis elements $\mathbf{x}_{f_i}$ of $X^{(n\epsilon_h+\epsilon_k)}$ is

$$\mathbf{x}_{f_i} \mapsto \begin{cases} q^{(j-2)a_{hh}+a_{kh}} \mathbf{x}_{f_{i+1}} & \text{if } 0 \le i \le j-2 \\ q^{i a_{hk}} \mathbf{x}_{f_0} & \text{if } i = j-1 \\ q^{(j-1)a_{hh}} \mathbf{x}_{f_i} & \text{if } j \le i \le n \ . \end{cases}$$

It follows that, for each $i = 0, \dots, n$,

$$\begin{aligned} b_1^{(n+1)} \cdot \mathbf{x}_{f_i} &= \Big(q^{(1-1)a_{hh}} + q^{(2-1)a_{hh}} + \dots + q^{(i-1)a_{hh}}\Big)\mathbf{x}_{f_i} + q^{i a_{hk}} \mathbf{x}_{f_0} + \\ &+ \Big(q^{(i+2-2)a_{hh}+a_{kh}} + q^{(i+3-2)a_{hh}+a_{kh}} + \dots + q^{(n+1-2)a_{hh}+a_{kh}}\Big)\mathbf{x}_{f_{i+1}} \\ &= [i]_h \mathbf{x}_{f_i} + q^i_{hk} \mathbf{x}_{f_0} + q_{kh} q^i_h [n-i]_h \mathbf{x}_{f_{i+1}} \ . \end{aligned}$$

Let $\mathbf{x} = \sum_{i=0}^n \mu_i \mathbf{x}_{f_i}$ be a general element of $X^{(n\epsilon_h+\epsilon_k)}$, where $\mu_i \in k$ are arbitrary scalars. Then

$$\begin{aligned} b_1^{(n+1)} \cdot \mathbf{x} &= \sum_{i=0}^n \mu_i [i]_h \mathbf{x}_{f_i} + \sum_{i=0}^n \mu_i q^i_{hk} \mathbf{x}_{f_0} + \sum_{i=0}^n \mu_i q_{kh} q^i_h [n-i]_h \mathbf{x}_{f_{i+1}} \\ &= \sum_{i=1}^n \Big(\mu_i [i]_h + \mu_{i-1} q_{kh} q_h^{i-1} [n-i+1]_h\Big)\mathbf{x}_{f_i} + \Big(\sum_{i=0}^n \mu_i q^i_{hk}\Big)\mathbf{x}_{f_0} \ . \end{aligned}$$

It follows that $\mathbf{x} \in \mathsf{Ker}\Big(b_1^{(n+1)}|_{X^{(n\epsilon_h+\epsilon_k)}}\Big)$ if and only if

(a) $\quad 0 = \mu_i [i]_h + \mu_{i-1} q_{kh} q_h^{i-1} [n-i+1]_h$ for each $i = 1, \dots, n$ and

(b) $\quad 0 = \sum_{i=0}^n \mu_i q^i_{hk}$.

Equation (a) determines μ_i in terms of μ_0, for $i = 1, \ldots, n$:

$$\mu_1 = -\mu_0 q_{kh}[n]_h$$
$$\mu_2 = -\mu_1 q_{kh} q_h \frac{[n-1]_h}{[2]_h} = \mu_0 q_{kh}^2 q_h \frac{[n]_h[n-1]_h}{[2]_h} = \mu_0 q_{kh}^2 q_h \begin{bmatrix} n \\ 2 \end{bmatrix}_h$$
$$\mu_3 = -\mu_2 q_{kh} q_h^2 \frac{[n-2]_h}{[3]_h} = -\mu_0 q_{kh}^3 q_h^3 \frac{\begin{bmatrix} n \\ 2 \end{bmatrix}_h [n-2]_h}{[3]_h} = -\mu_0 q_{kh}^3 q_h^3 \begin{bmatrix} n \\ 3 \end{bmatrix}_h$$

and in general, for $i = 1, \ldots, n$

$$\mu_i = (-1)^i \mu_0 q_{kh}^i q_i^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_h = \mu_0 \lambda_i \ ,$$

where λ_i is as defined in the statement of the proposition. This means that $\mathbf{x} = \mu_0 S_{hk}^n$. Thus, there are two possibilities for the kernel. If the λ_i satisfy equation (b) in place of the μ_i, then the kernel is one-dimensional spanned by S_{hk}^n, if not, the kernel is zero. But when we substitute μ_i for λ_i in the right hand side of (b) we get

$$\sum_{i=0}^{n} \lambda_i q_{hk}^i = \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_{q^{a_{hh}}} q^{i(a_{kh}+a_{hk})+\binom{i}{2}a_{hh}} \ ,$$

which, by lemma 4.10, is zero if and only if $-\frac{a_{hk}+a_{kh}}{a_{hh}} \in \{0, 1, 2, \ldots, n-1\}$. This completes the proof. □

The vectors $S_{hk}^n \in X^{\otimes(n+1)}$ defined in proposition 4.11 are called the quantum Serre relations (we will show in section 5.1 that they boil down to the usual quantum Serre relations for the case of Cartan matrices). We can now derive the main result of this section, which describes the nullspaces $K^{(\eta)}$, for η as above, in terms of the quantum Serre relations.

Corollary 4.12. *Assume that $a_{hh} \neq 0$ and let S_{hk}^n be as before. Then*

$$K^{(n\epsilon_h+\epsilon_k)} = \begin{cases} k\{S_{hk}^n\} & \text{if } -\frac{a_{hk}+a_{kh}}{a_{hh}} = n-1 \\ 0 & \text{otherwise} \ . \end{cases}$$

Proof. If $-\frac{a_{hk}+a_{kh}}{a_{hh}} \neq n-1$ then $K^{(n\epsilon_h+\epsilon_k)} = 0$ by lemma 4.8. Suppose that $-\frac{a_{hk}+a_{kh}}{a_{hh}} = n-1$. Then, in particular,

$$-\frac{a_{hk}+a_{kh}}{a_{hh}} \notin \{0, 1, \ldots, p-2\} \ \forall \ p = 1, \ldots, n \ .$$

Therefore, by proposition 4.11, $b_1^{(p)}$ is injective on $X^{((p-1)\epsilon_h+\epsilon_k)}$ for $p = 1, \ldots, n$. But then lemma 4.9 applies, to conclude in particular that

$$K^{(n\epsilon_h+\epsilon_k)} = \mathsf{Ker}\left(b_1^{(n+1)}|_{X^{(n\epsilon_h+\epsilon_k)}}\right) = k\{S_{hk}^n\} \ ;$$

the last equality by proposition 4.11 again. □

Finally, we summarize the information we have obtained on the nullspaces K and F for arbitrary matrices A. For the case of Cartan matrices, more precise information will be obtained in section 5.1, where it will be shown that our definition of the quantum group boils down to that of Drinfeld and Jimbo.

Corollary 4.13. *Let S be the ideal of $T(X)$ generated by the quantum Serre relations*

$$S_{hk}^n = \sum_{i=0}^{n}(-1)^i \left[\begin{array}{c} n \\ i \end{array} \right]_{q^{a_{hh}}} q^{i a_{kh} + \binom{i}{2} a_{hh}} \mathbf{x}_{f_i} \in X^{\otimes (n+1)}$$

for those h and k for which $a_{hh} \neq 0$ and $(n-1)a_{hh} + a_{hk} + a_{kh} = 0$. Then

$$S \subseteq \bar{K} \subseteq F \ .$$

Proof. By corollary 4.12, $S_{hk}^n \in K$ for those h and k, so $S \subseteq \bar{K}$. The rest is lemma 3.3. □

5. The case of Cartan matrices

In this section we specialize the constructions and results of sections 3 and 4 to the case when A is a symmetric Cartan matrix, or, more generally, the symmetrization of a symmetrizable Cartan matrix C. In section 5.1, it will be shown that the quantum group $U_q^0(A)$ introduced in section 3.2 is the usual one defined by Drinfeld and Jimbo. In section 5.2 we prove that, when C is of finite type, the points with non-negative integer coordinates on the hypersurface (H) from section 4.3 include the vertices of the zonotope corresponding to C, and the remaining integer points on (H) lie on the walls of the root system of C. This refines the results of section 4.4 and is one of the main results of the paper. The relevant notions on Cartan matrices and root systems will be reviewed as they are needed.

5.1. **Generalized Cartan matrices. Quantum groups.** A matrix $C = [c_{hk}] \in M_r(\mathbb{Z})$ is called a *(generalized) Cartan matrix* if

$$\begin{aligned} & c_{hh} = 2 \ \forall\ h = 1, \dots, r \ , \\ & c_{hk} \leq 0 \text{ for } h \neq k \text{ and} \\ & \text{if } c_{hk} = 0 \text{ then } c_{kh} = 0 \ . \end{aligned}$$

We will also assume, without further notice, that all Cartan matrices are *indecomposable*; this means that there is no permutation $\sigma \in S_r$ such that $[c_{\sigma(h)\sigma(k)}]$ splits as a direct sum.

An arbitrary matrix $C \in M_r(\mathbb{Z})$ is called *symmetrizable* if there is an invertible diagonal matrix $D \in M_r(\mathbb{Z})$ such that DC is symmetric.

If C is a symmetrizable Cartan matrix, then the diagonal matrix D is unique up to a constant factor, and all its entries have the same sign. The canonical symmetrization of C,

$$A := DC,$$

is the one corresponding to the choice of D with minimum positive integer entries. For more details on Cartan matrices the reader is referred to [Kac, chapters 1,2 and 4].

Associated to any symmetrizable generalized Cartan matrix C there is Lie algebra $g(C)$, called a Kac-Moody Lie algebra, and a quantum group (Hopf algebra) $U_q(g(C))$, defined by means of generators and relations [Jan, 4.3]. The latter were first defined by Drinfeld [D] and Jimbo [Jim]. We shall concentrate on the subalgebra $U_q^0(g(C))$, which is defined by generators x_h for $h = 1, \dots, r$ (usually denoted by E_h instead), subject to the so-called quantum Serre relations:

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i}_{q^{d_h}} x_h^i x_k x_h^{n-i} = 0 \text{ whenever } c_{hk} = 1 - n \ .$$

Here, c_{hk} are the entries of C, d_h are the (diagonal) entries of D and the q-binomial coefficient $\binom{n}{i}_q$ is that of [Kas, VI.1.6], *not* the q-binomial $\left[\begin{array}{c} n \\ i \end{array} \right]_q$ of previous sections. These q-binomials are related by the formula [Kas, VI.1.8][1]

$$\binom{n}{i}_q = q^{-i(n-i)} \left[\begin{array}{c} n \\ i \end{array} \right]_{q^2} .$$

Let us compare this definition with our definition of $U_q^0(A) = T(X)/\bar{K}$. The entries of A, C and D are related by $a_{hk} = d_h c_{hk}$; $c_{hh} = 2$ and $a_{hk} = a_{kh}$. It follows that

$$(n-1)a_{hh} + a_{hk} + a_{kh} = 0 \Leftrightarrow c_{hk} = 1 - n ,$$

and

$$(-1)^i \left[\begin{array}{c} n \\ i \end{array} \right]_{q^{a_{hh}}} q^{i a_{kh} + \binom{i}{2} a_{hh}} = (-1)^i \binom{n}{i}_{q^{d_h}} ,$$

so that

$$S_{hk}^n = \sum_{i=0}^{n} (-1)^i \left[\begin{array}{c} n \\ i \end{array} \right]_{q^{a_{hh}}} q^{i a_{kh} + \binom{i}{2} a_{hh}} \mathbf{x}_{f_i} = \sum_{i=0}^{n} (-1)^i \binom{n}{i}_{q^{d_h}} x_h^i x_k x_h^{n-i} .$$

In other words, the quantum Serre relations that were introduced in section 4.5 for arbitrary matrices A, boil down to the usual ones when A is the symmetrization of a Cartan matrix C.

To prove that our quantum group $U_q^0(A)$ coincides with $U_q^0(g(C))$ as defined above, we must show that the ideal $\bar{K}$ is generated by the quantum Serre relations, i.e. that $S = \bar{K}$. We know from corollary 4.13 that $S \subseteq \bar{K} \subseteq F$.

To prove that equality holds, we need to recall yet another definition of the quantum group, namely that of Lusztig, which is closer to ours. In Lusztig's book, $U_q^0(g(C))$ is defined as the quotient of $T(X)$ by the radical of a certain bilinear form on $T(X)$ [L, 1.2.5]. Lusztig proves, after developing the representation theory of his quantum group, that this ideal is generated by the quantum Serre relations [L, 33.1.5].

On the other hand, Schauenburg [Sch, example 3.1 and theorem 2.9] has noted that Lusztig's ideal coincides with F, the nullspace of the factorial braids as defined in section 4 (the fact that the braids appearing in Schauenburg's paper [Sch, definition 2.6] coincide with our factorial braids was already noted in [A2, section 7.1]).

Therefore, $S = \bar{K} = F$, and the proof that our definition boils down to that of Drinfeld and Jimbo (or Lusztig's) is complete.

5.2. **Cartan matrices of finite type. Zonotopes.** A Cartan matrix is of *finite type* if it is positive-definite. Such Cartan matrices are always symmetrizable. Finite-dimensional simple Lie algebras over $\mathbb{C}$ are in one-to-one correspondence with symmetrizable Cartan matrices of finite type. Good references for Cartan matrices, simple Lie algebras and root systems are [H, chapter III] and [Sam, chapters 2 and 3.1]. We will only need the basic information outlined below.

Let C be a Cartan matrix of finite type and $A = DC$ its canonical symmetrization. Let E be the *root space* of C. By definition, E is an Euclidean space of dimension r (the size of C) with a distinguished basis $\{\alpha_1, \dots, \alpha_r\}$ (whose elements are called the *simple roots*) and inner product

$$\langle \alpha_k, \alpha_h \rangle_E = a_{hk} = d_h c_{hk} .$$

Notice that the entries of C and D are given by

$$c_{hk} = 2\frac{\langle \alpha_k, \alpha_h \rangle_E}{\langle \alpha_h, \alpha_h \rangle_E} \text{ and } d_h = \frac{1}{2}\langle \alpha_h, \alpha_h \rangle_E$$

(this shows that our notation for the Cartan integers c_{hk} is the transpose of that of [H] and [Sam]).

[1] Warning: our $\binom{n}{i}_q$ is Kassel's $\left[\begin{array}{c} n \\ i \end{array} \right]_q$ and viceversa.

Consider the linear transformation $T : E \to E$ whose matrix in the basis of simple roots is C. Since C is invertible, so is T, and a second basis $\{\omega_1, \ldots, \omega_r\}$ of E is defined by $T(\omega_h) = \alpha_h$. The vectors ω_h are called the *fundamental weights*. Equivalently, we have

$$\alpha_h = \sum_{k=1}^{r} c_{hk}\omega_k \ . \tag{9}$$

The *lowest weight* is the vector

$$\delta = \sum_{h=1}^{r} \omega_h \in E \ . \tag{10}$$

Let us reformulate the results of section 4.4 in this context.

Proposition 5.1. *Let C be a Cartan matrix of finite type and $A = DC$ its canonical symmetrization.*

1. *With respect to the canonical inner product on the root space of C, (H) is the equation of the sphere of center δ (the lowest weight) that goes through the origin.*
2. *Suppose that C is symmetric. Then, if $n > 2\|\delta\|_E^2$, there are no solutions x to equation (H) satisfying at the same time $x_1 + x_2 + \ldots + x_r = n$. In particular, $K^{(n)} = 0$ for these n.*

Proof. By assumption, the diagonal of A is $d_A = 2Du$, where $u = (1, 1, \ldots, 1) \in \mathbb{R}^r$, so equation (H) for the matrix A becomes

$$\langle DCx, x\rangle = 2\langle Du, x\rangle \ .$$

Since C is positive definite, so is A. As explained above, if we identify the root space with $\mathbb{R}^r$ in such a way that the simple roots correspond to the canonical basis of $\mathbb{R}^r$, then the canonical inner product of the root space becomes the inner product $\langle Ax, y\rangle$ of $\mathbb{R}^r$. Therefore, proposition 4.5 applies, and (H) is the equation of the sphere of center $\frac{1}{2}A^{-1}d_A$ that goes through the origin, with respect to the canonical inner product on the root space. To complete the proof of 1, we must show that $\frac{1}{2}A^{-1}d_A = \delta$. Now,

$$\frac{1}{2}A^{-1}d_A = \frac{1}{2}C^{-1}D^{-1}2Du = C^{-1}u \ .$$

On the other hand, consider the linear transformation T as above. Under the identification in question between $\mathbb{R}^r$ and E, u corresponds to $\sum_{h=1}^{r} \alpha_h$, and C to T; therefore, $C^{-1}u$ corresponds to

$$\sum_{h=1}^{r} T^{-1}(\alpha_h) = \sum_{h=1}^{r} \omega_h \overset{(10)}{=} \delta \ .$$

Now assume that C is symmetric, i.e. $A = C$. In this case $d_A = 2u$ and the right hand side of the inequality in corollary 4.6 becomes simply $2\langle C^{-1}u, u\rangle$. But

$$\langle C^{-1}u, u\rangle = \langle C^{-1}u, C^{-1}u\rangle_E = \|C^{-1}u\|_E^2 = \|\delta\|_E^2 \ ,$$

so 2 is just a reformulation of corollary 4.6.

□

Example 5.1. For the root system A_r, corresponding to the Lie algebra $sl_{r+1}(\mathbb{C})$, the Cartan matrix is

$$C = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & 0 & -1 & 2 \end{bmatrix}$$

(a square matrix of size r).

From the proof above we know that $\delta = C^{-1}u$. Let

$$x_k = \frac{1}{2}k(r+1-k) \in \mathbb{R} \text{ for } k = 1, \dots, r \text{ and } x = (x_1, \dots, x_r) \in \mathbb{R}^r \ .$$

One easily computes $Cx = u$, from where $x = \delta$. Therefore,

$$||\delta||_E^2 = \langle C^{-1}u, u\rangle = \langle x, u\rangle = \sum_{k=1}^{r} x_k = \frac{1}{12}r(r+1)(r+2) \ .$$

Thus if $n > \frac{1}{6}r(r+1)(r+2)$ then $K^{(n)} = 0$. For instance if $r = 2$ then $K^{(n)} = 0$ for $n > 4$. The description of $K^{(n)}$ for $n \leq 4$ will be carried out in example 5.2 below.

Let W be the Weyl group of C, that is, the group of isometries of E generated by the reflections across the hyperplanes perpendicular to the simple roots $\alpha_h \in E$. Since C is of finite type, W is finite. The finite set

$$R = \{w(\alpha_i) \in E \ / \ w \in W \text{ and } i = 1, \dots, r\}$$

is called the root system of C, and its elements are called roots. For $\alpha \in R$, let $H_\alpha \subseteq E$ be the hyperplane perpendicular to α. These hyperplanes are called the walls of the root system. The connected components of

$$E \setminus \bigcup_{\alpha \in R} H_\alpha$$

are called the Weyl chambers. The closure $\overline{\mathcal{C}}$ of a Weyl chamber $\mathcal{C}$ is called a closed Weyl chamber.

As before, we identify $(E, \langle\ ,\ \rangle_E)$ with $(\mathbb{R}^r, \langle\ ,\ \rangle_A)$ by sending $\{\alpha_1, \dots, \alpha_r\}$ to the canonical basis $\{e_1, \dots, e_r\}$ of $\mathbb{R}^r$. The set of integer linear combinations of the simple roots gets then identified with $\mathbb{Z}^r \subseteq \mathbb{R}^r$; this is called the root lattice of C. The set of non-negative integer combinations of the simple roots is called the positive cone; it is identified with $\mathbb{N}^r$. The set of integer linear combinations of the fundamental weights is called the weight lattice. For instance the lowest weight δ belongs to the weight lattice (equation (10)). Equation (9) shows that the root lattice is included in the weight lattice.

A root $\alpha \in R$ is called positive if it lies in the positive cone $\mathbb{N}^r$. Let R^+ denote the set of positive roots. The zonotope of C is defined as the *Minkowski sum* of the positive roots:

$$\{\sum_{\alpha \in R+} \lambda_\alpha \alpha \in E \ /0 \leq \lambda_\alpha \leq 1\} \subseteq E \ .$$

For instance, the zonotope of the root system A_r is called the permutahedron [Z, examples 0.10 and 7.15]. Figures 4 and 5 show the zonotopes of A_2 and B_2 (below in this section).

The proof below will rely on the following facts [H, chapter III]:

(a) The set

$$\mathcal{C}_F = \{\lambda \in E \ / \ \langle \alpha_h, \lambda \rangle_E < 0 \ \forall \ h = 1, \dots, r\}$$

is a Weyl chamber (the opposite of the *fundamental* Weyl chamber).

(b) The chamber $\mathcal{C}_F$ coincides with the negative cone generated by the fundamental weights, that is,

$$\mathcal{C}_F = \{\sum_{h=1}^{r} a_h \omega_h \in E \ / \ a_h < 0 \ \forall \ h = 1, \dots, r\} \ .$$

(c) If $\mathcal{C}$ is a Weyl chamber and $w \in W$, then $w(\mathcal{C})$ is a Weyl chamber and $w(\overline{\mathcal{C}}) = \overline{w(\mathcal{C})}$.

(d) Given two Weyl chambers $\mathcal{C}$ and $\mathcal{C}'$, there is a unique $w \in W$ such that $w(\mathcal{C}) = \mathcal{C}'$.

(e) The closed Weyl chambers cover E.

(f) The root lattice $\mathbb{Z}^r$ is invariant under the action of W.

(g) Even though δ may not lie in the root lattice $\mathbb{Z}^r$, $\delta - w(\delta)$ lies in the positive cone $\mathbb{N}^r$ $\forall$ $w \in W$. (In fact, $\delta - w(\delta)$ is equal to the sum of those positive roots that become negative under w.)
(h) The set $\{\delta - w(\delta) \ / \ w \in W\}$ is precisely the set of vertices of the zonotope of C.

The following result describes the set of integer solutions of equation (H) in terms of the zonotope of C in some detail.The idea of the proof is to translate the hyperplanes and Weyl chambers from the origin to δ, and observe that there is only one solution in every translated chamber. According to (d) above, it suffices to look for solutions in the (translated) chamber $\delta + \mathcal{C}_F$. We will show that the only such integral solution is $\lambda = 0$ (part 1 of lemma 5.2). This will allow us to conclude that the set of integral solutions that do not lie on the walls (translated by δ) is precisely the set of vertices of the zonotope (theorem 5.3), which is the main result of this section. We will also show that $\lambda = 0$ is the only solution lying at the same time in the closed chamber $\delta + \overline{\mathcal{C}_F}$ and in the positive cone $\mathbb{N}^r$ (part 2 of lemma 5.2). Figure 3 reflects these assertions for the root system A_2.

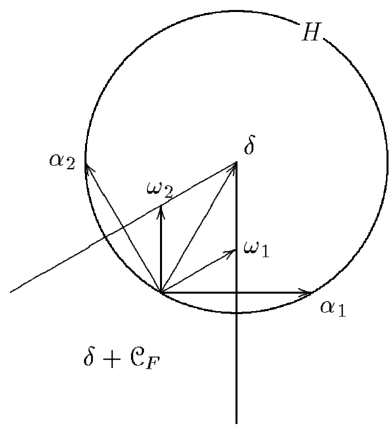


FIGURE 3. The translated fundamental chamber for A_2.

Lemma 5.2. *Let $\lambda \in \mathbb{Z}^r$ be a solution of equation (H) lying on the root lattice.*

1. *If $\lambda \in \delta + \mathcal{C}_F$ then $\lambda = 0$.*
2. *If $\lambda \in (\delta + \overline{\mathcal{C}_F}) \cap \mathbb{N}^r$ then $\lambda = 0$.*

Proof. 1. According to (b) we can write

$$\lambda - \delta = \sum_{h=1}^{r} a_h \omega_h \text{ with } a_h < 0 \ \forall \ h \ .$$

Recall that $\delta = \sum_{h=1}^{r} \omega_h$. Since the root lattice in included in the weight lattice, λ also belongs to the weight lattice. It follows that

$$a_h \leq -1 \ \forall \ h \ .$$

Since λ belongs to the sphere (H), we have that $||\lambda - \delta||_E = ||\delta||_E$. Hence,

$$\begin{aligned} ||\lambda||_E^2 &= ||\lambda - \delta + \delta||_E^2 = ||\lambda - \delta||_E^2 + 2\langle \lambda - \delta, \delta \rangle_E + ||\delta||_E^2 = 2||\delta||_E^2 + 2\langle \lambda - \delta, \delta \rangle_E \\ &= 2 \sum_{h,k=1}^{r} \langle \omega_h, \omega_k \rangle_E + 2 \sum_{h,k=1}^{r} a_h \langle \omega_h, \omega_k \rangle_E \leq 0 \ , \end{aligned}$$

and thus $\lambda = 0$ as needed.

2. Let

$$\lambda = \sum_{k=1}^{r} \lambda_k \alpha_k \text{ with } \lambda_k \in \mathbb{N}$$

be a solution in $(\delta + \overline{\mathcal{C}_F}) \cap \mathbb{N}^r$. Then $\lambda - \delta \in \overline{\mathcal{C}_F}$, so by (a)

$$0 \geq \langle \lambda - \delta, \alpha_h \rangle_E = \langle \lambda, \alpha_h \rangle_E - \langle \delta, \alpha_h \rangle_E = \sum_{k=1}^{r} \lambda_k d_h c_{hk} - d_h$$

from which it follows (since $d_h > 0 \ \forall \ h$) that

$$\text{(A)} \qquad \sum_{k=1}^{r} \lambda_k c_{hk} \leq 1 \ \forall \ h = 1, \ldots, r \ .$$

On the other hand, since λ lies on the sphere (H), we have

$$\text{(B)} \qquad ||\lambda - \delta||_E = ||\delta||_E \Rightarrow \langle \lambda, \delta \rangle_E = \frac{1}{2} \langle \lambda, \lambda \rangle_E \ .$$

Let us compute each side of (B) separately:

$$\langle \lambda, \delta \rangle_E = \langle \sum_{h=1}^{r} \lambda_h e_h, C^{-1} u \rangle_A = \sum_{h=1}^{r} \langle \lambda_h e_h, Du \rangle = \sum_{h=1}^{r} \lambda_h d_h$$

while

$$\langle \lambda, \lambda \rangle_E = \sum_{h,k=1}^{r} \lambda_k \lambda_h \langle \alpha_k, \alpha_h \rangle_E = \sum_{h,k=1}^{r} \lambda_k \lambda_h d_h c_{hk} = \sum_{h=1}^{r} \lambda_h d_h \Big(\sum_{k=1}^{r} \lambda_k c_{hk} \Big) \ .$$

Combining these with (A) and the fact that $\lambda_h \geq 0 \ \forall \ h$ we obtain

$$\langle \lambda, \lambda \rangle_E = \sum_{h=1}^{r} \lambda_h d_h \Big(\sum_{k=1}^{r} \lambda_k c_{hk} \Big) \leq \sum_{h=1}^{r} \lambda_h d_h = \langle \lambda, \delta \rangle_E \ .$$

Now using (B) we obtain

$$\langle \lambda, \delta \rangle_E = \frac{1}{2} \langle \lambda, \lambda \rangle_E \leq \frac{1}{2} \langle \lambda, \delta \rangle_E \ ,$$

which implies

$$\langle \lambda, \delta \rangle_E = \langle \lambda, \lambda \rangle_E = 0 \ \text{ i.e. } \ \lambda = 0 \ .$$

This completes the proof of the lemma.

□

Theorem 5.3. *Let C be a Cartan matrix of finite type. The set of solutions of equation (H) that lie in the root lattice $\mathbb{Z}^r$ but not on the walls of the root system of C (translated by δ) is precisely the set of vertices of the zonotope of C, or, equivalently, the set*

$$\{\delta - w(\delta) \ / \ w \in W\},$$

one different solution for each $w \in W$. In particular, these solutions lie in the positive cone $\mathbb{N}^r$.

Proof. Let $T_\delta : E \to E$ be the translation of vector δ,

$$T_\delta(\lambda) = \lambda + \delta \text{ for } \lambda \in E\ .$$

For each Weyl chamber $\mathcal{C}$, let $w_\mathcal{C} \in W$ be the only element of the Weyl group such that $w_\mathcal{C}(\mathcal{C}_F) = \mathcal{C}$, and let

$$\lambda_\mathcal{C} = \delta - w_\mathcal{C}(\delta) \in E\ .$$

We know from (g) that

$$\text{(A)} \qquad \lambda_\mathcal{C} \in \mathbb{N}^r\ .$$

We will show that:

$$\text{(B)} \qquad \lambda_\mathcal{C} \in T_\delta(\mathcal{C})$$

and

$$\text{(C)} \qquad \lambda_\mathcal{C} \text{ is the only solution in } T_\delta(\mathcal{C}) \cap \mathbb{Z}^r\ .$$

Since the closed Weyl chambers cover E and the open Weyl chambers are disjoint, this will prove that the set of solutions is as described.

Consider the action of W on E obtained by conjugating the canonical action by T_δ:

$$(*) \qquad w \cdot \lambda := \delta + w(\lambda - \delta) \ \text{ for } w \in W \text{ and } \lambda \in E.$$

Notice that $w_{\mathcal{C}_F} = \mathsf{id}_E$, so $\lambda_{\mathcal{C}_F} = 0 \in E$. Hence

$$\lambda_\mathcal{C} = \delta - w_\mathcal{C}(\delta) = \delta + w_\mathcal{C}(\lambda_{\mathcal{C}_F} - \delta) = w_\mathcal{C} \cdot \lambda_{\mathcal{C}_F}\ .$$

On the other hand, since $\mathcal{C} = w_\mathcal{C}(\mathcal{C}_F)$,

$$T_\delta(\mathcal{C}) = T_\delta \circ w_\mathcal{C}(\mathcal{C}_F) = w_\mathcal{C} \cdot T_\delta(\mathcal{C}_F)\ .$$

Therefore, assertion (B) is equivalent to $\lambda_{\mathcal{C}_F} \in T_\delta(\mathcal{C}_F)$. Since $\lambda_{\mathcal{C}_F} = 0$, this is in turn equivalent to $-\delta \in \mathcal{C}_F$, which is obvious from (b) and (10).

The canonical action of W is by isometries of E that fix the origin, while T_δ is an isometry that sends the origin to δ. Therefore, action $(*)$ is by isometries that fix δ. Hence this action preserves the sphere (H) of center δ. Also, by (f) and (g), $\mathbb{Z}^r$ is invariant under this action. Thus, the set of solutions to (H) in $\mathbb{Z}^r$ is invariant under $(*)$. Therefore, assertion (C) is equivalent to

$\lambda_{\mathcal{C}_F} = 0$ is the only solution in $T_\delta(\mathcal{C}_F) \cap \mathbb{Z}^r$.

Obviously $\lambda_{\mathcal{C}_F} = 0$ is a solution of (H). It is the only such precisely by part 1 of lemma 5.2. Thus (C) holds and the proof is complete. □

Remark 5.1. Any closed Weyl chamber $\mathcal{C}$ is a fundamental domain for the canonical action of W on E (this is the content of assertions (c), (d) and (e) above). This implies, as in the proof of theorem 5.3, that the set $T_\delta(\overline{\mathcal{C}}) \cap \mathbb{Z}^r \cap H$ is a fundamental domain for the action $(*)$ of W on the set of solutions in $\mathbb{Z}^r$ to (H). Part 1 of lemma 5.2 says that

$$T_\delta(\mathcal{C}_F) \cap \mathbb{Z}^r \cap H = \{0\}\ ,$$

i.e. $\lambda = 0$ is the only solution in the interior of this fundamental domain. Its W-orbit is of course the set of vertices of the zonotope, as in theorem 5.3. For root systems of rank 2, it is possible to prove by a geometric argument that

$$T_\delta(\overline{\mathcal{C}_F}) \cap \mathbb{Z}^r \cap H = \{0\}\ ,$$

i.e. the origin is the only solution in this fundamental domain, and hence the solutions in $\mathbb{Z}^r$ to (H) are *precisely* the vertices of the zonotope. In other words, for these root systems, there are no solutions lying on the translated walls. This will be verified directly in examples 5.2.

In general, however, there are other solutions in this fundamental domain (necessarily on the translated walls), and hence other solutions in $\mathbb{Z}^r$ besides the vertices of the zonotope. This was pointed out to the author by Dan Barbasch, who found the solution $\alpha_1+\alpha_2+\alpha_3-\alpha_5$ for the root system A_5. This solution lies in the root lattice and in one of the walls of $T_\delta(\overline{\mathcal{C}_F})$. In terms of coordinates, the equation of (H) is in this case

$$x_1^2+\ldots+x_5^2-x_1-\ldots-x_5-x_1x_2-x_2x_3-\ldots-x_4x_5=0 \ ,$$

and the above solution corresponds to $(1,1,1,0,-1) =$. Notice that the solution does not lie in the positive cone $\mathbb{N}^r$, as predicted by part 2 of lemma 5.2. However, some points in its W-orbit will lie in $\mathbb{N}^r$ (since one can find a chamber $\mathcal{C}$ such that $T_\delta(\overline{\mathcal{C}})\subseteq\mathbb{N}^r$ and then a w such that $w(\mathcal{C}_F)=\mathcal{C}$). This shows that even if we only look for solutions to (H) lying in $\mathbb{N}^r$, there are such solutions which are not vertices of the zonotope.

A simple description of these "extra" solutions appears difficult. Using Maple, we have computed the number of solutions in the fundamental domain $T_\delta(\overline{\mathcal{C}_F})\cap\mathbb{Z}^r\cap H$ for the root system A_r for $r\leq 11$. These numbers are displayed in table 5.1. In each case, $\lambda=0$ is the only solution that lies in $\mathbb{N}^r$ or that does not lie in any of the translated walls, as guaranteed by lemma 5.2. The first case when a non-zero solution appears is $r=5$.

r	1	2	3	4	5	6	7	8	9	10	11
$\#\left(T_\delta(\overline{\mathcal{C}_F})\cap\mathbb{Z}^r\cap H\right)$	1	1	1	1	3	9	27	80	255	847	2774

TABLE 1. The number of solutions in the fundamental domain for A_r.

Examples 5.2.

1. For the root system A_2 we have

$$C=A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \text{ and } \delta=C^{-1}u=(1,1)\ .$$

Equation (H) is simply

$$x^2+y^2=xy+x+y\ ;$$

from example 5.1 we no that there are no solutions with $x+y>4$. In fact, it is easy to see directly that the only solutions in $\mathbb{Z}^2$ are

$$(0,0),(1,0),(0,1),(1,2),(2,1) \text{ and } (2,2)\ .$$

These are the vertices of the zonotope of A_2, so in this case there are no other solutions than those predicted by theorem 5.3. Therefore,

$$\begin{aligned} K^{(n)} &=0\ \forall\ n\geq 5 \text{ or } n\leq 2;\\ K^{(3)} &=K^{(2,1)}\oplus K^{(1,2)} \text{ and}\\ K^{(4)} &=K^{(2,2)}\ . \end{aligned}$$

The first two non-trivial components are spanned by the quantum Serre relations, as described by corollary 4.12: notice that $-\frac{a_{12}+a_{21}}{a_{11}}=1=-\frac{a_{21}+a_{12}}{a_{22}}$; therefore,

$$K^{(2,1)}=k\{S_{12}^2\} \text{ where } S_{12}^2=x_{112}-(q+q^{-1})x_{121}+x_{211}$$

and

$$K^{(1,2)}=k\{S_{21}^2\} \text{ where } S_{21}^2=x_{112}-(q+q^{-1})x_{212}+x_{211}\ .$$

The remaining non-trivial component can be computed by hand; it turns out to be one-dimensional as well:

$$K^{(2,2)} = k\{R_{22}\} \text{ where } R_{22} = -x_{1122} + (q+q^{-1})x_{1212} - (q+q^{-1})x_{2121} + x_{2211} \ .$$

From section 5.1 we know that the relation at $(2,2)$ must be generated by the quantum Serre relations. In fact, in this case either Serre relation suffices to generate R_{22}, since we have that

$$R_{22} = -x_1 \otimes S_{12}^2 + S_{12}^2 \otimes x_1 = -S_{21}^2 \otimes x_2 + x_2 \otimes S_{21}^2 \ .$$

The zonotope is shown in figure 4.

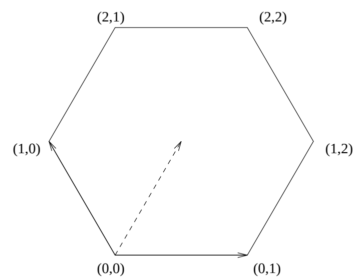


FIGURE 4. The zonotope of A_2.

2. For the root system B_2 we have

$$C = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{ so } A = DC = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \text{ and } \delta = C^{-1}u = (3/2, 2);$$

equation (H) is

$$4x^2 + 2y^2 - 4xy = 4x + 2y,$$

the only solutions in $\mathbb{Z}^2$ are

$$(0,0), (1,0), (0,1), (2,1), (1,3), (3,3), (2,4) \text{ and } (3,4),$$

and the zonotope is shown in figure 5.

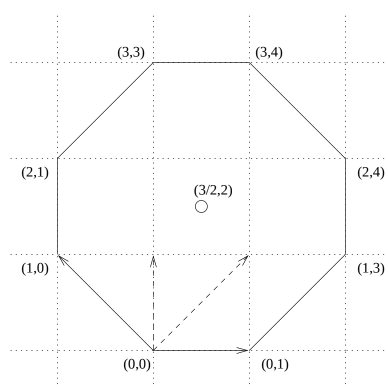


FIGURE 5. The zonotope of B_2.

The quantum Serre relations occur at the vertices $(2,1)$ and $(1,3)$.

3. For the root system G_2 we have

$$C = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \text{ so } A = DC = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix} \text{ and } \delta = C^{-1}u = (3,5);$$

equation (H) is

$$6x^2 + 2y^2 - 6xy = 6x + 2y,$$

the only solutions in $\mathbb{Z}^2$ are

$$(0,0), (1,0), (0,1), (2,1), (1,4), (4,4), (2,6), (5,6), (4,9), (6,9), (5,10) \text{ and } (6,10),$$

and the zonotope is shown in figure 6.

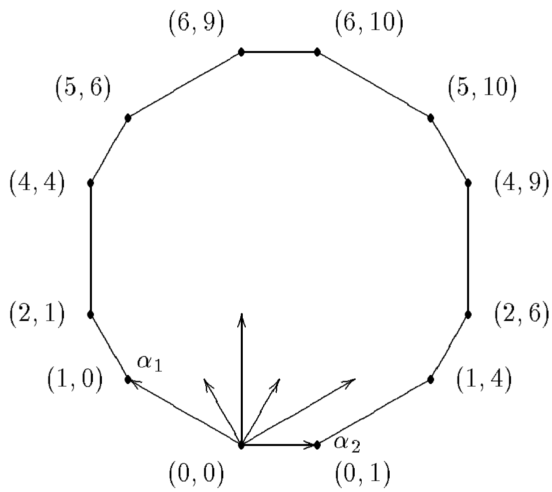


FIGURE 6. The zonotope of G_2.

6. FURTHER QUESTIONS AND RESULTS

Consider the action of the braid groups B_n on the tensor powers $X^{\otimes n}$ of a vector space X, defined by means of a fixed matrix $A \in M_r(\mathbb{Z})$ as in sections 3.1 and 3.2. In this paper the attention has been concentrated on the study of the corresponding nullspaces of the binomial braids, motivated by its relevance to quantum groups. In section 4.5, some components $K^{(\eta)}$ of these nullspaces were explicitly described. It may be possible to obtain similar descriptions of the other non-trivial components (and thus understand why the quantum Serre relations suffice to generate the ideal of relations for the case of Cartan matrices, without resorting to Lusztig's result as we did in section 5.1). A related question that arises naturally is whether $\bar{K} = F$ always (we know it holds for the case of Cartan matrices from section 5.1).

There are other interesting questions and results about this action. It is possible to obtain explicit expressions and factorizations for the determinants of some of the braid analogs discussed in this paper. For instance, one can show that for any $\sigma \in S_n$ and $\eta \in \mathcal{C}(n,r)$,

$$\det\left(s_\sigma^{(n)} : X^{(\eta)} \to X^{(\eta)}\right) = (-1)^{\binom{n}{\eta} - N(\sigma,\eta)} \cdot q^{\mathsf{inv}(\sigma) \frac{\binom{n}{\eta}}{n(n-1)}[Q_A(\eta) - D_A(\eta)]} , \tag{11}$$

where $\binom{n}{\eta} = \#\mathcal{S}(\eta)$ (the multinomial coefficient), $N(\sigma, \eta)$ is the number of orbits for the action of σ on $\mathcal{S}(\eta)$ and Q_A and D_A are as in section 4.3. It follows from here that

$$\det\Big(s_\sigma^{(n)} : X^{\otimes n} \to X^{\otimes n}\Big) = (-1)^{r^n - N(\sigma)} \cdot q^{\mathsf{inv}(\sigma) r^{n-2} \sum_{h,k} a_{hk}} , \tag{12}$$

where $N(\sigma)$ is the number of orbits for the action of σ on $\mathcal{F}(n, r)$. Explicit expressions for $N(\sigma, \eta)$ and $N(\sigma)$ can be obtained through Polya's theory; for instance for the n-cycle $\sigma = (1, 2, \dots, n)$,

$$N(\sigma, \eta) = \frac{1}{n} \sum_{d|D} \phi(d) \binom{n/d}{\eta/d} \text{ and } N(\sigma) = \frac{1}{n} \sum_{d|n} \phi(d) r^{n/d} ,$$

where ϕ is Euler's function and $D = \gcd(\eta)$ is the greatest common divisor of the components of η.

We have also obtained factorizations for the determinant of $b_1^{(n)} = [n]$ acting on some particular components $X^{(\eta)}$:

$$\det\Big([n+1] : X^{(n\epsilon_h + \epsilon_k)} \to X^{(n\epsilon_h + \epsilon_k)}\Big) = [n]!_{q^{a_{hh}}} \prod_{i=0}^{n-1} (1 - q^{a_{hk} + a_{kh} + i a_{hh}}) , \tag{13}$$

$$\begin{aligned} &\det\Big([n+2] : X^{(n\epsilon_h + 2\epsilon_k)} \to X^{(n\epsilon_h + 2\epsilon_k)}\Big) \\ &= \prod_{i=1}^{n} [i]!_{q^{a_{hh}}} (1 - q^{a_{hk} + a_{kh} + (n-i) a_{hh}})^i \prod_{i=0}^{n} \Big(1 + (-1)^i q^{a_{kk} + (n-i)(a_{hk} + a_{kh}) + \binom{n-i}{2} a_{hh}}\Big) \end{aligned} \tag{14}$$

and

$$\det\Big([r] : X^{(\epsilon_1 + \epsilon_2 + \dots + \epsilon_r)} \to X^{(\epsilon_1 + \epsilon_2 + \dots + \epsilon_r)}\Big) = \prod_{i=0}^{r-2} \prod_{I \in \mathcal{S}_i(r)} \Big(1 - q^{[Q_A(\eta_I) - D_A(\eta_I)]}\Big)^{(r-2-i)! i!} , \tag{15}$$

where $\mathcal{S}_i(r)$ denotes the set of all subsets I of $\{1, 2, \dots, r\}$ of cardinality i and $\eta_I \in \mathbb{N}^r$ has coordinates $\eta_{I,h} = \begin{cases} 1 & \text{if } h \notin I \\ 0 & \text{if } h \in I \end{cases}$. Notice that, by lemma 4.2, $Q_A(\eta_I) - D_A(\eta_I) = \sum_{\substack{h \neq k \\ h,k \notin I}} a_{hk}$.

From equation (15) one can deduce (by induction, using (1)) that

$$\det\Big(f^{(r)} : X^{(\epsilon_1 + \epsilon_2 + \dots + \epsilon_r)} \to X^{(\epsilon_1 + \epsilon_2 + \dots + \epsilon_r)}\Big) = \prod_{i=0}^{r-2} \prod_{I \in \mathcal{S}_i(r)} \Big(1 - q^{[Q_A(\eta_I) - D_A(\eta_I)]}\Big)^{(r-2-i)!(i+1)!} , \tag{16}$$

notice the slight difference between (15) and (16).

Equation (15) generalizes a formula of Hanlon and Stanley [HS, lemma 3.4]; their formula is the case $a_{hk} \equiv 1$ of ours. Other formulas in their paper can also be generalized to the context of braids.

The particular case when A is symmetric of equation (16) becomes a special case of a formula of Varchenko [V]. This author associates a certain matrix to any weighted hyperplane arrangement (that is, a hyperplane arrangement where a number has been chosen for every hyperplane) and obtains a formula for its determinant. In the special case of the braid arrangement $\mathcal{A}_{r-1} = \{H_{hk} \ / \ 1 \leq h < k \leq r\}$, where $H_{hk} = \{(x_1, \dots, x_r) \in \mathbb{R}^r \ / \ x_h = x_k\}$, weighted by numbers $q^{a_{hk}}$, Varchenko's matrix turns out to be the matrix of $f^{(r)} : X^{(\epsilon_1 + \epsilon_2 + \dots + \epsilon_r)} \to X^{(\epsilon_1 + \epsilon_2 + \dots + \epsilon_r)}$ with respect to the canonical basis of section 4.1, and his formula turns out to be precisely (16).

It would be interesting to obtain factorization formulas of this type on an arbitrary component $X^{(\eta)}$.

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